Exercises for a Course on Statistical Analysis of Econometric Models

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1. Preface

These exercises have been used during the course held at EUI 1996-2000. The topics taught have changed somewhat during the time and the exam questions therefore do not reflect the current curriculum.

S. J.

Firenze

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CHAPTER 1

Probability and Distributions

EXERCISE 1. *Let* x_1,x_2 *be jointly exponential with density*

$$p(x_1, x_2) = \lambda^2 \exp(-\lambda(x_1 + x_2)), x_1 > 0, x_2 > 0.$$

1. Find the marginal distribution of x_1 *and the conditional distribution of* $x_2 + x_1$ *given* x_1.

2. Find $E(x_1)$, *Var*(x_1), *Var*$(x_1 + x_2)$ *and Var*$(x_1 + x_2|x_1)$

EXERCISE 2. *Let* x_1, x_2, x_3 *be distributed as*

$$N_3\left(\left(\begin{array}{c}\xi_1\\ \xi_2\\ \xi_3\end{array}\right), \left(\begin{array}{ccc}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{array}\right)\right).$$

1. Find $E(x_1|x_2, x_3)$ *and Var*$(x_1|x_2, x_3)$.

2. Check the formula $Var(x_1) = E(Var(x_1|x_2, x_3)) + Var(E(x_1|x_2, x_3))$.

3. Find an expression for $E(x_1|x_2)$ *and compare with the coefficient of* x_2 *in the expression for* $E(x_1|x_2, x_3)$.

EXERCISE 3. *Let the random variables* y *and* x *be given by the equations*

$$\begin{aligned} y &= \beta x + \varepsilon_1, \\ x &= \varepsilon_2, \end{aligned}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2)'$ *are distributed as* $N_2(0, \Omega)$, *with*

$$\Omega = \left(\begin{array}{cc}\omega_{11} & \omega_{12}\\ \omega_{21} & \omega_{22}\end{array}\right).$$

1. Find $E(y)$, *Var*(y), $E(y|x)$, *Var*$(y|x)$, *and check that*

$$Var(y) \geq Var(y|x).$$

2. Find an expression for R^2, *and find for what values of* β *this is zero and for what values it is close to 1.*

EXERCISE 4. *Let* x_t *be a martingale, such that*

$$E(x_t|\mathcal{F}_{t-1}) = x_{t-1}.$$

Show by applying the rules for conditional expectations, that $\Delta x_t = x_t - x_{t-1}$ *is a martingale difference.*

Let $p(x_1, \ldots, x_t, \theta)$ *be the joint density of* T *stochastic variables.*

1. Show that under suitable regularity conditions

$$\frac{d \log p(x_1, \ldots, x_t, \theta)}{d\theta}$$

has expectation zero with respect to the density $p(x_1, \ldots, x_t, \theta)$.

2. Show that under suitable regularity conditions it holds that

$$\frac{d \log p(x_t|x_1, \ \ldots \ , \ x_{t-1}, \theta)}{d\theta},$$

has expectation zero with respect to the conditional distribution of x_t *given* $x_1, \ \ldots \ , \ x_{t-1}$, *and hence that* x_t *is a martingale difference sequence.*

3. Try to find examples of this result for independent Poisson variables, independent Gamma distributed variables etc.

The fact that the score function

$$\frac{d \log L(\theta)}{d\theta} = \sum_{t=1}^{T} \frac{d \log p(x_t|x_1, \ \ldots \ , \ x_{t-1}, \theta)}{d\theta},$$

is a martingale is the basic result that allows one to use the central limit theorem to prove asymptotic normality of the maximum likelihood estimators in a wide variety of cases.

Exercise 5. *Let* ε_t *be a sequence of independent identically distributed variables with expectation 0. Show that the random walk*

$$x_t = \sum_{i=1}^{t} \varepsilon_i$$

is a martingale with respect to

$$\mathcal{F}_t = \sigma(\varepsilon_1, \ \ldots \ , \ \varepsilon_t).$$

Exercise 6. *Let* x_t *be mixing, show that* x_t^2 *is mixing.*

Exercise 7. *Show the formulae for the variance and covariance function for the linear process* $x_t = \sum_{i=0}^{\infty} \theta_i \varepsilon_{t-i}$. *Show that* $v(h) \to 0$, *for* $h \to \infty$.

Exercise 8. *Let the* $p \times q$ *matrix* u *be distributed as* $N_{pq}(0, A \otimes B)$.

That is

$$a'ub \text{ is distributed as } N(0, a'Aab'Bb).$$

1. Show that by choosing a *and* b *to be unit vectors one can express the variance of the elements* u_{ij} *in terms of the entries of* A *and* B.

2. How can one choose the vectors a *and* b *to find the covariance of* u_{11} *and* u_{12} *expressed by the entries in* A *and* B *?*

See also Exam Questions January 2000, Q:2.

CHAPTER 2

Some concepts from statistics

EXERCISE 9. *Let $X_1, \ldots, X_T$ be independent identically distributed with the density*

$$p(x) = \lambda \exp(-\lambda x), x > 0, \lambda > 0.$$

1. On the basis of all data find an expression for the log likelihood function as a function of the parameter.

2.Find

2a) the score function $s_T(\lambda)$, that is, the first derivative of the log likelihood function and

2b) the information $i_T(\lambda)$, that is, minus the second derivative of the log likelihood function.

2c) Use this to find the maximum likelihood estimator $\hat{\lambda}$.

3. Determine by the law of large numbers the limits

$$p \lim T^{-1} s_T(\lambda)$$
$$p \lim \hat{\lambda}$$

and find the limit

$$\lim T^{-1} i_T(\lambda)$$

4. Find by the central limit theorem the asymptotic distribution

$$T^{-\frac{1}{2}} s_T(\lambda) \xrightarrow{w} ?$$

5. Find the asymptotic distribution of the maximum likelihood estimator

$$T^{\frac{1}{2}}(\hat{\lambda} - \lambda) \xrightarrow{w} ?$$

EXERCISE 10. *Let $x_1, \ldots, x_N$ be i.i.d. exponentially distributed with density*

$$p(x, \lambda) = \begin{cases} \lambda \exp(-x\lambda) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

1. Find an expression for the likelihood ratio test $(-2logQ(\lambda_0))$ of the hypothesis

$$H_0 : \lambda = \lambda_0,$$

and show that it is a function of the ratio $\frac{\hat{\lambda}}{\lambda_0}$.

