Solving DSGE models

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Outline

- Two stationary DSGE models.

- Solutions approaches: Bellman equation and Stochastic Lagrangian.

- Perturbation methods: First and second order approximations of optimality conditions.

- Measuring accuracy.

- Other approximation methods.

- A few tips.

- Perturbation methods for non-stationary models.

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1 Two benchmark stationary models

1) A Real Business cycle model

- Social planner:

$$\max_{\{c_t, N_t, K_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - N_t) \tag{1}$$

where c_t is consumption, N_t hours, K_t capital, $0 < \beta < 1$. The resource constraint is:

$$c_t + K_t + g_t \leq f(K_{t-1}, N_t, \zeta_t) + (1 - \delta)K_{t-1}$$

- Timing convention: K_{t-1} capital available at the beginning of time t.

- Endowments: one unit of time ($0 \leq N_t \leq 1$) and K_{-1} units of capital, depreciating at the rate $0 < \delta < 1$.

- Constraints: $K_t, c_t \geq 0$ all t.

- Shocks:

$$\ln \zeta_t = \rho_\zeta \ln \zeta_{t-1} + \epsilon_{1t} \tag{2}$$
$$\ln g_t = \rho_g \ln g_{t-1} + \epsilon_{2t} \tag{3}$$

- How is g_t financed? Either lump sum taxes or bonds. Why it does not matter which one? Ricardian equivalence!

- Technical requirements:

  - $E_0 \sum_t \beta^t U(c_t, 1 - N_t)$ is bounded (otherwise maximum does not exist).

  - $U(c_t, 1 - N_t)$ is twice continuously differentiable, strictly increasing and strictly concave in all arguments;

  - $f(K_{t-1}, N_t, \zeta_t)$ is twice continuously differentiable, strictly increasing and strictly concave in K_{t-1} and N_t.

  - $E_t U_c$, $E_t U_{(1-N)}$, $E_t f_K$, $E_t f_N$ exist and are bounded.

• Optimal $(c_t, N_t, K_t)_{t=0}^{\infty}$ from the social planner problem is the same as the following competitive problem (because of the second welfare theorem).
Households:

$$\max_{\{c_t, N_t, K_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - N_t) \tag{4}$$

$$c_t + K_t \leq w_t N_t + r_t K_{t-1} + (1-\delta) K_{t-1} + T_t \tag{5}$$

Firms:

$$\max_{\{N_t, K_{t-1}\}_{t=0}^{\infty}} f(K_{t-1}, N_t, \zeta_t) - w_t N_t - r_t K_{t-1} \tag{6}$$

Government: $G_t = T_t$.

Resource constraint $c_t + K_t - (1-\delta) K_{t-1} + g_t \leq f(K_{t-1}, N_t, \zeta_t)$

2) A New Keynesian sticky price model

Four agents; one monopolistic competitive.

- Consumers:

$$\max_{\{c_t, N_t, K_t, M_t\}} E_0 \sum_t \beta^t [U(c_t, 1 - N_t) + V(\frac{M_t}{p_t})] \tag{7}$$

Budget constraint:

$$c_t + K_t + \frac{M_t}{p_t} \leq \frac{M_{t-1}}{p_t} + w_t N_t + (r_t + (1 - \delta))K_{t-1} + \frac{prf_t}{p_t}$$

$prf_t =$ nominal profits from intermediate firms.

Endowments: One unit of time. M_{-1} units of money, K_{-1} units of capital.

- Final goods producing firms: competitive.

  Production: $gdp_t = (\int_0^1 inty_{it}^{\frac{1}{1+\varsigma_p}} di)^{1+\varsigma_p}$.

  Output price $p_t = (\int_0^1 p_{it}^{-\frac{1}{\varsigma_p}} di)^{-\varsigma_p}$, $\varsigma_p > 0$

  p_{it} = price of intermediate good i, $inty_{it}$ = quantity of intermediate good i.

  Profit $= (gdp_t p_t - \int_0^1 inty_{it} p_{it} di)$.

  Firms maximize profits is with respect to $inty_{it}$ for each i, t. Output price p_t and input prices p_{1t} are taken as given.

- Intermediate good producing firms: monopolistic competitive.

  Problem is split in two parts. Cost minimization:

$$\min_{\{K_{it-1}, N_{it}\}} (r_t K_{it-1} + \frac{W_t}{p_t} N_{it}) \tag{8}$$

  subject to $inty_{it} = \zeta_t K_{it-1}^{1-\eta} N_{it}^{\eta}$. Discounted profit maximization

$$\max_{\{p_{it^*}\}} \sum_j E_t(\beta^j \frac{U_{c,t+j}}{p_{t+j}} \gamma_p^j) prf_{it+j} \tag{9}$$

  subject to the demand function of the final good firms.

  At each t, only $(1 - \gamma_p)$ firms can change prices. $\frac{U_{c,t+1}}{p_{t+1}}$ = real value of a unit of profit to shareholders next period; $prf_{it+j} = (p_{it^*} - mc_{it+j}) inty_{it+j}$; mc_{it} = marginal costs.

- Monetary authority:

$$i_t = \kappa i_{t-1}^{\varpi_0} \pi_t^{\varpi_1} y_t^{\varpi_2} \epsilon_{2t} \tag{10}$$

where κ is a constant, $\varpi_0, \varpi_1, \varpi_2$ parameters, ϵ_{2t} a iid policy shock.

- Government budget

$$g_t - T_t = \frac{M_t - M_{t-1}}{p_t} \tag{11}$$

- Resource constraint:

$$k_t - (1-\delta)k_{t-1} + c_t + g_t = gdp_t \tag{12}$$

- Alternative formulation: no final good firms, demand functions for intermediate goods directly determined by consumers via expenditure optimization (now c_t is a basket of goods).

2 Solution methods

General idea:

1) Optimality conditions of the form $E_t f(y_t, \epsilon_t, \theta) = 0$, where ϵ_t are exogenous variables, y_t the endogenous variables and θ a vector of parameters.

2) Solution $y_t = h(\epsilon_t, \theta)$ must be such that $E_t f(h(\epsilon_t, \theta), \epsilon_t, \theta) = 0$.

How do we find h?

- Stochastic Lagrangian Multiplier (LM) approach.

- Dynamic Programming (DP) approach.

DP more restrictive than LM. It needs:

- Competitive equilibrium to be Pareto optimal (see Hansen and Prescott (1995) for an adaptation to suboptimal convex problems).

- Utility function to be time separable in the contemporaneous control and state variables (controls=variables you maximize w.r.t; states=predetermined and exogenous variables).

- Objective function and the constraints to be such that current decisions affect current and future utility but not past one. (IMPORTANT!!)

2.1 Stochastic Lagrangian approach

Set up the (stochastic) Lagrangian and maximize it.

Example 1 *(Two period consumption-saving decisions under certainty)*

$$\max_{\{c_1, c_2, k_1\}} U(c_1) + \beta U(c_2) \tag{13}$$

subject to

$$\begin{aligned} c_1 + k_1 &= \zeta_1 f(k_0) \\ c_2 &= \zeta_2 f(k_1) \end{aligned}$$

k_0 *given,* ζ_1, ζ_2 *known productivity disturbances.*

Lagrangean:

$$\max_{\{c_1,c_2,k_1\}} U(c_1) + \beta U(c_2) \quad - \\ \lambda_1(c_1 + k_1 - \zeta_1 f(k_0)) - \beta\lambda_2(c_2 - \zeta_2 f(k_1)) \tag{14}$$

Taking the FOC of (14) with respect to (c_1, c_2, k_1)*:*

$$U_1 = \lambda_1 \tag{15}$$

$$\beta U_2 = \beta\lambda_2 \tag{16}$$

$$\lambda_1 = \beta\lambda_2\zeta_2 f_1 \tag{17}$$

Combining (15)-(17) leads to

$$U_1 = \beta U_2 \zeta_2 f_1 = \beta U_2 r_1 \tag{18}$$

where $r_1 = \zeta_2 f_1$ *is the real rate,* $U_t = \frac{\partial U}{\partial c_t},\ t = 1, 2,\ f_1 = \frac{\partial f}{\partial k_1}$

Interpretation: Marginal utility of giving up one unit of consumption today = marginal benefit of investing and using one unit of k tomorrow. Along equilibrium path, agents must be indifferent.

If $U(c_t) = \frac{c_t^{(1-\sigma)}}{1-\sigma}$ (CRRA utility), (18) becomes:

$$1 = \beta(\frac{c_2}{c_1})^{-\sigma}\zeta_2 f_1 \quad (19)$$

1) Consumption growth must be related to the real rate (composed of the rate of time preferences and the return to capital).

2) If $\beta\zeta_2 f_1 = 1$ then $c_1 = c_2$, i.e. consumption must be constant along the optimal path.

Extensions

• Uncertainty about future production possibilities (ζ_t is a random variable)

$$U_1 = E_t \beta U_2 \zeta_2 f_1 = E_t \beta U_2 r_1 \tag{20}$$

Same Euler equation, but now it holds in expectations.

• Add infinitively lived agents.

(20) must hold for every possible t. No natural terminal condition for k_t. But, as $t \to \infty$, $\beta^t \lambda_t k_t \to 0$, where $\beta^t \lambda_t$ is the Arrow-Debreu price (transversality condition).

