Solutions to additional questions for Exercise 2.

1. Note first, that children are distinguishable, as they were born in some order. The exercise has an easy solution if we write first the sample space: $\{(boy, boy), (boy, girl), (girl, boy), (girl, girl)\}$. Now we see that there are two cases in which the older child is a boy, and one case in which both children are boys; hence, (a) the probability under question $p = 1/2$. Similarly, there are three cases in which there is at least one boy, and one case in which both children are boys; therefore, (b) the probability under question is $p = 1/3$.

2. As before the children are distinguishable. $P(A) = 2 * (1/2)^3 = 1/4$, as there are two cases to be considered: all the children are boys or all the children are girls, each event happening with probability $(1/2)^3$. Event B takes place if only girls are born (with probability $(1/2)^3$), or two girls and one boy (with probability $3 * (1/2)^3$, as the boy may be the first, second or third child in a family); so, $P(B) = 1/2$. To compute $P(C)$ note that the first two children may be of any sex, then if the first two children were boys, the third one must be a girls, if the first was a boy and then a girl, the third one may be either boy or girl and so on. Then by the Law of Total Probability and by the independence of births it is easy to calculate $P(C) = 3/4$.

$A \cap B$ means that only girls were born, $P(A \cap B) = 1/8 = 1/4 * 1/2 = P(A)P(B)$; therefore events A and B are independent. $B \cap C$ means that there is one boy and two girls, as the boy may be born as the first, second or third child, $P(B \cap C) = 3/8 = 1/2 * 3/4 = P(B)P(C)$; therefore B and C are also independent. Nevertheless A and C are not independent: $A \cap C = \emptyset$ so $P(A \cap C) = 0 \neq 1/4 * 3/4 = P(A)P(C)$.

3. Note first, that $P(A^R) = P(A^G) = P(A^B) = 1/2$ as for each colour there are two balls (out of total of four balls) in the urn with this colour on it. Next, for any two different colours $C1$ and $C2$ there is only one ball having both colours on it. Hence, $P(A^{C1} \cap A^{C2}) = 1/4 = 1/2 * 1/2 = P(A^{C1}) * P(A^{C2})$, and the events are pairwise independent.

There is only one ball with all the colours on it, so $P(A^R \cap A^G \cap A^B) = 1/4$. Hence, $P(A^R \cap A^G \cap A^B) = 1/4 \neq 1/8 = P(A^R) * P(A^G) * P(A^B)$, and the events are not (mutually or jointly) independent!

4. a) We know that if X has df f_X and if $Y = h(X)$, then under general conditions $f_Y(y) = f_X(h^{-1}(y))|J_{h^{-1}}(y)|$. In our case $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)$, $h(x) = ax + b$, and hence $h^{-1}(y) = \frac{y-b}{a}$ $(a \neq 0)$ and $|J_{h^{-1}}(y)| = \frac{1}{|a|}$. By the theorem quoted above we obtain

$$\begin{aligned} f_Y(y) &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\frac{y-b}{a})^2\right] \frac{1}{|a|} \\ &= \frac{1}{\sqrt{2\pi a^2}} \exp\left[-\frac{1}{2}\frac{(y-b)^2}{a^2}\right], \end{aligned}$$

which is the df of a $N(b, a^2)$ distribution.

b) For $t < 0$, $F_{X^2}(t) = 0$ and $f_{X^2}(t) = 0$. For $t \geqslant 0$ we have

$$F_{X^2}(t) = P(X^2 \leqslant t) = P(-\sqrt{t} \leqslant X \leqslant \sqrt{t}) = F_X(\sqrt{t}) - F_X(-\sqrt{t}),$$

$$f_{X^2}(t) = f_X(\sqrt{t})\frac{1}{2\sqrt{t}} + f_X(-\sqrt{t})\frac{1}{2\sqrt{t}} = \frac{1}{2\sqrt{t}}(f_X(\sqrt{t}) + f_X(-\sqrt{t})).$$

c) For $t < 0$, $F_{|X|}(t) = 0$ and $f_{|X|}(t) = 0$. For $t \geqslant 0$

$$F_{|X|}(t) = P(|X| \leqslant t) = P(X \leqslant t) - P(X \leqslant -t) = F_X(t) - F_X(-t),$$

$$f_{|X|}(t) = (F_{|X|})'(t) = 2f_X(t).$$

5. We will use again the theorem about the distribution of a transformation of a random vector. Let's note first, that

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{for} \quad \mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