2. Go through the proof that the asymptotic distribution of $-2logQ(\lambda_0)$ is $\chi^2(1)$, using the method of the notes and the results from Exercise ***9.***

EXERCISE 11. *1 Let $Z_i, i = 1, \ldots, n$, be independent identically distributed with values 1 and 0, such that*

$$P(Z_i = 1) = \alpha, i = 1, \ldots, n,$$

2 Find the likelihood function based on $Z_1, \ldots, Z_n$.

Let $Y_i, i = 1, \ldots, m$, *be independent identically distributed with values 1 and 0, such that*

$$P(Y_i = 1) = \beta, i = 1, \ldots, m.$$

Assume further that $Z_1, \ldots, Z_n$, *and* $Y_1, \ldots, Y_m$ *are independent.*

3 Find the likelihood function based on $Z_1, \ldots, Z_n, Y_1, \ldots, Y_m$.

4 Formulate the analysis as a special case of the logit regression model by choosing a suitable regressors $x_t = (x_{1t}, x_{2t})$, *and parameter* β_1, β_2.

5 Find the score function $s_{n+m,\alpha}(\alpha, \beta)$, $s_{n+m,\beta}(\alpha, \beta)$ *and the* (2×2) *information matrix with elements*

$$i_{n+m}(\alpha, \beta) = \begin{pmatrix} i_{n+m,\alpha\alpha}(\alpha, \beta) & i_{n+m,\alpha\beta}(\alpha, \beta) \\ i_{n+m,\beta\alpha}(\alpha, \beta) & i_{n+m,\beta\beta}(\alpha, \beta) \end{pmatrix}$$

and find the maximum likelihood estimator.

6 Show that if

$$n \to \infty,\ m \to \infty, \frac{n}{n+m} \to p \in]0, 1[$$

then the estimator of (α, β) *is asymptotically normal, and find the asymptotic variance matrix.*

7 Find the asymptotic distribution of the estimators for α, β.

See also Exam Questions January 1997, Q:2, January 2000, Q:3, June 2000, Q:1, Q:3.

CHAPTER 3

Regression models

EXERCISE 12. *As a continuation of Exercise 2 consider the three dimensional normal distribution with mean zero and variance matrix*

$$Var\begin{pmatrix} y \\ x_1 \\ x_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 1 & -2 \\ 1 & 3 & -2 \\ -2 & -2 & 4 \end{pmatrix}.$$

1. Show that if you want to explain y by x_1 the "regression coefficient" is $\frac{1}{3}$, that is, $E(y|x_1) = \frac{1}{3}x_1$, such that the marginal effect of x_1 on y is $\frac{1}{3}$.

2. Show then that the effect of x_1 on y corrected for x_2 is zero. In other words, given x_2 the variables y and x_1 are in fact independent.

EXERCISE 13. *In the model*

$$y_t = \beta' x_t + \varepsilon_t, t = 1, \dots, T$$

with ε_t i.i.d. $N(0, \sigma^2)$, and x_t fixed deterministic, show that $\hat{\beta}$ is independent of the residuals $y_t - \hat{\beta}' x_t$, and use this to show that the estimators of β and σ^2 are independent.

If $\beta' x_t = \beta_0 + \beta_1 (t - \bar{t})$, find a 95% confidence interval for the parameter β_0. Find a 95% confidence interval for the parameter β_1.

EXERCISE 14. *Define the 1-dimensional processes x_t and y_t by the equations*

$$y_t = \alpha x_t + \varepsilon_{1t},$$

$$x_t = \beta y_t + \varepsilon_{2t},$$

where ε_t is independent Gaussian $N_2(0, \Omega)$. Find $E(y_t|x_t)$ and determine which functions of α, β, Ω can be estimated. What will a regression of y_t on x_t estimate if $\alpha = 0$.

EXERCISE 15. *Let (y, x_1, x_2) be multivariate normally distributed with expectation 0 and variance Ω. We are interested in the hypothesis*

$$\Omega_{y1} = \Omega_{y2}\Omega_{22}^{-1}\Omega_{21}.$$

1. Show that a convenient parametrization is in terms of the marginal parameters for y and the conditional parameters of y given (x_1, x_2).

2. How can you estimate the parameters under the hypothesis if there are no further restrictions on the parameters.

EXERCISE 16. *Let*

$$y_t = \beta x_t + \varepsilon_t, t = 1, \dots, T$$

where ε_t are independent $N(0, \sigma^2)$ and x_t is a deterministic 1-dimensional variable.

1. Give the likelihood ratio test for the hypothesis $\beta = 0$, together with its distribution.

In the following we assume that the variances are known, since it only complicates the calculations to estimate them.

Now let

$$y_t = \beta_1 x_1 + \beta_2 x_2 + \varepsilon_t, t = 1, \dots, T$$

with the same assumptions about ε_t and x_t except that $x_t = (x_1, x_2)'$ is 2-dimensional.

2. Give the likelihood ratio test for the hypothesis $\beta_1 = \beta_2 = 0$, and its distribution.

3. In the situation in question 2 give the Wald test for the hypothesis $\beta_1 = 0$ and its distribution.

Now assume that x_t is stochastic, or more precisely assume that

$$y_t = \beta x_t + \varepsilon_t, t = 1, \dots, T,$$

where (ε_t, x_t) are independent Gaussian distributed as $N_2(0, \Omega)$, and that

$$\Omega = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma_{22} \end{pmatrix}.$$

4. Give the model for y_t given x_t and derive the likelihood ratio test for the hypothesis $\beta = 0$ based on the conditional likelihood function.

5. What is the conditional distribution of this test given the x ?

6. How can one from the conditional distribution of the test statistic derive the marginal distribution ?

7. What is the marginal distribution ?

Next assume that x_t is stochastic and multivariate, that is,

$$y_t = \beta_1 x_1 + \beta_2 x_2 + \varepsilon_t, t = 1, \dots, T,$$

where (ε_t, x_t) are independent 3-dimensional Gaussian $N_3(0, \Omega)$, and that

$$\Omega = \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma_{11} & \sigma_{12} \\ 0 & \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

8. Give the model for y_t given x_t and give the likelihood ratio test for the hypothesis $\beta = 0$ based on this conditional model.

9. What is the conditional distribution of this test given the x ?

10. What is the marginal distribution of the test ?

11. Find the test for the hypothesis $\beta_1 = 0$?

12. Give its distribution.

Finally we assume that x_t is a stochastic process, which generates the σ-algebra

$$\mathcal{F}_t = \sigma(x_{t+1}, x_t, \dots, x_1, \varepsilon_t, \dots, \varepsilon_1)$$

and that ε_t given $\mathcal{F}_{t-1}$ is Gaussian $N(0, \sigma^2)$. A special case of this is where the x are the lagged y.

