MICROECONOMICS III

Information Economics and Contract Theory

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SOLUTION to PROBLEM SET 1

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1 Adverse Selection & Monopoly Screening

A monopolist can produce a good of different qualities. The cost of production of one unit of quality q is

$$c(q) = \frac{1}{2}q^2 \tag{1}$$

Consumers buy at most one unit. The utility of a consumer of type θ is

$$u(q \mid \theta) = \theta q \tag{2}$$

if she consumes a unit of quality q and 0 otherwise. There are $N \geq 3$ consumers and consumer n has preference θ_ns.t. $\theta_1 > \theta_2 > ... > \theta_N$. The proportion of type θ_n consumers is π_n.

1.1 Perfect Price Discrimination (First Best Allocation)

There is symmetric information. The monopolist is perfectly informed about the buyer's characteristics and will treat each consumer separately i.e. offer a type-specific contract. The monopolist maximizes profits

$$\max_{p_n, q_n}(p_n - \frac{1}{2}q_n^2) \tag{3}$$

s.t. participation

$$\theta_n q_n - p_n \geq 0 \tag{4}$$

where $n = 1, ..., N$.

As information is symmetric, full rent extraction implies that the participation constraint binds

$$\theta_n q_n = p_n$$

for all types of consumers. So we

$$\max_{q_n}(\theta_n q_n - \frac{1}{2}q_n^2)$$

which gives the efficient provision of quality

$$q_n^* = \theta_n \tag{5}$$

It follows that the monopoly price is

$$p_n^* = q_n^{*2} \tag{6}$$

and profits from type n equal

$$\pi_n \left[q_n^{*2} - \frac{1}{2} q_n^{*2} \right] = \frac{1}{2} \pi_n q_n^{*2} \tag{7}$$

1.2 Assymetric Information (Second Best) and Menu of Contracts

The monopolist cannot observe the consumers' types but can offer different qualities at different prices. Assume in equilibrium consumer n choses quality q_n at price p_n for $n = 1..N$. Write the incentive compatibility constraints $IC_{n,n'}$ that type n does not deviate to n' and the individual rationality constraints (or participation constraints) IR_n. How many constraints are there? Write the monopoly profit maximizing problem.

a) Participation and incentive compatibility: The individual rationality constraint IR_n reads

$$\theta_n q_n - p_n \geq 0 \tag{8}$$

stating that in order for a type n consumer to accept the contract, her utility from consuming one unit of q must at least outweigh the cost p (weakly positive utility).

The incentive compatibility constraints in Screening problems are also known as the "truth telling" constraints as they should make sure that agents (here consumers) pick the contract designed for their true type and do not pretend to be of another type by choosing the "wrong" contract from the monopolist perspective.

The $IC_{n,n'}$ reads

$$\theta_n q_n - p_n \geq \theta_n q_{n'} - p_{n'} \tag{9}$$

and

$$\theta_{n'}q_{n'} - p_{n'} \geq \theta_{n'}q_n - p_n$$

b) How many constraints are there?

The structure of these constraints remains the same for all pairs $n, n' \in [1, N]$. So first we have N participation constraints, one for each consumer. Furthermore each consumer has $N-1$ ICs to make sure she prefers the one contract designed for her type. As there are N consumers the number of ICs is $N(N-1)$. So the overall number of constraints in this maximization problem is

$$N + N(N-1) = N^2$$

c) The profit maximization problem of the monopolist is to choose $\{(q_n, p_n); n = 1, ..., N\}$ from among all feasible menus of contracts, which solve the program

$$\max_{\{(q_n,p_n)\}} \sum_{n=1}^{N} (p_n - \frac{1}{2}q_n^2)\pi_n \tag{10}$$

s.t.

$$\theta_n q_n - p_n \geq 0 \ \forall n \tag{11}$$

$$\theta_n q_n - p_n \geq \theta_n q_{n'} - p_{n'} \ \forall n, n' \tag{12}$$

1.3 Show that q_n is decreasing in n.

There are $N \geq 3$ consumers and consumer n has preference θ_ns.t. $\theta_1 > \theta_2 > ... > \theta_N$. In equilibrium she choses $q_{n.}$ This simply states that high types (i.e. consumers that value quality more) have to consume higher quality than lower types. Notice here that the higher is n, the lower the type, hence quality is decreasing in n.

So we add

$$\theta_n q_n - p_n \geq \theta_n q_{n'} - p_{n'}$$

where

$$\theta_n > \theta_{n'} \text{ and } n' > n$$

and

$$\theta_{n'}q_{n'} - p_{n'} \geq \theta_{n'}q_n - p_n$$

which yields

$$\theta_n(q_n - q_{n'}) \geq \theta_{n'}(q_n - q_{n'})$$

The resulting inequality reads

$$(\theta_n - \theta_{n'})(q_n - q_{n'}) \geq 0 \tag{13}$$

Hence for this to hold and given that

$$\theta_n - \theta_{n'} > 0$$

it must follow that

$$q_n - q_{n'} \geq 0 \Longrightarrow q_n \geq q_{n'} \longleftrightarrow q_n \geq q_{n+1} \tag{14}$$

In other words, an incentive compatible contract must be such that

$$q_n \geq q_{n+1} \text{ whenever } \theta_n > \theta_{n+1}$$

that is, consumption increases monotonically in type θ when the single crossing condition holds. To check, take utility function of the consumer

$$u(q, p \mid \theta) = \theta q - p$$

and check whether it satisfies the Spence Mirrlees single crossing condition

$$\frac{\partial}{\partial \theta}\left[-\frac{\partial u/\partial q}{\partial u/\partial p}\right] > 0 \tag{15}$$

This condition tells us that the Marginal Utility of consumption relative to the MU of money is increasing in θ. Here:

$$\frac{\partial}{\partial \theta}\left[-\frac{\theta}{-1}\right] = 1 > 0$$

single crossing of indifference curves is satisfied. Furthermore the higher type indifference curve crosses the lower type indifference curve always from below as can be seen from

$$\frac{dp}{dq} = \theta_n$$

i.e. the slope of type θ_n is higher than that of type θ_{n+1} as $\theta_n > \theta_{n+1}$. (Notice also Linearity from $\frac{d^2p}{d^2q} = 0$)

1.4 Show that if $IC_{n,n+1}$ bind for $n = 1..N-1$ and IR_N binds, and q_n is decreasing in n, then all the other constraints automatically hold.

We use the single crossing condition to further reduce the set of constraints in order to increase simplicity.

DOWNWARD CONSTRAINTS: We start out with the **Local Downward Incentive Constraints** (I denote them as downward because types are decreasing in n.):

$LDICs$:

$$\begin{aligned} \theta_{n-1}q_{n-1} - p_{n-1} &\geq \theta_{n-1}q_n - p_n \text{ for } n = 1, ..., N-1 \\ \theta_n q_n - p_n &\geq \theta_n q_{n+1} - p_{n+1} \text{ etc.} \end{aligned}$$

which we initially assume to be binding.

To show that the **Global Downward Constraints** hold as well, we add $IC_{n,n+1}$and $IC_{n+1,n+2}$ to get

$$\theta_n q_n - p_n \geq (\theta_n - \theta_{n+1)} q_{n+1} + \theta_{n+1} q_{n+2} - p_{n+2}$$

Then

$$\theta_n q_n - p_n \geq \underbrace{(\theta_n - \theta_{n+1)}(q_{n+1} - q_{n+2})}_{\geq 0} + \theta_n q_{n+2} - p_{n+2} \tag{16}$$

We see that

$$\theta_n q_n - p_n \geq \theta_n q_{n+2} - p_{n+2} \tag{17}$$

Furthermore we can show $IC_{n,n+3}$:

$$\theta_n q_n - p_n \geq (\theta_n - \theta_{n+1)} q_{n+1} + \theta_{n+1} q_{n+2} - p_{n+2}$$

As we know from above ($IC_{n,n+2}$):

$$\theta_{n+1} q_{n+1} - p_{n+1} = \theta_{n+1} q_{n+2} - p_{n+2} \geq \theta_{n+1} q_{n+3} - p_{n+3} \tag{18}$$

By replacing above we get

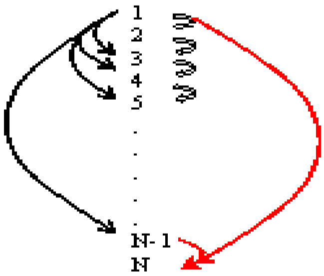
$$\theta_n q_n - p_n \geq (\theta_n - \theta_{n+1)} q_{n+1} + \theta_{n+1} q_{n+3} - p_{n+3} \tag{19}$$

$$\theta_n q_n - p_n \geq (\theta_n \underset{(+)}{-} \theta_{n+1)} q_{n+1} + (\theta_{n+1} \underset{(-)}{-} \theta_n) q_{n+3} + \theta_n q_{n+3} - p_{n+3} \tag{20}$$

But we need monotonicity, so we know that $q_{n+1} > q_{n+3}$ and $\left|\theta_n - \theta_{n+1})\right| = |\theta_{n+1} - \theta_n|$, hence

$$\theta_n q_n - p_n \geq \theta_n q_{n+3} - p_{n+3} \tag{21}$$

This tells us that if for each type θ_n the IC with respect to the next lower type θ_{n+1} binds until type θ_{N-1}, then all other downward ICs are also satisfied:



UPWARD CONSTRAINTS: The same rationale as above holds true for the Upwards Constraints (i.e. type n does choose the contract offered to the next higher type $n-1$):

$$\theta_n q_n - p_n \geq \theta_n q_{n-1} - p_{n-1} \tag{22}$$

Proof.

$$\begin{aligned} \theta_n q_n - p_n &= \theta_n q_{n-1} + \theta_n(q_n - q_{n-1}) - p_n \\ &= \theta_n q_{n-1} + (\theta_n - \theta_{n-1})(q_n - q_{n-1}) + \boldsymbol{\theta}_{n-1}(\mathbf{q}_n - q_{n-1}) - \mathbf{p}_n \end{aligned}$$