Example 2 *(Two periods price decisions in NK model with log utility, half of the firms changing price at each $t, g_t = 0$). Choose prices to maximize expected profis subject to demand functions. Lagrangen*

$$\max_{\{p_{it^*}\}} \frac{1}{c_t p_t} prf_t + E_t[\beta \frac{1}{c_{t+1} p_{t+1}} prf_{it+1}] \quad - \\ \lambda_{1t}(\frac{inty_{it}}{gdp_t} - (\frac{p_{it^*}}{p_t})^{-\frac{1+\varsigma_p}{\varsigma_p}}) \quad - \\ E_t[\lambda_{2t}(\frac{inty_{it+1}}{gdp_{t+1}} - (\frac{p_{it+1^*}}{p_{t+1}})^{-\frac{1+\varsigma_p}{\varsigma_p}})] \tag{21}$$

Optimality implies (all firms are identical so drop i subscript)

$$\frac{p_t^*}{p_t} = E_t(1+\varsigma_p)(\frac{U_{c,t} c_t w_t + \beta U_{c,t+1} c_{t+1} w_{t+1} \pi_{t+1}^{\frac{1+\varsigma_p}{\varsigma_p}}}{U_{c,t} c_t + \beta U_{c,t+1} c_{t+1} \pi_{t+1}^{\frac{1}{\varsigma_p}}}) \tag{22}$$

p_t^* *is the optimal price,* w_t *the wage rate and* $\pi_t = \frac{p_{t+1}}{p_t}$ *the inflation rate.*

- Ideally firms would like to charge a price which is a constant markup $(1 + \varsigma_p)$ *over marginal (labor) costs. However, because individual prices are set for two periods, firms can't do this and when prices are allowed to be changed, they are set as a constant markup over current and expected future marginal costs.*

- If there are no shocks, $\pi_{t+1} = 1, w_{t+1} = w_t, c_{t+1} = c_t$ *and* $\frac{\tilde{p}_t}{p_t} = (1 + \varsigma_p)w_t$.

Back to the RBC Model, Lagrangean:

$$\max_{\{c_t,N_t,K_t\}} E_0 \sum_t \beta^t \; [U(c_t, 1-N_t) \quad - \quad \lambda_t(c_t + K_t + g_t - f(K_{t-1}, \zeta_t, N_t) - (1-\delta)K_{t-1})] \tag{23}$$

Let $f_{K_t} = \frac{\partial f}{\partial K_t}; f_{N_t} = \frac{\partial f}{\partial N_t}; U_{c,t} = \frac{\partial U(c_t,1-N_t)}{\partial c_t}; U_{N,t} = \frac{\partial U(c_t,1-N_t)}{\partial N_t}$. The FOC are

$$E_t(\beta \frac{U_{c,t+1}}{U_{c,t}}[f'_{K_t} + (1-\delta)] - 1) \quad = \quad 0 \tag{24}$$

$$\frac{U_{c,t}}{U_{N,t}} - \frac{1}{f'_{N_t}} \quad = \quad 0 \tag{25}$$

$$c_t + K_t + g_t - f(K_{t-1}, N_t, \zeta_t) - (1-\delta)K_{t-1} \quad \leq \quad 0 \tag{26}$$

Solution for $(c_t, N_t, K_t)_{t=0}^{\infty}$, given k_{-1} and $(g_t, \zeta_t)_{t=0}^{\infty}$ is found using (24), (25), (26) and transversality condition $\lim_{t\to\infty} E_0(\beta^t \lambda_t K_t) = 0$, where λ_t is the Lagrangian multiplier.

Problem complicated! Need to solve a vector of nonlinear difference equations involving expectations subject to initial and terminal conditions.

In the sticky price model, when $g_t = 0$, optimality conditions are

$$1 = E_t[\beta \frac{U_{c,t+1}}{U_{c,t}}(r_{t+1} + (1-\delta))] = E_t[\beta \frac{U_{c,t+1}}{U_{c,t}} \frac{i_t}{\pi_{t+1}}] \quad (27)$$

$$0 = \frac{U_{c,t}}{U_{N,t}} - \frac{1}{f'_{N_t}} \quad (28)$$

$$0 = V_{\frac{M_t}{p_t}} - U_{c,t} + \beta E_t \frac{U_{c,t+1}}{\pi_{t+1}} \quad (29)$$

$$0 = E_t \sum_j (\beta^j \gamma_p^j \frac{U_{c,t+j}}{p_{t+j}})[\frac{p_t}{1+\varsigma_p} - mc_{t+j}] inty_{t+j} \quad (30)$$

We need to solve these four equations, the monetary policy rule and the production function for the six unknowns $(c_t, N_t, K_t, m_t, \pi_t, i_t)$ under the transversality condition, given K_{-1}, M_{-1}. Same complication!

2.2 Dynamic Programming approach

Consider the RBC model. Approach works in two steps:

• Transform the constrained problem into an unconstrained one (substitute constraints into utility).

• Split the infinite dimensional problem in a sequence of one period problems (recursive formulation).

1) Use the resource constraint into utility. Let $\psi_t = (\zeta_t, g_t)$. Then

$$\max_{\{c_t, K_t, N_t\}} E_0 \sum_t \beta^t U(c_t, 1 - N_t) =$$

$$\max_{\{K_t, N_t\}} E_0 \sum_t \beta^t U([f(K_{t-1}, \zeta_t, N_t) + (1 - \delta)K_{t-1} - K_t - g_t], 1 - N_t) \equiv$$

$$\max_{\{K_t, N_t\}} E_0 \sum_t \beta^t U(K_{t-1}, K_t, N_t, \psi_t)$$

2) Recursive formulation: rather than choosing $\{N_t, K_t\}_{t=0}^{\infty}$, choose (N_t, K_t) at each t assuming that $(N_{t+i}, K_{t+i}), i \geq 1$ will be optimally selected. Possible because:

a) U is time separable in the states and controls, i.e. we can rewrite the maximization problem as

$$\max_{\{K_t,N_t\}} E_0 \sum_{t=0} \beta^t U(K_{t-1}, K_t, N_t, \psi_t) =$$

$$\max_{K_0,N_0} U(K_{-1}, K_0, N_0, \psi_0) + \max_{K_1,N_1} \beta U(\bar{K}_0, K_1, N_1, \psi_1) +$$

$$\max_{\{K_2,N_2\}} E_0 \beta^2 U(\bar{K}_1, K_2, N_2, \psi_2) + \max_{\{K_t,N_t\}} E_0 \sum_{t=3}^{\infty} \beta^t U(\bar{K}_{t-1}, K_t, N_t, \psi_t)$$

given K_{-1} and separately optimize at each t since (K_0, N_0) enter only the first maximization, (K_1, N_1) the second, etc.

b) If agents are faced with the same states $(\bar{K}_{t-1}, \psi_t)$ they will choose the same optimal (K_t, N_t) at each t (recursive problem).

What are the states? What are the controls?

States $y_{2t} \equiv (K_{t-1}, \psi_t)$, controls $y_{1t} \equiv (N_t, K_t)$.

- Time invariant setup (Bellman equation):

$$\begin{aligned} V(y_2) &= \max_{\{K^+,N\}} \{U[f(K,N,\psi) + (1-\delta)K - K^+], 1-N\} \\ &+ \beta E[V(y_2^+)|y_2]\} \end{aligned} \tag{31}$$

$E(V|*) =$ conditional expectation; a "$+$" indicates future values.

- You need expectations since the future is unknown.

- Original problem depends on t. Bellman equation is time invariant.

How do we solve (31)? Complicated in general.

- There is an infinite number V functions, one for each value of y_{2t}.

- There is an infinite number of unknowns (V unknown).

- There are expectations of future variables.

Special cases when Bellman equation can be solved analytically:

- The time horizon is finite (e.g. natural resource extraction problem).

- The number of states is finite.

- Utility is logarithmic or quadratic and the constraints are linear.

Example 3 *(Finite time horizon) Suppose T=2, and that we want to minimize* $\sum_{t=0}^{1}(x_t^2 + v_t^2) + x_2^2$ *subject to* $x_{t+1} = 2x_t + v_t, x_0 = 1$ *given, by choices of* v_t*. Recursive formulation:*

a) Stage 1: $\min_{v_1}(x_1^2 + v_1^2 + x_2^2)$ *subject to* $x_2 = 2x_1 + v_1, x_1 = \bar{x}_1$*, some* $\bar{x}_1$*. Using the constraint into the objective function*

$$V(x_1, 1) = \min_{v_1}(\bar{x}_1^2 + v_1^2 + (2\bar{x}_1 + v_1)^2) \tag{32}$$

which implies: $v_1 = -x_1$ *and* $V(x_1, 1) = 3x_1^2$*.*

b) Stage 2: $\min_{v_0}(x_0^2 + v_0^2 + V(x_1, 1))$ *subject to* $x_1 = 2x_0 + v_0, x_0 = 1$*. Using the constraint into the objective function*

$$V(x_0, 2) = \min_{v_0}(1 + v_0^2 + 3(2 + v_0)^2) \tag{33}$$

which implies: $v_0 = -1.5$ *and* $V(x_0, 2) = 4.0$*. Using* v_0 *into the law of motion of* x_t *we have* $x_1 = 0.5, x_2 = 0.5$ *and* $V(x_1, 1) = 0.75$*.*

Example 4 *(Finite number of states) A supermarket chain needs to allocate six liters of milk to three outlets. If milk is sold gives 2 Euros, if unsold 0.5 Euros.*

Store	Demand for Milk	Probability	Allocation of Milk	Value of Allocation
1	1	0.6	1	2.0
	2	0.0	2	3.1
	3	0.4	3	4.2
2	1	0.5	1	2.0
	2	0.1	2	3.25
	3	0.4	3	4.35
3	1	0.4	1	2.0
	2	0.3	2	3.4
	3	0.3	3	4.35

Allocation value calculated as follows: if 1 liter allocated to store 1: 0.6 2+ 0.4 * 2=2.0; if 2 liters to store 1: 0.6* 2.0 + 0.4* (2 x2) +0.6*0.5 = 3.1; if 3 liters to store 1: 0.6* 2+ 0.4*(2 x 3)+ 0.6 *(2 x 0.5)=4.2, etc.*

9 possible allocations-stores combinations (allocations of 0 or greater than 3 unfeasible). Work backward from store 3 (the far away one). Choice set for store 3: $\{$1,2,3$\}$; for stores 2 and 3: $\{$3,4,5$\}$; for all stores: $\{$6$\}$.

