

# Structural Vector Autoregressions and Asymptotic Theory for Time Series Econometrics

## Part I. Structural Vector Autoregressions

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- 1 Session 1. SVARs
  - The A-Model
  - The B-Model
  - The AB-Model
- 2 Session 2. SVARs with Long-Run Restrictions
  - Blanchard-Quah Model
  - Structural VECM
- 3 Session 3. Further Issues
  - Factor Augmented SVARs
  - Sign Restrictions
  - Identification via Statistical Data Properties

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# Reduced form VAR

## VAR( $p$ )

$$y_t = A_1 y_{t-1} + \dots + A_p y_{t-p} + u_t$$

## MA representation

$$y_t = u_t + \Phi_1 u_{t-1} + \Phi_2 u_{t-2} + \dots$$

$$\Phi_s = \sum_{j=1}^s \Phi_{s-j} A_j, \quad s = 1, 2, \dots \text{ with } \Phi_0 = I_K$$

## Alternative MA representation

$$y_t = \Theta_0 \varepsilon_t + \Theta_1 \varepsilon_{t-1} + \Theta_2 \varepsilon_{t-2} + \dots$$

$$\Theta_j = \Phi_j B \quad (j = 0, 1, 2, \dots)$$

$$\varepsilon_t = B^{-1} u_t \sim (0, \Sigma_\varepsilon = B^{-1} \Sigma_u B'^{-1})$$

# A-Model I

## Structural form VAR( $p$ )

$$Ay_t = A_1^* y_{t-1} + \dots + A_p^* y_{t-p} + \varepsilon_t$$

- $A_j^* := AA_j$  ( $j = 1, \dots, p$ )
- $\varepsilon_t := Au_t \sim (0, \Sigma_\varepsilon = A\Sigma_u A')$
- $\Sigma_\varepsilon$  diagonal matrix

## Identification problem

$\Sigma_\varepsilon = A\Sigma_u A'$  represents  $K(K-1)/2$  independent equations,  
 $K(K+1)/2$  more needed to solve uniquely for all  $K^2$   
 elements of  $A$

## Typical identifying assumption

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ a_{21} & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ a_{K1} & a_{K2} & \dots & 1 \end{bmatrix}$$

# A-Model II

## More generally

- $A^{-1}\Sigma_{\varepsilon}A'^{-1} = \Sigma_u$
- $C_A \text{vec}(A) = c_A$
- $C_{\sigma} \text{vech}(\Sigma_{\varepsilon}) = 0$

## Local uniqueness condition

$$\text{rk} \begin{bmatrix} -2\mathbf{D}_K^+(\Sigma_u \otimes A^{-1}) & \mathbf{D}_K^+(A^{-1} \otimes A^{-1})\mathbf{D}_K \\ C_A & 0 \\ 0 & C_{\sigma} \end{bmatrix}$$

$$= K^2 + \frac{1}{2}K(K+1)$$

$\mathbf{D}_K$  is a  $(K^2 \times \frac{1}{2}K(K+1))$  duplication matrix,  
 $\mathbf{D}_K^+ := (\mathbf{D}'_K \mathbf{D}_K)^{-1} \mathbf{D}'_K$

# B-Model

## Structural form VAR( $p$ )

$$y_t = A_1 y_{t-1} + \dots + A_p y_{t-p} + B \varepsilon_t$$

- $\varepsilon_t = B^{-1} u_t \sim (0, \Sigma_\varepsilon = B^{-1} \Sigma_u B'^{-1})$

- $\Sigma_\varepsilon = I_K$

## Identification problem

$\Sigma_u = BB'$  represents  $K(K+1)/2$  independent equations,  
 $K(K-1)/2$  more needed to solve uniquely for all  $K^2$   
elements of B