We know (from the LDICs) that

$$\theta_{n-1} q_{n-1} - p_{n-1} = \boldsymbol{\theta}_{n-1}\mathbf{q}_n - \mathbf{p}_n$$

which we plug into the above equation to get

$$\theta_n q_n - p_n = \theta_n q_{n-1} - p_{n-1} + (\theta_n - \theta_{n-1})(q_n - q_{n-1}) \geq \theta_n q_{n-1} - p_{n-1}$$

which shows that all upwards constraints hold locally. (same logic as above with LDICs) ■

To show that the **Global UICs** (≥ 2 steps) hold we take

$$\begin{aligned}
\theta_n q_n - p_n &\geq \theta_n q_{n-1} - p_{n-1} \\
\theta_n q_n - p_n &\geq \theta_n q_{n-2} + \theta_n (q_{n-1} - q_{n-2)} - p_{n-1} \\
\theta_n q_n - p_n &\geq \theta_n q_{n-2} + (\theta_n - \theta_{n-1})(q_{n-1} - q_{n-2)} + \boldsymbol{\theta}_{n-1}\mathbf{q}_{n-1} - \mathbf{p}_{n-1} - \theta_{n-1} q_{n-2}
\end{aligned}$$

Now we know from the local upward constraints that

$$\theta_{n-1} q_{n-1} - p_{n-1} \geq \theta_{n-1} q_{n-2} - p_{n-2} \tag{23}$$

which we use to get (by replacing it in the previous line)

$$\theta_n q_n - p_n \geq \theta_n q_{n-2} - p_{n-2} + (\theta_n \underset{(-)}{-} \theta_{n-1})(q_{n-1} \underset{(-)}{-} q_{n-2}) \tag{24}$$

$$\theta_n q_n - p_n \geq \theta_n q_{n-2} - p_{n-2} + (.)$$

which shows that the constraint holds globally

$$\theta_i q_i - p_i \geq \theta_i q_j - p_j \tag{25}$$

for $i \neq j$, $j < i - 1$ and $i, j \in [1, N]$

Proof. for ≥ 2 steps:

$$\begin{aligned}
\theta_n q_n - p_n &= \theta_n (q_n - q_{n-3}) - p_n + \theta_n q_{n-3} = && (26) \\
&= (\theta_n - \theta_{n-1})(q_n - q_{n-3}) - \mathbf{p}_n + \theta_n q_{n-3} + \boldsymbol{\theta}_{n-1}(\mathbf{q}_n - q_{n-3}) && (27)
\end{aligned}$$

Further we know from $IC_{n-1,n}$ and $IC_{n-1,n-3}$ that

$$\theta_{n-1} q_n - p_n = \theta_{n-1} q_{n-1} - p_{n-1} \geq \theta_{n-1} q_{n-3} - p_{n-3} \tag{28}$$

so we replace to get

$$\theta_n q_n - p_n = (\theta_n - \theta_{n-1})(q_n - q_{n-3}) + \theta_n q_{n-3} - p_{n-3} \geq \theta_n q_{n-3} - p_{n-3} \tag{29}$$

etc. ■

As the Participation constraint for the lowest type is binding, $\theta_n > \theta_N$ $\forall n = 1, ..., N - 1$ implies that all other PCs hold automatically.

Proof.

$$\theta_n q_n - p_n = \theta_n q_{N-1} - p_{N-1} \geq \theta_n q_N - p_N > \theta_N q_N - p_N = 0 \ \forall n = 1, ..., N - 1$$

implies directly

$$\theta_n q_n - p_n > 0$$

■

1.5 Show that $IC_{n,n+1}$ and IR_N have to bind in equilibrium.

The monopolist does not have to give any surplus to the low type N.

Lemma 1 *IR_N has to bind in equilibrium.*

Proof. Suppose NOT:

$$\theta_N q_N - p_N > 0$$

This would imply that

$$\theta_n q_n - p_n > 0 \; \forall n = 1, ..., N-1$$

hold as well with strict inequality.

Then consider a deviation

$$\widetilde{p_N} = p_N + \varepsilon$$

and

$$\widetilde{pn} = p_n + \varepsilon$$

respectively, which would not affect any other constraint such as e.g. the IC_N

$$\theta_N q_N - p_N \geq \theta_N q_{N-1} - p_{N-1}$$

As a matter of fact this would allow the monopolist to increase his profits up until the point that the participation constraint of the lowest type binds. Further increase in prices is not possible anymore as from then onwards incentive compatibility would be violated. ■

Lemma 2 *All $IC_{n,n+1}$(Local downward incentive constraints) have to bind in the optimum.*

Proof. Suppose not, that is

$$\theta_n q_n - p_n > \theta_n q_{n+1} - p_{n+1}$$

then the monopolist can raise p_{n+1} by ε s.t.

$$\theta_n q_n - p_n = \theta_n q_{n+1} - (p_{n+1} + \epsilon)$$

without affecting any other constraint, but making her better off. ■

So all types are indifferent between their contract and the contract of the next to below type except for the lowest type N.

1.6 Solve for the profit maximizing allocation.

The monopolist's problem finally reduces to

$$\max_{\{(q_n,p_n\}} \sum_{n=1}^{N} (p_n - \frac{1}{2}q_n^2)\pi_n$$

subject to

$$\begin{aligned} \theta_N q_N - p_N &= 0 \\ \theta_n q_n - p_n &= \theta_n q_{n+1} - p_{n+1} \ \forall n \\ q_n &> q_{n+1} \text{ where } \theta_n > \theta_{n+1} \end{aligned}$$

In order to solve for the profit maximizing allocation we derive an expression for p_n in the following way:

$$\mathbf{IR}_N : \theta_N q_N = p_N$$

Using the fact that all $IC_{n,n+1}$ have to bind, we take $IC_{N-1,N}$ as a starting point and procees iteratively upwards (in terms of types):

$$\begin{aligned} \theta_{N-1}q_{N-1} - p_{N-1} &= \theta_{N-1}q_N - p_N \\ p_{N-1} &= \theta_{N-1}q_{N-1} - \theta_{N-1}q_N + \underbrace{p_N}_{=\theta_N q_N} \\ p_{N-1} &= \theta_{N-1}q_{N-1} + q_N(\theta_N - \theta_{N-1}) \end{aligned}$$

Now consider $IC_{N-2,N-1}$:

$$\begin{aligned} p_{N-2} &= \theta_{N-2}q_{N-2} - \theta_{N-2}q_{N-1} + p_{N-1} \\ p_{N-2} &= \theta_{N-2}q_{N-2} - \theta_{N-2}q_{N-1} + \theta_{N-1}q_{N-1} - \theta_{N-1}q_N + \theta_N q_N \end{aligned}$$

which gives after reordering

$$p_{N-2} = \theta_{N-2}q_{N-2} + q_{N-1}(\theta_{N-1} - \theta_{N-2)} + q_N(\theta_N - \theta_{N-1})$$

We are considering finally $IC_{N-3,N-2}$ in order to see the emerging pattern clearly.

$$p_{N-3} = \theta_{N-3}q_{N-3} + q_{N-2}(\theta_{N-2} - \theta_{N-3}) + q_{N-1}(\theta_{N-1} - \theta_{N-2)} + q_N(\theta_N - \theta_{N-1})$$

From here we can state the following general expression for p_n:

$$p_n = \theta_n q_n + \sum_{i=n+1}^{N} [q_i(\theta_i - \theta_{i-1})] \text{ for } n = 1, ..., N-1 \tag{30}$$

Define the Cumulative Distribution Function as

$$\Pi_{N-1} = \sum_{n=1}^{N-1} \pi_n \tag{31}$$

This simply can be understood as the fraction of all consumers from highest θ_1 to type θ_{N-1}. So we can rewrite monopoly profits in the following way:

$$\begin{aligned} MP \;=\; & q_N \left[\pi_N \theta_N + (\theta_N - \theta_{N-1})\Pi_{N-1}\right] - \pi_N \frac{1}{2} q_N^2 + \\ & + q_{N-1}\left[\pi_{N-1}\theta_{N-1} + (\theta_{N-1} - \theta_{N-2})\Pi_{N-2}\right] - \pi_{N-1}\frac{1}{2} q_{N-1}^2 + \\ & + \ldots + \\ & + q_2\left[\pi_2\theta_2 + (\theta_2 - \theta_1)\Pi_1\right] - \pi_2 \frac{1}{2} q_2^2 + \\ & + q_1 \pi_1 \theta_1 - \pi_1 \frac{1}{2} q_1^2 \end{aligned} \tag{32}$$

So when solving for all the different q_n we get

$$\begin{aligned} \frac{\partial MP}{\partial q_N} & \Longrightarrow q_N = \theta_N + (\theta_N - \theta_{N-1})\frac{\Pi_{N-1}}{\pi_N} \\ \frac{\partial MP}{\partial q_n} \forall n & > 1 \Longrightarrow q_n = \theta_n + (\theta_n - \theta_{n-1})\frac{\Pi_{n-1}}{\pi_n} \\ \frac{\partial MP}{\partial q_1} & \Longrightarrow q_1^* = \theta_1 \end{aligned} \tag{33}$$

(Compare this result with Lemma 6 in the lecture notes and note that $\Pi_{n-1} = \sum_{n=1}^{n-1} \pi_n \equiv 1 - \pi$). We can write the monopoly profits from any type $n < 1$ consumer in the following way:

$$MP^* = \pi_n \left[\underbrace{q_n\theta_n - \frac{1}{2} q_n^2}_{\text{SOCIAL SURPLUS}} \quad - \quad \underbrace{q_n(\underset{(+)}{\theta_{n-1} - \theta_n})\frac{\Pi_{n-1}}{\pi_n}}_{\text{INFORMATIONAL RENT}} \right] \tag{34}$$