Let $u_i(x_i)$ be the value of allocating x_i liters to store i. Then:

$$V_3(x_3) = u_3(x_3) \tag{34}$$

$$V_2(x_2) = \max_{x_2+x_3 \in [3,4,5]} u_2(x_2) + V_3(x_3) \tag{35}$$

$$V_1(x_1) = u_1(6 - x_3 - x_2) \tag{36}$$

Solving Belman equation involves calculating the total value $V_1(x_1) + V_2(x_2) + V_3(x_3)$ for 9 possible combinations of (x_1, x_2, x_3).

Optimal decision: $\{$1,3,2$\}$. Return function is 9.75 Euros.

Note: Decision $\{$2,2,2$\}$ also gives 9.75 Euros. Why it is not chosen?

Example 5 *(log utility) Let* $U(c_t, 1-N_t) = \ln(c_t)$, $\delta = 1$, $y_t = \zeta_t K_{t-1}^{1-\eta}$, $g_t = 0,\ \forall t$, $\ln z_t = \rho \ln z_{t-1} + e_t$

If $V^0(K,\zeta) = 0$, *Bellman equation maps logarithm functions into logarithmic ones, and the limit* $V(K,\zeta)$ *will be logarithmic. Solution found as follows:*

(a) Guess a solution: $V(K,\zeta) = V_0 + V_1 \ln K_{t-1} + V_2 \ln \zeta_t$.

(b) Substitute the guess in (31);

$$V(K,\zeta) = \max_{K_t} \ln(K_{t-1}^{1-\eta}\zeta_t - K_t) + \beta E_t(V_0 + V_1 \ln K_t + V_2 \ln \zeta_{t+1}) \quad (37)$$

(c) The FOC of the perfect foresight problem is: $\frac{1}{K_{t-1}^{1-\eta}\zeta_t - K_t} = \frac{\beta V_1}{K_t}$ *or*

$$K_t = \frac{\beta V_1}{1+\beta V_1} K_{t-1}^{1-\eta}\zeta_t \quad (38)$$

(d) Using (38) in (37)we have

$$V_0 + V_1 \ln K_{t-1} + V_2 \ln \zeta_t = \ln(K_{t-1}^{1-\eta}\zeta_t - \frac{\beta V_1}{1+\beta V_1} K_{t-1}^{1-\eta}\zeta_t) + \beta V_0 + \beta V_1 \ln(\frac{\beta V_1}{1+\beta V_1} K_{t-1}^{1-\eta}\zeta_t) + V_2 \beta\rho \ln \zeta_t \quad (39)$$

(e) Collecting terms and matching coefficients $V_1 = \frac{(1-\eta)}{1-\beta(1-\eta)}$. *Inserting* V_1 *into (39) we get* $V_0 = \frac{\ln(1-\beta(1-\eta))+\beta V_1 \ln(\beta(1-\eta))}{1-\beta}$ *and* $V_2 = \frac{1+\beta V_1}{1-\beta\rho}$.

(f) Using the expression for V_1 *in (38) we have* $K_t = \beta(1-\eta)\zeta_t K_{t-1}^{1-\eta}$ *and from the budget constraint* $c_t = (1-\beta(1-\eta))\zeta_t K_{t-1}^{1-\eta}$.

Note : with a "guess and verify" approach, you need to check that the solution satisfies the first order conditions and the resource constraint.

• Solving the Bellman equation analytically is generally impossible.

• With the assumptions made on utility and production, if $\beta < 1$, we know V exists and is unique (see Lucas and Stokey (1989)), so we can compute V numerically (brute force approach).

Algorithm 2.1

1. *Choose an (initial) differentiable and concave* $V^0(K,\psi)$. *Select parameters of the model* (β, δ, *etc.*), *set a tolerance level* ι.

2. *Compute* $V^1(K,\psi) = \max_{\{K^+,N\}}\{U[f(K,N,\psi)+(1-\delta)K - K^+ - g], 1-N\} + \beta E[V^0(K^+,\psi^+)|K,\psi]\}$

3. *For each* $j = 2, 3, \ldots$, *set* $V^0 = V^{j-1}$ *and repeat step 2. until* $||V^j - V^{j-1}|| < \iota$.

4. *When* $V^j \approx V^{j-1}$, *compute* $K^+ = h_1(K,\psi)$, $N = h_2(K,\psi)$ *maximizing* $\{U[f(K,N,\psi)+(1-\delta)K-g-K^+], 1-N\}+\beta E[V^j(K^+,\psi^+)|K,\psi]\}$ *for* K^+, N. *Get* c *from the budget constraint.*

Problems with a brute force approach:

- Iterations complicated when number of states is large or the horizon is infinite (we need to find a function in a high dimensional space).

- Unless V^0 is appropriately chosen, the iteration process is time and computationally consuming (it may take a long time to converge).

Example 6 *(A case where DP can't be used) Two stage game, goverment dominant player.*

Stage 1: Given (T_t, p_t)*, agents choose* (c_t, M_t)*, to maximize*

$$E_0 \sum_t \beta^t (\ln c_t + \gamma \ln \frac{M_t}{p_t})$$

subject to

$$c_t + \frac{M_t}{p_t} \leq w_t + \frac{M_{t-1}}{p_t} + T_t$$

where M_t *are nominal assets,* w_t *is labor income and* T_t *lump sum taxes.*

- *Optimality implies*

$$\frac{1}{c_t p_t} = \beta \frac{1}{c_{t+1} p_{t+1}} + \frac{\gamma}{M_t} \tag{40}$$

• *Solving forward and using the resource constraint* $c_t = w_t - g_t$

$$\frac{1}{p_t} = \gamma(w_t - g_t) \sum_{j=0}^{\infty} \beta^j \frac{1}{M_{t+j}} \tag{41}$$

• *The government budget constraint is* $g_t = \frac{M_t}{p_t} - \frac{M_{t-1}}{p_t} + T_t$ *where* g_t *is random. Government chooses* M_t *to maximize agents' welfare subject to (41) and the resource constraint.*

Substituting (41) into the utility we have

$$\max_{M_t} \sum_t \beta^t (\ln c_t + \gamma \ln(M_t \gamma(w_t - g_t) \sum_{j=0}^{\infty} \beta^j \frac{1}{M_{t+j}}) \tag{42}$$

Future M_t *affects current utility. Government problem is not recursive; it can not be cast into a Bellman equation.*

Summary

• Solving dynamic general equilibrium model is hard: there are mathematical and numerical complications.

• Without a solution, there is little we can do (we can not estimate, we can not forecast, we can not do policy analyses, etc.).

• So what do we do? Calculate an approximate solution to the problem which is valid under in certain circumstances.

3 Approximation Methods

• Many approximation algorithms are available.

- Some locally valid (around some point).

- Other globally valid (for the whole domain of the states).

Older approaches:

- Quadratic approximations of the utility (solves a linear problem).

- Discrete approximations of shocks and state space (solve a finite set of equations).

- Approximation of expectations in the optimality conditions.

3.1 Perturbation methods, basic ideas.

Let x_t be a scalar and θ a vector of (known) parameters. Suppose we want to solve $f(x_{t+2}, x_{t+1}, x_t, \theta) = 0$, where the functional form of f is known. Here there is no uncertainty; we will add uncertainty later on.

(Think about this as a first order condition of an optimization problem when the constraints have been substituted in).

We seek a solution of the form $x_{t+1} = h(x_t, \theta)$. Then, it must be that:

$$f(h(h(x_t, \theta)), h(x_t, \theta), x_t, \theta) \equiv F(x_t, \theta) = 0 \tag{43}$$

Taking a Taylor expansion of $h(x_t)$ around some $x_t = \bar{x}$ yields:

$$\begin{aligned} h(x_t, \theta) &= h(\bar{x}, \theta) + h'(\bar{x}, \theta)(x_t - \bar{x}) + 0.5h''(\bar{x}_t, \theta)(x_t - \bar{x})^2 + \ldots \\ &\equiv h_0 + h_1(x_t - \bar{x}) + 0.5h_2(x_t - \bar{x})^2 + \ldots \end{aligned} \tag{44}$$

To compute an approximate solution we need to find $h_0, h_1, h_2, \ldots$. Perturbation methods give a sequential (recursive) approach to calculate these coefficients.

Strategy:

- To find $h_0 \equiv h(\bar{x}, \theta)$ use a numerical solver (h_0 is typically the steady state solution of the model) or the first best solution (e.g. flexible price equilibrium in price distorted economies).

- To find h_1, note that if $F(\bar{x}, \theta) = 0$, $F^j(\bar{x}, \theta) = 0$, $j = 1, 2, \ldots$. Then

$$\begin{aligned} F'(x_t, \theta) &= \frac{\partial f}{\partial x_{t+2}} \frac{\partial h}{\partial x_{t+1}} \frac{\partial h}{\partial x_t} + \frac{\partial f}{\partial x_{t+1}} \frac{\partial h}{\partial x_t} + \frac{\partial f}{\partial x_t} \\ &= f_1 h_1^2 + f_2 h_1 + f_3 = 0 \end{aligned} \tag{45}$$

where $f_1 = \frac{\partial f}{\partial x_{t+2}}$; $f_2 = \frac{\partial f}{\partial x_{t+1}}$; $f_3 = \frac{\partial f}{\partial x_{t+2}}$ are evaluated at $x_t = \bar{x}$ and

$$\frac{\partial h(x_t)}{\partial x_t} = h_1 + h_2(x_t - \bar{x}) + \ldots = h_1 \tag{46}$$

when evaluated at $x_t = \bar{x}$.

