MICROECONOMICS III: Information and Contract Theory

PROBLEM SETS 2006

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MICROECONOMICS III: Information and Contract Theory

PROBLEM SET 1: QUESTIONS AND ANSWERS

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Problem 1: The market for lemons

A group of 200 car owners are looking to sell their car. Each of the cars has a different quality θ, which is observed privately by its owner only. The quality θ is distributed uniformly on the interval $[0, 1]$. Let p denote the price on the car market. Car owners are willing to sell only if $p \geq \theta$. There are 100 type A buyers, who value a car of expected quality $\hat{\theta}$ at $2\hat{\theta}$, and 100 type B buyers who value such a car at $\frac{3}{2}\hat{\theta}$.

a) Find the average quality of the cars supplied as a function of the price.

b) Specify the demand for cars.

c) Is there any price at which trade can take place? How many cars are traded and what is their average quality?

Answer to Problem 1

Assume sellers sell and buyers buy when they are indifferent.

a) A car owner of type $\theta \in [0,1]$ is willing to sell his car if $p \geq \theta$. Let μ denote the average quality of the cars on sale for a given price p, then

$$\hat{\theta} = \left\{ \begin{array}{cc} \frac{p}{2} & \text{for } 0 \leq p \leq 1 \\ \frac{1}{2} & \text{for } p > 1 \end{array} \right\}$$

Note for $0 \leq p \leq 1$: $\hat{\theta} = E[\theta \mid \theta < p] = \frac{\int_0^p \theta dF(\theta)}{F(p)} = \frac{1}{p}\frac{p^2}{2} = \frac{p}{2}$. The supply of cars is

$$S(p) = \left\{ \begin{array}{cc} 200p & \text{for } 0 \leq p \leq 1 \\ 200 & \text{for } p > 1 \end{array} \right\}$$

b) Type A buys if $p \leq 2\hat{\theta}$. Type B buys if $p \leq \frac{3}{2}\hat{\theta}$. Hence market demand is

$$D(p) = \begin{array}{cl} 0 & \text{if } p > 2\hat{\theta} \\ 100 & \text{if } \frac{3}{2}\hat{\theta} < p \leq 2\hat{\theta} \\ 200 & \text{if } p \leq \frac{3}{2}\hat{\theta} \end{array}$$

c) Let p^* denote the equilibrium price. Suppose $p^* > 1$. Then supply is 200 but demand is 100. Hence p^* is not a market clearing price. Now suppose $0 \leq p^* \leq 1$. The average quality in the market is $p/2$. Then demand is 100. At a price $p^* = \frac{1}{2}$ there are exactly 100 suppliers willing to sell. Hence $p^* = \frac{1}{2}$ is a market clearing price, at which 100 cars are traded. The average quality of all cars sold is $\frac{1}{4}$.

Problem 2: A model of positive selection

There is a measure one continuum of workers, each with productivity level θ. Let $[\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$ be the set of possible worker productivity levels. The density of workers of type θ is given by $f(\theta)$, with $f(\theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Let $r(\theta)$ be the opportunity cost of a worker of type θ of accepting employment. $r(\theta)$ is a continuous, *strictly decreasing* function of θ. Price taking, risk neutral firms seek to maximize profits. Let w be the wage offered by the firms. Finally, assume that individual productivity levels are unobservable by the firms.

a) Define a competitive equilibrium.

b) If $r(\theta) > \theta$ for all θ, the competitive equilibrium is Pareto efficient. TRUE or FALSE? Provide a graphical example of such an equilibrium as in MWG CH13.B.

c) Suppose there exists a $\hat{\theta}$ such that $r(\theta) < \theta$ for $\theta > \hat{\theta}$ and $r(\theta) > \theta$ for $\theta < \hat{\theta}$. Any competitive equilibrium with strictly positive employment involves too much employment relative to the Pareto optimum. TRUE or FALSE? Provide a graphical example of such an equilibrium as in MWG CH13.B.

d) Any competitive equilibrium level of employment must be unique. TRUE of FALSE?

Answer to Problem 2

a) See MWG Definition 13.B.1.

b) When $r(\theta) > \theta$, all workers are more productive at home. Hence, the Pareto efficient outcome is one with no firm employment. A competitive equilibrium with a strictly positive measure of workers employed requires $w \geq r(\theta)$ for $\theta \in \Theta$ where Θ is a subset of $[\underline{\theta}, \overline{\theta}]$ with strictly positive measure . But since $r(\theta) > \theta$ for every $\theta \in \Theta$, firms would always expect to make a loss. Thus, no one works in the competitive equilibrium either, and the statement is TRUE.

c) A Pareto efficient outcome requires all workers of type $\theta \in [\hat{\theta}, \overline{\theta}]$ to work in a firm and workers of type $\theta \in [\underline{\theta}, \hat{\theta}]$ to stay home, where $\underline{\theta} < \hat{\theta} < \overline{\theta}$. For that to be the case the wage must be $w = \hat{\theta} = r(\hat{\theta})$. We know that for $r(\theta) \leq w$, $\theta > \hat{\theta}$. But in that case, $E[\theta \mid r(\theta) \leq w] > \hat{\theta}$ such that firms will demand more workers then there are in supply. So we have established that any competitive equilibrium cannot be the Pareto efficient one.