## Typical identifying assumption

B is triangular

# AB-Model

## Structural form VAR( $p$ )

$$Ay_t = A_1^*y_{t-1} + \cdots + A_p^*y_{t-p} + B\varepsilon_t$$

## Residuals

$$Au_t = B\varepsilon_t, \quad \varepsilon_t \sim (0, I_K)$$

## Identification problem

$\Sigma_u = A^{-1}BB'A^{-1}$  represents  $K(K+1)/2$  independent equations,

$2K^2 - K(K+1)/2$  more needed to solve uniquely for all  $2K^2$  elements of A and B

## Typical identifying assumptions

- $\text{vec}(A) = R_A\gamma_A + r_A$
- $\text{vec}(B) = R_B\gamma_B + r_B$

# Example

$$u_t^q = -a_{12}u_t^i + b_{11}\varepsilon_t^{IS} \quad (\text{IS curve}),$$

$$u_t^i = -a_{21}u_t^q - a_{23}u_t^m + b_{22}\varepsilon_t^{LM} \quad (\text{inverse LM curve}),$$

$$u_t^m = b_{33}\varepsilon_t^m \quad (\text{money supply rule}).$$

$$\begin{bmatrix} 1 & a_{12} & 0 \\ a_{21} & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix} u_t = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} \varepsilon_t$$

# ML Estimation of AB-Models

## log likelihood function

$$\begin{aligned}
 \log l(A, A, B) &= -\frac{KT}{2} \log 2\pi - \frac{T}{2} \log |A^{-1}BB'A'^{-1}| \\
 &\quad - \frac{1}{2} \text{tr}\{(Y - AX)'[A^{-1}BB'A'^{-1}]^{-1}(Y - AX)\} \\
 &= \text{constant} + \frac{T}{2} \log |A|^2 - \frac{T}{2} \log |B|^2 \\
 &\quad - \frac{1}{2} \text{tr}\{A'B'^{-1}B^{-1}A(Y - AX)(Y - AX)'\}
 \end{aligned}$$

## Concentrated log likelihood function

$$\begin{aligned}
 \log l_c(A, B) &= \text{constant} + \frac{T}{2} \log |A|^2 - \frac{T}{2} \log |B|^2 - \frac{T}{2} \text{tr}(A'B'^{-1}B^{-1}A\tilde{\Sigma}_u)
 \end{aligned}$$

$$\text{with } \tilde{\Sigma}_u = T^{-1}(Y - \hat{A}X)(Y - \hat{A}X)'$$

# Testing Over-identifying Restrictions

## LR test statistic

$$\lambda_{LR} = T(\log |\tilde{\Sigma}'_u| - \log |\tilde{\Sigma}_u|)$$

- $\tilde{\Sigma}_u = T^{-1}(Y - \hat{A}X)(Y - \hat{A}X)'$
- $\tilde{\Sigma}'_u = \tilde{A}^{-1}\tilde{B}\tilde{B}'\tilde{A}'^{-1}$

## Asymptotic distribution under $H_0$

$$\lambda_{LR} \xrightarrow{d} \chi^2(df)$$

$df$  = no of over-identifying restrictions

= no of independent restrictions imposed on A and B minus  
 $2K^2 - \frac{1}{2}K(K+1)$

# Estimating Impulse Responses

## Impulse response coefficients

$$\Theta_j = \Phi_j A^{-1} B, \quad j = 0, 1, 2, \dots, \text{ where}$$
$$\Phi_j = \Phi_j(A_1, \dots, A_p)$$

## Estimated impulse response coefficients

$$\hat{\Theta}_j = \hat{\Phi}_j \hat{A}^{-1} \hat{B}, \quad j = 0, 1, 2, \dots, \text{ where}$$
$$\hat{\Phi}_j = \Phi_j(\hat{A}_1, \dots, \hat{A}_p)$$

## Properties

$$\sqrt{T} \text{vec}(\hat{\Theta}_j - \Theta_j) \xrightarrow{d} \mathcal{N}(0, \Sigma_{\hat{\Theta}_j})$$

can be used to set up confidence intervals etc.