13. Derive on the basis of the partial likelihood a likelihood ratio test for the hypothesis $\beta = 0$, and give under suitable assumptions on the x the asymptotic distribution.

CHAPTER 4

Statistical analysis of Gaussian autoregressive models

EXERCISE 17. *Show that the autocovariance function for a p−dimensional stationary process satisfies*

$$v(h) = v(-h)' \,, h = 0,\ 1,\ 2,\ \ldots$$

EXERCISE 18. *Show that the difference equation*

$$x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2},$$

has solution

$$x_t = c_1 z_1^{-t} + c_2 z_2^{-t},$$

where z_1 and z_2 are the solutions of the equation

$$1 = \rho_1 z + \rho_2 z^2,$$

and determine the coefficients c_1 and c_2 as functions of the initial values.

EXERCISE 19. *Show that the process determined by the equations*

$$y_t = \rho y_{t-1} + \varepsilon_t,\ t = 1, \ldots, T,$$

and the initial value y_0, is stationary if we choose the distribution of y_0 to be $N(0, \frac{\sigma^2}{1-\rho^2})$.

EXERCISE 20. *Let y_t be stationary AR(1) process given by*

$$y_t = \rho y_{t-1} + \mu + \varepsilon_t,\ t = 1,\ \ldots,\ T.$$

Take expectations on both sides and use this to find $E(y_t)$. Next take the variance on both sides and use this to find the variance of y_t.

EXERCISE 21. *Find which of the following processes that are stationary. Throughout the ε are independent identically distributed $N(0,\sigma^2)$.*

$$\begin{array}{llll}
(a) & y_t & = & \varepsilon_t + \varepsilon_{t-1} \\
(b) & y_t & = & y_{t-1} + \varepsilon_t \qquad\qquad y_0 = 0 \\
(c) & y_t & = & \varepsilon_{1t} + t\varepsilon_{2t} \\
(d) & y_t & = & y_{t-1} - 0.5 y_{t-1} + \varepsilon_t \\
(e) & \multicolumn{3}{l}{y_t = \varepsilon_t \text{ for } t \text{ odd and } y_t = 2\varepsilon_t \text{ for } t \text{ even.}}
\end{array}$$

EXERCISE 22. *Consider the process z_t defined by*

$$z_t = \varepsilon_1 \sin(t) + \varepsilon_2 \cos(t),$$

where ε_1 and ε_2 are independent normally distributed stochastic variables with mean 0 and variance 1.

a) Find the autocovariance function for z_t.

b) Is z_t stationary ?

EXERCISE 23. *Let* y_t *be given by*

$$y_t = \rho y_t + \varepsilon_t,\ t = 1,\ \ldots,\ T,$$

and assume further that the initial value y_0 *is* $N(0, \frac{\sigma^2}{1-\rho^2})$, *and independent of* $\varepsilon_1, \ldots, \varepsilon_T$ *and finally that* $|\rho| < 1$, *and* $\sigma^2 > 0$.

Derive the likelihood equations to determine $\hat{\rho}$ *and* $\hat{\sigma}^2$ *on the basis of the simultaneous density for the variables* $y_0,\ \ldots,\ y_t$.

This analysis is important if one has a large number of short times series, such that too much information is lost by conditioning on the initial values.

EXERCISE 24. *For a stationary AR(1) process* y_t *find*

$$Cov(y_t, y_{t+2}|y_{t+1})$$

and

$$Cov(y_t, y_{t+3}|y_{t+1}, y_{t+2}).$$

EXERCISE 25. *The linear process*

$$y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1},$$

is called an MA(1) or moving average process. Find the autocovariance function and find

$$Cov(y_t, y_{t+2}|y_{t+1})$$

and

$$Cov(y_t, y_{t+3}|y_{t+1}, y_{t+2}).$$

EXERCISE 26. *Let* ε_t *be independent* $N(0, \sigma^2)$ *and define the empirical autocorrelation function*

$$\hat{\gamma}(h) = \frac{\frac{1}{T-h}\sum_{t=h+1}^{T} \varepsilon_t \varepsilon_{t+h}}{\frac{1}{T}\sum_{t=1}^{T} \varepsilon_t^2}.$$

Show that

$$\sqrt{T}\hat{\gamma}(h) \xrightarrow{w} N(0,1). \tag{0.1}$$

Show the same result if ε_t *is replaced by the residuals from the model*

$$y_t = \rho y_{t-1} + \varepsilon_t.$$

That is, ε_t *is replaced by*

$$\varepsilon_t = y_t - \hat{\rho} y_{t-1},$$

where

$$\hat{\rho} = \frac{\sum_{t=1}^{T} y_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2}.$$

In the proof of the result (0.1) it is assumed that $\rho = 0$, *such that* $y_t = \varepsilon_t$.

EXERCISE 27. *Show that the autocorrelation function* $\rho(h)$ *for a stationary process satisfies*

$$|\rho(h)| \leq 1.$$

EXERCISE 28. *Let the process y_t be given by*

$$y_t = \rho y_{t-1} + \alpha t + \varepsilon_t.$$

Discuss the representation of the process by a doubly infinite sequence of ε. That is, how does the term αt influence the results that are valid for the process without this term. Find the mean of y_t and the autocovariance function. Discuss the estimation of ρ, α and σ^2.

Find the asymptotic distribution of $(\hat{\rho}, \hat{\alpha})$ if $\rho = 0$.

[Hint. It pays to investigate how the matrix

$$\begin{pmatrix} \sum_{t=1}^T \varepsilon_{t-1}^2 & \sum_{t=1}^T t\varepsilon_{t-1} \\ \sum_{t=1}^T t\varepsilon_{t-1} & \sum_{t=1}^T t^2 \end{pmatrix}$$

shall be normed to convergence.]

EXERCISE 29. *Show that the autocovariance function for a stationary AR(2) process, is given recursively by the equations*

$$v(h) = \rho_1 v(h-1) + \rho_2 v(h-2), \quad h = 2, 3, \dots$$

and that $v(1)$ can be calculated from

$$v(1) = \rho_1 v(0) + \rho_2 v(1),$$

such that $v(h)$ can be calculated recursively from $v(0) = \sigma^2 \sum_{j=0}^\infty \vartheta_j^2$.