   Suppose we try a lower wage $w < \hat{\theta} = r(\hat{\theta})$. Define θ^* as the value for θ for which $r(\theta^*) = w$ holds. Since $r(\theta)$ is decreasing, $\theta^* > \hat{\theta}$ and all $\theta > \theta^*$ will work. Hence it must be the case that $E[\theta \mid r(\theta) \leq w] > w$. Lowering the wage below $\hat{\theta}$ only increases the average supplied productivity, such that firms will still demand more workers then they are in supply.

   Suppose we try a higher wage $w > \hat{\theta} = r(\hat{\theta})$. This increases the number of employed workers and lowers expected productivity. Since $E[\theta \mid r(\theta) \leq w]$ is strictly decreasing in w, a fixed point $w = E[\theta \mid r(\theta) \leq w]$ always implies overemployment. The statement is TRUE.

d) TRUE. See Graphs in class.

Problem 3

Consider the model of problem 2, but now assume that $r(\theta)$ is a continuous, *non decreasing* function of θ. Note that the model is now identical to MWG CH13.B.

If and only if $r(\overline{\theta}) > E[\theta]$, any competitive equilibrium must be inefficient. TRUE or FALSE?

Answer to Problem 3

FALSE. Counterexample: Suppose $r(\theta) > \theta$ for all θ.

MICROECONOMICS III: Information and Contract Theory

PROBLEM SET 2: QUESTIONS AND ANSWERS

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Problem 1: Screening

Consider the screening model discussed in the lecture notes and described in MWG 13.D. Let (θ_H, t_H) and $(\theta_L, 0)$ be the pair of contracts of the candidate SPNE of the screening game as given in Proposition 13.D.2 in MWG. (i.e. t_H satisfies $\theta_H - c(t_H, \theta_L) = \theta_L - c(0, \theta_L)$).
Now consider a new pair of contracts $(\tilde{\theta}_H, \tilde{t}_H) = (\theta_H - \delta_H^w, t_H - \delta_H^t)$ and $(\tilde{\theta}_L, 0) = (\theta_L + \delta_L^w, 0)$ that constitutes a *local* deviation from the candidate equilibrium. In addition, suppose the following tie-breaking rule applies: If the workers are indifferent between the old and the new contract, they accept the new contract.

1.a) When do workers of type θ_L accept $(\tilde{\theta}_L, 0)$ rather than $(\theta_L, 0)$ or $(\tilde{\theta}_H, \tilde{t}_H)$?

1.b) When do workers of type θ_H accept $(\tilde{\theta}_H, \tilde{t}_H)$ rather than (θ_H, t_H) or $(\tilde{\theta}_L, 0)$?

1.c) When is the new pair of contracts a profitable deviation for a firm?

2) Show that (θ_H, t_H) and $(\theta_L, 0)$ is not a SPNE of the screening game if

$$\frac{\lambda}{1-\lambda} > \left[\frac{\partial c(t_H, \theta_L)}{\partial t} - \frac{\partial c(t_H, \theta_H)}{\partial t}\right]\left[\frac{\partial c(t_H, \theta_H)}{\partial t}\right]^{-1}$$

3) Argue briefly why the following statement is true: If there exists a profitable deviation from the candidate equilibrium described in MWG Proposition 13.D.2, there does not exist any SPNE equilibrium.

Answer to Problem 1

1.a) Low type workers will prefer $(\tilde{\theta}_L, 0)$ over $(\theta_L, 0)$ if

$$\theta_L + \delta^w_L - c(0, \theta_L) \geq \theta_L - c(0, \theta_L) \tag{IRC1}$$

Low type workers will prefer $(\tilde{\theta}_L, 0)$ over $(\tilde{\theta}_H, \tilde{t}_H)$ if

$$\theta_L + \delta^w_L - c(0, \theta_L) \geq \theta_H - \delta^w_H - c(t_H - \delta^t_H, \theta_L) \tag{ICC1}$$

1.b) High type workers will prefer $(\tilde{\theta}_H, \tilde{t}_H)$ over (θ_H, t_H) if

$$\theta_H - \delta^w_H - c(t_H - \delta^t_H, \theta_H) \geq \theta_H - c(t_H, \theta_H) \tag{IRC2}$$

High type workers will prefer $(\tilde{\theta}_H, \tilde{t}_H)$ over $(\tilde{\theta}_L, 0)$ if

$$\theta_H - \delta^w_H - c(t_H - \delta^t_H, \theta_H) \geq \theta_L + \delta^w_L - c(0, \theta_H) \tag{ICC2}$$

1.c) The new pair of contracts is profitable for the firm if

$$\lambda \delta^w_H - (1 - \lambda)\delta^w_L > 0 \tag{PR}$$

2) Recall that in the candidate equilibrium (θ_H, t_H), $(\theta_L, 0)$, the following is true:

$$\theta_H - c(t_H, \theta_L) = \theta_L - c(0, \theta_L) \tag{1}$$

From IRC2

$$c(t_H, \theta_H) - c(t_H - \delta^t_H, \theta_H) \geq \delta^w_H \tag{2}$$

Combining (1) and ICC1, we get

$$\theta_H - c(t_H, \theta_L) + \delta^w_L \geq \theta_H - \delta^w_H - c(t_H - \delta^t_H, \theta_L)$$

$$\Leftrightarrow c(t_H, \theta_L) - \delta^w_L \leq \delta^w_H + c(t_H - \delta^t_H, \theta_L)$$

$$\Leftrightarrow c(t_H, \theta_L) - c(t_H - \delta^t_H, \theta_L) - \delta^w_L \leq \delta^w_H \tag{3}$$