ECONOMIC INTUITION: The highest type, here θ_1 gets the efficient quality q_1^* and rent $q_n(\theta_{n-1} - \theta_n)$from all other n types. The Social Surplus is the surplus gained from increasing the quality of any $n < 1$ type. However, we see that this increase in product quality for a "lower" type comes at a cost, which we call Informational Rent. If the monopolist increases type n quality, he has to compensate all higher types $n-1$

until 1. The rent obviously increases with types and all types are indifferent between their contract and the one offered to the next type below. All other types get quality $q_n < q_n^*$ which is suboptimal (distorted downward)We see this by looking at

$$q_n = \theta_n + \underset{(-)}{(\theta_n - \theta_{n-1})}\frac{\Pi_{n-1}}{\pi_n}$$

. Finally notice that the expression for the informational rent is divided by π_n, which simply states the amount per consumer in the cumulation $\Pi_{n-1.}$

To be completely safe we should check whether our result from above, i.e.

$$q_n > q_{n+1}\forall n \tag{35}$$

really holds. So take any q_n and $q_{n+1}$and start from our result to write

$$q_n = \theta_n + (\theta_n - \theta_{n-1})\frac{\Pi_{n-1}}{\pi_n} \geq \theta_{n+1} + (\theta_{n+1} - \theta_n)\frac{\Pi_n}{\pi_{n+1}} = q_{n+1} \tag{36}$$

Further assume that the types θ_n are equally close to each other over the whole interval $\theta_1...\theta_N$, and that they are distributed with equal probabilty, i.e. $\pi_1 = \pi_2 = \pi_n$ for all $n = 1...N$. So we simplify to

$$\underset{(+)}{(\theta_n - \theta_{n+1})} + \underset{(-)}{(\theta_n - \theta_{n-1})}\frac{\Pi_{n-1}}{\pi_n} + \underset{(+)}{(\theta_n - \theta_{n+1})}\frac{\Pi_n}{\pi_{n+1}} \geq 0 \tag{37}$$

$$(\theta_n - \theta_{n+1})\frac{\Pi_n}{\pi_{n+1}} - (\theta_{n-1} - \theta_n)\frac{\Pi_{n-1}}{\pi_n} \geq 0 \tag{38}$$

As $\pi_n = \pi_{n+1}$and $(\theta_n - \theta_{n+1}) = (\theta_{n-1} - \theta_n)$ we have to see whether

$$\Pi_n \geq \Pi_{n-1} \tag{39}$$

which is true by definition of CDF

$$\Pi_n = \sum_{i=1}^{n} \pi_i > \Pi_{n-1} = \sum_{i=1}^{n-1} \pi_i \tag{40}$$

as long as $\pi_i \geq 0$, which is also true.

2 MGW Chapter 13-B

Assume $r(\theta)$ to be continuous and non decreasing in θ. Furthermore we assume that $\theta \geq r(\theta)$. However, here we want to show that

$$\text{Any competitive equilibrium allocation is pareto inefficient, IFF } r(\overline{\theta}) > E\theta$$

STEP 1: Define a competitive equilibrium as Def. 13.B.1 (MGW p.439) to be a wage rate w^* and a set Θ^* of types of workers who accept employment s.t.

$$\Theta^* = \{\theta : r(\theta) \leq w^*\}$$

and

$$w^* = E\left[\theta \mid \theta \in \Theta^*\right] \text{ (rational expectations)}$$

STEP 2: So if $r(\overline{\theta}) > E\theta$, then the high productivity workers $\overline{\theta}$ do not work anymore as the wage offered is too low for them. This in turn decreases the average productivity of the workforce, lowering the wage that firms are willing to pay further. In an efficient equilibrium with $\theta \geq r(\theta)$ however, workers in general are more productive at work and as a matter of fact should ideally work. So any equilibrium where the high types $\overline{\theta}$ do not enter employment is pareto inefficient.

STEP: 3Assume $r(\overline{\theta}) \leq E\theta$. This implies that all workers $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ are accepting the contract and work. Now we see that given $\theta \geq r(\theta)$ in this case all equilibria are efficient. The wage firms are willing to pay based on their expectation is sufficient to attract even the high type workers

$$w^* \geq r(\overline{\theta})$$

and hence the outcome is efficient.

For a similar kind of exercise see last years Problem Set 1 Exercise 3: "Iff $r(\overline{\theta}) > E\theta$, then any equilibrium is inefficient" and see that this can be proven FALSE by providing a counterexample, i.e. the case of $r(\theta) > \theta$.

Microeconomics III

Information Economics and Contract Theory

SOLUTION to PROBLEM SET 2

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3 Competitive Screening in the Insurance Market

A risk neutral company offers insurance against the risk of a car accident. There are two types of agents, characterized by their probability to have an accident. Careful drivers (G) face a low probability of accident π_G while risky drivers (B) are more likely to have an accident with probabilty $\pi_B > \pi_G$. Denote the proportion of G type agents with q. If no accident occurs, the value of the car is $\omega^{NA} = w$, while if an accident occurs the damaged car values $\omega^A = w - d$. Agents are risk averse with Von Neumann Morgenstern utility $u(.)$. Agents pay a premium $\alpha_1 = pz$ and receive z in case of an accident We denote the net payoff in case of an accident $\alpha_2 = z - pz$.

3.1 What is the expected utility of agent $i = G, B$ when offered insurance (α_1, α_2)?

Utility for type $i = G, B$ in case of an accident is

$$u_i^A = u(w - d + \alpha_2) \quad (41)$$

$$u_i^{NA} = u(w - \alpha_1) \quad (42)$$

Expected utility is

$$EU_i = (1 - \pi_i)u_i^{NA} + \pi_i u_i^A \quad (43)$$

Assume for the moment that each agent can choose her optimal coverage amount.

3.2 What are the expected profits of the firm?

Expected profits are: under pooling

$$E\Pi = q\left[(1 - \pi_G)\alpha_1 - \pi_G\alpha_2\right] + (1 - q)\left[(1 - \pi_B)\alpha_1 - \pi_B\alpha_2\right] \quad (44)$$

under separation for types $i = G, B$:

$$E\Pi_i = (1 - \pi_i)\alpha_1^i - \pi_i\alpha_2^i \tag{45}$$

So total profits are

$$E\Pi = qE\Pi_G + (1 - q)E\Pi_B \tag{46}$$

3.3 First Best: Write down the agent's problem for $i = G, B$ and solve for the optimal coverage z_i.

Assume for the moment that the price per unit $p = const$. This would imply that each agent can choose her optimal coverage amount. The problem for types $i = B, G$ is

$$\max_{z_i} \{\pi_i u(w - d + \alpha_2) + (1 - \pi_i)u(w - \alpha_1\} \tag{47}$$

The first order condition then reads

$$\frac{u'(w - d + z_i - pz_i)}{u'(w - pz_i)} = \frac{(1 - \pi_i)p}{\pi_i(1 - p)} \tag{48}$$

Furthermore we know that

$$\frac{(1 - \pi_G)}{\pi_G} > \frac{(1 - \pi_B)}{\pi_B} \tag{49}$$

which in turn requires that

$$z_B > z_G \tag{50}$$

This states that the riskier types optimally require a higher coverage than the low risk agents.Alternatively we could just assume **symmetric information** and treat each agent separately, i.e. we just show the results for the representative agent i : First it is important to notice that $E\Pi$ of the firm must be 0.

$$E\Pi_i = (1 - \pi_i)\alpha_1^i - \pi_i\alpha_2^i = (1 - \pi_i)p_i z_i + \pi_i(p_i z_i - z_i) = 0 \tag{51}$$

$$p_i = \pi_i \tag{52}$$

The optimal price will equal the probability of default of the respective type of agent. We can now solve again the agent's problem as above to get the FOC

$$\frac{u'(w - d + z_i - pz_i)}{u'(w - pz_i)} = 1 \tag{53}$$

$$\implies z_i^* = d \tag{54}$$

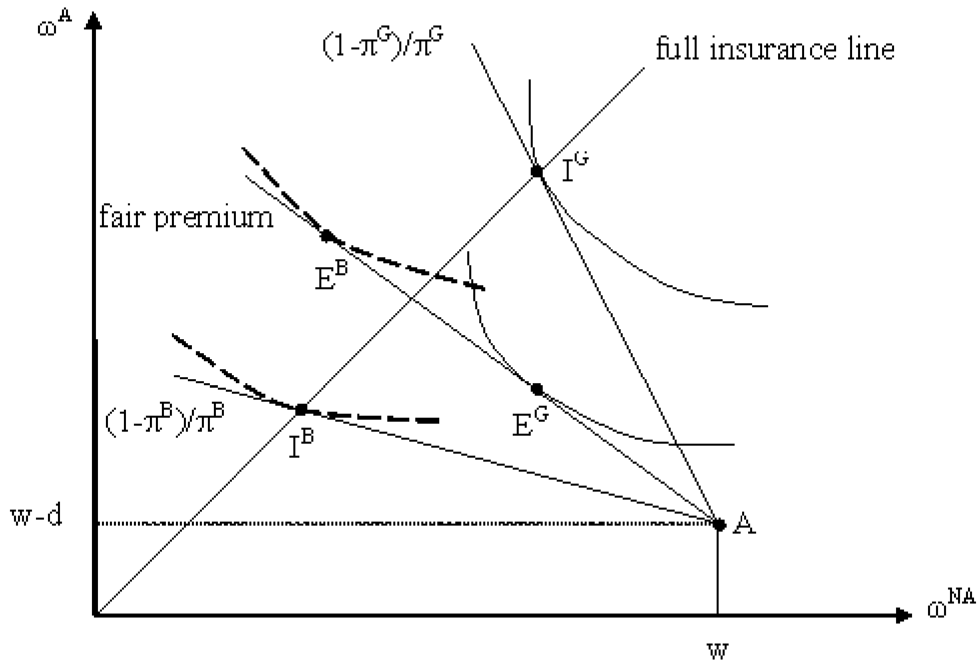
which shows that in the first best case the coverage amount equals the loss in case of accident. SO independent of the state of the world occuring, the agent will have the same flow of income, i.e. full insurance:

$$\{(\alpha_1^*, \alpha_2^*\} = \{(\pi_i d), (d - \pi_i d)\} \tag{55}$$

3.4 Describe the situation graphically. (Hint: use the ω_A, ω_{NA} space.)

Distinguish the case when there are two different prices (π_i) and just one price $\pi_G < p < \pi_B$. Concerning the latter case of just one price, this graph shows what may happen if the insurance company offers a "fair" contract with a premium of

$$p = q\pi_G + (1 - q)\pi_B \tag{56}$$



At the optimal contract(s), the slopes of the indifference curves coincide with the slopes of the corresponding iso-expected profit lines of the firm:

$$-\frac{(1 - \pi_i)u'(\omega_{NA})}{\pi_i u'(\omega_A)} = -\frac{(1 - \pi_i)}{\pi_i} \tag{57}$$

$_1$Again we see that we are on the 45° line of full insurance as for this to be true the MRS must equal 1. We see that I^G is not offered as bad agents would have an incentive to cheat on their types. A result such that only high risk agents (B) are insured at all might occur. Hence expected losses might occur. See the case of the fair premium: It can be seen that the bad type B would overinsure as $p < \pi_B$ while the G type will underinsure.

