STATISTICS: Problem Set 4-Maximum Likelihood Estimation(I)

1. Let X_1, X_2,..., X_T be iid with the following density $p_{X_i}(x_i) = \lambda e^{-\lambda x_i}$, $x_i > 0$, $\lambda > 0$.
(a) Find an expression for the log-likelihood function as a function of the parameter λ, conditioned on the available data.

(b) Evaluate the score function $s_T(\lambda)$, that is the first derivative of the log-likelihood function.
(c) Work out the maximum likelihood estimator $\hat{\lambda_{ML}}$.
(d) Try to work out its variance(Do not spend too much time on that question...)
(e) Evaluate the Information Matrix $i_T(\lambda)$.
(f) Is $\hat{\lambda_{ML}}$ an efficient estimator of λ? Comment.

2. Let X_1, X_2,..., X_T be iid with normal density $N(\mu, \sigma^2)$.
(a) Find an expression for the log-likelihood function as a function of the parameter $\theta = (\mu, \sigma^2)'$, conditioned on the available data.
(b) Evaluate the score function $s_T(\theta)$.
(c) Work out the maximum likelihood estimator $\hat{\theta_{ML}}$.
(d) Work out its variance.
(e) Evaluate the Information Matrix $i_T(\theta)$.
(f) Is the maximum likelihood estimator of σ^2 an unbised estimator of σ^2? Does is achieve the Cramer-Rao lower band? Comment.

3.
(a) Define a maximum likelihood estimator.
(b) Let X_1, X_2,..., X_T be iid with poisson distribution given by:

$$f_{X_i}(x_i) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

Derive the log-likelihood function for the sample, and compute the *Cramer-Rao* lower bound.
(c) Derive the maximum likelihood estimator for λ.
(d) Derive the mean and variance of the ML estimator and show whether or not its variance achieves the Cramer-Rao bound.

4. Consider the process:

$$\begin{pmatrix} y_t \\ z_t \end{pmatrix} \tilde{} IN_2 \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{matrix} \sigma^2 & \varphi \\ \varphi & \omega^2 \end{matrix} \right] \text{ for } t = 1, 2..., T.$$

(a) You want to esimate the parameter β in a conditional linear model relating y_t to z_t. How can this be justified? Derive β as a function of the parameters of the process above.

(b) Write down the likelihood function for the bivariate process (use the assumption of independence) and find the maximum likelihood estimator $\widehat{\beta}$ of β.

5. Let

$$y_t = \beta z_t + u_t,$$

where

$$u_t = \rho u_{t-1} + \varepsilon_t$$

with $|\rho| < 1$ and $\varepsilon_t \tilde{} IN(0,1)$.

You may also assume that the z_t are determined outside the model, and can be taken as fixed in repeated samples.

Obtain the score and information matrix for the problem and show that the maximum likelihood estimators of β and ρ are independent of each other (i.e. the information matrix is diagonal).

Briefly describe how you would calculate the estimators.