Combining (2) and (3),

$$\left[c(t_H, \theta_H) - c(t_H - \delta^t_H, \theta_H)\right] - \left[c(t_H, \theta_L) - c(t_H - \delta^t_H, \theta_L)\right] \geq -\delta^w_L \tag{4}$$

Let's consider the expression

$$\frac{\lambda}{1-\lambda} > \left[\frac{\partial c(t_H, \theta_L)}{\partial t} - \frac{\partial c(t_H, \theta_H)}{\partial t}\right] \left[\frac{\partial c(t_H, \theta_H)}{\partial t}\right]^{-1}$$

Rearranging, we get

$$\underbrace{\lambda \frac{\partial c(t_H, \theta_H)}{\partial t}}_{>0} + (1-\lambda) \underbrace{\left[\frac{\partial c(t_H, \theta_H)}{\partial t} - \frac{\partial c(t_H, \theta_L)}{\partial t}\right]}_{<0} > 0$$

Define

$$\hat{\delta}^w_H = \lim_{\delta \to 0} \frac{c(t_H, \theta_H) - c(t_H - \delta, \theta_H)}{\delta} = \frac{\partial c(t_H, \theta_H)}{\partial t} > 0$$

$$\hat{\delta}^w_L = -\lim_{\delta \to 0} \frac{[c(t_H, \theta_H) - c(t_H - \delta, \theta_H)] - [c(t_H, \theta_L) - c(t_H - \delta, \theta_L)]}{\delta} = -\left[\frac{\partial c(t_H, \theta_H)}{\partial t} - \frac{\partial c(t_H, \theta_L)}{\partial t}\right] > 0$$

Recall the assumption that under the tie-breaking rule, if the workers are indifferent between the old and the new contract, they accept the new contract. Since $\hat{\delta}^w_L > 0$, IRC1 is satisfied. Clearly, $\hat{\delta}_H^w$ satisfies (2) and therefore (IRC2) is satisfied. Also $\hat{\delta}_L^w$ satisfies (4) so ICC1 is satisfied. Verify that provided that $c(t_H, \theta_L) - c(t_H, \theta_H)$ is large enough, (ICC2) is also satisfied. Finally

$$\lambda\hat{\delta}_H^w - (1-\lambda)\hat{\delta}_L^w > 0$$

and there exists a profitable deviation.

3) As there exists a profitable deviation to the unique candidate SPNE equilibrium, there cannot exist a SPNE equilibrium.

Problem 2: Monopoly Screening

Consider a single firm with the option of producing one product with low quality v_L and one with high quality v_H at marginal cost c_L and c_H respectively. There is a measure N consumers, each of whom obtains an indirect utility of $\theta v_j - p_j$ from buying $j = L, H$, where p_j is the price of the good. Consumers consume at most one product. The parameter θ is a consumer-specific taste parameter, measuring the marginal willingness to pay for quality. It is distributed uniformly on $[\underline{\theta}, \overline{\theta}]$. Normalize such that $\overline{\theta} - \underline{\theta} = N$.

1) Suppose first that the firm can only sell one type of quality. Derive the profit maximizing price and quantity.

2) Suppose the firm can sell both qualities. Given the prices p_j, derive the demand for both qualities. What are the profit maximizing prices and quantities?

3) Now suppose, $c_L = c_H = 0$. What are the profit maximizing prices and quantities now?

Answer to Problem 2

1) Suppose the firm decides to sell quality j. Note that a consumer buys if $\theta v_j - p_j \geq 0$. The market demand $D(p_j)$ is given by

$$D(p_j) = \begin{cases} 0 & \text{if} \quad p_j > \overline{\theta} v_j \\ \overline{\theta} - \frac{p_j}{v_j} & \text{if} \quad \underline{\theta} v_j \leq p_j < \overline{\theta} v_j \\ N & \text{if} \quad p_j \leq \underline{\theta} v_j \end{cases}$$

The firm's profits are given by

$$\pi = (p_j - c_j) D(p_j)$$

First note that if $c_j > \overline{\theta}_j v_j$, the firm does not sell good j. In an interior solution the optimal price and quantity is

$$p_j^* = \frac{\overline{\theta} v_j + c_j}{2} \ , \ q^* = \frac{\overline{\theta} v_j - c_j}{2v_j}$$

Whenever $\left(2\underline{\theta} - \overline{\theta}\right) v_j > c_j$, the firm sells to all consumers at a price $\underline{\theta} v_j$. Focusing on interior solutions, optimal firm profits from selling good j are

$$\pi_j = \left(\frac{\overline{\theta} v_j + c_j}{2} - c_j\right)\left(\frac{\overline{\theta} v_j - c_j}{2v_j}\right) = \frac{(\overline{\theta} v_j - c_j)^2}{4v_j}$$

The firm chooses j that yields the highest profits.

2) Define the consumer $\hat{\theta}$ who is indifferent between buying the low and the high quality products, i.e.

$$\hat{\theta} v_H - p_H = \hat{\theta} v_L - p_L$$

$$\Rightarrow \hat{\theta} = \frac{p_H - p_L}{v_H - v_L}$$

Let's assume that it is optimal to have $\underline{\theta} \leq \hat{\theta} \leq \overline{\theta}$. If this is not the case, the firm sells either only one of the goods or none. If the firm sells only one good, optimal prices and quantities are those computed in 1).