3.5 Competition: Assume there are $n \geq 2$ firms in the market competing for agents. Show that firms earn 0-Profits.

We want to show that

$$E\Pi = 0 \tag{58}$$

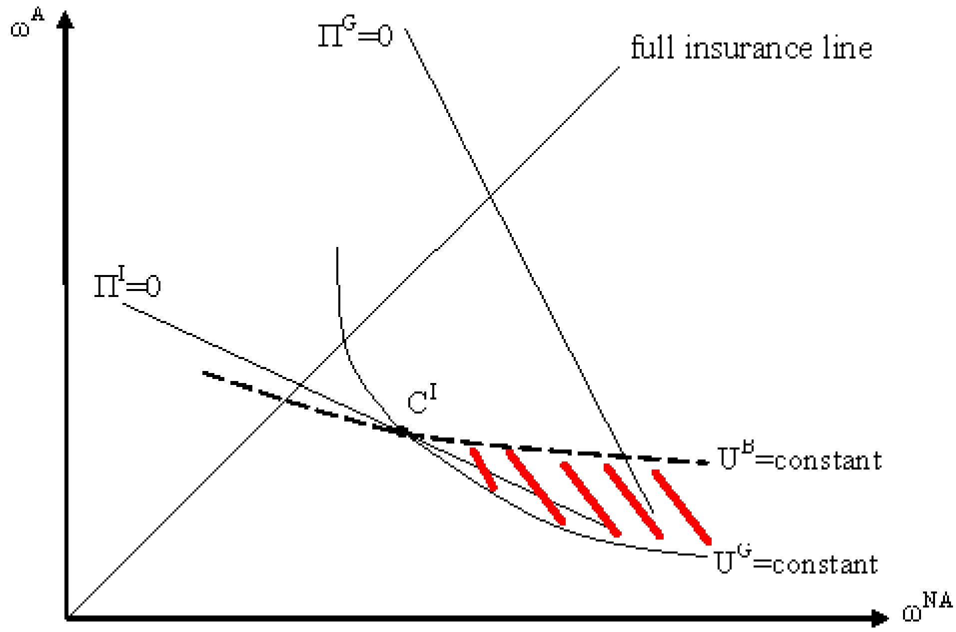
Proof. Suppose that $E\Pi > 0$: Another firm could offer a contract $\{\alpha_1 - \varepsilon\Pi, \alpha_2 + \varepsilon\Pi\}$ where $\varepsilon \in [0, 1]$, all agents would accept as they were better off, while the firm still makes profits of $(1-\varepsilon)\Pi$. (Bertrand kind of argument). This rationale can be applied to both the separating and the pooling case. ■

In general an equilibrium is based on the fact that no firm can add a contract that would give positive expected profits from the type(s) of agent that prefer this new contract to the old one (eithe $B, G, or\ both$). This implies zero expected profits.

3.6 Can a pooling contract (α_1, α_2) exist? Why (not)?

NO. A pooling contract is NOT robust to competition. Consider the graph below. Denote a pooling contract

$$C^I = \left(\alpha_1^I, \alpha_2^I\right) \tag{59}$$



Any pooling equilibrium must be on the zero profit pooling line

$$\Pi^I = q[(1-\pi_G)\alpha_1 - \pi_G\alpha_2] + (1-q)[(1-\pi_B)\alpha_1 - \pi_B\alpha_2] = 0 \tag{60}$$

$$= p^I\alpha_1 + (1-p^I)\alpha_2 \tag{61}$$

where we denote

$$p^I = q(1-\pi_G) + (1-q)(1-\pi_B) \tag{62}$$

So p^I is the probability that there will be no accident when the firm does not know which type the agent is. So the slope of this isoprofit line would then be

$$-\frac{1-p^I}{p^I} \tag{63}$$

Then any competitor firm could offer a contract in the shaded area, i.e. between the high and low type indiffernce curves. This contract would only attract H types (good drivers or low risk guys with steeper indiffernce curves) while all B types would not be willing to take it. This contract would however be below the $\Pi^G = 0$ line and hence imply $E\Pi > 0$ for the competitor. So we could expect some other firm to be willing to offer such a contract right away. This would leave the expected profits for the "incumbent" firm to be negative as they only attract B types anymore. So we conclude that a pooling equilibrium cannot exist.

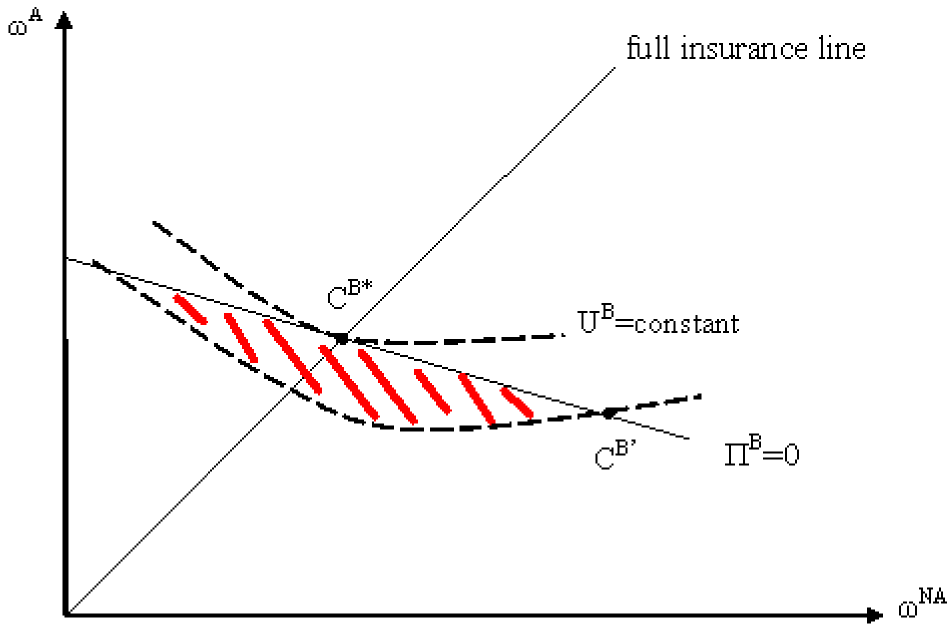
3.7 Consider a separating equilibrium candidate $\{(\alpha_1^G, \alpha_2^G), (\alpha_1^B, \alpha_2^B)\}$.

3.7.1 Show that in any equilibrium $(\alpha_1^B, \alpha_2^B) = (\alpha_1^{B*}, \alpha_2^{B*})$

Denote a separating equilibrium candidate

(64) $$C^S = \{(\alpha_1^G, \alpha_2^G), (\alpha_1^B, \alpha_2^B)\} = \{C^G, C^B\}$$

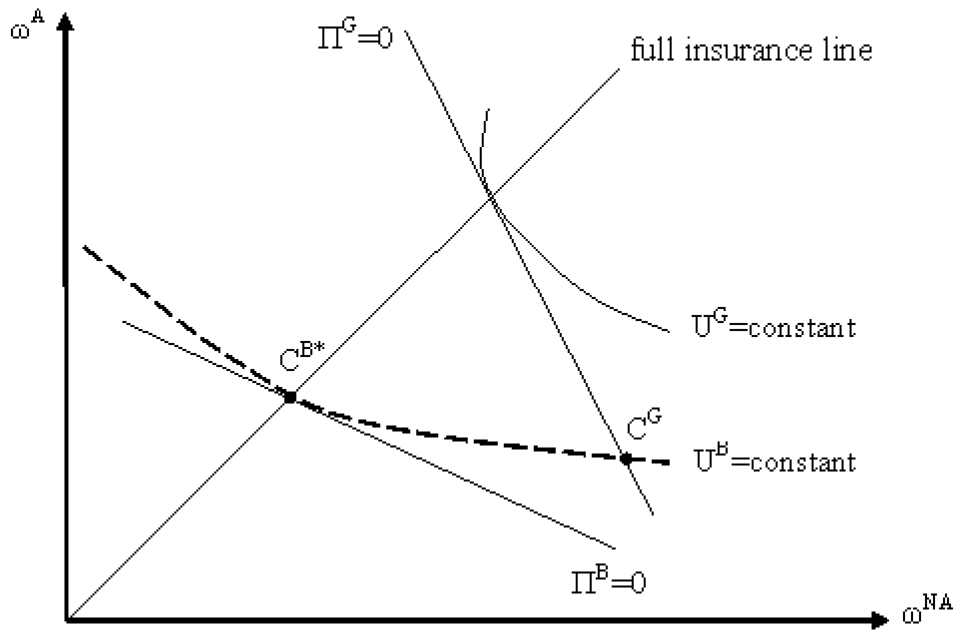
This statement says that the contract offered in any equilibrium to the bad type B must be equivalent to the symmetric information contract (first best). Intuitively this must be true as the problem here is that the type B agents have an incentive to pretend to be of type G, while this is not true vice versa.



Suppose $C^B \neq C^{B*} \implies C^{B'}$, then any contract in the shaded area could be strictly preferred by both type B agents (higher utility) and the firm (below the =-profits line, i.e. $E\Pi > 0$). So the Pareto optimal contract offered to type B must be at C^{B*}, implying full insurance for the high risk agents.