# Bootstrap Confidence Intervals for Impulse Responses I

## Quantity of interest

$$\theta = \theta(A_1, \dots, A_p, A, B)$$

## Residual based bootstrap estimates

- Estimate SVAR model to get estimate  $\hat{\theta}$ .
- Get bootstrap estimates  $\hat{\theta}_1^*, \dots, \hat{\theta}_N^*$  for some large  $N$ .

## Standard percentile interval

$$CI_S = \left[ s_{\gamma/2}^*, s_{(1-\gamma/2)}^* \right],$$

where  $s_{\gamma/2}^*$  and  $s_{(1-\gamma/2)}^*$  are the  $\gamma/2$ - and

$(1 - \gamma/2)$ -quantiles, respectively, of  $\hat{\theta}_1^*, \dots, \hat{\theta}_N^*$ .

# Bootstrap Confidence Intervals for Impulse Responses II

## Hall's percentile interval

$$CI_H = \left[ \hat{\theta} - t_{(1-\gamma/2)}^*, \hat{\theta} - t_{\gamma/2}^* \right],$$

where  $t_{\gamma/2}^*$  and  $t_{(1-\gamma/2)}^*$  are the  $\gamma/2$ - and

$(1 - \gamma/2)$ -quantiles, respectively, of  $(\hat{\theta}_n^* - \hat{\theta})$  and the interval is obtained by using that  $\sqrt{T}(\hat{\theta} - \theta) \approx \sqrt{T}(\hat{\theta}_n^* - \hat{\theta})$ .

## Hall's studentized interval

$$CI_{SH} = \left[ \hat{\theta} - t_{(1-\gamma/2)}^{**} \sqrt{\widehat{\text{Var}}(\hat{\theta})}, \hat{\theta} - t_{\gamma/2}^{**} \sqrt{\widehat{\text{Var}}(\hat{\theta})} \right],$$

where  $t_{\gamma/2}^{**}$  and  $t_{(1-\gamma/2)}^{**}$  are the relevant quantiles from the distribution of  $(\hat{\theta}_n^* - \hat{\theta}) / (\widehat{\text{Var}}(\hat{\theta}_n^*))^{1/2}$ .

# Residual Based Bootstrap

Given a sample  $y_1, \dots, y_T$  plus presample values proceed as follows:

- ① Estimate the parameters of the model VAR( $p$ ) under consideration. Let  $\hat{u}_t$ ,  $t = 1, \dots, T$ , be the estimation residuals.
- ② Get bootstrap residuals  $u_1^*, \dots, u_T^*$  by randomly drawing with replacement from the centered (mean-adjusted) estimation residuals.
- ③ Bootstrap time series are computed recursively as

$$y_t^* = \hat{\nu} + \hat{A}_1 y_{t-1}^* + \dots + \hat{A}_p y_{t-p}^* + u_t^*, \quad t = 1, \dots, T,$$

where the same initial values may be used for each generated series,  $(y_{-p+1}^*, \dots, y_0^*) = (y_{-p+1}, \dots, y_0)$ .

- ④ Reestimate the VAR parameters based on the bootstrap time series.
- ⑤ Calculate a bootstrap version of the statistic of interest.
- ⑥ Repeat these steps  $N$  times, where  $N$  is a large number.