If both goods are sold however, market demand is given by

$$D^H(p_H, p_L) = \overline{\theta} - \hat{\theta}$$

$$D^L(p_H, p_L) = \begin{cases} \hat{\theta} - \frac{p_L}{v_L} & \text{if } \frac{p_L}{v_L} \geq \underline{\theta} \\ \hat{\theta} - \underline{\theta} & \text{if } \frac{p_L}{v_L} < \underline{\theta} \end{cases}$$

The firms profits are given by

$$\pi = (p_H - c_H) D^H(p_H, p_L) + (p_L - c_L) D^L(p_H, p_L)$$

– Suppose that it is optimal to cover the whole market. Firms profits are

$$\pi = (p_H - c_H)\left[\overline{\theta} - \frac{p_H - p_L}{v_H - v_L}\right] + (p_L - c_L)\left[\frac{p_H - p_L}{v_H - v_L} - \underline{\theta}\right]$$

The FOC w.r.t. p_H yields

$$p_H = p_L + \frac{c_H - c_L}{2} + \overline{\theta}\frac{v_H - v_L}{2}$$

The optimal prices are

$$p_H^* = \underline{\theta} v_L + \frac{c_H - c_L}{2} + \overline{\theta}\frac{v_H - v_L}{2}$$

$$p_L^* = \underline{\theta} v_L$$

The optimal quantities are

$$q_H^* = \overline{\theta} - \frac{p_H^* - p_L^*}{v_H - v_L} = \frac{\overline{\theta}}{2} - \frac{c_H - c_L}{2(v_H - v_L)}$$

$$q_L^* = \frac{p_H^* - p_L^*}{v_H - v_L} - \underline{\theta} = \frac{c_H - c_L}{2(v_H - v_L)} + \frac{\overline{\theta}}{2} - \underline{\theta}$$

– Suppose it is not optimal to cover the whole market. Firms profits are

$$\pi = (p_H - c_H)\left[\overline{\theta} - \frac{p_H - p_L}{v_H - v_L}\right] + (p_L - c_L)\left[\frac{p_H - p_L}{v_H - v_L} - \frac{p_L}{v_L}\right]$$

The FOCs w.r.t. p_H and p_L yield the optimal prices

$$p_H^* = \frac{\overline{\theta} v_H + c_H}{2}$$

$$p_L^* = \frac{\overline{\theta} v_L + c_L}{2}$$

and optimal quantities

$$q_H^* = \overline{\theta} - \frac{p_H^* - p_L^*}{v_H - v_L} = \frac{\overline{\theta}}{2} - \frac{c_H - c_L}{2(v_H - v_L)}$$

$$q_L^* = \frac{p_H^* - p_L^*}{v_H - v_L} - \frac{p^*}{v_L} = \frac{c_H - c_L}{2(v_H - v_L)} + \frac{\overline{\theta}}{2} - \frac{\overline{\theta} v_L + c_L}{2v_L}$$

Note that if $(2\underline{\theta} - \overline{\theta})v_L > c_L$, the firm sells to the whole market.

3) Suppose $c_j = 0$ for $j = H, L$. If the firm sells both types of the good profits are

$$\pi_1 = \left[\underline{\theta} v_L + \overline{\theta}\tfrac{v_H - v_L}{2}\right]\left[\tfrac{\overline{\theta}}{2}\right] + \underline{\theta} v_L \left[\tfrac{\overline{\theta}}{2} - \underline{\theta}\right] \quad \text{if the whole market is covered}$$
$$\pi_1' = \tfrac{\overline{\theta} v_H}{2}\left[\tfrac{\overline{\theta}}{2}\right] + \tfrac{\overline{\theta} v_L}{2}\left[\tfrac{\overline{\theta}}{2} - \tfrac{\overline{\theta}}{2}\right] = \tfrac{\overline{\theta}^2 v_H}{4} \quad \text{if the market is not completely covered}$$

Hence, in the case where the market is not completely covered, the firm sells only the high quality good.

Now consider the case where the whole market is covered. If the firm only sells one of the goods, it will sell the high quality good and profits are $\pi_2 = \frac{\overline{\theta}^2 v_H}{4}$. After some algebra, one can show that $\pi_1 > \pi_2$ if

$$\underline{\theta}\overline{\theta} - \frac{\overline{\theta}^2}{4} - \underline{\theta}^2 > 0$$

So if this inequality holds, selling both qualities remains the optimal strategy.

MICROECONOMICS III: Information and Contract Theory

PROBLEM SET 3: QUESTIONS AND ANSWERS

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Problem 1: Mechanism Design

Suppose agent A possesses an indivisible object and agent B is the potential buyer. Each agent i has a valuation θ_{ij} for the object where j can take on two values, high (H) and low (L). For agent A the valuation is θ_{AH} or θ_{AL} with probability π, $1-\pi$ respectively. Similarly, for agent B the valuation is θ_{BH} or θ_{BL} with probability π, $1-\pi$ respectively. Now assume that $\theta_{AH} > \theta_{BH} > \theta_{AL} > \theta_{BL}$. If B does not get the good, his reservation value is zero.