• Given $f_i, i = 1, 2, 3$, (45) is a quadratic equation in h_1. If the utility function and the production function are concave, one of the two solutions will be greater than one and one will be less than one (saddle path). A method to solve this quadratic equation when x is a vector is given below.

- To find h_2 differentiate $F'(x_t, \theta) = 0$ to obtain

$$\begin{aligned} F''(x_t,\theta) &= (f_{11}h_1^2 + f_{12}h_1 + f_{13})h_1^2 + f_1(h_1h_2 + h_2h_1^2) \\ &+ (f_{21}h_1^2 + f_{22}h_1 + f_{23})h_1 + (f_{31}h_1^2 + f_{32}h_1 + f_{33}) \\ &+ f_2h_2 = 0 \end{aligned} \tag{47}$$

where $f_{ij} = \frac{\partial^2 f}{\partial x_{t+i}\partial x_{t+j}}$ and $\frac{\partial^2 h(x_t)}{\partial x_t^2} = h_2$ when evaluated at $x_t = \bar{x}$. Given h_1, f_i, f_{ij}, this is a linear equation in h_2. Easy to solve.

- Third order approximations. Differentiate $F'''(x_t, \theta)$ and solve for h_3

- For approximation higher than $j = 2$ need to solve linear equations to find $h_j, j = 3, 4, etc.$.

• If there is uncertainty, let σ be a scalar controlling the uncertainty in the equation. Then, the Taylor expansion is valid around $\sigma = 0$ (see later on or den Haan (2009a)).

• Procedure gives intuitive results (e.g. certainty equivalence for first order approximations, etc.).

• Perturbation techniques approximate arbitrary well a function at a point away from $\bar{x}$. To do this, the function h needs to be smooth (differentiable many times) and the Taylor expansion large enough.

For a fixed Taylor expansion (say, first order), the approximation away from $\bar{x}$ can be arbitrarily bad (see below).

- Perturbation methods rely on the **implicit function** theorem.

Let $H(x, y)$ be a function mapping $R^n \times R^m \rightarrow R^m$. Assume

i) $H_y(x, y)$ is not singular.

ii) $H(\bar{x}, \bar{y}) = 0$, some $(\bar{x}, \bar{y})$.

iii) H is differentiable a sufficiently large number of times.

Then there exists a $y = h(x)$ such that $H(x, h(x)) = 0$ and the derivatives of $h(x)$ can be obtained by implicit differentiation.

3.2 Log-linear Approximations

- Give the same result as linear perturbation applied to the log of the FOCs.

• Linear vs. log-linear approximations: which one to choose?

i) When the problem is mildly non-linear, a log-linear approximation is preferable.

ii) A Log-linear approximation has the interpretation of percent deviation from the steady state.

Mechanics of log-linear approximations:

Let $x_t = \log(X_t) - \log(\bar{X})$, X_t is a vector and $\bar{X}$ a pivot point. There are two types of FOCs of a DSGE model:

$$1 = E_t[g(x_{t+1}, x_t)] \tag{48}$$

$$1 = f(x_t, x_{t-1}) \tag{49}$$

with $f(x_t = 0, x_{t-1} = 0) = 1$; $g(x_{t+1} = 0, x_t = 0) = 1$.

A first order expansion of (48)-(49) around $(x_t, x_{t-1}) = (0, 0)$ gives:

$$0 \approx E_t[g_{t+1}x_{t+1} + g_t x_t)] \tag{50}$$

$$0 \approx f_t x_t + f_{t-1}x_{t-1} \tag{51}$$

where $f_j = \frac{\partial f}{\partial x_j}$ and $g_j = \frac{\partial g}{\partial x_j}$.

• (50) and (51) are a system of linear expectational equations. We know how to solve these since at least Sargent (1978).

• The approximation is valid only around $\bar{X}$. **The solution can not be generally used to study problems which involve large deviations from $\bar{X}$.** In particular: can't study e.g. dynamics due to unit roots. Can't study dynamics that lead to another regime, etc.

• Certain models do not have the setup of (48)-(49); e.g. Rotemberg and Woodford (1997): consumption at time t depends on the expectations of variables dated at $t + 2$. Need to redefine the problem.

Example 7 *In an RBC model with no goverment expenditure, log utility, where output is produced with capital and there one period lag in transforming investment into capital, the Euler equation is:*

$$c_t^{-1} = \beta E_t[c_{t+1}^{-1}(1-\delta) + \beta c_{t+2}^{-1}(\eta K_{t+1}^{\eta-1}\zeta_{t+2})] \quad (52)$$

Define $c_{t+1}^* = c_{t+2}$ *and assume* $\zeta_t = \rho\zeta_{t-1} + e_t, E_t(e_t) = 0$. *Solve using*

$$c_t^{-1} = \beta E_t[c_{t+1}^{-1}(1-\delta) + \beta(c_{t+1}^*)^{-1}(\eta(K_{t+1})^{\eta-1}(\rho\zeta_{t+1} + e_{t+2})]$$

$$c_{t+1}^* \equiv c_{t+2}$$

Rule: transform higher order processes into vectors of AR(1) processes.

Example 8 *(Example of a log linearization of a FOC)*

$$\frac{U_{c,t}}{U_{N,t}} = \frac{1}{f'_N} \tag{53}$$

Let $U(c, N) = \ln c_t + \ln(1 - N_t)$; $F(N, K, \zeta) = N_t^{\eta} K_{t-1}^{1-\eta} \zeta_t$. *Then (53) is*

$$\frac{1 - N_t}{c_t} = \frac{1}{\eta N_t^{\eta-1} K_{t-1}^{1-\eta} \zeta_t} \tag{54}$$

Taking logs

$$\log(1 - N_t) - \log c_t + \log \eta - (1 - \eta)(\log N_t - \log K_{t-1}) + \log \zeta_t = 0 \tag{55}$$

First order expansion of $\log(1 - N_t) \equiv h(N)$ *is* $h(\bar{N}) + h(\bar{N})'(N - \bar{N})$. *So*

$$\log(1 - N_t) = \log(1 - \bar{N}) + \frac{-1}{1 - \bar{N}}(N_t - \bar{N}) \quad =$$

$$\log(1 - \bar{N}) + \frac{-\bar{N}}{1 - \bar{N}} \frac{N_t - \bar{N}}{\bar{N}} \quad \equiv$$

$$\log(1 - \bar{N}) + \frac{-\bar{N}}{1 - \bar{N}} \hat{N}_t \tag{56}$$

Repeating the derivation for other terms and noting that values in the steady states cancel out we have

$$\frac{-\bar{N}}{1 - \bar{N}} \hat{N}_t + \hat{c}_t - (1 - \eta)(\hat{N}_t - \hat{K}_{t-1}) + \hat{\zeta}_t = 0 \tag{57}$$

Tricks to log-linearize without differentiation (Uhlig (1999)):

Replace X_t by $\bar{X}e^{x_t}$, x_t is small. Then for a_0 a constant and y_t small:

- $e^{x_t + a_0 y_t} \approx 1 + x_t + a_0 y_t$.

- $x_t y_t \approx 0$.

- $E_t[a_0 e^{x_{t+1}}] \propto E_t[a_0 x_{t+1}]$.

Example 9 *Suppose the budget constraint is* $C_t + G_t + INV_t = GDP_t$. *Then* $\bar{C}e^{c_t} + \bar{G}e^{g_t} + i\bar{n}ve^{inv_t} = G\bar{D}Pe^{gdp_t}$; *use rule i) to get* $\bar{C}(1 + c_t) + \bar{G}(1+g_t)+i\bar{n}v\ (1+inv_t)-G\bar{D}P(1+gdp_t) = 0$. *Since* $\bar{C}+\bar{G}+i\bar{n}v = G\bar{D}P$ *this implies that* $\bar{C}c_t + \bar{G}g_t + i\bar{n}v\ inv_t - G\bar{D}Pgdp_t = 0$.

Solution to a system of log-linear equations can be found in two ways:

- with the method of the undetermined coefficients.

- finding the saddle-path (Vaughan's method). This is applicable only to a restricted number of cases, see King and Watson (2002).

3.2.1 Method of undetermined coefficients

Write log-linearized first order conditions as:

$$
\begin{aligned}
0 &= AAy_{2t} + BBy_{2t-1} + CCy_{1t} + DDy_{3t} && (58)\\
0 &= E_t[FFy_{2t+1} + GGy_{2t} + HHy_{2t-1} + JJy_{1t+1} + KKy_{1t} + LLy_{3t+1} + MMy_{3t}] \\
0 &= y_{3t+1} - NNy_{3t} - \epsilon_t && (59)
\end{aligned}
$$

$AA, \ldots, MM$, are matrices involving the parameters of the model and NN has only stable eigenvalues, y_{2t} are the states, y_{1t} are the controls, y_{3t} are the innovations in the shocks. Assume a solution of the form:

$$
\begin{aligned}
y_{2t} &= PPy_{2t-1} + QQy_{3t} && (60)\\
y_{1t} &= RRy_{2t-1} + SSy_{3t} && (61)
\end{aligned}
$$

• Uhlig (1999) shows that (60)-(61) exists and is unique and how PP, QQ, RR, SS can be found.

