

# Statistics and Econometrics

Friday 28 October 2005

10:00 - 12:00

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You have two hours to answer this question paper. Answer ALL questions, as completely as you can, taking care to include all the steps necessary. Marks allocated to each question and to each sub-part of each question are indicated.

## PART A

1.  
X. (a) Define the probability density function of a discrete random variable  $X$ .  
2  
(b) Let  $B(n, k, p)$  be a probability density function given by

$$B(n, k, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}; k = 0, 1, 2, \dots, n$$

and let  $\tilde{P}(k, \lambda)$  be another probability density function given by

$$\tilde{P}(k, \lambda) = \frac{e^{-\lambda} \lambda^k}{k!}, k = 0, 1, 2, \dots$$

Show that

$$\lim_{\substack{n \rightarrow \infty \\ np = \lambda}} B(n, k, p) = \tilde{P}(k, \lambda).$$

- 20  
(c) Now consider showing the result in (b) in another way:

(i) Derive the moment generating function for the random variable with probability density function  $B(n, k, p)$  and show that this moment generating function is given by

$$M(t) = (q + pe^t)^n, \text{ where } q = 1 - p.$$

10  
(ii) Now show that the moment generating function for the random variable with probability density function given by  $\tilde{P}(k, \lambda)$  is given by

$$M(t) = e^{[\lambda(e^t - 1)]}$$

10  
(iii) Finally show that under the same limiting process as in (b), i.e.  $n \rightarrow \infty$  with  $np = \lambda$ , the two moment generating functions coincide.

- 10  
(iv) Explain why this proves the same result as (b). 3

PART B

2.

Suppose that the joint probability density function of the bivariate random variable  $(X, Y)$  is given by

$$f_{X,Y}(x, y) = [1 - \alpha(1 - 2x)(1 - 2y)], 0 \leq x \leq 1, 0 \leq y \leq 1$$

- (a) Work out  $E(XY)$ . 5
- (b) Work out the marginal density functions  $f_X(x)$  and  $f_Y(y)$  and hence  $E(X)$  and  $E(Y)$ . 5
- (c) Attempt to prove or disprove the following statement:  
"The variables  $X$  and  $Y$  in this example are independent if and only if they are uncorrelated." 5

- 3.
- (a) Define a 'maximum' likelihood estimator. 2
- (b) Let  $X_1, X_2, \dots, X_n$  be a random sample from the density  $f(\cdot; \lambda)$  given by  $\frac{\exp(-\lambda)\lambda^x}{x!}$  for  $x = 0, 1, \dots$ . Derive the log likelihood function for the sample, and compute the Cramer-Rao lower bound. 3
- (c) Derive the maximum likelihood (ML) estimator for  $\lambda$ . 5
- (d) Derive the mean and variance of the ML estimator and show whether or not its variance achieves the Cramer-Rao lower bound. 5

4.

For any two events  $A$  and  $B$ , define the conditional probability  $P(A/B)$  as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Prove or disprove the following statements:

- (a) If  $P(A/B) > P(A)$  then  $P(B/A) > P(B)$  3
- (b) If  $P(A) > P(B)$  then  $P(A/C) > P(B/C)$ , where  $C$  is another event 3
- (c) If  $P(A/B) = P(B)$  then the events  $A$  and  $B$  are independent 3
- (d) Let  $P(A) = a$  and  $P(B) = b$ .  
Show that  $P(A \cap B) \geq a + b - 1$  6

5.

(a) What are the properties of a joint probability density function? 2

(b) Let  $f_X(x)$  and  $f_Y(y)$  be two probability density functions with corresponding cumulative density functions given by  $F_X(x)$  and  $F_Y(y)$ .

Show that for all  $\alpha$ , such that  $-1 \leq \alpha \leq 1$ , the function given by

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)\{1 + \alpha[2F_X(x) - 1][2F_Y(y) - 1]\}$$

characterizes an infinite family of density functions. 13