# 5. Incentives and Mechanism Design

This section considers allocation problems where agents have private information about their preferences.

**A.** Mechanism Design (MWG Ch 23A-B)

- Social Choice Function
- Implementation

**B.** Bayesian Implementation (MWG Ch 23.D)

- The Revelation Principle
- Incentive Compatibility

**C.** Application to Auctions

# Economic Issues

- Symmetric versus asymmetric information
- Preferences are typically privately known (does this matter in G.E. analysis?)
- Evaluate welfare properties of institutions used allocate goods (e.g. first price sealed bid auction)
- Applications: Public choice (e.g. enviromental standards), monopoly pricing, auctions
- Tools: Mechanism design

# Mechanism Design Problem

- $i = 1..I$ agents
- $x \in X$ alternative
- $\theta_i \in \Theta_i$ type of agent $i$,privately observed by $i$
- $u_i(x, \theta_i)$ utility of agent $i$ when alternative $x$ is chosen
- $\Phi(\theta)$ probability density on $\Theta = \Theta_1 x..x\Theta_I$
- $\{u_i, \Phi, X\}$ is common knowledge

Questions:

1. Efficient choice of alternative $x$ conditional on realization of $\theta$

2. Implementation through agreed-upon mechanism (institution)

   - Which choice functions are 'implementable'?
   - Efficiency properties of commonly used mechanisms

3. Is it possible to get agents to reveal truthfully their types?

# Clarke-Groves Mechanism (1/3)

- Example: Allocation of right to play music with privately known preferences

  1. Music lover $(ML)$ gets benefit $b$ from playing music
  2. Neighbour $(N)$ suffers $c$ from music annoyance
  3. $b$ and $c$ are privately known and distributed $g(b)$ and $f(c)$

- What is the effficient allocation?

# Take-it-or-leave-it Offer (2/3)

- Assume $N$ has the right to a quite environment and makes a take-it-or-leave-it offer of the type "you can play if you pay me $t$"

  1. $ML$ accepts offer $t$ with probability $1 - G(t)$
  2. $N$ chooses $t$ to maximize $(1 - G(t))(t - c)$
  3. FOC implies $g(t)(t - c) = 1 - G(t)$
  4. The optimal offer is such that $t > c$

- The equilibrium allocation is not always efficient (Coase theorem fails)!

# Gloves-Clarke Mechanism (3/3)

- Consider the following game
- Both $ML$ and $N$ first announce their private preference $\widehat{b}$ and $\widehat{c}$ to a third party
- The third party then choose the following allocation
  1. $ML$ is allowed to play iff $\widehat{b} > \widehat{c}$
  2. If $\widehat{b} > \widehat{c}$, $ML$ pays $\widehat{c}$ and $N$ receives $\widehat{b}$
- Truth telling is wealky dominant startegy (why?)
- The equilibrium allocation is efficient
- Problem: not budget balanced!

# Main Concepts

- Definition: A social choice function (SCF) is a mapping from types to alternative $f(\theta) \in X$ (example, $f(b,c) = \{y_b, y_c; t_b, t_c\}$, with $y = 0,1$ and $t \in \mathbb{R}$)

- Definition: $f$ is ex-post efficient if there does not exist $(x, \theta_i)$ such that $u_i(x, \theta_i) \geq u_i(f(\theta), \theta_i)$ for $i = 1..I$ with at least one inequality strict

- Definition: A mechanism $\Gamma = (S_1, .., S_I, g())$ is a collection of $I$ strategy sets and an outcome function which maps each strategy vector into an alternative $g(s) \in X$ where $S = (s_1, .., s_I) \in S_1 x..x S_I$ (example: Clark-Groves mechanism)

- Formally, $(\Gamma, \Theta, u_i, \Phi)$ defines a Bayesian game of incomplete information (still need to define equilibrium concept)

- Definition: Mechanism $\Gamma$ implement SCF $f$ if there exists a strategy $(s_1^*(), .., s_I^*())$ of the game induced by $\Gamma$ such that $g(s_1^*(\theta), .., s_I^*(\theta)) = f(\theta)$ for any $\theta \in \Theta$ (for example, the Clark-Groves mechanism implements the first-best allocation)

- Definition: A direct revealation mechanism is a mechanism in which $S_i = \Theta_i$ (for example, an English auction, as defined in later slide, is not a direct mechanism)

- Definition: SCF $f$ is truthfully implementable (or incentive compatible IC) if the direct revealation mechanism $\Gamma = (\Theta_1, .., \Theta_I; f(\theta))$ has equilibrium $(s_1^*(), .., s_I^*())$ such that $s_1^*(\theta_i) = \theta_i$, that is, truthtelling is an equilibrium of $\Gamma$ (for example, the first best allocation with transfer $t = \frac{b+c}{2}$ is not IC, why?)