3.7.2 Show that in any equilibrium $U(\alpha_1^{B*}, \alpha_2^{B*}) = (1-\pi_B)U(w-\alpha_1^G) + \pi_B U(w-d+\alpha_2^G)$ has to hold.

The next step now is to determine what contract to offer to type G agents, i.e. C^G. Consider the following graphical illustration:



First we know that again the firm must make 0-profits, i.e. any equilibrium contract for type G in the separating case must lie on the iso expected profits line $\Pi^G = 0$ with slope

$$-\frac{(1-\pi_G)}{\pi_G}$$

Next we consider incentive compatability, i.e contract C^G cannot be preferred by type B over C^{B*}, which implies that C^G must be located on the indiffernce curve of type B that makes her indifferent between the two contracts (Tie breaking: As agents are risk averse they prefer the full insurance contract over another contract with the same expected utility) Hence we find that

$$U(\alpha_1^{B*}, \alpha_2^{B*}) = (1-\pi_B)U(w-\alpha_1^G) + \pi_B U(w-d+\alpha_2^G) \tag{65}$$

from the IC_B : $(1-\pi_B)U(w-\alpha_1^B) + \pi_B U(w-d+\alpha_2^B) \geq (1-\pi_B)U(w-\alpha_1^G) + \pi_B U(w-d+\alpha_2^G)$.

Proof. Any contract on the line $\Pi^G = 0$ below C^G is strictly dominated by C^G as the agent would derive higher utility from C^G, which would allow a rival firm to offer a contract that catches all agents of type G while allowing for positive profits and keep agents of type B at their contract. On the other hand if it would be above C^G a contract would not be incentive compatible anymore as type B would strictly prefer it over C^{B*}. ■

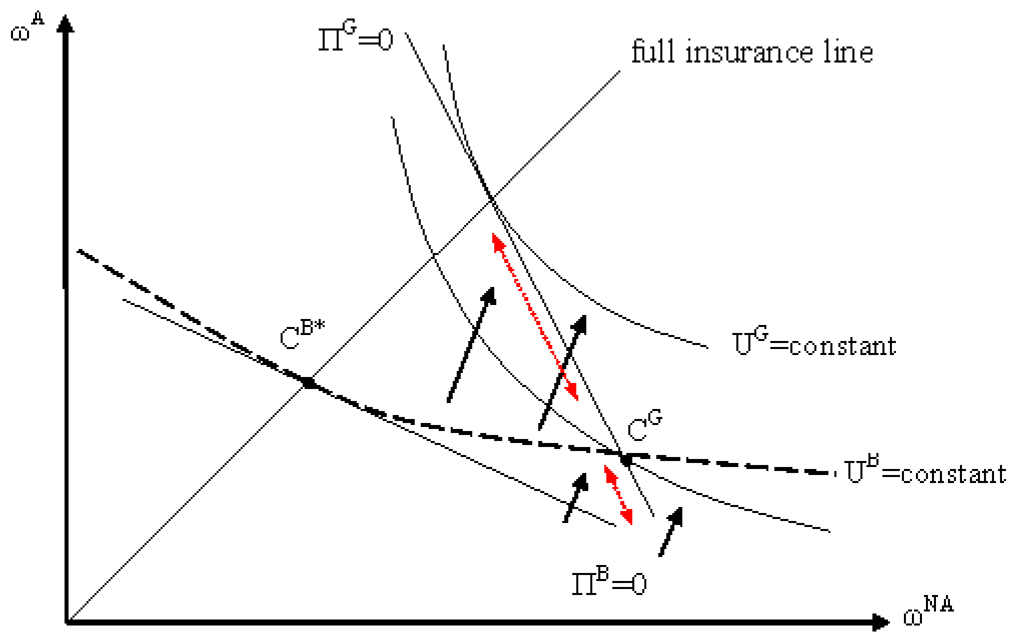
3.7.3 Characterize and describe graphically the set of restrictions that any separating equilibrium has to satisfy.

Summary 1 *What have we shown so far: The separating equilibrium candidate contract pair is* $\{C^G, C^{B*}\}$*, where* C^G *is defined by the following system of equations:*

(66) $$U(\alpha_1^{B*}, \alpha_2^{B*}) = (1 - \pi_B)U(w - \alpha_1^G) + \pi_B U(w - d + \alpha_2^G)$$

(67) $$\Pi^G = 0$$

For graphical analysis see this summary graph:



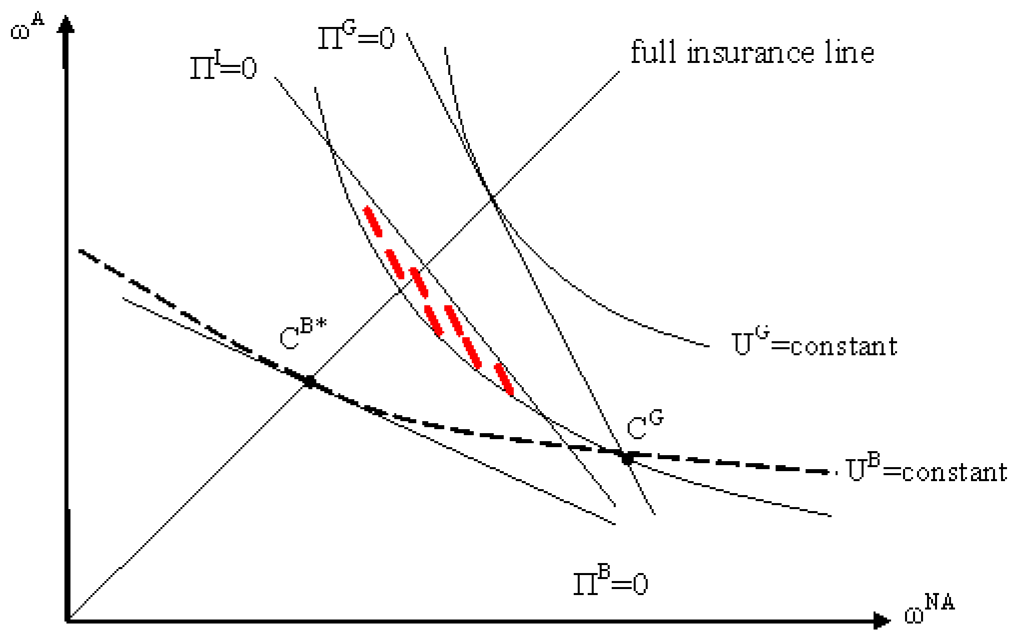
3.8 Show graphically under which conditions a separating equilibrium does exist. What role does q play?

As the question is suggesting, q (the share of H types) plays the crucial role in the question of existence of any (here the only possible is a separating) equilibrium. After the previous analysis it remains to be checked /verified that no other contract exists that is preferred by BOTH types of agents to their respective contracts and that would give positive expected profits to the firm offering it. So we again want to compare to a pooling contract below the line $\Pi^I = 0$ and see whether it can be that it dominates the separating case. As can be seen here this equation crucially depends on the fraction of hih types q. Recall

(68) $$\Pi^I = q\left[(1-\pi_G)\alpha_1 - \pi_G\alpha_2\right] + (1-q)\left[(1-\pi_B)\alpha_1 - \pi_B\alpha_2\right] = 0$$

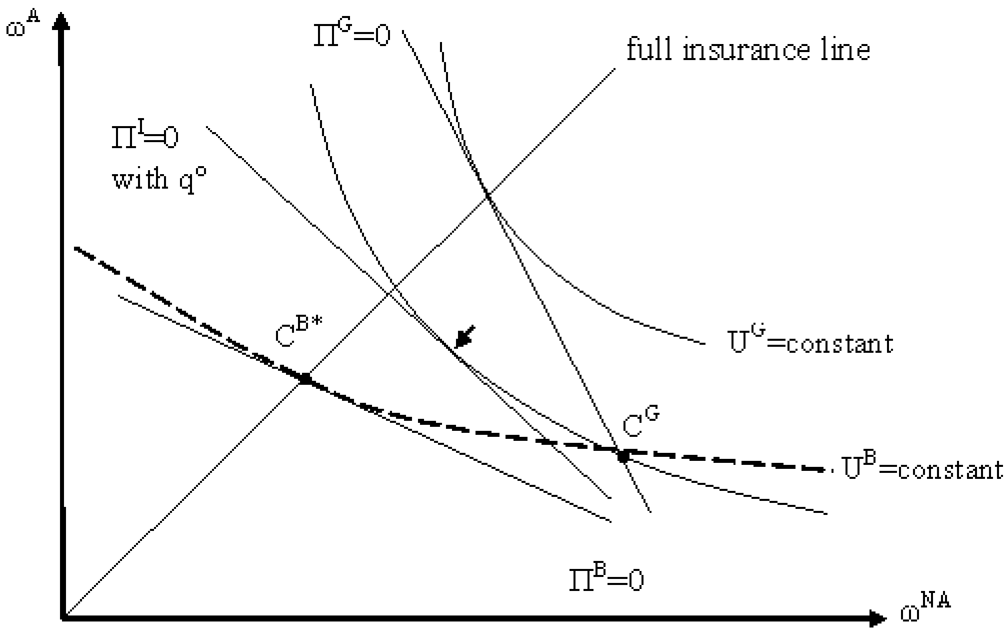
We compare two cases:

1 : q is relatively HIGH.



Here the pooling 0-profits line will intersect the indifference curve of the low risk (type H) agents that passes through C^G. Any contract offered in the shaded area will be accepted by H and B types and gives strictly positive expected profits to the firm offering it. So in this case, our separating candidate equilibrium cannot actually be an equilibrium. It follows that in this case there does not exist any equilibrium at all.

1 : q is relatively LOW.



In this case type G agents will just NOT be ready to take the pooling candidate contract as their indifference curve is slightly above the $\Pi^I = 0$ line. So our separating equilibrium candidate actually constitutes an equilibrium.

INTUITION: If there are relatively few H type people in the total population, the incentive for a firm to offer a contract that all agents will sign is low. Such a contract should attract all agents, hence its price should be low enough. However, most clients will be bad type B, which would make the expected profits of the firm negative (no compensation from G types).