# Bayesian Estimation

Parameter vector of interest  $\beta$

Prior probability density  $g(\beta)$

Sample probability density  $f(y_1, \dots, y_T | \beta)$

Posterior probability density

$$g(\beta | y_1, \dots, y_T) = f(y_1, \dots, y_T | \beta) g(\beta) / f(y_1, \dots, y_T)$$

(Bayes' Theorem)

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# Blanchard-Quah Long-run Restrictions

## Total impact matrix

$$\Xi_{\infty} = \sum_{i=0}^{\infty} \Theta_i = (I_K - A_1 - \dots - A_p)^{-1} A^{-1} B$$

## Identifying restrictions

Typically zero restrictions on  $\Xi_{\infty}$  which imply restrictions for A and/or B

## Example

$$y_t = (q_t, ur_t)' \text{ with}$$

- $q_t$  - output growth
- $ur_t$  - unemployment rate

$$\varepsilon_t = (\varepsilon_t^s, \varepsilon_t^d)'$$

$\varepsilon_t^d$  (demand shock) has only transitory effect on  $q_t$

$\Rightarrow \Xi_{\infty}$  is lower triangular

# Estimation of Blanchard-Quah Model

## A useful relation

$$\Xi_{\infty} \Xi'_{\infty} = (I_K - A_1 - \dots - A_p)^{-1} \Sigma_u (I_K - A'_1 - \dots - A'_p)^{-1}$$

## ML estimator of $\Xi_{\infty}$

estimation by Choleski decomposition of

$$(I_K - \hat{A}_1 - \dots - \hat{A}_p)^{-1} \tilde{\Sigma}_u (I_K - \hat{A}'_1 - \dots - \hat{A}'_p)^{-1}$$

if VAR model stable

## Estimation of B in B-Model

$$\tilde{B} = (I_K - \hat{A}_1 - \dots - \hat{A}_p) \tilde{\Xi}_{\infty}$$

## Estimation of A in A-Model

$$\tilde{A} = [(I_K - \hat{A}_1 - \dots - \hat{A}_p) \tilde{\Xi}_{\infty}]^{-1}$$

# Structural VECM Setup

## SVECM

$$\Delta y_t = \alpha \beta' y_{t-1} + \Gamma_1 \Delta y_{t-1} + \dots + \Gamma_{p-1} \Delta y_{t-p+1} + B \varepsilon_t$$

## Beveridge-Nelson MA representation

$$y_t = \Xi \sum_{i=1}^t u_i + \sum_{j=0}^{\infty} \Xi_j^* u_{t-j} + y_0^*$$

## Matrix of long-run effects

$$\Xi B = \beta_{\perp} \left[ \alpha'_{\perp} \left( I_K - \sum_{i=1}^{p-1} \Gamma_i \right) \beta_{\perp} \right]^{-1} \alpha'_{\perp} B$$

## Rank of long-run effects matrix

$$\text{rk}(\alpha) = \text{rk}(\beta) = r \Rightarrow \text{rk}(\Xi B) = K - r$$

# Accounting of Identifying Restrictions

## Covariance matrix restrictions

$$\Sigma_u = BB' \Rightarrow K(K + 1)/2 \text{ restrictions}$$

## Cointegration restrictions

$$\begin{aligned} \text{rk}(\Xi B) = K - r &\Rightarrow \text{at most } r \text{ zero columns of } \Xi B \\ &\Rightarrow \text{at most } r \text{ transitory shocks} \end{aligned}$$

## Restrictions for transitory shocks

$r$  transitory shocks count for  $r(K - r)$  restrictions due to reduced rank of  $\Xi B$

## Restrictions on permanent shocks

Impose  $(K - r)(K - r - 1)/2$  restrictions on long-run effects of permanent shocks

## Restrictions on instantaneous effects

Impose  $r(r - 1)/2$  restrictions on instantaneous effects of transitory shocks

## Total number of restrictions

$$\frac{1}{2}K(K + 1) + r(K - r) + \frac{1}{2}r(r - 1) + \frac{1}{2}(K - r)(K - r - 1) = K^2$$