Let $x(\theta_{Aj}, \theta_{Bj})$ denote the probability that agent B receives the good given types θ_{Aj} and θ_{Bj} and let $t(\theta_{Aj}, \theta_{Bj})$ be the payment from agent B to agent A given types θ_{Aj} and θ_{Bj}. This implies that utility conditional on the realization of the types is

$$\text{agent A:} \quad u_A(\theta_{Aj}, \theta_{Bj}) = (1 - x(\theta_{Aj}, \theta_{Bj}))\theta_{Aj} + t(\theta_{Aj}, \theta_{Bj})$$

$$\text{agent B:} \quad u_B(\theta_{Aj}, \theta_{Bj}) = x(\theta_{Aj}, \theta_{Bj})\theta_{Bj} - t(\theta_{Aj}, \theta_{Bj})$$

1) Find $x(\theta_{Aj}, \theta_{Bj})$ such that the allocation is (ex post) efficient.

2) Write down the agent's incentive compatibility constraints.

3) Write down the individual rationality constraints.

4) Does there exist an (ex post) efficient contract that both agents are willing to sign ex ante?

Answer to Problem 1

1) Ex post efficiency implies that $x(\theta_{AL}, \theta_{BH}) = 1$ as there are gains from trade in this case and $x(\theta_{AL}, \theta_{BL}) = x(\theta_{AH}, \theta_{BH}) = x(\theta_{AH}, \theta_{BL}) = 0$ as there are no gains from trade in these cases.

2) A mechanism is incentive compatible if it is such that no agent has an incentive to lie about his type for any given type of the other agent. Truthtelling requires

   - for type A:

$$\pi\left(\theta_{AH} + t_{HH}\right) + (1-\pi)\left(\theta_{AH} + t_{HL}\right) \geq \pi t_{LH} + (1-\pi)\left(\theta_{AH} + t_{LL}\right) \qquad \text{(ICC AH)}$$

$$\pi t_{LH} + (1-\pi)\left(\theta_{AL} + t_{LL}\right) \geq \pi\left(\theta_{AL} + t_{HH}\right) + (1-\pi)\left(\theta_{AL} + t_{HL}\right) \qquad \text{(ICC AL)}$$

   - for type B:

$$-\pi t_{HH} + (1-\pi)\left(\theta_{BH} - t_{LH}\right) \geq -\pi t_{HL} - (1-\pi) t_{LL} \qquad \text{(ICC BH)}$$

$$-\pi t_{HL} - (1-\pi) t_{LL} \geq -\pi t_{HH} + (1-\pi)\left(\theta_{BL} - t_{LH}\right) \qquad \text{(ICC BL)}$$

   Notation: $t_{ij} = t(\theta_{Aj}, \theta_{Bj})$.

3) The (interim) participation constraints are

   - for type A:

$$\pi\left(\theta_{AH} + t_{HH}\right) + (1-\pi)\left(\theta_{AH} + t_{HL}\right) \geq \theta_{AH} \qquad \text{(IRR AH)}$$

$$\pi t_{LH} + (1-\pi)\left(\theta_{AL} + t_{LL}\right) \geq \theta_{AL} \qquad \text{(IRR AL)}$$

   - for type B:

$$-\pi t_{HH} + (1-\pi)\left(\theta_{BH} - t_{LH}\right) \geq 0 \qquad \text{(IRR BH)}$$

$$-\pi t_{HL} - (1-\pi)t_{LL} \geq 0 \qquad \text{(IRR BL)}$$

[Note: Alternatively, you could have assumed that participation is decided *before* each agent knows its own type. (see class)]

4) YES. First, recall that ex post efficiency requires $x(\theta_{AL}, \theta_{BH}) = 1$, $x(\theta_{AL}, \theta_{BL}) = x(\theta_{AH}, \theta_{BH}) = x(\theta_{AH}, \theta_{BL}) = 0$. Next, it is natural to consider a contract with no payment unless there is trade, i.e. $t_{LL} = t_{HH} = t_{HL} = 0$ and $t_{LH} > 0$. Let's check whether such a contract can satisfy all constraints specified above.

(ICC AH)	$\Rightarrow \theta_{AH} \geq \pi t_{LH} + (1-\pi)\theta_{AH}$	(IRR AH)	$\Rightarrow \theta_{AH} \geq \theta_{AH}$
(ICC AL)	$\Rightarrow \pi t_{LH} + (1-\pi)\theta_{AL} \geq \theta_{AL}$	(IRR AL)	$\Rightarrow \pi t_{LH} + (1-\pi)\theta_{AL} \geq \theta_{AL}$
(ICC BH)	$\Rightarrow (1-\pi)(\theta_{BH} - t_{LH}) \geq 0$	(IRR BH)	$\Rightarrow (1-\pi)(\theta_{BH} - t_{LH}) \geq 0$
(ICC BL)	$\Rightarrow 0 \geq (1-\pi)(\theta_{BL} - t_{LH})$	(IRR BL)	$\Rightarrow 0 \geq 0$

Note that (IRR BL) and (IRR AH) hold trivially. We can also drop (IRR BH) and (IRR AL), as they are identical to (ICC BH) and (ICC AL). In other words, if all the ICC's hold, then all the IRR's hold. From (ICC AH) and (ICC AL) we get that $\theta_{AL} \leq t_{LH} \leq \theta_{AH}$. From (ICC BH) and (ICC BL) we get that $\theta_{BL} \leq t_{LH} \leq \theta_{BH}$. Hence, a contract satisfies all constraints if $\theta_{AL} \leq t_{LH} \leq \theta_{BH}$.