To summarize, there exists a crucial level of H types, q° such that if

$$q > q^\circ \tag{69}$$

there exists no equilibrium and if

$$q < q^\circ \tag{70}$$

the above separating equilibrium candidate constitutes an equilibrium. When comparing with the first best case we see that the B types always get the first best contract with full insurance. On the other hand H types must pay the consequences of the information assymetry. Even though the insurance price is fair, they are unable to fully cover their risk, which as risk averse agents they would like to do.

4 Monopoly Price Discrimination

A monopolist can sell two types of products l and h to two types of consumers L and H. .There are n_H and n_L consumers of type H and L respectively. A consumer of type $A \in \{L, H\}$ gets utility $v(A, a)$ if she consumes product $a \in \{l, h\}$ and 0 if she does not consume any product. The cost of producing product type $a \in \{l, h\}$ is c_a. We have $v(H, h) \geq v(L, h) \geq c_h$, $v(H, l) \geq v(L, l) \geq c_l$, $v(H, h) - v(H, l) \geq v(L, h) - v(L, l)$, and $v(L, h) - c_h \geq v(L, l) - c_l$. Let $\pi(a, b)$ denote the profits from selling product a to H and b to L.

(71) $$v(H, h) - v(H, l) \geq v(L, h) - v(L, l)$$

simply states that consumers, who are willing to pay the most for a low quality product l (more precisely the H types, which can be seen from

(72) $$v(H, l) \geq v(L, l) \geq c_l$$

are also those consumers that are willing to pay the most to increase the quality from low l to high h. This single crossing property guarantees that price discrimination is feasible.

4.1 Write down $\pi(h, l)$, $\pi(h, h)$, and $\pi(h, \varnothing)$.

One quality, h, for both types implies one price such that the low types will participate, i.e. $PC(L, h)$ binds:

(73) $$v(L, h) - p = 0$$

(74) $$\pi(h, h) = (v(L, h) - c_h)(n_H + n_L)$$

Assume the monoploist just wants to offer good h to the high types, i.e. to make sure that the low types do not participate.The highest price to be charged would imply full rent extraction from the high types, i.e. $PC(H,h)$ binding:

$$\pi(h,\varnothing) = (v(H,h) - c_h)\, n_H \tag{75}$$

Finally the monopolist could decide to offer both product types, h to the high types H and l to the low types L. Here it is important to assure incentive compatability between types, i.e.

$$IC_H \;:\; v(H,h) - p_H \geq v(H,l) - p_L \tag{76}$$

$$IC_L \;:\; v(L,l) - p_L \geq v(L,h) - p_H \tag{77}$$

We can proof that PC_L has to bind, i.e.

$$p_L = v(L,l) \tag{78}$$

Proof. Suppose

$$v(L,l) > p_L \tag{79}$$

Then the monopolist could increase prices (both $p_{L,H}$) by ε without violating any other constraint, but making higher profits. ■

Furthermore we can show that IC_H ha sto bind for a similar argument as above:

$$v(H,h) - p_H = v(H,l) - v(L,l) \tag{80}$$

$$p_H = v(H,h) - v(H,l) + v(L,l) \tag{81}$$

Proof. Suppose not: The monopolist could again increase the price p_H by an ϵ and so increase his profits without violating any other constraint. Note that we can show that PC_L and $IC_H \Longrightarrow PC_H$. ■

CHECK: from $IC_H : p_H \leq v(H,h) - v(H,l) + v(L,l)$ and from $IC_L : p_H \geq v(L,h) \Longrightarrow v(H,h) - v(H,l) + v(L,l) \geq v(L,h)$ which yields after rearranging

$$v(H,h) - v(H,l) \geq v(L,h) - v(L,l) \tag{82}$$

which is true by assumption. So it follows that

$$\pi(h,l) = (v(H,h) - v(H,l) + v(L,l) - c_h)n_H + (v(L,l) - c_l)n_L \tag{83}$$

4.2 Show that no other product allocation dominates the 3 allocations considered in the previous question (e.g. $\pi(l,l)$, $\pi(\varnothing,h)$, ... are dominated by one of the above profits).

Here it is enough to show that any other allocation is dominated by just one out of the three proposed allocation: So we need to show that $\pi(l,l)$ and $\pi(l,\varnothing)$ are dominated.

$$\pi(l,l) = (v(L,l) - c_l)(n_H + n_L) \tag{84}$$

where PC_L is binding (as otherwise again the monopolist could increase the price - note here of course just one price as quality is the same for both types - and increase profits. No higher price would be feasible as type L would not participate anymore.

$$\pi(l,\varnothing) = (v(H,l) - c_l)n_H \tag{85}$$

Note that all other options are not feasible as they would lead to contradictions regarding all constraints:

$$\pi(\varnothing,h) \tag{86}$$

as $v(H,h) \geq v(L,h)$: The higest price possible to assure participation of the L type $(v(L,h))$ still would not keep the H type out, i.e. not participating.

$$\pi(\varnothing,l) \tag{87}$$

for the same argument as above just for the l quality as $v(H,l) \geq v(L,l)$.

$$\pi(\varnothing,\varnothing) \tag{88}$$

obviously is not interesting. Finally

$$\pi(l,h) \tag{89}$$

would not allow for incentive compatability, a the high types always would want to take the contract designed for the low types due to $p_L \leq v(L,h) \leq v(H,h)$.

Now we show that

$$\boldsymbol{\pi}(\mathbf{h},\mathbf{l}) \geq \boldsymbol{\pi}(\mathbf{l},\mathbf{l}) \tag{90}$$

We write

$$(v(H,h) - v(H,l) + v(L,l) - c_h)n_H + (v(L,l) - c_l)n_L \geq (v(L,l) - c_l)(n_H + n_L) \tag{91}$$

We see that the terms multiplied by n_L cancel which leaves

$$(v(H,h) - v(H,l) + v(L,l) - c_h)n_H \geq (v(L,l) - c_l)n_H \tag{92}$$

We need to show that

$$v(H,h) - v(H,l) \geq c_h - c_l \tag{93}$$

which follows from

$$v(H,h) - v(H,l) \geq v(L,h) - v(L,l) \geq c_h - c_l \tag{94}$$

where the latter inequality comes from

$$v(L,h) - c_h \geq v(L,l) - c_l \tag{95}$$

Next we show that

$$\boldsymbol{\pi}(\mathbf{h}, \varnothing) \geq \boldsymbol{\pi}(\mathbf{l}, \varnothing) \tag{96}$$

We write

$$\begin{aligned} (v(H,h) - c_h)n_H &\geq (v(H,l) - c_l)n_H && (97) \\ v(H,h) - v(H,l) &\geq c_h - c_l && (98) \end{aligned}$$

which again can be shown by

$$v(H,h) - v(H,l) \geq v(L,h) - v(L,l) \geq c_h - c_l \tag{99}$$

4.3 Show that $\pi(h,l)$ dominates $\pi(h,h)$, and $\pi(h,\varnothing)$ only if $$\frac{v(H,h) - c_h}{v(H,l) - c_l} \geq \frac{v(L,h) - c_h}{v(L,l) - c_l}$$

STEP 1:

$$\boldsymbol{\pi}(\mathbf{h}, \mathbf{l}) \geq \boldsymbol{\pi}(\mathbf{h}, \mathbf{h}) \tag{100}$$

$$(v(H,h)-v(H,l)+v(L,l)-c_h)n_H+(v(L,l)-c_l)n_L \geq (v(L,h)-c_h)(n_H+n_L) \tag{101}$$

Next we rewrite just collectin all terms multiplying $n_{H,L}$:

$$\underbrace{[v(H,h)-v(H,l)+v(L,l)-v(L,h)]}_{B} n_H \geq \underbrace{[v(L,h)-c_h-v(L,l)+c_l]}_{A} n_L \tag{102}$$

So we have

$$Bn_H \geq An_L \iff \frac{A}{A+B} \leq \frac{n_H}{n_H+n_L} \tag{103}$$

Proof.

$$A(n_H+n_L) \leq (A+B)n_H \Longrightarrow An_L \leq Bn_H \tag{104}$$

■

We write this equation as

$$\frac{\overbrace{v(L,h)-c_h-v(L,l)+c_l}^{A}}{\underbrace{v(H,h)-v(H,l)+c_l-c_h}_{A+B}} \leq \frac{n_H}{n_H+n_L} \tag{105}$$

STEP 2:

$$\boldsymbol{\pi}(\mathbf{h},\mathbf{l}) \geq \boldsymbol{\pi}(\mathbf{h},\varnothing) \tag{106}$$

$$(v(H,h)-v(H,l)+v(L,l)-c_h)n_H+(v(L,l)-c_l)n_L \geq (v(H,h)-c_h)n_H \tag{107}$$

which gives

$$\underbrace{[v(L,l)-c_l]}_{C} n_L \geq \underbrace{[v(H,l)-v(L,l)]}_{D} n_H \tag{108}$$

See above for

$$\frac{C}{C+D} \geq \frac{n_H}{n_H+n_L} \tag{109}$$

We rewrite to obtain

$$\frac{v(L,l)-c_l}{v(H,l)-c_l} \geq \frac{n_H}{n_H+n_L} \tag{110}$$

STEP 3:

$$\frac{C}{C+D} \geq \frac{n_H}{n_H+n_L} \geq \frac{A}{A+B} \Longrightarrow \frac{A+B}{C+D} \geq \frac{A}{C} \tag{111}$$

We have

$$\underbrace{\frac{v(H,h)-v(H,l)+c_l-c_h}{v(H,l)-c_l}}_{RHS} \geq \frac{v(L,h)-c_h-v(L,l)+c_l}{v(L,l)-c_l} \tag{112}$$

We simplify

$$RHS+1 \geq \underbrace{\frac{v(L,h)-c_h}{v(L,l)-c_l}}_{LHS} \tag{113}$$

$$\frac{v(H,l)-c_l+v(H,h)-v(H,l)-c_h+c_l}{v(H,l)-c_l} \geq LHS \tag{114}$$

which is exactly the INCREASING PERCENTAGE DIFFERENCES RULE we wanted to find:

$$\frac{v(H,h)-c_h}{v(H,l)-c_l} \geq \frac{v(L,h)-c_h}{v(L,l)-c_l} \tag{115}$$

4.4 Discuss the implication for when one should expect to observe price discrimination.

The above condition implies that both products h and l are offered only if the ratio of the high type's total surplus to the low type's total surplus is increasing in quality. Equivalently we could say that both products are offered only (i.e. price discrimination is profitable) if the marginal surplus from an increase in quality as a percentage of total surplus is increasing in the type of consumer. This makes the authors call this the increasing percentage differnces condition. For a detailed analysis and discussion you can have a look at

"Integrating Models of Price Discrimination" (Anderson, Dana 2006) - Northwestern University.