- Questions: Can we restrict to truthfully implementable SCF?

# Bayesian Implementation

- Let $\theta_{-i} = (\theta_1, .., \theta_{i-1}, \theta_{i+1}, .., \theta_I)$, $s_{-i} = (s_1, .., s_{i-1}, s_{i+1}, .., s_I)$, $s = (s_i, s_{-i})$ is a strategy, and $s() = (s_i(), s_{-i}())$ is a strategy profile

- Definition: $s^*(\theta) = (s_1^*(\theta), .., s_I^*(\theta))$ is a Bayesian Nash equilibrium (BNE) of $\Gamma$ if $\forall i, \forall \theta$

  $E_{\theta_{-i}}\left[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i\right] \geq E_{\theta_{-i}}\left[u_i(g(\widehat{s}_i, s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i\right]$
  for any $\widehat{s}_i \in S_i$

- Definition: $\Gamma$ implements SCF $f$ in BNE if there exists a BNE of, $\Gamma$, such that $g(s^*(\theta)) = f\ (\theta)$ for any $\theta \in \Theta$

- Definition: SCF $f$ is truthfully implementable in BNE (or IC) if $\forall i, \forall \theta_i$

  $E_{\theta_{-i}}\left[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i\right] \geq E_{\theta_{-i}}\left[u_i(f(\widehat{\theta}_i, \theta_{-i}), \theta_i)|\theta_i\right]$ for any $\widehat{\theta}_i \in \Theta$

# Revelation Principle

- Proposition: If $\Gamma$ implements SCF $f$ in BNE then $f$ is truthfully implementable in BNE.

- Intuition: Under $\Gamma$, $\theta_i$ finds it optimal to reveal $s_i^*(\theta_i)$. Assume $\theta_i$ reveals its own type, $\theta_i$, to an automaton who then plays $s_i^*(\theta_i)$ on her behalf. It is optimal to reveal $\theta_i$ to the automaton.

- Conclusion: Can we restrict without loss of generality to truthfully implementable SCF.

# Application to Auctions

- $x = \{y_1, .., y_I; t_1, .., t_I\}$ where $y_i = 0, 1$ and $\sum_i t_i \leq 0$
- $u_i(x, \theta_i) = \theta_i y_i + t_i$, where $\theta_i$ is type $i$'s valuation
- A SCF is ex-post efficient iff $y_i(\theta_i)(\theta_i - Max_i \theta_i) = 0$ and $\sum_i t_i(\theta_i) = 0$
- $i = 2$ corresponds to bilateral trade.
- Auctions is a special case of unit-good allocation problems: assuming that agent $0$ is the auctioneer and receives $\sum_i t_i$
- We assume that the $\theta_i$ are independently distributed with density $f(\theta)$ and distribution $F(\theta)$ with support $[\underline{\theta}, \overline{\theta}]$

# General Direct Mechanism

- Denote a direct mechanism $\{y(\theta), t(\theta)\}$
- Consider the following SCF: $y_i(\theta_i)(\theta_i - Max_i\theta_i) = 0$ and $y_i(\theta_i)t_i(\theta_i) = \theta_i$
- It it incentive compatible?

## • Indirect Mechanism 1: Second Price Sealed Bid Auction (SPSBA)

- Institutional Design

1. All bidders make sealed bid
2. The highest bid wins the good and pays an amount corresponding to the second highest bid

- Equilibrium and direct implementation

1. What is a (weakly dominant strategy) equilibrium of the bidding game? Is the allocation efficient?
2. What SCF is (indirectly implemented) by the SPSBA?
3. What direct revealation mechanism implements this SCF?

## • Indirect Mechanism 2: First Price Sealed Bid Auction (FPSBA)

- Institutional Design

  1. All bidders make sealed bid
  2. The highest bid wins the good and pays an amount corresponding to her bid

- Equilibrium and direct implementation (Assume $f$ is uniform on $[0, 1]$)

  1. What is the Bayesian Nash equilibrium of the bidding game? Is the allocation efficient?
  2. What SCF is (indirectly implemented) by the FPSBA?
  3. What direct revealation mechanism implements this SCF?