[Note: Assuming a participation decision before agents know their own type does not change the answer (see class).]

Problem 2: Signaling

Consider a measure one continuum of entrepreneurs of two types: a fraction π is of type θ^H and a fraction $1-\pi$ is of type θ^L with $\theta^H > \theta^L$. Each entrepreneur can invest in a project of fixed size normalized to one. Each entrepreneur has wealth $W_0 > 1$ and could therefore in principle self finance the project entirely. However, each investment project yields a random net return $R(\tilde{\theta})$ that is normally distributed with mean θ^j, $j = H, L$ and variance σ^2. Each entrepreneur's utility is given by $u(w) = -e^{-\rho w}$ where w denotes final wealth and $\rho > 0$. This means entrepreneurs are *risk averse.*

There is a large number of *risk neutral* investors to whom the entrepreneurs can sell (parts of) the project at a per unit price P. "Selling a part of the project" can be interpreted as raising equity.

1) Suppose θ is perfectly observable. This means that each entrepreneur can make P contingent on θ. Show that in that case each entrepreneur would sell of his entire project to the entrepreneurs at a price $P = \theta$ and would therefore be perfectly insured. Explain why this is efficient.

2) Suppose θ is unobservable by the investors. Let $\hat{\theta}$ be such that an entrepreneur for which $\theta < \hat{\theta}$ raises equity. Find $\hat{\theta}$ as a function of P.

3) Show that efficiency under asymmetric information is only possible if $(1-\pi)(\theta^H - \theta^L) \leq \frac{1}{2}\rho\sigma^2$ and explain.

4) Suppose the condition for efficiency specified in 3) is not satisfied. High type entrepreneurs can choose to self-finance a fraction α of the project to signal the project type. Write down the incentive compatibility condition.

5) Find the Pareto dominating signaling equilibrium.

Answer to Problem 2

1) First note that the market price of project P^j must be θ^j, $j = H, L$. At this price, the risk neutral investors expect to break-even. By selling the project to the market, the expected utility of entrepreneur j is

$$u^1 = u(W_0 + \theta^j)$$

If the entrepreneur chooses to self finance instead, his expected utility is

$$u^2 = u\left(W_0 + \theta^j - \frac{1}{2}\rho\sigma^2\right)$$

You can find the derivation of expected utility in a technical note below. Since entrepreneurs are risk averse $u^1 > u^2$. The risk-neutral investors provide perfect insurance for the entrepreneurs and the equilibrium outcome is efficient.

2) If both types decide to go to the financial market, the price P of equity must be the same for all entrepreneurs. However, the entrepreneur's outside option (not going to the market) yields utility u^2. Therefore, an efficient equilibrium with both selling their entire project is only possible if

$$\theta^j \leq \hat{\theta} = P + \frac{1}{2}\rho\sigma^2 \ , \ j = H, L \tag{1}$$

[Note: Alternatively, you could also have allowed for partial self-finance at this stage. The condition would have been $\theta^j \leq \hat{\theta} = P + \rho\sigma^2$.]

3) Efficiency is only possible if $(\theta^L <)\theta^H < \hat{\theta}$. In that case the price of equity must equal

$$P = \pi\theta^H + (1-\pi)\theta^L$$

Plugging this into equation 1, the condition for efficiency is

$$\pi\theta^H + (1-\pi)\theta^L + \frac{1}{2}\rho\sigma^2 \geq \theta^H$$

$$\Leftrightarrow \frac{1}{2}\rho\sigma^2 \geq (1-\pi)(\theta^H - \theta^L)$$

If this condition is not satisfied, the high-type entrepreneurs prefer to self finance, which is inefficient.

[Note: Allowing for partial finance, the condition for efficiency is $\rho\sigma^2 \geq (1-\pi)(\theta^H - \theta^L)$]

4) The high type entrepreneurs can signal the quality of their project by self-financing a fraction α of the project. With signalling the prices are $P^j = \theta^j$. The ICC for the low type is

$$u(W_0 + \theta_L) \geq u\left(W_0 + (1-\alpha)\theta^H + \alpha\theta^L - \frac{1}{2}\rho\sigma^2\alpha^2\right)$$

$$\Leftrightarrow \theta_L \geq (1-\alpha)\theta^H + \alpha\theta^L - \frac{1}{2}\rho\sigma^2\alpha^2$$

$$\Leftrightarrow \frac{\alpha^2}{1-\alpha} \geq \frac{2(\theta^H - \theta^L)}{\rho\sigma^2} \tag{2}$$

The ICC for the high type is

$$u(W_0 + \theta_L) \leq u\left(W_0 + (1-\alpha)\theta^H + \alpha\theta^H - \frac{1}{2}\rho\sigma^2\alpha^2\right)$$

$$\Leftrightarrow \alpha^2 \leq \frac{2(\theta^H - \theta^L)}{\rho\sigma^2} \tag{3}$$

5) The level of self-finance is perfectly observed and there is a continuum of signalling equilibria with $P^j = \theta^j$ and a value for α that satisfies (2) and (3). For the investors and low type entrepreneurs the outcome is always the same as the perfect information (efficient) equilibrium. The utility level of the high type entrepreneurs is

$$u(W_0 + \theta^H - \frac{1}{2}\rho\sigma^2\alpha^2)$$

which is strictly decreasing in the level of self-finance α. Therefore, the Pareto dominant signalling equilibrium is the one in which α is the lowest possible, i.e. the solution to

$$\frac{\alpha^2}{1-\alpha} = \frac{2(\theta^H - \theta^L)}{\rho\sigma^2}$$

Technical Note: Expected utility in the case of normal returns and negative exponential utility.