In general notice that the building blocks of this condition are changes in social surplus, which is the difference between consumer benefits and firm costs when there is an increase in quality from l to h. So price discrimination is profitable (and hence will be observed) if and only if the percentage change in social surplus from product upgrades is increasing in consumers' willingness to pay (here types).

Microeconomics III

Information Economics and Contract Theory

SOLUTION to PROBLEM SET 3

Markus Kitzmueller

EUI 2007

5 Signaling and Corporate Finance

A risk neutral firm wants to finance a project. The initial cost to start the project is 1 euro. The project can be good (G) with probability $0 \leq \alpha \leq 1$ or bad (B) with probability $1-\alpha$. The probability of success for project $i = G, B$ is p_i leading to profits $\Pi > 0$. In case of failure profits are 0. Assume $p_G > p_B$. Assume that the firm does not have any initial funds, so it needs to borrow from competitive and risk neutral banks. The firm knows the type of project it wants to finance while banks do not. The amount R , which is specified in a loan contract, should be repaid to the bank per unit of the initial loan. In the case of project failure the firm defaults and the bank receives 0. Let

$$p_G\Pi > 1 + r > p_B\Pi \quad (116)$$

where r is the risk free rate of interest. Initially assume the following timing: The bank first offers R and then the firm either accepts R or does not finance the project at all. (We consider SPNE.)

5.1 Show that in equilibrium both projects are financed or neither is.

As the bank does not know the type of firm and it is risk neutral, there will be one R offered to all firms, taking into account the probabilities of a firm being either type. The profits in case of success are the same for both types of projects (firms) $i = G, B$, namely Π. So a firm will decide to finance the project given the R offered by the bank if

$$p_i(\Pi - R) \geq 0 \quad (117)$$

where $i = G, B$. So the decision of both types of firm depends on the relationship between Π and R only such that the project is financed if $\Pi \geq R$ and vice versa. So

$$p_B(\Pi - R) \geq 0 \wedge p_G(\Pi - R) < 0 \tag{118}$$

and

$$p_G(\Pi - R) \geq 0 \wedge p_B(\Pi - R) < 0 \tag{119}$$

is not feasible as there does not exist any R that could achieve this discrimination given that $p_i > 0$.

5.2 Show that the bank earns $0 - profits$.

The banking sector is competitive and banks are risk neutral. So when projects are financed, i.e. $\Pi_{B,G}^{firm} > 0$, a bank makes profits

$$\Pi^{Bank} = \alpha\,(p_G R) + (1-\alpha)(p_B R) - (1+r) = 0 \tag{120}$$

Proof. Suppose

$$\Pi^{\circ} = \frac{\Pi_{total}^{Bank}}{N} > 0 \tag{121}$$

where N is the number of banks in the market. A bank then can earn Π° or less and could set $R' = R - \varepsilon$ and both types of firms will be attracted. As ε is marginally small the profits for the bank would be very close to $\Pi_{total}^{Bank} > \Pi^{\circ}$. This incentive to deviate will discipline banks until R is such that expected profits of the bank are 0.

■

5.3 Derive the equilibrium.

We start from the 0−profit condition of the banks:

$$\alpha\,(p_G R) + (1-\alpha)(p_B R) - (1+r) = 0 \tag{122}$$

$$\implies R^{Pool} = \frac{1+r}{(\alpha p_G + (1-\alpha)p_B)} \tag{123}$$

Recall that we have

$$\frac{1+r}{p_B} > \Pi > \frac{1+r}{p_G} \tag{124}$$

so we see that α plays a crucial role in deciding whether both projects are financed or none is.

5.4 When are projects financed and how does this depend on p_i, α, r and Π?

Projects are financed if

$$\Pi \geq R^{Pool} = \frac{1+r}{(\alpha p_G + (1-\alpha)p_B)} \tag{125}$$

given that banks have 0−expected profits. We have that

$$\frac{\partial R^{Pool}}{\partial p_i} < 0 \tag{126}$$

so an increase in the probability of success of either type of project reduces R.

$$\frac{\partial R^{Pool}}{\partial \alpha} < 0 \tag{127}$$

so an increase in the share of type G projects reduces R. An increase in p_i and α would hence increase the likelihood of financing of both projects. On the other hand

$$\frac{\partial R^{Pool}}{\partial r} > 0 \tag{128}$$

so an increase in the risk free interest rate increases R and reduces the probability of both projects being financed. Obviously an increase in Π (exogenous) will increase the likelihood of project finance.

Suppose now that the firm can finance fraction $z \in [0,1]$ out of its own budget. The effective cost of doing so is $(1+l)z$ where $l > r$. l can be interpreted as the cost of giving up liquidity.

5.5 Write the firm's pay-off as a function of R, z and project type.

$$E\Pi_i = p_i\left[\Pi - (1-z)R\right] - (1+l)z = \tag{129}$$

$$= p_i(\Pi - R) - z\left[(1+l) - p_i R\right] \tag{130}$$

Assume the following sequence of events: First the firm offers to self finance z. Second, banks respond by offering a contract R. Last, the firm decides to do the planned project or to drop it.

5.6 Discuss existence and characterize the pareto dominant separating equilibrium.

Now there exsits a devise that allows the firm to signal and hence the banks to distinguish between types before making their choice about R. So in a separating equilibrium the bank knows the project type and from the $o-$profit condition we get that

$$R_i p_i - 1 - r = 0 \Longrightarrow R_i = \frac{1+r}{p_i} \tag{131}$$

for $i = G, B$. By assumption we know that

$$\frac{1+r}{p_G} = R_G \leq \Pi \leq R_B = \frac{1+r}{p_B} \tag{132}$$

STAGE 1: First we assume the following beliefs of banks: Beliefs are such that

$$\mu_G(z^*) = 1 \tag{133}$$

i.e. when banks observe z^* they assign probability 1 to the firm being type G. Further

$$\mu_G(z \neq z^*) = 0 \tag{134}$$

banks assign probability 0 of being type H for all other z, or vice versa they believe that those firms signaling $z \neq z^*$ are of type B with probability 1.This implies

$$1 + r = \mu(z^*) p_G R_G + \mu(z \neq z^*) p_B R_B \tag{135}$$

From the participation constraint of type G we get

$$E\Pi_G(z) = p_G(\Pi - R_G) - z\left[(1+l) - p_G R_G\right] \geq 0 \tag{136}$$

$$\Rightarrow \quad z \leq \frac{p_G(\Pi - R_G)}{[(1+l) - p_G R_G]} \tag{137}$$

$$\Longrightarrow \quad \overline{z} = \frac{p_G(\Pi - R_G)}{[(1+l) - p_G R_G]} \tag{138}$$

We can then say that any $z \in [z^*, \overline{z}]$ could constitute an equilibrium, but the pareto dominant one will be the one with z^* given the above beliefs. Let's consider the following equilibrium:

$$E\Pi_B(z) = p_B(\Pi - R_G) - z\left[(1+l) - p_B R_G\right] \leq 0 \tag{139}$$

i.e. there is no incentive for type B to pretend to be a type G and signal z. It follows that

$$z \geq \frac{p_B(\Pi - R_G)}{(1+l) - p_B R_G} = \frac{p_B\Pi - \frac{p_B}{p_G}(1+r)}{(1+l) - \frac{p_B}{p_G}(1+r)} \tag{140}$$

$$\Rightarrow z^* = \frac{p_B\Pi - \frac{p_B}{p_G}(1+r)}{(1+l) - \frac{p_B}{p_G}(1+r)} \tag{141}$$

Then type B does choose $z_B = 0$.

Proof. Suppose $z_B > 0$. Given the beliefs of the banks, the z that would make the bank believe that a firm is of type G should be z^*. However, z^* would yield zero profits hence any possible $z_B < z^*$ will result in the same offer R_B, so the one that maximizes profits is $z_B = 0$. ■

It follows then that the IC(B) has to bind

$$p_B(\Pi - R_B) = p_B(\Pi - R_G) - z\left[(1+l) - p_B R_G\right] \tag{142}$$

as

$$p_B(\Pi - R_B) \leq 0 \tag{143}$$

by definition.

Hence type B faces zero expected payoff

$$E\Pi_B = p_B(\Pi - R_B) = p_B\Pi - (1+r) < 0 \tag{144}$$

and in a separating equilibrium, bad projects B will not be financed at all, i.e. type B firms do not accept R_B.

STAGE 2: The bank offers

$$\frac{1+r}{p_G} = R_G < \Pi < R_B = \frac{1+r}{p_B} \tag{145}$$

STAGE 3: Type B firms choose NO FINANCE as $\Pi \leq R_B$. Type G firms face the following expected profit

$$\begin{aligned}
E\Pi_G(z) &= p_G(\Pi - R_G) - z\left[(1+l) - p_G R_G\right] = && (146)\\
&= p_G(\Pi - \frac{1+r}{p_G}) - z\left[(1+l) - p_G\frac{1+r}{p_G}\right] = && (147)\\
&= p_G\Pi - (1+r) - z(1+l) + z(1+r) = && (148)\\
&= \underbrace{p_G\Pi - (1+r)}_{\text{profits}} + \underbrace{z(r-l)}_{\text{cost of signaling}} = && (149)\\
&= p_G\Pi - (1+r) + \frac{\left[p_B\Pi - \frac{p_B}{p_G}(1+r)\right](r-l)}{(1+l) - \frac{p_B}{p_G}(1+r)} && (150)
\end{aligned}$$

So the separating equilibrium candidate is characterized by

$$z^* = \begin{cases} 0 \text{ if } i = B \\ z = \frac{p_B(\Pi - R_G)}{(1+l) - p_B R_G} \text{ if } i = G \end{cases} \tag{151}$$

and Banks offer R_B to type B and R_G to type G. Bad projects won't be financed while good projects will at minimum necessary signaling cost (pareto optimum) as any $z_G > z^*$ would reduce the expected profits of type H and any $z_G < z^*$ would not be a separating equilibrium.