[Hint: The equation

$$x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2} + \varepsilon_t, t = \dots, -1, 0, 1, \dots$$

can be multiplied by x_{t-1}, $x_{t-2}, \dots$, respectively and then one can take expectations.]

EXERCISE 30. *Let x_t be defined by the equations*

$$x_t = \sum_{i=1}^k \rho_i x_{t-i} + \varepsilon_t, t = 1, \dots, T$$

with fixed initial values $y_{-k+1}, \dots, y_0$.

Now rewrite this autoregressive process of order k in 1 dimension as a multivariate process, but autoregressive of order 1. The representation is called the companion form. We define

$$\tilde{y}_t' = (y_t, \dots, y_{t-k+1})$$

$$\tilde{\varepsilon}_t' = (\varepsilon_t, 0, \dots, 0)$$

$$R = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_{k-1} & \rho_k \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Thus the initial value for $\tilde{y}_t$ is $\tilde{y}_0' = (y_0, \dots, y_{-k+1})$, and we see that y_t is determined by the equation

$$\tilde{y}_t = R\tilde{y}_{t-1} + \tilde{\varepsilon}_t, t = 1, \dots, T.$$

a) Show that the eigenvalues for R, that is, the solutions to the equations

$$\det(\lambda I - R) = 0$$

are the reciprocal roots of the polynomial

$$A(z) = 1 - \sum \rho_i z^i.$$

[Hint: Perform column operation on the last column in $(R - \lambda I)$].

b) Find an expression for the stationary process $\tilde{y}_t^*$ when the roots in $A(z)$ have modulus > 1.

EXERCISE 31. *Consider the two processes c_t and y_t given by the equations*

$$\Delta c_t = \gamma \Delta y_t - \alpha(c_{t-1} - y_t) + \varepsilon_{1t}, \quad t = 1, \ldots, T.$$

Assume also that y_t is a random walk $y_t = \sum_{i=0}^{t} \varepsilon_{2i}$. Show that for $0 < \alpha < 2$ the process c_t is a non-stationary process of the form

$$c_t = \sum_{i=0}^{t} \varepsilon_{2i} + \textit{stationary process}.$$

From this it follows that c_t and y_t are non-stationary, but that $c_t - y_t$ is stationary, since the common random walk can be eliminated by taking differences.

[Hint: Let $u_t = c_t - y_t$ and $v_t = y_t$ and solve the equations for u_t.] This model can be applied to introduce the static theory model for consumption

$$C = aY$$

into a dynamic statistical model for the observed series for $c_t = \log(C_t)$ and $y_t = \log(y_t)$. The processes c_t and y_t are said to be cointegrated.

EXERCISE 32. *Let the process y_t and x_t satisfy the equation*

$$y_t = \alpha y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + \varepsilon_t,$$

where $\varepsilon_t, t = \ldots, -1, 0, 1, \ldots$ are independent normally distributed with mean 0 and variance σ^2, and $\beta_0 \neq 0$.

Show that if $\alpha = -\beta_1/\beta_0$ then the process $y_t - \beta_0 x_t$ is an AR(1) process. Give the condition for this to be stationary.

EXERCISE 33. *Let the processes u_t, v_t and r_t be given by the equations*

$$u_t = \alpha r_t + \beta + \varepsilon_{1t} \tag{0.2}$$

$$v_t = \kappa(u_t - u_{t-1}) + \varepsilon_{2t} \tag{0.3}$$

$$r_t = u_t + v_t. \tag{0.4}$$

Here u_t is consumption in tth period, r_t is income and v_t is investments, such that (0.2) is the consumption function, (0.3) is investment function and finally (0.4) is an identity between consumption, investment and income.

a) Find (u_t, v_t) expressed by (u_{t-1}, v_{t-1}) and ε_t.

b) Find the conditional expectation of u_t and v_t given the lagged values of the processes u and v.

c) Find the conditions for the process (u_t, v_t) to be stationary.

d) Find consistent estimators for α, β and κ.

EXERCISE 34. *In economic theory one meets the notion of expectations. One way of specifying the expectation formation is by adaptive expectations: Let y_t be a times series and y_t the expectation of this time series based on observations of the process up to time $t-1$. The expectations to the process change from period to period by an amount that is proportional to the last observed deviation between expected and actual value*

$$y_t^e = y_{t-1}^e + \beta(y_{t-1} - y_{t-1}^e), \quad t = 1, \ldots, T, \ 0 \leq \beta \leq 1.$$

1. Show, that

$$y_t^e = \beta y_t + \beta(1-\beta)y_{t-1} + \cdots + \beta(1-\beta)^{t-2}y_1 + (1-\beta)^{t-1}(\beta y_0 + (1-\beta)y_0^e)$$

expresses y_t as function of lagged values of y_t and y_0^e. [Hint: start for example by y_2^e and y_3^e to get a feeling for the result.]

We want to find out when the expectation y_t corresponds to the conditional mean y_t, given the past. In this case one can say that the expectation formation is model based or rational. To investigate this we are forced to specify the stochastic process y_t. We will assume that y_t has the representation:

$$y_t = \varepsilon_t + \sum_{i=1}^{t-1} w_i \varepsilon_{t-i} + y_0, \ t = 1, \ldots, T,$$

where ε are independent normally distributed with mean 0 and variance σ^2 and y_0 is fixed.

2. Show that the $\sigma-$fields generated by the y is the same as that generated by the ε :

$$\sigma(y_1, \ldots, y_t) = \sigma(\varepsilon_1, \ldots, \varepsilon_t).$$

This $\sigma-$algebra is called $\mathcal{F}_t$.

3. Find $E(y_t|\mathcal{F}_{t-1})$.

4. Show that if we choose $\omega_1 = \cdots = \omega_t = \beta$ then

$$y_t^e = E(y_t|\mathcal{F}_{t-1}) + (1-\beta)^t(y_0^e - y_0).$$

It is seen that apart from the discounted initial values of the expectation and the process the expectations are model based. If you guess correctly at time 0, $y_0^e = y_0$, then the expectations will be rational.

Note that the process y_t is non-stationary, if the expectations are to be adaptive.

EXERCISE 35. *The purpose of this exercise is to investigate a test that the residuals from an AR(1) process are independent against the alternative that they follow an AR(1) process. We define the process y_t by:*

$$y_t = \rho y_t + \varepsilon_t, \quad t = 1, \ldots, T,$$

where $\varepsilon_1, \ldots, \varepsilon_t$ are independent normally distributed with mean 0 and variance σ^2. It is assumed that $|\rho| < 1$, such that the process is stationary.