# Example (King, Plosser, Stock, Watson, 1991, AER)

## Variables

$y_t = (q_t, c_t, i_t)'$  with

- $q_t$  - output
- $c_t$  - consumption
- $i_t$  - investment

## Restrictions

2 cointegration relations

2 transitory shocks

$$\Xi B = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

# ML estimation of structural VECM

## Step 1

Estimate reduced form VECM by reduced rank regression

## Step 2

Determine  $\tilde{\Xi} = \tilde{\beta}_\perp \left[ \tilde{\alpha}'_\perp \left( I_K - \sum_{i=1}^{p-1} \tilde{\Gamma}_i \right) \tilde{\beta}_\perp \right]^{-1} \tilde{\alpha}'_\perp$

## Step 3

Maximize concentrated log likelihood

$$\log l_c(B) = \text{constant} - \frac{T}{2} \log |B|^2 - \frac{T}{2} \text{tr}(B'^{-1} B^{-1} \tilde{\Sigma}_u)$$

subject to restrictions on  $\tilde{\Xi}B$  and  $B$

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# Factor Augmented SVARs I

## Reference

Bernanke, Boivin, Eliasch (2005)

## Dilemma

The number of parameters in a VAR increases with the square of the number of variables included. Hence, estimation precision suffers from including many variables. Also, more variables may make identification of shocks more difficult. On the other hand, if important variables are omitted from a VAR, impulse responses will be distorted (omitted variables bias).

## Potential solution

Include factors which summarize many variables.

# Factor Augmented SVARs II

## Model setup

$$A(L) \begin{bmatrix} f_t \\ y_t \end{bmatrix} = v_t$$

where  $A(L) = A_0 + A_1L + \dots + A_pL^p$  is a  $((M + K) \times (M + K))$  VAR operator,

$f_t$  is a vector of  $M$  factors composed of  $N$  “informational” variables  $x_t$  and

$v_t$  is a  $(K + M)$ -dimensional white noise vector.

## Estimation of factors

Suppose

$$x_t = \Lambda^f f_t + \Lambda^y y_t + e_t,$$

where  $e_t$  is zero mean white noise. Use this to estimate factors by principle components or ML.

# Sign Restrictions for Impulse Responses

## References

Faust (1998), Canova and De Nicoló (2002), Uhlig (2005), Fry and Pagan (2007)

## Example

A monetary policy shock (increase in interest rate) reduces GDP and inflation (possibly with some delay)

## How to choose the shocks and impulse responses

- 1 Determine all shocks which result in impulse responses satisfying the restrictions
- 2 Determine the median of the impulse responses satisfying the restrictions or determine quantiles from them to determine confidence bands

## Problem

The final impulse responses may not come from the same shocks for all leads/lags

# Computations for Sign Restricted Impulse Responses

## Problem

Find all  $B$  such that  $\Sigma_u = BB'$  and the corresponding impulse responses satisfy the sign restrictions

## Step 1

Choose some  $W$  such that  $\Sigma_u = WW'$ , e.g., by a Choleski decomposition

## Step 2

For a large number of (all) orthogonal matrices  $Q$  choose  $B = WQ$ .

## Step 3

Maintain those  $B$  matrices which lead to admissible impulse responses



# $(3 \times 3)$ Orthogonal Matrices

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & \sin \theta_1 \\ 0 & -\sin \theta_1 & \cos \theta_1 \end{bmatrix} \\ \times \begin{bmatrix} \cos \theta_2 & \sin \theta_2 & 0 \\ -\sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \times \begin{bmatrix} \cos \theta_3 & 0 & \sin \theta_3 \\ 0 & 1 & 0 \\ -\sin \theta_3 & 0 & \cos \theta_3 \end{bmatrix}$$

Draw  $\theta_1, \theta_2, \theta_3$  from  $U[0, \pi]$  to choose a large number of orthogonal matrices.

# Identification via Statistical Data Properties

## References

Rigobon (2003), Lanne and Lütkepohl (2008, 2010), Lanne, Lütkepohl, Maciejowska (2010)

## Problem

Too little information to identify all shocks or to compare alternative just-identified models

## Potential solution

Use data properties such as residual heteroskedasticity, nonnormality, Markov switching in residual volatility