Exercise 5

1. Let $X_1, X_2, ..X_T$ be *i.i.d.* random variables with the following density $f_{X_i}(x_i) = \lambda^{-1}\exp(-\lambda^{-1}x_i)$, $x_i > 0, \lambda > 0$, $i = 1, 2, ...T$.

(a) Work out the maximum likelihood estimator $\widehat{\lambda}$. What is the finite sample distribution of this estimator?

(b) From (a) above, where you have worked out the score vector $q(\lambda)$, show that $E(q(\lambda_P)) = 0$, where λ_P is the true (or population value) of the parameter. Show also that $var(q(\lambda_P)) = \frac{T}{\lambda^2}$. Hence work out the mean and variance of $\sqrt{T}\overline{q}(\lambda_P)$, where $\overline{q}(\lambda_P)$ is defined as the average score and given by the expression $\frac{q(\lambda_P)}{T}$.Using a central limit theorem can you work out the large sample or asymptotic distribution of $\sqrt{T}\overline{q}(\lambda_P)$?

(c) Finally, write

$$\sqrt{T}(\widehat{\lambda} - \lambda_P) \simeq \left[-\frac{Q(\lambda_P)}{T}\right]^{-1} \sqrt{T}\overline{q}(\lambda_P)$$

Using a law of large number, work out the probability limit of $-\frac{Q(\lambda_P)}{T}$, and hence (following all the usual steps) derive the asymptotic distribution of $\sqrt{T}(\widehat{\lambda} - \lambda_P)$.

(d) Can you construct a chi-squared test (Wald) of the null hypothesis $H_o : \lambda = \lambda_0$,

given your result in (c)?

2. Let $X_1, X_2, ..X_T$ be an *i.i.d.* random sample from a Poisson distribution with parameter λ. Show that

$$e^{-\overline{X}_T} \rightarrow_P P(X_1 = 0)$$

where $\overline{X}_T$ is the sample mean. What theorem did you use?

3. (OPTIONAL)

An investigator asserts the model

$$y_t = \alpha z_t + \epsilon_t$$

where it is claimed that $\epsilon_t \tilde{} IN(0, \sigma_\epsilon^2)$ and $E(z_t \epsilon_s) = 0 \forall t, s$.

(a) Derive the log-likelihood function for $\theta' = (\alpha, \, \sigma_\epsilon^2)$, its first and second derivatives $q(\theta)$ and $H(\theta)$, and the maximum likelihood estimator $\widehat{\theta}_{ML}$ of θ.

(b) In fact suppose that the investigator has misspecified the model. Suppose that the process ϵ_t is generated as a moving average process of order 1 - i.e.

$$\epsilon_t = u_t - \lambda u_{t-1}$$

$u_t \tilde{} IN(0, 1)$.

Assume also that z_t are determined outside the model and therefore, may be considered as fixed.

Compute $E(q(\theta_p))$ and $E(H(\theta_p))$ for the model asserted in (a) when (b) holds. θ_p is the population value of the parameter θ. Can you check if the information matrix identity holds? Why not?