The Durbin–Watson test (DW) is a misspecification test for independence of the shocks ε_t *against the alternative that* ε_t *is an AR(1) process:*

$$\varepsilon_t = \alpha\varepsilon_{t-1} + u_t, \quad t = 2, \ \dots\ , \ T,$$

where $u_1, \ \dots\ , \ u_t$ *are independent normally distributed with mean 0 and variance* τ^2*. The DW-test statistic is calculated by first regressing* y_t *on* y_{t-1} *to form residuals* ε_t*. Next we check the independence by regressing* ε_t *on* ε_{t-1}*. The test statistics is given by*

$$DW = \frac{\sum_{t=2}^{T}(\hat{\varepsilon}_t - \hat{\varepsilon}_{t-1})^2}{\sum_{t=1}^{T}\hat{\varepsilon}_t^2}\ , \quad \text{where } \hat{\varepsilon}_t = y_t - \hat{\rho}y_{t-1} \text{ and } \hat{\rho} = \frac{\sum_{t=1}^{T} y_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2}.$$

In the following it is assumed that $\alpha = 0$*, such that the* ε_t *are independent identically distributed.*

1. Show that DW and $2(1-\hat{\alpha})$ *have the same asymptotic distribution, where*

$$\hat{\alpha} = \frac{\sum_{t=2}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_{t-1}}{\sum_{t=2}^{T}\hat{\varepsilon}_{t-1}^2}$$

is the estimated coefficient in the regression of $\hat{\varepsilon}_t$ *on* $\hat{\varepsilon}_{t-1}$*.*

The test accepts for values of DW close to 2 corresponding to $\hat{\alpha}$ *close to 0. What is close should of course be determined by the quantiles in the limit distribution.*

2. Show that $\hat{\alpha}$ *converges in probability towards zero when* $\alpha = 0$*.*

We next want to show that $\rho \neq 0$*, we have*

$$\sqrt{T}\hat{\alpha} = \frac{\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_{t-1}}{\frac{1}{T}\sum_{t=2}^{T}\hat{\varepsilon}_{t-1}^2} \xrightarrow{w} N(0, \rho^2).$$

3. Show that for $\varepsilon_t = y_t - \hat{\rho}y_{t-1} = (\rho - \hat{\rho})y_{t-1} + \varepsilon_t$ *it holds that*

$$\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_{t-1} \tag{0.5}$$

and

$$\begin{aligned} &\tfrac{1}{\sqrt{T}}\textstyle\sum_{t=2}^{T}\varepsilon_t\varepsilon_{t-1} - (\hat{\rho}-\rho)\tfrac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_{t-1}y_{t-1} \\ &= \tfrac{1}{\sqrt{T}}\textstyle\sum_{t=2}^{T}\varepsilon_t\varepsilon_{t-1} - \frac{\sum_{t=1}^{T}y_{t-1}\varepsilon_{t-1}}{\sum_{t=1}^{T}y_{t-1}^2}\tfrac{1}{\sqrt{T}}\sum_{t=2}^{T}y_{t-1}\varepsilon_t \end{aligned} \tag{0.6}$$

have the same asymptotic distribution, that is the difference converges to zero in probability.

[Hint: Insert the expression for $\hat{\varepsilon}$ *and use that* $\sqrt{T}(\hat{\rho}-\rho)$ *converges in distribution, see Theorem 2.1.]*

4. Show that (0.6) has the same asymptotic distribution as

$$\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_{t-1}\varepsilon_t - (1-\rho^2)\frac{1}{\sqrt{T}}\sum_{t=2}^{T}y_{t-1}\varepsilon_t$$

since

$$\frac{\sum_{t=1}^{T}y_{t-1}\varepsilon_{t-1}}{\sum_{t=1}^{T}y_{t-1}^2} \xrightarrow{P} 1-\rho^2.$$

5. Show that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\begin{pmatrix}\varepsilon_{t-1}\\ y_{t-1}\end{pmatrix}\varepsilon_t \xrightarrow{w} N_2(0,\Sigma),\ \ \textit{hvor}\ \Sigma=\begin{pmatrix}\sigma^4 & \sigma^4\\ \sigma^4 & \frac{\sigma^4}{1-\rho^2}\end{pmatrix}$$

and use this to find the asymptotic distribution of (0.5).

6. What is the asymptotic distribution of $\sqrt{T}\hat{\alpha}$ *and* $\sqrt{T}(DW-2)$ *?*

See also Exam Questions January 1997, Q:3, May 1997 Q: 1, January 1998, Q:1, June 1998, Q:1, Q:2, January 1999, Q:1, Q:2, Q:3, Q:4.

CHAPTER 5

Weak, strong and super exogeneity

EXERCISE 36. *Let the bivariate processes* (y_t, x_t) *be given by the equations*

$$\begin{aligned} \Delta y_t &= \alpha_1(y_{t-1} - \beta x_{t-1}) + \varepsilon_{1t}, \\ \Delta x_t &= \alpha_2 x_{t-1} + \varepsilon_{2t}. \end{aligned}$$

Here $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ *are i.i.d.* $N_2(0, \Omega)$, *with*

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}.$$

1. Show that

$$E(\Delta y_t | \Delta x_t, y_{t-1}, x_{t-1}) = \alpha_1 y_{t-1} - (\alpha_1 \beta + \omega \alpha_2) x_{t-1} + \omega \Delta x_t,$$

where $\omega = \omega_{12}\omega_{22}^{-1}$.

Consider the statistical model given by the parameters

$$(\alpha_1, \alpha_2, \beta, \Omega)$$

varying freely except for the condition that $\alpha_1 \neq 0$.

2. Show that x_t *is weakly exogenous for the parameter of interest* α_1 *but not for the parameter of interest* β.

3. Discuss the reparametrization of the model we get by using the parameters from the conditional model for Δy_t *given* Δx_t *and the past,* θ_c, *and the parameters from the marginal model for* Δx_t *given the past,* θ_m.

4. Derive the maximum likelihood estimators of the original parameters from the maximum likelihood estimators of the parameters θ_c *and* θ_m.

See also Exam Questions January 1997, Q:1, May 1997 Q: 2, Janary 1998 Q:2, January 2000, Q:1, June 2000, Q:2.

