Statistics and Econometrics

Friday 27 October 2006

10:00 - 12:00

You have two hours to answer this question paper. Answer ALL questions, as completely as you can, taking care to include all the steps necessary. Marks allocated to each question and to each sub-part of each question are indicated.

PART A

1.

(a) Define (i) the probability density function (pdf); and (ii) moment generating function (mgf) of a continuous random variable X. 5

(b) Let a random variable X have pdf given by

$$\begin{aligned} f_X(x) &= \lambda e^{-\lambda x},\ x \geqq 0 \\ &= 0,\ \ x < 0 \end{aligned}$$

Find the mgf for X. 5

(c) Now consider another random variable Y with pdf given by
Show that

$$\begin{aligned} f_Y(y) &= \lambda^n y^{n-1} e^{-\lambda y}/(n-1)!,\ y > 0 \\ &= 0\ \ y < 0 \\ n &> 1 \end{aligned}$$

Find the mgf for Y. You will need to use the result that for n integer, the gamma function is defined as

$$\begin{aligned} \Gamma(n) &= \int_0^\infty w^{n-1} e^{-w} dw \\ \Gamma(n) &= (n-1)! \end{aligned}$$

10

(d) Finally, define a random variable Z as follows:

$$Z = X_1 + X_2 + ... + X_n$$

where

$$X_i, i = 1, 2, ...n,\ \text{is i.i.d. with pdf given by (b).}$$

Using your results in (b) and (c) above, show that Y and Z are the same random variable. 15

2.

For $\lambda > 0$ and $y > 0$, consider the cumulative distribution function (cdf) of a random variable distributed exponentially:

$$
\begin{aligned}
F_Y(y) &= 1 - e(-\lambda y), y > 0 \\
&= 0 \;\; y < 0
\end{aligned}
$$

(a) Show that $F_Y(y)$ satisfies the requirements of a cdf. Derive the corresponding pdf and work out the mean and variance of y. 10

(b) Now suppose a random variable X has pdf given by

$$
\begin{aligned}
f_X(x) &= 1, 0 \leq X \leq 1 \\
& \quad 0 \text{ everywhere else}
\end{aligned}
$$

Show that if we define a random variable V which is a transformation of X, given by

$$
\begin{aligned}
V &= \frac{-\log(1 - X)}{\lambda} \\
\lambda &> 0
\end{aligned}
$$

the random variable Y and V have the same pdf. 10

3.

(a) Define a 'maximum' likelihood estimator. 2

(b) Let $X_1, X_2, ...X_n$ be a random sample from a binomial $B(m;p)$ given by

$$P(X_i = x_i) = \left(\begin{array}{c} m \\ x_i \end{array} \right) p^{x_i}(1-p)^{m-x_i}, \ x_i = 0, 1, .., m.$$

Derive the log likelihood function for the sample, and compute the Cramer-Rao lower bound. 3

(c) Derive the maximum likelihood (ML)estimator for p. 5

(d) Derive the mean and variance of the ML estimator and show whether or not it variance achieves the Cramer-Rao lower bound. 5

4.

(a) For any two events A and B, define independence. 5

(b) Three dice (each with six faces, numbered 1 to 6) are rolled. Define the events

$$\begin{aligned} A &= \text{The first and second show the same number} \\ B &= \text{The second and third show the same number} \\ C &= \text{The first and third show the same number} \end{aligned}$$

Show that the events A, B and C are pairwise independent but are not independent - i.e. A and B are independent,.B and C are independent, A and C are independent, but A, B and C are not independent. Attempt an intuitive explanation for this simple result. 10

5.

Let X_1, X_2, X_3 and X_4 be four mutually stochastically independent random variables, each with pdf $f_{X_i}(x_i) = 3(1 - x_i)^2$, $0 < x_i < 1$.

(a) If Y_4 denotes the minimum of these four variables, find the cdf and hence the pdf of Y_4. 5

(b) If instead Y_1 denotes the maximum of these four variables, work out the cdf and pdf of Y_1. 5

(c) Can you try to work out the joint cdf of Y_1 and Y_4? 5