• PP, QQ, RR, SS are nonlinear combinations of the parameters entering the $AA, \ldots, NN$ matrices, which in turn are functions of the structural parameters θ.

• Solution is a state space system: (60) is a transition equation, (61) is a measurement equation.

• Solution is also a restricted VAR(1). Let

$$B_0 = \begin{bmatrix} SS \\ QQ \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & RR \\ 0 & PP \end{bmatrix}$$

Then $Y_t = B_1 Y_{t-1} + B_0 y_{3t}$, where $Y_t = [y_{1t}, y_{2t}]'$.

Example 10 *(Solving an RBC model with undetermined coefficients)*

$$\max_{\{c_t,K_t\}} E_0 \sum_t \beta^t \frac{c_t^{1-\varphi}}{1-\varphi} \tag{62}$$

$$c_t + K_t - (1-\delta)K_{t-1} - K_{t-1}^{1-\eta}\zeta_t = 0 \tag{63}$$

Let $E\zeta_t = \zeta^{ss}$; $\hat{\zeta} \equiv (\zeta_t - \zeta^{ss})/\zeta^{ss} = \rho\hat{\zeta}_{t-1} + e_t,\ e_t \sim (0, \sigma^2)$.

FOC:

$$\begin{aligned} E_t[c_t^{-\varphi} - \beta c_{t+1}^{-\varphi}((1-\eta)K_t^{-\eta}\zeta_{t+1} + (1-\delta))] &= 0 \\ c_t + K_t - (1-\delta)K_{t-1} - K_{t-1}^{1-\eta}\zeta_t &= 0 \end{aligned} \tag{64}$$

or

$$\begin{aligned} E_t[(-K_t + (1-\delta) + K_{t-1}^{1-\eta}\zeta_t)^{-\varphi} &\quad - \\ \beta(-K_{t+1} + (1-\delta) + K_t^{1-\eta}\zeta_{t+1})^{-\varphi}((1-\eta)K_t^{-\eta}\zeta_{t+1} + (1-\delta))] &\equiv \\ E_t[v(K_{t+1}, K_t, K_{t-1}, \zeta_{t+1}, \zeta_t)] &= 0 \end{aligned} \tag{65}$$

Solution: A $K_t = g(K_{t-1}, \zeta_t)$ *such that (65) is satisfied.*

Finding $g(K_{t-1}, \zeta_t)$ *is difficult. Compute an approximate solution around the steady states.*

a) Compute steady states: given ζ^{ss} *solve the following for* (c^{ss}, K^{ss})

$$(c^{ss})^{-\varphi} - \beta(c^{ss})^{-\varphi}((1-\eta)(K^{ss})^{-\eta}\zeta^{ss} + (1-\delta)) = 0 \quad (66)$$

$$c^{ss} + K^{ss} - (1-\delta)K^{ss} - (K^{ss})1 - \eta\zeta^{ss} = 0 \quad (67)$$

(These are the FOCs once we eliminate expectations and time subscript.)

Solution: $K^{ss} = (\beta^{-1} - (1-\delta))/(1-\eta)\zeta^{ss})^{1/\eta})$, $c^{ss} = \delta K^{ss} + (K^{ss})^{1-\eta}\zeta^{ss}$.

Note that $v(K^{ss}, K^{ss}, K^{ss}, \zeta^{ss}, \zeta^{ss}) = 0$ *(the steady state solution satisfies the first order conditions).*

b) Log-linear approximation around the steady state:

$$\begin{aligned} E_t v(K_{t+1}, K_t, K_{t-1}, \zeta_{t+1}, \zeta_t) &\approx \\ E_t(a_1\hat{K}_{t+1} + a_2\hat{K}_t + a_3\hat{K}_{t-1} + a_4\hat{\zeta}_{t+1} + a_5\hat{\zeta}_t) &= 0 \end{aligned} \tag{68}$$

where $\hat{K}_t = (K_t - K^{ss})/K^{ss}$, $a_1 = v_1 K^{ss}, a_2 = v_2 K^{ss}, a_3 = v_3 K^{ss}, a_4 = v_4 \zeta^{ss}, a_5 = v_5 \zeta^{ss}$ *and, e.g.,* $v_2 = \partial v(K^{ss}, K^{ss}, K^{ss}, \zeta^{ss}, \zeta^{ss})/\partial K_t$.

c) Use the method of undetermined coefficients to solve (68). That is:

i) Assume $\hat{K}_t = A\hat{K}_{t-1} + B\hat{\zeta}_t$.

ii) Find A, B *such that, for any* $\hat{K}_{t-1}, \hat{\zeta}_t$*, the following is satisfied*

$$\begin{aligned} E_t[a_1(A^2\hat{K}_{t-1} + AB\hat{\zeta}_t + B\rho\hat{\zeta}_t + Be_{t+1}) &+ \\ a_2(A\hat{K}_{t-1} + B\hat{\zeta}_t) &+ \\ a_3\hat{K}_{t-1} + a_4\rho\hat{\zeta}_t + a_4 e_{t+1} + a_5\hat{\zeta}_t] &= 0 \end{aligned} \tag{69}$$

or taking expectations of future variables

$$\begin{aligned} [a_1A^2 + a_2A + a_3]\hat{K}_{t-1} + (a_1AB + a_1B\rho + a_2B + a_4\rho + a_5)\hat{\zeta}_t &\equiv \\ F_1(A)\hat{K}_{t-1} + F_2(A,B)\hat{\zeta}_t &= 0 \end{aligned} \tag{70}$$

This equation is true if and only if $F_1(A) = 0$ *and* $F_2(A,B) = 0$*. These are two equations in two unknowns* A, B*. The solution is recursive; solve* $F_1(A) = 0$ *for* A*, plug solution into* $F_2(A,B) = 0$*, find* B*.*

Since $a_1 A^2 + a_2 A + a_3 = 0$*, the solution for* A *is*

$$A = -\frac{a_2 \pm (a_2^2 - 4a_1 a_3)^{0.5}}{2 * a_1} \tag{71}$$

Two roots for A. Possibilities:

- one root greater and one less than one (saddle point) $\longrightarrow$ *keep the one less than one. In this case the equilibrium is stable.*

- both greater than one $\longrightarrow$ *all equilibria are unstable. Linear approximation ceases to be useful.*

- both less than one: $\longrightarrow$ *multiple equilibria. Shocks may move economy across different equilibria, both of which are stable.*

Complication 1: if some control variables are chosen prior to the realization of shocks need to modify the equations. Suppose

$$\max E_0 \sum_t \beta^t \frac{c_t^{1-\varphi}}{1-\varphi} + \log(1 - N_t) \tag{72}$$

$$c_t + K_t - (1-\delta)K_{t-1} = N_t^{\eta} K_{t-1}^{1-\eta} \zeta_t \tag{73}$$

and suppose investment decision made after the shock but hours decision before the shock is realized. The FOC are

$$E_t U_k(K_{t+1}, N_{t+1}, K_t, N_t, K_{t-1}, \zeta_{t+1}, \zeta_t) = 0 \tag{74}$$

$$U_N(K_t, N_t, K_{t-1}, \zeta_t) = 0 \tag{75}$$

The log linearized FOC are:

$$E_t[a_1 z_{t+1} + a_2 z_t + a_3 z_{t-1} + a_4 s_{t+1} + a_5 s_t] = 0$$

where $E_t z_t = \begin{bmatrix} E_t(N_t|\hat{\zeta}_{t-1}) \\ E_t(K_t|\hat{\zeta}_t) \end{bmatrix}$, $s_t = \begin{bmatrix} \hat{\zeta}_t \\ \hat{\zeta}_{t-1} \end{bmatrix}$. Define $P \equiv \begin{bmatrix} \rho & 0 \\ 1 & 0 \end{bmatrix}$,
and let $a_1 = \begin{bmatrix} U_k^1 K^{ss} & U_k^2 N^{ss} \\ 0 & 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} U_k^3 K^{ss} & U_k^4 N^{ss} \\ U_N^1 K^{ss} & U_N^2 N^{ss} \end{bmatrix}$,
$a_3 = \begin{bmatrix} U_5^k K^{ss} & 0 \\ U_1^N K^{ss} & 0 \end{bmatrix}$, $a_4 = \begin{bmatrix} U_k^6 \zeta^{ss} & 0 \\ 0 & 0 \end{bmatrix}$, $a_5 = \begin{bmatrix} U_k^7 \zeta^{ss} & 0 \\ U_n^4 \zeta^{ss} & 0 \end{bmatrix}$
where U_i^j is the derivative of U_i with respect to its j-th element.

Assume $z_t = Az_{t-1} + Bs_t$. Solution requires:

$$F_1(A) = 0$$

$$0 = \begin{bmatrix} 0 & \rho F_{11} + F_{12} \\ F_{21} & F_{22} \end{bmatrix} s_t \equiv \tilde{F}_2 s_t = 0$$

Solve the first equation for A, the second for B once A is obtained. Note that $\tilde{F}_2$ is consistent with the information structure since
$B = \begin{bmatrix} 0 & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$.

Complication 2: if there is more than one control variable $F_1(A) = 0$ is a matrix equation so the solution is a matrix polynomial. There exist a unique stable solution to the problem under the conditions of Uhlig (1999).

- Solving the matrix quadratic equation $F_1(A) = 0$ is difficult. Typically cast the problem into a generalized eigenvalue-eigenvector setup; use the generalized Schur decomposition (see Uhlig (1999, Klein (2001)). Other methods: King and Watson (1998), Sims (2001). For standard problems, all methods give roughly the same results (see later).