$$
\begin{aligned}
E\left[u(x)\right] =& - \frac{1}{\sigma\sqrt{2\pi}} \int \exp(-\rho x) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right) dx \\
=& - \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(-\frac{x^2 - 2\mu x + \mu^2 + 2\sigma^2 \rho x}{2\sigma^2} \right) dx \\
=& - \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(-\frac{x^2 - 2(\mu - \sigma^2\rho)x + \mu^2}{2\sigma^2} \right) dx \\
=& - \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(-\frac{x^2 - 2(\mu - \sigma^2\rho)x + \mu^2 - 2\sigma^2\rho\mu + \rho^2\sigma^4 + 2\sigma^2\rho\mu - \rho^2\sigma^4}{2\sigma^2} \right) dx \\
=& - \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(-\frac{x^2 - 2(\mu - \sigma^2\rho)x + (\mu^2 - 2\sigma^2\rho\mu + \rho^2\sigma^4) + 2\sigma^2\rho\mu - \rho^2\sigma^4}{2\sigma^2} \right) dx \\
=& - \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(-\frac{\left[x - (\mu - \sigma^2\rho)\right]^2}{2\sigma^2} - \rho\left(\mu - \rho\frac{\sigma^2}{2}\right) \right) dx \\
=& - \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(-\frac{\left[x - (\mu - \sigma^2\rho)\right]^2}{2\sigma^2} \right) dx \exp\left(-\rho\left(\mu - \rho\frac{\sigma^2}{2}\right) \right) \\
=& - \exp\left(-\rho\left(\mu - \rho\frac{\sigma^2}{2}\right) \right) \\
=& u\left(\mu - \rho\frac{\sigma^2}{2}\right)
\end{aligned}
$$

MICROECONOMICS III: Information and Contract Theory

PROBLEM SET 4: QUESTIONS AND ANSWERS

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Problem 1: Moral Hazard

An Agent (A) needs to complete a task for a Principal (P). A can choose between working hard, $a = 1$ or shirking $a = 0$. If A accepts the task, he is paid a wage w is his utility is $u(w) - a$, where $u(\cdot)$ is strictly concave. A can always reject the task and obtain an outside option $\underline{u}$. The only thing P can observe is whether A succeeds or fails his task. He can make the wage w contingent on that observation. If $a = 1$, the probability of success is p_H. If he $a = 0$, the probability of success is p_L. The pay-off to P is x_H in case of succes and x_L in case of failure. P's total pay-off, conditional on his observation, is $x_j - w_j$, $j = H, L$. Assume $x_H > x_L$ and $p_H > p_L$.

1) Suppose P wants A to work hard, write down the incentive compatibility constraint (ICC). Also write down $A's$ individual rationality constraint (IRR).

2) Argue that the optimal (for P) wage schedule (w_H, w_L) implies both constraints holding with equality.

3) Let W_1 denote $P's$ expected payoff if he makes A work hard. Off course it might also be optimal for P to have $a = 0$, in which case the wage is w and his expected pay-off is W_0.

When is $W_1 > W_0$?

4) Show that $p_L w_H + (1 - p_L) w_L > w$.

5) Show that if $x_H - w_H \leq x_L - w_L$, then $W_1 < W_0$ and explain.

Answer to Problem 1

1) The ICC constraint is

$$p_H(u(w_H) - 1) + (1 - p_H)(u(w_L) - 1) \geq p_L(u(w_H)) + (1 - p_L)u(w_L) \tag{ICC}$$

The IRR constraint is

$$p_H(u(w_H) - 1) + (1 - p_H)(u(w_L) - 1) \geq \underline{u} \tag{IRR}$$

2) Verify first that (ICC) can be rewritten as

$$(p_H - p_L)(u(w_H) - u(w_L)) \geq 1$$

P's expected payoff is $p_H(x_H - w_H) + (1 - p_H)(x_L - w_L)$.

Suppose (IRR) does not hold with equality. Then P can lower both w_L and w_H such that ICC is left unchanged. The payoff for P is then strictly higher. Hence, IRR must bind.

Suppose (ICC) does not hold with equality. Consider a deviation

$$w_H - \frac{(1 - p_H)\epsilon}{u'(w_H)}$$

$$w_L + \frac{p_H \epsilon}{u'(w_L)}$$

For small ϵ, such a deviation satisfies both (ICC) and (IRR) and decreases wage payments for P. Hence (ICC) must bind.

