Some warm-up exercises

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1 Introduction

The first year of your PhD program in Florence will be very busy and full of new concepts and techniques. Therefore, you will not have much time to spend on learning math as taught at the undergraduate math courses for economists. The background course in mathematics to be given in September is meant to refresh the mathematical concepts taught for undergrads in economics and introduce you to some of the mathematics you will be exposed to during your first year at the EUI.

The following exercises will not be graded as will be the case with all exercises given during the course. However solutions will be provided. Try to solve as many of these exercises as possible during August. This should give you some hint what is your knowledge of mathematics.

Wish you nice vacations in August and see you in September.

2 Logic, proofs and topology

Exercise 1. *Let p be the following statement: $\forall x\ \exists y:\ y = f(x)$. State the negation of this statement, i.e. give $\neg\, p$.*

Exercise 2 (Proof by construction)**.** *Prove the following theorem: If $\max \mathcal{X}$ exists then*

$$\max \mathcal{X} = \sup \mathcal{X}$$

Below you can find the definitions you may need to prove the theorem:

- *Def.* *Take $\mathcal{X} \subset \mathbb{R}$. Then we say $a \in \mathbb{R}$ is an* ***upper bound*** *for $\mathcal{X}$ if $a \geq x$, $\forall x \in \mathcal{X}$.*
- *Def.* *We say $a \in \mathbb{R}$ is a* ***least upper bound*** *of $\mathcal{X}$, or the* ***supremum*** *of $\mathcal{X}$, if a is an upper bound, and if a' is also an upper bound then $a' \geq a$. We write $a = \sup \mathcal{X}$.*
- *Def.* *We say that $a \in \mathbb{R}$ is a* ***maximum*** *of $\mathcal{X}$, if $a \in \mathcal{X}$ and $a \geq x$, $\forall x \in \mathcal{X}$. We write $a = \max \mathcal{X}$.*

Exercise 3 (Proof by induction). *Show that*

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$$

Exercise 4 (Direct proof). *Prove de Morgan laws which are stated below:*

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Hint: To prove that two sets are equal e.g. $X = Y$, you might want to prove that $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, you should show that $x \in X \Rightarrow y \in Y$.

Exercise 5. *State whether each of the following sets is open, closed, compact:*

a) $A = \{(x, y) \in \mathbb{R}^2 | \ -1 < x < 1,\ y = 0\}$,

b) $B = \{(x, y) \in \mathbb{R}^2 | \ y \leq 0\}$,

c) $C = \{(x, y) \in \mathbb{R}^2 | \ x = 0 \vee y = 0\}$,

d) $D = \{(x, y) \in \mathbb{R}^2 | \ x = 0 \wedge y = 0\}$,

e) $E = \{1, 2, 3\}$.

Give example of a set which is both closed and open.

Exercise 6. *Check whether function e^x is convex and/or concave and/or quasi-convex and/or quasi-concave.*

3 Linear algebra

Exercise 7. *Find the rank of the following matrix:*

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 1 & -3 & -1 \\ -2 & 0 & -4 \\ 4 & 6 & 14 \end{pmatrix}.$$

Exercise 8. *Solve the system $Ax = b$, given by:*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 1 & 3 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}.$$

Exercise 9. *Given the quadratic form $x^2 + 2y^2 + 3z^2 + 4xy - 6yz + 8xz$ find matrix A of $v'Av$, where v' denotes the transpose of v (v is a vector and x, y, z are its coordinates)*

Exercise 10. *Find the eigenvalues and corresponding eigenvectors of the following matrix:*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Is this matrix positive definite? If yes, why? Is this matrix non-singular, what is its rank?

4 Calculus

Exercise 11. *Find the following limits:*

a) $\lim_{x \to -2} \frac{x^2+x-2}{x^2+2x}$

b) $\lim_{x \to 1} \frac{x-1}{\ln x}$

c) $\lim_{x \to +\infty} x^n \cdot e^{-x}$

d) $\lim_{x \to 0} x^x$.

Exercise 12. *Show that a parabola* $y = \frac{x^2}{2e}$ *is tangent to the curve* $y = \ln x$ *and find the tangent point.*

Exercise 13. *Find* $y'(x)$ *if*

a) $y = \ln \sqrt{\frac{e^{4x}}{e^{4x}+1}}$

b) $y = \ln \left(\sin x + \sqrt{1 + \sin^2 x} \right)$.

Exercise 14. *Show that the function* $y(x) = x(\ln x - 1)$ *is a solution to the differential equation* $y''(x) x \ln x = y'(x)$.

Exercise 15. *Find* $\frac{d^2y}{dx^2}$, *given*

a) $\begin{cases} x = t^2, \\ y = t + t^3. \end{cases}$

b) $\begin{cases} x = e^{2t}, \\ y = e^{3t}. \end{cases}$

c) $\begin{cases} x = e^{2t}, \\ y = e^{3}t. \end{cases}$

Exercise 16. *Given* $y(x) = x^2|x|$, *find* $y'(x)$ *and draw the graph of the function.*

Exercise 17. *Find extrema of the following functions:*

a) $z = y\sqrt{x} - y^2 - x + 6y$

b) $z = e^{x/2}(x + y^2)$.