- For stability: number of unstable roots = the number of jump (forward looking) variables. With some parametrization: less unstable roots than jump variables $\rightarrow$ multiple equilibria (many paths converge into the same steady state). This may occur for economic reasons (strategic complementarities, government policies, sunspots etc.), inappropriate choice of states or parameters.

Example 11 *(Sticky price model) Assume* $U(c_t, N_t, m_t) = \ln c_t + \ln(1 - N_t) + \frac{m_t^{1-\epsilon}}{1-\epsilon}$*, and the production function* $y_t = k_t^{\alpha} N_t^{1-\alpha} \zeta_t$*. Set* $N^{ss} =$ 0.33, $\pi^{ss} =$ 1.005, $\beta =$ 0.99, $(\frac{C}{GDP})^{ss} =$ 0.7, $\epsilon =$ 7 *(consumption elasticity of money demand),* $\gamma_p =$ 0.75 *(on average firms change prices every three quarters). Persistence of technology disturbances=0.95; persistence of monetary policy shocks=0.75 Parameters of policy rule:* $\varpi_1 =$ 0.5; $\varpi_2 =$ 1.6*. The decision rules are:*

$$\begin{bmatrix} \widehat{\pi}_t \\ \widehat{k}_t \\ \widehat{c}_t \\ \widehat{y}_t \\ \widehat{N}_t \\ \widehat{w}_t \\ \widehat{i}_t \\ \widehat{m}_t \end{bmatrix} = \begin{bmatrix} 0.12 & 0.02 \\ 1.36 & 0.90 \\ 0.80 & 0.53 \\ 16.95 & -0.51 \\ 26.49 & -1.37 \\ 0.80 & 0.53 \\ 10.07 & -0.25 \\ 1.30 & -0.11 \end{bmatrix} \begin{bmatrix} \widehat{\pi}_{t-1} \\ \widehat{k}_{t-1} \end{bmatrix} + \begin{bmatrix} -0.03 & 0.01 \\ 0.20 & -0.01 \\ 0.44 & -0.01 \\ 2.73 & -0.19 \\ 2.70 & -0.31 \\ 0.44 & -0.01 \\ 1.36 & 0.90 \\ 0.12 & 0.12 \end{bmatrix} \begin{bmatrix} \widehat{\epsilon}_{1t} \\ \widehat{\epsilon}_{2t} \end{bmatrix}$$

$\widehat{\epsilon}_{1t} =$ *technological disturbance;* $\widehat{\epsilon}_{2t} =$ *monetary disturbance.*

Interpretation:

- Monetary disturbances have little contemporaneous impact on all variables except interest rates (small real effects of monetary policy).

- Technology shocks explain the majority of the variance of consumption, output, hours and real wages at most horizons.

- Limited effect from lagged to current inflation (low persistence). Stronger effect from lagged to current capital.

- ϵ_{1t}*: contemporaneously* $y \uparrow, \ \pi \downarrow$. ϵ_{3t}*: contemporaneously* $y \downarrow, \ \pi \uparrow$. *Monetary and technology shocks have similar contemporaneous* y, π *correlations because of Taylor rule.*

Some tips

- Decision rules contain all the information you need about contemporaneous and lagged dynamics of the system. Learn to stare at them.

- Dynamics depend on the parameterization. Difficult to know how a parameter affects the decision rules: the mapping between the parameters θ and the matrices PP, QQ, RR, SS is unknown (**we only have a numerical approximation**).

- To learn about the properties of your model change θ and see how the decision rules change. If they do not, it will be difficult to estimate the parameters, even if the model is the true DGP.

- Is the solution accurate? Can we trust it?

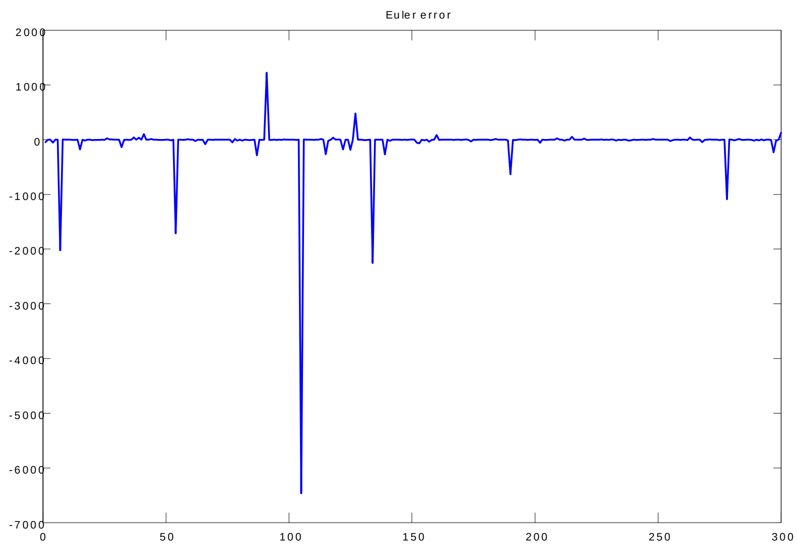
Recall: A good approximate solution must satisfy the first order conditions.

i) Simulated data for the approximate solution.

ii) Plug-in simulated data in the original optimality conditions (without expectations).

iii) Check properties of the errors (difference between right hand side and left hand side of the equations).

iv) Repeat 1)-iii) many times and average the errors over simulations.



Approximation error using log-linear solution in the Euler equation of a 3 equations NK model

Mean = -0.17*e+06; standard deviation 4.29*e+06.

3.3 Comparing methods

1) Uhlig (2001) (solving.m). System should be of the form:

$$0 = AAy_{2t} + BBy_{2t-1} + CCy_{1t} + DDy_{3t} \quad (76)$$
$$0 = E_t[FFy_{2t+1} + GGy_{2t} + HHy_{2t-1} + JJy_{1t+1} + KKy_{1t} + LLy_{3t+1} + MMy_{3t}]$$
$$0 = y_{3t+1} - NNy_{3t} - \epsilon_t \quad (77)$$

Solution is:

$$y_{2t} = PPy_{2t-1} + QQy_{3t} \quad (78)$$
$$y_{1t} = RRy_{2t-1} + SSy_{3t} \quad (79)$$

2) Klein (2001)([F,P]=solab(A,B,ns)). System should be of the form:

$$AE_t y_{t+1} = By_t \tag{80}$$

where $y_t = [x_t, y_{1t}], x_t = [z_t, y_{2t}]$; z_t is the vector of exogenous states and ns is the total number of states $(= dim(x_t))$.

The solution is:

$$y_{1t} = Fy_{2t} \tag{81}$$

$$y_{2t} = Py_{2t-1} + y_{3t} \tag{82}$$

Bonaldi (2009): Ulhig and Klein solutions are equivalent.

3) Sims (2001):([G4,c,G5,f1,f2,f3,gev,eu] = gensys(G0,G1,C,G2,G3,div)). System should be of the form

$$G_0 y_t = C + G_1 y_{t-1} + G_2 y_{3t} + G_3 \eta_t \tag{83}$$

where $y_t = [y_{1t}, y_{2t}]'$ and $\eta = y_{t+1} - E_t[y_{t+1}]$ is a vector of expectational errors.

Solution is:

$$y_t^* = c + G_4 y_{t-1}^* + G_5 y_{3t} + f_3 (I - f_1 \ L^{-1})^{-1} f_2 \ y_{3t+1}. \tag{84}$$

where $y_t^* = S y_t$, S is a selection matrix of rank equal to the number of stable eigenvalues of $[G_0, G_1]$ and η_t is assumed to be a linear function of the structural errors i.e. $\eta = V y_{3t}$, L is generated inside the program. Typically, div is omitted from argument list.

- If y_{3t} is iid solution is

$$y_t = c + G_4 y_{t-1} + G_5 y_{3t} \tag{85}$$

Note:

- eu(1)=1:existence; eu(2)=1 uniqueness; eu=[-2,-2] coincident zeros.

- Solution uses: 1) QZ decomposition of G_0, G_1, 2) eigenvalue-eigenvector decomposition of matrices of the system.

- Allows general form of serial correlation of y_{3t}, singularity of G_0, and it is valid both in discrete and continuous time.

- Can be used to calculate both determinate and indeterminate solutions.

- No distinction between states and controls (jump and predeterminated variables). Simply attach a forecast error to the variables with expectations.

3.4 Computing statistics from the solution

- Once you have the (linear) solution, it is easy to simulate data, compute moments, or trace out impulse responses.

Recall, the solution is of the form:

$$y_{2t} = PP(\theta)y_{2t-1} + QQ(\theta)y_{3t} \quad (86)$$

$$y_{1t} = RR(\theta)y_{2t-1} + SS(\theta)y_{3t} \quad (87)$$

• We need to choose the inputs: θ (the parameters), $\{y_{3t}\}_{t=1}^{T}$(the path for the shocks) and y_{20} (the initial conditions).

• To simulate data: with (86) and (87) produce time paths for $\{y_{1t}, y_{2t}\}_{t=1}^{T}$.

• To compute moments (in population): solve (86) for y_{2t}. Plug the solution in (87). Then

$$y_{1t} = RR(\theta)(I - PP(\theta))^{-1}QQ(\theta)y_{3t-1} + SS(\theta)y_{3t} \equiv H(L,\theta)y_{3t} \quad (88)$$

and $Ey_{1t} = H(L,\theta)E(y_{3t})$, var$(y_{1t}) = H(L,\theta)$ var$(y_{3t})H(L,\theta)'$, etc. (For sample moments: use the simulated data - careful since statistics depend on the draws of y_{3t}).