3) P's expected payoff when A works hard is

$$W_1 = p_H(x_H - w_H) + (1 - p_H)(x_L - w_L)$$

where (w_H, w_L) are the values for which the ICC and IRR hold with equality:

$$\begin{aligned} w_H &= u^{-1}\left(\underline{u} + \frac{1 - p_L}{p_H - p_L}\right) \\ w_L &= u^{-1}\left(\underline{u} - \frac{p_L}{p_H - p_L}\right) \end{aligned}$$

P's expected payoff when A shirks is

$$W_0 = p_L(x_H - w) + (1 - p_L)(x_L - w)$$

where w is the minimum wage for which the worker is willing to accept the task, i.e. w solves

$$w = u^{-1}(\underline{u})$$

$W_1 > W_0$ requires

$$p_H(x_H - w_H) + (1 - p_H)(x_L - w_L) - p_L(x_H - w) - (1 - p_L)(x_L - w) > 0$$

4) Using Jensen's inequality, strict concavity of $u(\cdot)$ and :

$$\begin{aligned} p_L w_H + (1 - p_L)w_L =& p_L\left[u^{-1}\left(\underline{u} + \frac{1 - p_L}{p_H - p_L}\right)\right] + (1 - p_L)\left[u^{-1}\left(\underline{u} - \frac{p_L}{p_H - p_L}\right)\right] \\ p_L w_H + (1 - p_L)w_L >& p_L\left[u^{-1}\left(\underline{u}\right)\right] + (1 - p_L)\left[u^{-1}\left(\underline{u}\right)\right] \\ p_L w_H + (1 - p_L)w_L >& p_L w + (1 - p_L)w \\ p_L w_H + (1 - p_L)w_L >& w \end{aligned}$$

5) Proof:

$$\begin{aligned}
x_H - w_H &\leq x_L - w_L \\
(p_H - p_L)(x_H - x_L) &\leq (p_H - p_L)(w_H - w_L) \\
(p_H - p_L)(x_H - x_L) &\leq p_H w_H - p_L w_H - p_H w_L + p_L w_L \\
(p_H - p_L)(x_H - x_L) &\leq p_H w_H + (1 - p_H) w_L - (p_L w_H + (1 - p_L) w_L)
\end{aligned}$$

Using the inequality in (4):

$$\begin{aligned}
(p_H - p_L)(x_H - x_L) - p_H w_H - (1 - p_H) w_L + w &< 0 \\
(p_H - p_L) x_H - p_H w_H + (p_L - p_H) x_L - (1 - p_H) w_L + p_L w + (1 - p_L) w &< 0 \\
p_H(x_H - w_H) + (1 - p_H)(x_L - w_L) - p_L(x_H - w) - (1 - p_L)(x_L - w) &< 0 \\
W_1 - W_0 &< 0
\end{aligned}$$

If the difference in payoff is not sufficiently high relative to the difference in wages necessary to induce high effort, P prefers not to incentivate A.

Problem 2: Moral Hazard and Credit Market Rationing

Assume an investor can choose between two projects, A and B. Both projects require a fixed investment of I. The pay-off $\tilde{X}_j$ for $j = A, B$ is

$$\tilde{X}_i = \begin{cases} X_j & \text{with probability} \quad p_j \\ 0 & \text{with probability} \quad 1 - p_j \end{cases}$$

Assume $p_A X_A > p_B X_B > I$, $1 > p_A > p_B > 0$ and $X_B > X_A$. The investor must borrow the amount I from a bank. Assume the gross interest payment R will only be paid if the project is successful. I.e. the bank cannot get payments from a bankrupt investor. The expected payoff to the investor is $U_j(R) = p_j(X_j - R)$. The bank's expected profit is $\pi_j = p_j R - I$.

1) Assume that R cannot be made contingent on the project choice. Define $\hat{R}$ such that the investor chooses project A if $R \leq \hat{R}$. Find $\hat{R}$.

2) Suppose the bank is a monopolist and there is a single investor. What is the bank's optimal interest policy?

3) Suppose now there are N identical investors. Also suppose the bank has a fixed amount L of loanable funds, with $I < L < NI$. Discuss why and when *credit rationing* might arise, i.e demand for loanable funds necessarily exceeds the supply of loanable funds.

Answer to Problem 2

1) Given a payment R, the investor will choose A if

$$p_A(X_A - R) \geq p_B(X_B - R) \Leftrightarrow R \leq \frac{p_A X_A - p_B X_B}{p_A - p_B}$$

So $\hat{R}$ is the value for which this expression holds with equality:

$$\hat{R} = \frac{p_A X_A - p_B X_B}{p_A - p_B} > 0$$

If $R \leq \hat{R}$, investors choose project A. If $\hat{R} < R \leq X_B$, investors choose project B.

2) The bank's profits as a function of R are

$$\pi(R) = \begin{cases} p_A R - I & \text{if} \quad 0 \leq R \leq \hat{R} \\ p_B R - I & \text{if} \quad \hat{R} < R \leq X_B \end{cases} \tag{1}$$

The profit function has two local maxima: $p_A \hat{R}$ and $p_B X_B$. So the optimal interest rate policy is

$$R^* = \begin{cases} \hat{R} & \text{if} \quad p_A\hat{R} > p_B X_B \\ X_B & \text{if} \quad p_A\hat{R} < p_B X_B \end{cases}$$

3) **Credit rationing** occurs when a lender refuses to make a loan to an investor, even though the investor is willing to pay (more than) the market interest rate.

   Consider first the situation where it is optimal for the banks to set $R^* = X_B$. In that case the expected payoff to the investors is

$$p_B(X_B - X_B) = 0$$

   So at this interest rate, investors are indifferent between taking out a loan or not. Hence, there is no credit rationing.

   Now suppose that it is optimal for the banks to set $R^* = \hat{R}$. The expected payoff to the investors is

$$p_A(X_A - \hat{R}) > 0$$

   This means that all N investors will ask for a loan and the demand for loanable funds is NI. However, the supply is only $L < NI$. Hence there is credit rationing.