STATISTICS: Problem Set 2: Multivariate Distributions

1. Let X_1, X_2, X_3 and X_4 be four mutually stochastically independent random variables, each with pdf $f_{X_i}(x_i) = 3(1 - x_i)^2$, $0 < x_i < 1$. If Y is the minimum of these four variables, find the cdf and the pdf of Y.
(*Hint:* $F_Y(y) = P(Y \leq y) = 1 - P(Y > y)$ *and* $P(Y > y) = P(X_i > y,\ i = 1, 2, 3, 4)$).

2. Let X_1 and X_2 be jointly exponential with density function given by
$p_{X_1,X_2}(x_1, x_2) = \lambda^2 e^{-\lambda(x_1+x_2)}$, $x_1 > 0$, $x_2 > 0$.
(a) Find the marginal distribution of X_1
(b) Find the conditional distribution of $Y = X_1 + X_2$ given X_1
(c) Find $E(X_1)$, $Var(X_1)$ and $Var(Y|X_1)$

3. Consider the joint density function of the random variables X and Y given by
$f_{X,Y}(x, y) = 8xy$, $0 < x < y < 1$; 0 elsewhere.
Show that:
(a) The marginal density $f_X(x) = 4x(1 - x^2)$, $0 < x < 1$
(b) The marginal density $f_Y(y) = 4y^3$, $0 < y < 1$
(c) Find the conditional densities $f_{X|Y}(x, y)$ and $f_{Y|X}(y, x)$, paying attention to the interval over which the functions are defined.
(d) Show that $E(Y|X = x) = \frac{2}{3}\left(\frac{1-x^3}{1-x^2}\right)$
(e) Show that $E(X|Y = y) = \frac{2}{3}y$

4. Using the result according to which if two random variables X and Y are independent, then the mgf of $Z = X + Y$ is the product of the mgf of both X and Y, show that the

sum of two poisson is poisson distributed.

5. Let X_1, X_2,..., X_T be iid with normal density $N(\mu, \sigma^2)$. Find the distribution of:

$$Y_n = \frac{\sum_{k=1}^{n} kX_k - \mu \sum_{k=1}^{n} k}{\sum_{k=1}^{n} k^2}$$

6. Let $Y_1 = \frac{1}{2}(X_1 - X_2)$, where X_1 and X_2 have the following joint pdf:

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{4} e^{-\frac{x_1+x_2}{2}} \mathbb{I}_{x_1>0} \mathbb{I}_{x_2>0}$$

Find the pdf of Y_1.

7. Suppose that the joint probability density function of the bivariate random variable (X,Y) is given by:

$$f_{X,Y}(x, y) = [1 - \alpha(1 - 2x)(1 - 2y)] \mathbb{I}_{0\leq x\leq 1} \mathbb{I}_{0\leq y\leq 1}$$

(a) Work out $E(XY)$
(b) Work out the marginal density functions $f_X(x)$ and $f_Y(y)$ and hence $E(X)$ and $E(Y)$.
(c) Attempt to prove or disapprove the following statement:
"In this example, the variables X and Y are independant if and only if they are uncorrelated."

8. Let X and Y be random variables with $E(X) = E(Y) = 0$. Assuming that $E(XY)$ exists and that $E(X|Y) = 0$, show that X and Y are uncorrelated.

9. The joint density of 2 random variables X and Y is given by:

$$f_{X,Y}(x, y) = \frac{1}{y} e^{-\frac{x}{y}} e^{-y} \mathbb{I}_{x>0} \mathbb{I}_{y>0}$$

Compute E(X|Y=y).