CHAPTER 6

Exam questions

Department of Economics
European University Institute
January 1997

Econometrics and Statistics

Question 1

Let the bivariate processes (y_t, x_t) be given by the equations

$$\begin{aligned} \Delta y_t &= \alpha_1(y_{t-1} - \beta x_{t-1}) + \varepsilon_{1t}, \\ \Delta x_t &= \alpha_2 x_{t-1} + \varepsilon_{2t}. \end{aligned}$$

Here $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ are i.i.d. $N_2(0, \Omega)$, with

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}.$$

1. Show that

$$E(\Delta y_t | \Delta x_t, y_{t-1}, x_{t-1}) = \alpha_1 y_{t-1} - (\alpha_1 \beta + \omega \alpha_2) x_{t-1} + \omega \Delta x_t,$$

where $\omega = \omega_{12}\omega_{22}^{-1}$.

Consider the statistical model given by the parameters

$$(\alpha_1, \alpha_2, \beta, \Omega)$$

varying freely except for the condition that $\alpha_1 \neq 0$.

2. Show that x_t is weakly exogenous for the parameter of interest α_1 but not for the parameter of interest β.

3. Discuss the reparametrization of the model we get by using the parameters from the conditional model for Δy_t given Δx_t and the past, θ_c, and the parameters from the marginal model for Δx_t given the past, θ_m.

4. Derive the maximum likelihood estimators of the original parameters from the maximum likelihood estimators of the parameters θ_c and θ_m.

Question 2

Let $(y_t, x_t)'$ be i.i.d. $N_2(0, \Omega)$, with

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}.$$

The following questions concern the derivation of a test for the hypothesis $H_0 : \omega_{12} = \frac{1}{2}\omega_{22}$.

1. Derive the conditional density of $y_t, \ldots, y_T$ given $x_1, \ldots, x_T$.

2. Reparametrize the model using the parameters of the conditional and the marginal distribution and express the hypothesis H_0 in terms of the new parameters.

3. Show that the likelihood ratio test of H_0 can be based upon the conditional likelihood function, and that the test can be performed as a $t-$test.

Question 3

Let the univariate process y_t be given by the equation

$$y_t = \alpha y_{t-1} + 2\alpha y_{t-2} + \varepsilon_t, \; t = 1, \dots, T. \tag{0.7}$$

Here ε_t are i.i.d. $N(0, \sigma^2)$, and the initial values are y_0 and y_{-1}.

1. Show that for

$$-\frac{1}{2} < \alpha < \frac{1}{3},$$

the process y_t can be made stationary.

Next assume α is so chosen and that we let the initial values be distributed as

$$N_2\left(0, \tau^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right),$$

where

$$\rho = \frac{\alpha}{1-2\alpha},$$

and

$$\tau^2 = \sigma^2(1 - 5\alpha^2 - 4\alpha^2\rho)^{-1}.$$

In this case one can show that the process becomes stationary. This result can be applied without proof.

2. Find the maximum likelihood estimator based upon the conditional likelihood function given the initial values y_0 and y_{-1}, when it is assumed that the parameters α and σ^2 vary freely, and show that

$$T^{\frac{1}{2}}(\hat{\alpha} - \alpha) \overset{w}{\to} N(0, \kappa^2),$$

where

$$\kappa^2 = \frac{1}{2}(1 - 2\alpha - 5\alpha^2 + 6\alpha^3)$$

when it is assumed that y_t is a stationary process.

Department of Economics
European University Institute
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Econometrics and Statistics

Question 1

Consider the process

$$X_t = \frac{3}{4}X_{t-1} - \frac{1}{8}X_{t-2} + \varepsilon_t,\ t = 1, \dots, T, \tag{0.8}$$

where ε_t are i.i.d. $N(0, 1)$.

1. Show that the process can be made stationary for a suitable choice of the initial distribution. The variance of the stationary process is in this case equal to $\frac{64}{35}$. This result you need not prove, show instead that the correlation $\rho = Cov(X_t, X_{t-1})/V(X_t) = \frac{2}{3}$.

Next consider the statistical model

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \varepsilon_t,\ t = 1, \dots, T, \tag{0.9}$$

where ε_t are i.i.d. $N(0, \sigma^2)$. The parameters are $(\alpha_1, \alpha_2, \sigma^2)$, which vary unrestrictedly, such that $\sigma^2 > 0$.

2. Give the parameter values of $(\alpha_1, \alpha_2, \sigma^2)$ which correspond to the process (0.8) and give the maximum likelihood estimator $\hat{\alpha}_1$ for the parameter α_1. Finally show that the asymptotic distribution for $\hat{\alpha}_1$ is given by

$$T^{\frac{1}{2}}(\hat{\alpha}_1 - \alpha_1) \xrightarrow{w} N(0, \frac{63}{64}),$$

for the process given by (0.8).

Next consider the statistical model

$$X_t = \frac{1}{8}(6\alpha X_{t-1} - \alpha X_{t-2}) + \varepsilon_t,\ t = 1, \dots, T. \tag{0.10}$$

with parameters (α, σ^2), which vary freely.

3. Give the value of the parameters in this model which correspond to the process (0.8) and find the maximum likelihood estimator. Show that the asymptotic distribution of $\hat{\alpha}$ is given by

$$T^{\frac{1}{2}}(\hat{\alpha} - \alpha) \xrightarrow{w} N(0, \frac{35}{29}),$$

for the process (0.8).

4. Show finally that if we want to estimate the coefficient to X_{t-1}, then we get a smaller variance from the estimator in model (0.10) than from the estimator in model (0.9).

Question 2

It is generally assumed that log consumption C_t depends on log income Y_t through the equation

$$C_t = \alpha + \beta Y_t.$$

In practice this model is too simple and an attempt to describe the dynamic adjustment of consumption to income is given by a so-called partial adjustment model of the form

$$\Delta C_t = (1-\lambda)(\alpha + \beta Y_t - C_{t-1}) + \varepsilon_{1t}, \tag{0.11}$$

where, for the present exercise, it will be assumed that log income is generated by

$$Y_t = \rho Y_{t-1} + \varepsilon_{2t}. \tag{0.12}$$

Moreover it is assumed that $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ are i.i.d. $N_2(0, \Omega)$, where

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}.$$

The parameters $(\alpha, \beta, \lambda, \rho, \Omega)$ in the statistical model are assumed to vary freely, such that Ω is positive definite.