• To compute impulse responses:

i) Set $y_{2t-1} = 0, y_{3t} = 1(\sigma_e)$. Compute $y_{1t} = RR(\theta)y_{2t-1} + SS(\theta)y_{3t} = SS(\theta)y_{3t}$.

ii) Set $y_{3t+\tau} = 0$ for all $\tau > t$, and compute

$$\begin{aligned} y_{2t+\tau} &= PP(\theta)y_{2t+\tau-1} & (89) \\ y_{1t+\tau} &= RR(\theta)y_{2t+\tau} & (90) \end{aligned}$$

3.5 Second order approximations

First order approximations insufficient, when evaluating welfare across policies that do not have direct effects on steady state, when risk considerations are important (e.g. in asset pricing problems), etc.

Second order approximations intermediate between first order and global (nonlinear) approximations.

- Let $y_t = [y_{1t}, y_{2t}]'$. Write the FOC of the optimization problem as:

$$E_t \mathfrak{I}(y_{t+1}, y_t, \sigma\epsilon_{t+1}) = 0$$

$\mathfrak{I}$ is a $n \times 1$ vector of functions, y_t a $n \times 1$ vector of endogenous variables, ϵ_t a $n_1 \times 1$ vector of shocks $\sim (0, 1)$ and σ a parameter controlling the uncertainty in the shocks.

- First order approximation:

$$E_t[\mathfrak{J}_1 dy_{t+1} + \mathfrak{J}_2 dy_t + \mathfrak{J}_3 \sigma d\epsilon_{t+1}] = 0 \tag{91}$$

where dx_t is the deviation of x_t from some pivotal point.
This is computed assuming $y_{t+1} = h(y_t, \sigma\epsilon_{t+1}, \sigma)$, linearly expanding it around $h(y^{ss}, 0, 0)$, substituting the linear expression in (91) and matching coefficients.

- Second order approximation:

$$\begin{aligned} E_t[\mathfrak{J}_1 dy_{t+1} + \mathfrak{J}_2 dy_t + \mathfrak{J}_3 \sigma d\epsilon_{t+1} &+ \\ 0.5 \times \mathfrak{J}_{11} dy_{t+1} dy_{t+1} + \mathfrak{J}_{12} dy_{t+1} dy_t + \mathfrak{J}_{13} dy_{t+1} \sigma d\epsilon_{t+1} &+ \\ \mathfrak{J}_{22} dy_t dy_t + \mathfrak{J}_{23} dy_t \sigma d\epsilon_t + \mathfrak{J}_{33} \sigma^2 d\epsilon_{t+1} d\epsilon_{t+1})] &= 0 \end{aligned} \tag{92}$$

The solution principle is the same as in the linear case, since second order terms enter linearly in the specification.

Thus, assume $y_{t+1} = h(y_t, \sigma\epsilon_{t+1}, \sigma)$, take a second order expansion around $h(y^{ss}, 0, 0)$, substitute the expansion into (92) and match coefficients.

Schmitt Grohe and Uribe (2004): problem can be solved sequentially - find the first order terms first and the second order ones later (application of perturbation methods); see also Kim et al. (2008).

Example 12 *Two country RBC model, identical population and weights in social planner problem, capital adjustment costs are zero, output is produced with local capital only.*

The planner objective is $\max E_0 \sum_t \beta^t(\frac{c_{1t}^{1-\varphi}}{1-\varphi}+\frac{c_{2t}^{1-\varphi}}{1-\varphi})$*, the resource constraint is* $c_{1t}+c_{2t}+k_{1t}+k_{2t}-(1-\delta)(k_{1t-1}+k_{2t-1})=\zeta_{1t}k_{1t-1}^{1-\eta}+\zeta_{2t}k_{2t-1}^{1-\eta}$ *and* $\ln\zeta_{it},\ i=1,2$ *is iid with mean zero and variance* σ^2*.*

In equilibrium $c_{1t}=c_{2t}$ *and Euler equations for capital accumulations in the two countries are identical. Setting* $\varphi=2, \delta=0.1, 1-\eta=0.3, \beta=0.95$*, the steady state is* $(k_i,\zeta_i,c_i)=(2.62,1,00,1.07), i=1,2$*. The first order policy function for* $i=1,2$ *is:*

$$\left[\ k_{it}\ \right]=\left[\begin{array}{cccc} 0.444 & 0.444 & 0.216 & 0.216 \\ & & & \end{array}\right]\left[\begin{array}{c} k_{1t-1} \\ k_{2t-1} \\ \zeta_{1t} \\ \zeta_{2t} \end{array}\right] \tag{93}$$

The second order policy function is

$$
\begin{aligned}
k_{it} \;&=\; \begin{bmatrix} 0.444 & 0.444 & 0.216 & 0.216 \\ & & & \end{bmatrix} \begin{bmatrix} k_{1t-1} \\ k_{2t-1} \\ \zeta_{1t} \\ \zeta_{2t} \end{bmatrix} - 0.83\sigma^2 \\
&+\; 0.5 \begin{bmatrix} k_{1t-1} & k_{2t-1} & \zeta_{1t} & \zeta_{2t} \end{bmatrix} \begin{bmatrix} 0.22 & -0.18 & -0.02 & -0.08 \\ -0.18 & 0.22 & -0.08 & -0.02 \\ -0.02 & -0.08 & 0.17 & -0.04 \\ -0.08 & -0.02 & -0.04 & 0.17 \end{bmatrix} \begin{bmatrix} k_{1t-1} \\ k_{2t-1} \\ \zeta_{1t} \\ \zeta_{2t} \end{bmatrix}
\end{aligned}
\tag{94}
$$

- *In second order solution, σ^2 matters. When technology shocks are volatile, less capital will be accumulated.*

- *Entries in second order terms small. Curvature of the solution small.*

3.6 Other approaches

i) Perfect foresight approach.

- Solves non-linear equations of the system (no log-linearization needed).

- Assumes that future values of y_t are known.

- Needs very good initial conditions (steady states could be used if they can be easily calculated).

ii) Global methods.

- Projection methods/Parametrizing expectations.

- Collocation methods.

3.7 A Comparison of methods

JBES (1991): Special issue on the topic.

Marimon and Scott (1999): full array of methods

Ruge Murcia (2006): comparison of solution/estimation

Fernandez Villaverde and Rubio Ramirez (2006): linear vs. nonlinear approximations

Caldara, Fernandez, Rubio, Yao (2009): need non-linear approximations with non-expected utility models.

3.8 Additional Tips

- Try first order approximations. Check later if higher order (or non-linear) approximations give similar results.

- Second order approximation necessary for welfare calculations. Higher order approximations needed if risk variations needs to be considered or if there are kinks in the model (occasionally binding constraints).

- Approximate the first order conditions (which allow for distortions if sufficiency is satisfied) rather than the value function.

- Careful with sunspot, imaginary eigenvalues.

- First order approximations inappropriate for problems involving **large** shocks (what if oil prices go to 200 dollars per barrel?), structural policy changes (what happens if we switch from targeting exchange rates to targeting inflation?), transition economies.

- What to do in these situations is problem dependent. Higher approximations work better. Putting a unit root in technology is not necessarily the best solution.

- Dynare can solve nonlinear models directly using perfect foresight.

- Latest version of Dynare (above 4.3) have built in first, second and third order solutions. Use symbolic rather than numerical computations.

4 Perturbation methods in non-stationary models

• How do you use perturbation methods when the model feature non-stationary shocks? Problem! Steady state does not exist.

• Strategy:

1) Scale the model by the non-stationary shock. This transforms a non-stationary problems into a stationary one.

2) Specify the optimality conditions for the scaled model.

3) Apply perturbation methods to the optimality conditions of the scaled model.

Example 13 *Take a textbook New Keynesian model (see Gali(2008))*

- Add habit in consumption, a preference shock and a stochastic elasticity of variety of goods.

- The optimality conditions of the model are:

$$0 = \chi_t(C_t - hC_{t-1})^{-\sigma_c} - \lambda_t = \chi_t(C_t - hC_{t-1})^{-\sigma_c} - P_t Q_t$$

$$0 = N_t^{\sigma_n} - \lambda_t \frac{W_t}{P_t}$$

$$1 = E_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \frac{R_t}{\Pi_{t+1}} \right]$$

$$0 = E_t \left(\sum_{k=0}^{\infty} \frac{Q_{t+k}}{Q_t} (\beta\zeta_p)^k \left[1 - \epsilon_{t+k} + MC_{t+k}^r \epsilon_{t+k} \left(\frac{\widetilde{P}_t}{P_{t+k}} \right)^{\frac{\alpha - 1 - \alpha\epsilon_{t+k}}{1-\alpha}} \right] Y_{t+k}(j) \right)$$

$$1 = \left(\zeta_p \left(\frac{P_{t-1}}{P_t} \right)^{1-\epsilon_t} + (1-\zeta_p) \left(\frac{\widetilde{P}_t}{P_t} \right)^{1-\epsilon_t} \right)^{\frac{1}{1-\epsilon_t}}$$

$$Y_t = C_t$$

$$N_t = \left(\frac{Y_t}{Z_t} \right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{\frac{-\epsilon_t}{1-\alpha}} dj$$

$$MC_t^r = \frac{W_t}{P_t} \left(\frac{1}{Z_t} \right)^{\frac{1}{1-\alpha}} Y_t^{\frac{\alpha}{1-\alpha}} \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{\frac{-\epsilon_t \alpha}{1-\alpha}} dj$$

λ_t *is the Lagrangian on the budget constraint*

$R_t = 1 + r_t = 1/b_t$ *is the gross nominal rate of return on bonds*

$\frac{Q_{t+k}}{Q_t} = \frac{\lambda_{t+1}/P_{t+1}}{\lambda_t/P_t}$ *is the stochastic discount factor*