$\det(\mathbf{A}) = 1 \neq 0$, therefore if f is df of $(X_1, X_2)'$, the df of $(Y_1, Y_2)'$ is given by

$$g = \frac{1}{|\det(\mathbf{A})|} f \circ \mathbf{A}^{-1} \quad \text{where} \quad \mathbf{A}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Using independence of X_1 and X_2 we get

$$\begin{aligned} f_{(Y_1,Y_2)'}(y_1, y_2) = f_{(X_1,X_2)'}(y_1 + y_2, y_2) = f_{X_1}(y_1 + y_2) f_{X_2}(y_2) = \\ \lambda \exp(-\lambda(y_1 + y_2)) \lambda \exp(-\lambda y_2) 1_{(0,\infty)}(y_1 + y_2) 1_{(0,\infty)}(y_2) = \\ \lambda^2 \exp(-\lambda(y_1 + 2y_2)) 1_{(0,\infty)}(y_1 + y_2) 1_{(0,\infty)}(y_2) \end{aligned}$$

Given the joint df above, the df of Y_1 we compute by integrating (for $y_1 \geqslant 0$)

$$\begin{aligned} f_{Y_1}(y_1) = \int_{-\infty}^{\infty} \lambda^2 \exp(-\lambda(y_1 + 2y_2)) 1_{(0,\infty)}(y_1 + y_2) 1_{(0,\infty)}(y_2) dy_2 = \\ \lambda^2 \int_0^{\infty} \exp(-\lambda(y_1 + 2y_2)) dy_2 = \ldots y_1 + 2y_2 = t \ldots = \\ \frac{\lambda^2}{2} \int_{y_1}^{\infty} \exp(-\lambda t) dt = \frac{\lambda^2}{2} [-\frac{1}{\lambda} \exp(-\lambda t)]_{y_1}^{\infty} = \frac{\lambda}{2} \exp(-\lambda y_1) \end{aligned}$$

Similarly we can calculate $f_{Y_1}(y_1)$ for $y \leqslant 0$ obtaining

$$f_{Y_1}(y_1) = \frac{\lambda}{2} \exp(\lambda y_1)$$

To sum up, for $y_1 \in \mathbb{R}$ we get the Laplace's distribution $f_Y(y) = \frac{1}{2}\lambda \exp(-\lambda|y|)$.

6. Denote by F common cdf, and by f common df. Then by independence of $\{X_i\}$ we get

$$\begin{aligned} F_{Y_1}(y_1) &= P(Y_1 \leq y_1) = 1 - P(Y_1 > y_1) = 1 - P(X_1 > y_1, \ldots, X_n > y_1) = \\ &= 1 - \prod_{i=1}^{n} P(X_i > y_1) = 1 - \prod_{i=1}^{n} (1 - F(y_1)) = 1 - (1 - F(y_1))^n \end{aligned}$$

then $f_{Y_1}(y_1) = n(1 - F(y_1))^{n-1} f(y_1)$.

For Y_2 the calculations are even simpler.

$$\begin{aligned} F_{Y_2}(y_2) &= P(Y_2 \leq y_1) = 1 - P(X_1 \leq y_1, \ldots, X_n \leq y_1) = \\ &= \prod_{i=1}^{n} P(X_i \leq y_1) = 1 - \prod_{i=1}^{n} F(y_1) = (F(y_1))^n, \end{aligned}$$

and thus $f_{Y_2}(y_2) = n(F(y_2))^{n-1} f(y_2)$.