1. Find the condition on the parameters for the processes C_t and Y_t defined by (0.11) and (0.12) to be stationary.

2. Show that for $\omega = \omega_{12}\omega_{22}^{-1}$ we have from (0.11) and (0.12) that

$$E(\Delta C_t | Y_t, C_{t-1}, Y_{t-1}) = (1-\lambda)\alpha + ((1-\lambda)\beta + \omega)Y_t - (1-\lambda)C_{t-1} - \omega\rho Y_{t-1}$$

and use this result to show that Y_t is weakly exogenous for the parameters of interest (α, λ), but not or β.

3. Suggest a restriction on the parameter space that makes Y_t weakly exogenous for the parameters (α, λ, β).

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European University Institute
January 1998

Econometrics and Statistics

Question 1

If x_t is investment and y_t is production in period $t, (t = 0, 1, \ldots, T)$ then x_{t-1} is a leading indicator for y_t if

$$y_t = \beta x_{t-1} + \varepsilon_{1t},\ t = 1, \ldots, T.$$

We assume further that x_t is given by

$$x_t = \rho x_{t-1} + \varepsilon_{2t},\ t = 1, \ldots, T,$$

and that the ε are i.i.d. $N_2(0, \Omega)$.

1. Find the characteristic polynomial for the process (y_t, x_t) and find the condition for the process to be stationary.

Consider the statistical model we get by letting the parameters vary freely, that is,

$$\beta \in R,\ \rho \in R,\ \Omega \text{ positive definite}$$

2. Discuss if x_t is weakly exogenous for β and find the maximum likelihood estimator for β.

3. Find the asymptotic distribution of $\hat{\beta}$, when the process is assumed stationary.

Question 2

Consider the model

$$\begin{aligned} y_t &= \beta x_t + \varepsilon_{1t},\ t = 1, \ldots, T, \\ x_t &= \alpha y_{t-1} + \varepsilon_{2t},\ t = 1, \ldots, T. \end{aligned}$$

Here ε are i.i.d. $N_2(0, \Omega)$ and the parameters are freely varying such that Ω is positive symmetric, $\beta \in R$, and $\alpha \neq 0$.

1. Show that the conditional model for y_t given x_t and (y_{t-1}, x_{t-1}) has the form

$$y_t = \theta_1 x_t + \theta_2 y_{t-1} + \theta_3 x_{t-1} + \varepsilon_{1.2t},$$

and determine the conditional parameters $\theta_{cond} = (\theta_1, \theta_2, \theta_3, \omega_{11.2})$ as functions of the parameters (α, β, Ω)

2. Show that the conditional and the marginal parameters are variation free and express the hypothesis $\omega_{12} = 0$ in terms of the conditional parameters.

3. How would you test the independence of ε_{1t} and ε_{2t} using the results in 1. and 2. ?

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Econometrics and Statistics

Question 1.

Let the bivariate process y_t, x_t be given by the equations

$$\begin{aligned} y_t &= \pi_{11} y_{t-1} + \pi_{12} x_{t-1} + \mu_{1t} + \varepsilon_{1t} \\ x_t &= \pi_{21} y_{t-1} + \pi_{22} x_{t-1} + \mu_{2t} + \varepsilon_{2t} \end{aligned}$$

$t = 1, \ldots, T = 2N$, where ε_t are i.i.d. $N_2(0, \Omega)$, and

$$\mu_{it} = \begin{cases} \mu_i & t = 1, \ldots, N \\ \nu_i & t = N+1, \ldots, 2N \end{cases}$$

such that a break occur at time $t = N$, in the middle of the period. equivalently we can write

$$\mu_t = \mu + (\nu - \mu) I\{t > N\},$$

where the function $I\{t > N\}$ is 0 for $t = 1, \ldots, N$ and 1 for $t = N+1, \ldots, 2N$.

1. Find the conditional model for y_t given x_t and the past and show that the condition for the conditional model not to have a break is

$$\mu_1 - \omega \mu_2 = \nu_1 - \omega \nu_2, \omega = \omega_{12} \omega_{22}^{-1} \tag{0.13}$$

2. Describe the regressions that one can perform in order to find the maximum likelihood estimator of the parameters of the model under the assumption that there is no break in the conditional model, but otherwise no restrictions on the parameters.

Question 2

Let the process (y_t, x_t) be given by the equations

$$\begin{aligned} y_t &= \alpha x_{t-1} + \varepsilon_{1t} \\ x_t &= \beta y_{t-1} + \varepsilon_{2t} \end{aligned}$$

$t = 1, \ldots, T$, where ε_t are i.i.d. $N_2(0, \Omega)$.

1. Find the condition for the process to be stationary expressed in terms of the parameters.

2. Find the variance matrix

$$Var \begin{pmatrix} y_t \\ x_t \end{pmatrix}$$

3. If we further assume $\omega_{12} = 0$, and no other restrictions on the parameters, find the maximum likelihood estimator for the parameter α and its asymptotic distribution.

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European University Institute
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Question 1

The bivariate vector autoregressive process is given by

$$\begin{aligned} y_t &= \alpha_1(y_{t-1} - \theta x_{t-1}) + \varepsilon_{1t} \\ x_t &= \alpha_2(y_{t-1} - \theta x_{t-1}) + \varepsilon_{2t} \end{aligned}$$

with $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ i.i.d. $N_2(0, \Omega)$, $t = 1, \ldots, T$.

1. Find the conditional model for y_t given x_t, y_{t-1}, x_{t-1}.

2. In the statistical model with parameters $\alpha_1 \neq 0$, $\alpha_2 = 0$, $\theta \in R$, Ω unrestricted, show that x_t is weakly exogenous for θ.

3. Find the maximum likelihood estimator for θ under the assumption that $\alpha_2 = 0$.

4. What is the probability limit of this estimator if $\alpha_2 \neq 0$.

Question 2

The bivariate vector autoregressive model is given by

$$\begin{aligned} \Delta y_t &= \alpha_1(y_{t-1} - x_{t-1}) + \varepsilon_{1t} \\ \Delta x_t &= \alpha_2(y_{t-1} - x_{t-1}) + \varepsilon_{2t} \end{aligned}$$

with $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ i.i.d. $N_2(0, \Omega)$.

1. Find the characteristic polynomial, and its roots and show that the process is not stationary.

2. Show that $v_t = y_t - x_t$ is an autoregressive process satisfying an equation of the form

$$v_t = \rho(\alpha_1, \alpha_2) v_{t-1} + \delta_t$$

for $\delta_t = \delta(\varepsilon_{1t}, \varepsilon_{2t})$.

