Additional questions for Exercise 1

Combinatorics (questions 1 and 2 of the original sheet)

1. In how many different ways can you put in a row n balls, of which r_i are of colour i, $i=1,2,\ldots,k$.

2. There are 2 white and 4 black balls in an urn. Which event is more probable: drawing 2 balls of the same colour, or drawing 2 balls of different colour? Explain.

3. 10 boys and 10 girls sit at a round table. What is the probability that no two boys or two girls sit next to each other?

Understanding probability (reinforces questions 3 and 4 of the original sheet)

4. Prove the 'Total Probability law' and Bayes' Theorem (you may use simplifying examples if you wish.)
5. Show that the 'sure thing principle', i.e. if $A > B$ given C and $A > B$ given C^C, then $A > B$. (You will need a suitable definition for '>'. Now consider 'Simpson's paradox': drug 1 gave a positive result in 81 out of 87 male patients and in 192 out of 265 female patients. On the other hand drug 2 gave a positive result in 243 out of 270 male patients and 55 out of 80 female patients. Which drug is preferred in the male sub-population? Which in the female sub-population? Which one gives the better result overall? How do you reconcile the 'sure thing principle' with 'Simpson's paradox'?

Probability models (reinforces question 5 of the original sheet)

6. Y_1 and Y_2 be two independent Bernoulli-distributed random variables with $p = 0.5$.

(a) Find the possible outcomes for the new random variable $\bar{Y} = \frac{Y_1 + Y_2}{2}$.

(b) Work out the density and distribution functions for $\bar{Y}$.

(c) Now let $Y_i, i = 1,2,\ldots N$ be N independent Bernoulli-distributed random variables with $p = 0.5$ for each of these random variables. Define $\bar{Y} = \frac{Y_1 + Y_2}{n}$ and work out its density function.

Additional questions for Exercise 2

Joint, conditional and marginal densities

1. A family has 3 children. What is the probability that they are both boys if we know that: (a) the older one is a boy; b) there is at least one boy?

2. A family has 3 children. Denote by A=children of the same sex, B=at most one boy, C=at least one boy and one girl. Show that A and B are independent events and that B and C are independent; are A and C independent?

3. There are 4 balls in an urn; Red, Green, Blue and one with RGB dots. We draw one ball. Denote by A^C the event that the drawn ball has colour C on it (C=R, G, B). Show that events A^C are pair-wise independent but are not mutually independent.

4. (a) Show using whichever method you wish (transformation of random variables, distribution functions, moment generating functions) that if $X \sim N(0, 1)$ then $Y = aX + b \sim N(b, a^2)$.
   (b) What is the distribution of $Z=X^2$?
   (c) And of W=|X|?

5. Let X_i, $i=1,2$ be exponentially distributed random variables, independent of each other and each with parameter λ. Work out the **joint** density function of (Y_1 , Y_2) where $Y_1= X_1 – X_2$ and $Y_2 = X_2$. Hence work out the marginal density function of Y_1 and show that it has the so-called Laplace distribution.

6. If $X_1, X_2,…X_n$ is a set of n random variables which are independently and identically distributed, each with distribution function given by $F(X_i)$ and corresponding density function given by $f(X_i)$, find the density functions for the order statistics Y_1=min($X_1, X_2,…X_n$) and Y_n = max($X_1, X_2,…X_n$).