More notation...

- $\bar{t}_i(\widehat{\theta}_i) = E_{\theta_{-i}} t_i(\widehat{\theta}_i, \theta_{-i})$ expected payment under $\widehat{\theta}_i$
- $\overline{y}_i(\widehat{\theta}_i) = E_{\theta_{-i}} y_i(\widehat{\theta}_i, \theta_{-i})$ expected probability of receiving the good under $\widehat{\theta}_i$
- $u_i(\theta_i) = \theta_i \overline{y}_i(\theta_i) - \bar{t}_i(\theta_i)$ equilibrium expected utility
- $\widetilde{u}_i(\theta_i, \widetilde{\theta}_i) = \theta_i \overline{y}_i(\widetilde{\theta}_i) - \bar{t}_i(\widetilde{\theta}_i)$ is the expected utility of type $\theta_i$ if she claims to be type $\widetilde{\theta}_i$

# Implication of IC

Proposition: Assume $\{y(\theta), t(\theta)\}$ is IC. (a) $y_i$ is non-decreasing, (b) $u_i(\theta_i) = u_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{y}_i(s)ds$ for $i = 1..I$.

- Proof (sketch): Type $\theta_i$ maximizes $\widetilde{u}_i(\theta_i, x)$ over $x$. FOC implies $\theta_i \overline{y}_i'(\theta_i) - \overline{t}_i'(\theta_i) = 0$. Taking full derivative of $\widetilde{u}_i(\theta_i, \theta_i)$ with respect to $\theta_i$ and plugging in FOC implies $u_i'(\theta_i) = \overline{y}_i(\theta_i)$.

- Intuition: Higher types are more likely to get the good. Expected utility is only a function of the utility of the lowest type and the allocation probability.

# Revenue Equivalence Theorem

- Proposition: (Revenue Equivalence Theorem) Any auction mechanism in which (a) the good is always allocated to the highest bidder, and (b) any bidder with the lowest valuation $\underline{\theta}$ gets $0$ expected surplus, yields the same expected revenue for the seller, and results in a buyer with valuation $\theta$ making the same expected payment.

- Intuition: $u_i(\theta_i)$ depends only on $\overline{y}_i(\theta_i)$ and $u_i(\underline{\theta}_i)$. Therefore expected payment of $\theta_i$ depends only on $\overline{y}_i(\theta_i)$ and $u_i(\underline{\theta}_i)$.

- Interpretation: Consider the following four auction mechanisms

  1. First price sealed bid auction
  2. Second price sealed bid auction
  3. English auction (the price increases by unit increment from $\underline{\theta}$ to $\overline{\theta}$ until one bidder bids)
  4. Dutch auction (the price decreases by unit increment from $\overline{\theta}$ to $\underline{\theta}$ until one bidder bids)

- If in equilibrium the bidder with the highest realized valuation always gets the good and a bidder with valuation $\underline{\theta}$ gets zero surplus, then the seller gets is indifferent (ex-ante) between these 4 schemes

# Application to Uniform Case

- Assume the valuations are uniformly distributed between $\underline{\theta}$ and $\overline{\theta}$
- $F(\theta) = \frac{1}{\overline{\theta}-\underline{\theta}}$ and $F(\theta) = \frac{\theta-\underline{\theta}}{\overline{\theta}-\underline{\theta}}$
- The $k^{th}$ highest expected value drawn among $n$ independently drawn values is $\underline{\theta} + \frac{n+1-k}{n+1}(\overline{\theta} - \underline{\theta})$
- The expected revenue of any auction that achieves efficiency and leave no surplus to the lowest valuation is $\underline{\theta} + \frac{n-1}{n+1}(\overline{\theta} - \underline{\theta})$
- Can compute bidding strategies using revenue equivalence theorem
  1. Under FPSBA, $u_i(\theta_i) = \theta_i \overline{y}_i(\theta_i) - \overline{t}_i(\theta_i) = (\theta_i - b_i)\overline{y}_i(\theta_i)$
  2. This implies $b_i = \frac{\overline{t}_i(\theta_i)}{\overline{y}_i(\theta_i)} = \underline{\theta} + \frac{n-1}{n}(\theta_i - \underline{\theta})$