$MC^r_{t+k} = \frac{MC_{t+k}}{P_{t+k}}$ *is the real (aggregate) marginal cost.*

C_t *is aggregate consumption,* Y_t *is aggregate output,* W_t *the real wage,* P_t *the aggregate price level,* Π_t *the inflation rate,* N_t *hours worked.*

χ_t *a preference shock,* ϵ_t *a markup shock,* Z_t *a TFP shock.*

h *a habit parameter,* σ_n *the Frish elasticity,* ζ_p *the probability of non-changing prices,* $1 - \alpha$ *the share of labor in production,* β *the discount factor,*

Non stationary technology shock

- *Assume the preference shock is* $\ln \chi_t = \rho_\chi \ln \chi_{t-1} + \epsilon_t^\chi$ *where* $\epsilon_t^\chi \sim N(0, \sigma_\chi^2)$ *and the markup shock be iid.*

- *Assume technology process is*

$$Z_t = Z_t^c Z_t^T$$
$$\ln Z_t^T = bt + e_t^{Z,T}$$
$$\ln Z_t^c = \rho_z \ln Z_{t-1}^c + e_t^{Z,c}$$

- *The equilibrium conditions need to be rescaled by* Z_t^T. *Let* $\widehat{Y}_t = \frac{Y_t}{Z_t^T}$, $\widehat{C}_t = \frac{C_t}{Z_t^T}$ *and* $\widehat{W}_t = \frac{W_t}{Z_t^T}$, $\widehat{\mathcal{L}}_t = \mathcal{L}_t (Z_t^T)^{\sigma_c}$, $\widehat{Q}_{t+k} = \widehat{\mathcal{L}}_{t+k} P_{t+k}$. *Then:*

$$
\begin{aligned}
N_t &= \left(\frac{\widehat{Y}_t}{Z_t^c}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{\frac{-\epsilon_t}{1-\alpha}} dj \\
MC_t^r &= \frac{W_t}{P_t}\left(\frac{1}{Z_t}\right)^{\frac{1}{1-\alpha}} Y_t^{\frac{\alpha}{1-\alpha}} \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{\frac{-\epsilon_t\alpha}{1-\alpha}} dj \\
&= \frac{W_t}{P_t}\left(\frac{1}{Z_t^c}\right)^{\frac{1}{1-\alpha}} \left(\frac{1}{Z_t^T}\right)^{\frac{1-\alpha+\alpha}{1-\alpha}} Y_t^{\frac{\alpha}{1-\alpha}} \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{\frac{-\epsilon_t\alpha}{1-\alpha}} dj \\
&= \frac{\widehat{W}_t}{P_t}\left(\frac{1}{Z_t^c}\right)^{\frac{1}{1-\alpha}} \widehat{Y}_t^{\frac{\alpha}{1-\alpha}} \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{\frac{-\epsilon_t\alpha}{1-\alpha}} dj \\
\widehat{C}_t &= \widehat{Y}_t \\
\lambda_t(1/Z_t^T)^{-\sigma_c} &= \chi_t(\widehat{C}_t - h\widehat{C}_{t-1}\frac{Z_{t-1}^T}{Z_t^T})^{-\sigma_c} \\
\widehat{\lambda}_t &= \chi_t(\widehat{C}_t - h\widehat{C}_{t-1}\exp\{-b + e_{t-1}^{Z,T} - e_t^{Z,T}\})^{-\sigma_c} \\
\widehat{N}_t^{-\sigma_n} &= -\widehat{\lambda}_t\frac{\widehat{W}_t}{P_t}
\end{aligned}
$$

- Since $\widehat{N}_t = \frac{N_t}{(Z_t^T)^{\frac{\sigma_C - 1}{\sigma_n}}}$*, consistency is insured if* $\sigma_c = 1$*. This makes hours worked is stationary.*

-The Euler equation is

$$1 = E_t \left[\beta \frac{\widehat{\lambda}_{t+1}}{\widehat{\lambda}_t} \exp\{-b - e_{t+1}^{Z,T} + e_t^{Z,T}\} \frac{R_t}{\Pi_{t+1}} \right]$$

- The firm optimal condition when $\sigma_c = 1$ *is*

$$0 = E_t \left(\sum_{k=0}^{\infty} \frac{\widehat{Q}_{t+k}}{\widehat{Q}_t} (\beta \zeta_p)^k \left[1 - \epsilon_{t+k} + MC_{t+k}^r \epsilon_{t+k} \left(\frac{\widehat{P}_t}{P_{t+k}} \right)^{\frac{\alpha - 1 - \alpha \epsilon_{t+k}}{1-\alpha}} \right] \widehat{Y}_{t+k}(j) \right)$$

These are then the FOC of the transformed model. Perturbation methods can now be applied.

Non stationary preference shock

- *Assume that the technology shock is* $\ln z_t = \rho_z \ln z_{t-1} + \epsilon_t^z$ *where* $\epsilon_t^z \sim N(0, \sigma_z^2)$ *and the markup shock is iid.*

- *Assume that the preference process is*

$$\chi_t = (\chi_t^T)^{1+\sigma_n} \chi_t^c$$
$$\ln \chi_t^T = \ln \chi_{t-1}^T + e_t^{T,\chi}$$
$$\ln \chi_t^c = \rho_\chi \ln \chi_{t-1}^c + e_t^{c,\chi}$$

where $e_t^{j,\chi} \sim N(0, \sigma_{j,\chi}^2)$ *with* $j = T, c$.

- *Assume that* $\sigma_c = 1$ *and* $\alpha = 0$ *(* **CRS in production is needed***).*

- *Define* $\widehat{C}_t = C_t/\chi_t^T$, $\widehat{Y}_t = Y_t/\chi_t^T$, $\widehat{N}_t = N_t/\chi_t^T$, $\widehat{\mathcal{L}}_t = \lambda_t (\chi_t^T)^{-\sigma_n}$, $\widehat{Q}_{t+k} = \widehat{\lambda}_{t+k} P_{t+k}$.

- *The equilibrium conditions are then*

$$\widehat{\lambda}_t = \frac{\chi_t^c}{\widehat{C}_t - h\widehat{C}_{t-1}\,\mathsf{exp}(-e_t^{T,\chi})}$$

$$0 = \widehat{N}_t^{\sigma_n} + \widehat{\lambda}_t \frac{W_t}{P_t}$$

$$1 = \beta E_t \left[\frac{\widehat{\lambda}_{t+1}}{\widehat{\lambda}_t} R_t \frac{P_t}{P_{t+1}}\,\mathsf{exp}(\sigma_n \epsilon_{t+1}^{T,\chi}) \right]$$

$$\widehat{N}_t = \frac{\widehat{Y}_t}{Z_t} \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_t} dj$$

$$MC_t^r = \frac{W_t}{P_t} \frac{1}{Z_t}$$

$$0 = E_t \sum_{k=0}^{\infty} \frac{\widehat{Q}_{t+k}}{\widehat{Q}_t}\,\mathsf{exp}\left[(1+\sigma_n) \sum_{j=0}^{k-1} \epsilon_{t+k-1}^{\chi,P} \right] (\beta\zeta_p)^k \left[1 - \epsilon_{t+k} + MC_{t+k}^r \epsilon_{t+k} \left(\frac{\widetilde{P}_t}{P_{t+k}} \right)^{\frac{\alpha-1-\alpha\epsilon_{t+k}}{1-\alpha}} \right] \widehat{Y}_{t+k}(j)$$

Perturbation methods can be used with this set of equations.

There are problems allowing for non-stationary shocks in a model:

1) The FOCs depend on which shock is assumed to be non-stationary.

2) Consistency requires log utility.

3) With non-stationary preference shocks, we also need constant returns to scale in production.

5 Exercises

Exercise 1 *In example 4 show that the allocation (3,2,1) is not optimal.*

Exercise 2 *Repeat the steps of example 5 when* $\delta \neq 1$.

Exercise 3 *Consider the model* $\max E_0 \sum_t \beta^t \frac{c_t^{1-\varphi}}{1-\varphi} + \log(1 - N_t)$

$$c_t + k_{t+1} - (1-\delta)k_t = N_t^{\eta} k_t^{1-\eta} \zeta_t \tag{95}$$

$E\zeta_t = \zeta^{ss}$; $\hat{\zeta} \equiv (\zeta_t - \zeta^{ss})/\zeta^{ss} = \rho\hat{\zeta}_{t-1} + e_t,\ e_t \sim (0, \sigma^2)$.*Find* $F_1(A)$ *and* $F_2(A, B)$. *Find the decision rules when* $\varphi = 2, \beta = 0.99, \eta = 0.65, \delta = 0.02$.

Exercise 4 *Consider the model with a lag in delivering capital* $\max E_0 \sum_t \beta^t \frac{c_t^{1-\varphi}}{1-\varphi} + \log(1 - N_t)$

$$c_t + k_{t+2} - (1-\delta)k_{t+1} = N_t^{\eta} k_t^{1-\eta} \zeta_t \tag{96}$$

$E\zeta_t = \zeta^{ss}$; $\hat{\zeta} \equiv (\zeta_t - \zeta^{ss})/\zeta^{ss} = \rho\hat{\zeta}_{t-1} + e_t,\ e_t \sim (0, \sigma^2)$ *and the capital accumulation equation is* $k_{t+1} = (1-\delta)k_t + inv_{t-1}$. *Find* $F_1(A)$ *and* $F_2(A, B)$. *Find the decision rules when* $\varphi = 2, \beta = 0.99, \eta = 0.65, \delta = 0.02$.