3. Show that the maximum likelihood estimator for $\rho(\alpha_1, \alpha_2)$ based upon $\{y_t, x_t\}_{t=1}^T$ is the same as the maximum likelihood estimator based on the observations of the process $v_1, \ldots, v_T$.

4. Find the asymptotic distribution of $\hat{\rho}(\alpha_1, \alpha_2)$ when it is assumed that the parameters α_1, α_2 have been chosen such that v_t is stationary.

Question 3

Let y_t and x_t be given by

$$\begin{aligned} y_t &= (\rho + \lambda) y_{t-1} + (\rho - \lambda) x_{t-1} + \varepsilon_{1t} \\ x_t &= (\rho - \lambda) y_{t-1} + (\rho + \lambda) x_{t-1} + \varepsilon_{2t} \end{aligned}$$

with $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ i.i.d. $N_2(0, \Omega)$.

1. Find the characteristic polynomial $A(z)$ for the process and its roots and the vectors

$$v_1 = \begin{pmatrix} 1 \\ \gamma_1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ \gamma_2 \end{pmatrix}$$

which satisfy

$$A(z_i) v_i = 0.$$

2. Find the autoregressive model for the new variables

$$\begin{aligned} u_t &= (y_t, x_t)v_1 = y_t + \gamma_1 x_t \\ w_t &= (y_t, x_t)v_2 = y_t + \gamma_2 x_t \end{aligned}$$

and find the condition for the processes to be stationary.

Question 4

Let

$$\begin{aligned} y_t &= \beta x_t + \varepsilon_{1t} + \theta \varepsilon_{1t-1} \\ x_t &= \rho x_{t-1} + \varepsilon_{2t} \end{aligned}$$

with $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ i.i.d. $N_2(0, \Omega)$.

This is not an autoregressive process and maximum likelihood estimation is more difficult. Here is another way of solving the estimation problem.

1. Show that

$$E(y_t x_{t-2}) = \beta E(x_t x_{t-2}) = \beta \rho^2 \frac{\omega_{22}}{1-\rho^2}.$$

The so called instrumental variable estimator has the form

$$\hat{\beta}_{IV} = \frac{\sum_{t=1}^{T} y_t x_{t-2}}{\sum_{i=1}^{T} x_t x_{t-2}}$$

2. Show that $\hat{\beta}_{IV}$ is a consistent estimator for β.

Department of Economics
European University Institute
January 2000

Econometrics and Statistics

Question 1

Consider the model

$$\begin{aligned} y_t &= \alpha x_{t-1}^2 + \varepsilon_{1t} \\ x_t &= \rho y_{t-1} + \varepsilon_{2t} \end{aligned}$$

where $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ are i.i.d. Gaussian with mean zero and variance Ω.

1. Find the conditional model for y_t given x_t, x_{t-1}, y_{t-1}.

2. Show that the parameters in the conditional model for y_t given x_t, x_{t-1}, y_{t-1} are not variation independent of the parameters in the marginal model for x_t given x_{t-1} and y_{t-1}.

Question 2

Let $x_1, \ldots, x_T$ be i.i.d. Poisson variables with mean λ.

1. Find

$$E(\sum_{t=1}^{T} t x_t)$$

2. Show that

$$Var(\frac{\sum_{t=1}^{T} t x_t}{\sum_{t=1}^{T} t}) \to 0$$

Question 3

Let $x_1, \ldots, x_T$ be independent exponential variables with density

$$p(x_t) = \frac{1}{\theta t} \exp(-\frac{x_t}{\theta t}), t = 1, \ldots, T,$$

where $\theta > 0$ is a parameter.

1. Find the likelihood function, the score function, and the information.

2. Find the asymptotic distribution of the maximum likelihood estimator for the parameter θ.

Department of Economics Econometrics and Statistics
European University Institute
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Question 1

Consider the regression model for the univariate stochastic variable y_t :

$$y_t = \beta x_t + \varepsilon_t, t = 1, \ldots, 2T,$$

where ε_t are i.i.d. Gaussian with mean zero and variance σ^2, and the deterministic regressor x_t is given by

$$x_t = \begin{cases} -1 & t = 1, \ldots, T, \\ 1 & t = T+1, \ldots, 2T. \end{cases}$$

Hence the first half of the observations have mean $-\beta$, and the last half have mean β.

1. Show that the maximum likelihood estimator or the ordinary least squares estimator is

$$\hat{\beta} = \frac{-\sum_{t=1}^{T} y_t + \sum_{t=T+1}^{2T} y_t}{2T}.$$

2. Find expressions for the mean and variance of the estimator $\hat{\beta}$ for this choice of regressor x_t.

Question 2

Consider the model for the bivariate process (y_t, z_t) defined by the equations

$$\begin{array}{l} y_t = \alpha y_{t-1} + \lambda\beta z_{t-1} + \varepsilon_{1t} \\ z_t = \lambda z_{t-1} + \varepsilon_{2t} \end{array} \quad t = 1, \ldots, T$$

where $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ are i.i.d. $N_2(0, \Omega)$. The parameters are $(\alpha, \beta, \lambda, \Omega)$, which vary freely.

1. Find the conditional expectation of y_t and z_t, given the past, that is, in this case y_{t-1} and z_{t-1}.
2. Find the conditional expectation of y_t given the past and z_t.
3. Show that z_t is weakly exogenous for α, but not for β.

Question 3

Consider the regression model for the univariate process

$$y_t = e^{\theta t} + \varepsilon_t,\ t = 1, \dots, T$$

where ε_t are i.i.d. Gaussian with mean zero and variance $\sigma^2 = 1$. This model is not so easy to estimate, so we want instead to derive a score (or Lagrange multiplier) test for the hypothesis $\theta = 0$.

1. Show that the score and information are given by

$$\begin{aligned} s_T(\theta) &= \textstyle\sum_{t=1}^T (y_t - e^{\theta t}) t e^{\theta t} \\ i_T(\theta) &= -\textstyle\sum_{t=1}^T (y_t - e^{\theta t}) t^2 e^{\theta t} + \sum_{t=1}^T t^2 e^{2\theta t} \end{aligned}$$

2. Show that for $\theta = 0$,

$$\left. \frac{i_T(\theta)}{\sum_{t=1}^T t^2} \right|_{\theta=0} \xrightarrow{P} 1$$

3. Show finally that

$$\left. \frac{s_T(\theta)}{\sqrt{\sum_{t=1}^T t^2}} \right|_{\theta=0}$$

is distributed as $N(0, 1)$.