5.7 Compare the equilibria in 1) and 2). Discuss when good and bad projects are financed and whether efficiency is achieved.

By comparing the pooling and the separating equilibrium we adress two issues:

1. Is it worth to signal for type G?

   This depends on the difference in expected profits under the two equilibria:
   (152)

   $$\Pi^{Diff} = \underbrace{\left(p_G \Pi - (1+r) + \frac{\left[p_B \Pi - \frac{p_B}{p_G}(1+r)\right](r-l)}{(1+l) - \frac{p_B}{p_G}(1+r)} \right)}_{\text{Separating}} - \underbrace{p_G \left[\Pi - \frac{1+r}{(\alpha p_G + (1-\alpha) p_B)} \right]}_{\text{Pooling}} > 0$$

   A positive difference here implies that firm G decides to signal and a separating equilbrium occurs. We see that a marginal increase in α would reduce this difference and hence the more type G firms there are, the less likely a type G firm will decide to signal its type (as profits are approaching those of the pooling outcome). In other words, the bigger α the more likely a pooling equilibrium occurs.

2. Now suppose that $\Pi^{Diff} < 0$. Then we have that both projects are financed if

   $$\Pi > R^{Pool} \tag{153}$$

   and none is if

   $$\Pi < R^{Pool} \tag{154}$$

   In the pooling case an increase in α increases the likelihood of projects to be financed as it reduces R^{Pool}.

• Efficiency:	SEPARATING - CASE 1	only type G financed	only type G financed
	POOLING - CASE 2	A) no project financed	B) both, B and G projects fina
	(fraction of type H): α	$\alpha < \alpha^*$	$\alpha \geq \alpha^*$

where α^* is defined as the α that makes

$$R^{Pool} = \frac{1+r}{\alpha^* p_G + (1-\alpha^*) p_B} = \Pi \tag{155}$$

So all αs above α^* are reducing R^{Pool} below Π such that both projects are financed and vice versa. Case 2 A and B are both not efficient as either B is financed or G is not financed. In the separating case we have a wasteful signal z that reduces type G's payoff.

6 Equity Financing

A firm has a project that requires an investment $I = 20$ at t=0 for a sure return of 30 at t=1. There is no discounting. I has to be raised from the financial market via a new equity issue. Potential new investors are uncertain about the value of the firm's assets in place: $A \in \{A_B = 50, A_G = 100\}$ with $\Pr[A = 100] = 0.1$.

6.1 Assume that investors believe that both types of firms invest. What fraction of the firm's equity has to be issued to new investors? What are the payoffs to existing shareholders (i.e the firm before issueing new equity) if they undertake the project? Can these beliefs be part of a Perfect Bayesian Equilibrium?

Here investors are unable to infer types of firms, so they assume an expected (average) value of the firm

$$EA^{Pool} = 0.9 * 50 + 0.1 * 100 = 55 \tag{156}$$

before investment and

$$EA_I^{Pool} = 0.9 * 80 + 0.1 * 130 = 85 \tag{157}$$

after investment. So the share of the firm to be issued is

$$s * EA_I = 20 \tag{158}$$

$$s = \frac{20}{85} = \frac{4}{17} \tag{159}$$

The payoffs of the existing shareholders (i.e. those who own the firm before the issue) are

$$\Pi(A_B) = (1-s)80 - A_B = \frac{13 * 80}{17} - 50 = 11.2 > 0 \tag{160}$$

$$\Pi(A_G) = (1-s)130 - A_G = -0.6 < 0 \tag{161}$$

So the bad firm would invest while the good type firm would not. Hence these beliefs cannot be part of a PBE. (See definition of a PBE in MWG Ch9, page 285)

6.2 Asume that investors believe that only firms of type A_B issue new equity. Adress the same questions as in 1.

So here investors value firms G and B separately (i.e. they assume that it is for sure a type$B$ firm that issues equity) at

$$EA_I^B = 80 < 130 = EA_I^G \tag{162}$$

and the share issued is

$$s_B = \frac{20}{80} = \frac{1}{4} \tag{163}$$

The payoffs are

$$\Pi(A_B) = (1-s)80 - A_B = 10 \tag{164}$$

$$\Pi(A_G) = (1-s)130 - A_G = -2.5 < 0 \tag{165}$$

Now beliefs can be part of the PBE.

6.3 Now shareholders commit at t=0 to a wasteful advertising campaign at t=1, i.e. after the project return is realized. The advertising expenditure is irreversible and results in a drop in profits of K. The size of the expenditure is a choice variable. Can a good firm signal its type via such an expenditure? Discuss.

The question is whether K can be used as a successful signal for type G firms to distinguish themselves from type B. One can distinguish two subcases of which we will focus on the more realistic, i.e. K has to be paid in t=1 by all shareholders, old and new ones (as opposed to the cas where old shareholders commit to paying K out of their profit share ex post). Hence we get the following set of equations:

$$s_B = \frac{I}{(A_B + 30 - K_B)} = \frac{20}{(80 - K_B)} \tag{166}$$

$$s_G = \frac{20}{(130 - K_G)} \tag{167}$$

Let

$$\mu(K_G) = 1 \wedge \mu(K \neq K_G) = 0 \tag{168}$$

be again the probability assigned by potential new shareholders to a firm to be of type A_G and type A_B respectively. In a similar fashion than in Exercise 1 the separating equilibrium value of K will be just high enough to assure that type B has no incentive anymore to set any $K > 0$. Notice that here we assume that if firm B is indifferent between signaling and not signaling it prefers not to signal.

In any separating PBE equilibrium firm B sets $K_B = 0$.

Proof. Suppose $K_B^{\circ} > 0$: Then profits in t+1 will be reduced. The signal cannot be high enough to signal type G and investors will value the firm at $(A_B^I - K_B)$ and ask for a bigger share

$$\frac{20}{(80 - K_B)} > \frac{20}{80} \tag{169}$$

which gives an incentive to reduce K_B down to 0. ■

Now we have to characterize the optimal K_G that is just making the B types indifferent between signaling and not signaling and hence keeps them from advertising. The starting point is the incentive compatability constraint of type B

$$80(1-\frac{1}{4})-A_B=(80-K_G)(1-s_G)-A_B \tag{170}$$

Furthermore we consider the participation constraint of type G:

$$(130-K_G)(1-s_G)-100\geq 0 \Longrightarrow 10-K_G\geq 0 \Longrightarrow K_G\leq 10 \tag{171}$$

From the ICC we get

$$60(130-K_G) = (130-K_G-20)(80-K_G) \tag{172}$$

$$K_G^2-130K_G+1000 = 0 \tag{173}$$

So we apply the quadratic formula to get

$$K_G=\frac{130+/-\sqrt{130^2-4000}}{2} \Longrightarrow K_G=\sim 8.2<10 \tag{174}$$

so the participation constraint of type G holds (while the alternative solution to this quadratic equation would violate the PC). So finally we can compute the shares

$$s_B=\frac{1}{4} \tag{175}$$

and

$$s_G=\frac{20}{121.8}=0.165 \tag{176}$$

Payoffs to the existing (old) shareholders are then

$$\Pi(A_B) = (1-s)80-A_B=10 \tag{177}$$

$$\Pi(A_G) = (1-0.165)121.2-100=1.2>0 \tag{178}$$

but we notice that

$$\Pi(A_G)<\Pi(A_B) \tag{179}$$

So we notice the basic trade off the type G firm faces: On the one hand by setting $K>0$ it can signal its type and distinguish itself from type B but at the same time it reduces its expected value and the share asked for by new shareholders, s_G, increases, which in turn reduces profits for existing shareholders. Now we compare type G's profits under separating with those under PBE consistent beliefs of question 2

$$1.2=\Pi(A_G)>\Pi^2(A_G)=-2.1 \tag{180}$$

and the same for types B:

(181) $$10 = \Pi^2(A_B) = \Pi(A_B) = 10$$

So we conclude that in a separating equilibrium type G can signal its type and make positive expected profits, while type B also faces positive expected profits even higher than type G. So both projects are financed in this equilibrium, while in the case of question 2 only type B projects are financed, where the payoff is the same as under separation, i.e. 10.

Microeconomics III

Information Economics and Contract Theory

SOLUTION to PROBLEM SET 4

Markus Kitzmueller

EUI 2007

7 Moral Hazard and Motivation

A risk neutral principal (P) hires a risk neutral agent (A) to carry out a project. The project's outcome can be high or low such that $Y_H > Y_L \geq 0$ and $\Pr(Y_H) = e \in [0, 1]$, where e is agent's effort. The cost of exerting effort is $c(e) = \frac{1}{2}e^2$. Effort cannot be observed. The minimum wage paid to the agent has to be positive ($\underline{w} \geq 0$), because the agent benefits from limited liability. A's reservation utility is $\overline{u} \geq 0$ and P's reservation payoff is 0. A contract between P and A has two components: the fixed wage w (independent of outcome) and the bonus payment $b \geq 0$ paid if Y_H occurs. If the project outcome is high, P gets profit $\pi > 0$. Agents care about something more than just money and receive a non-pecuniary benefit θ if the project outcome is Y_H.

7.1 Assume effort is contractible. Solve for the first best effort level.

In order to focus on the interior solution for effort, we assume that

$$\pi + \theta < 1 \tag{182}$$

We consider again the first best (symmetric information) outcome as a benchmark. Effort then is chosen to maximize the joint payoff of P and A. When proceeding we refer to θ as agent motivation. The objective function then is

$$\max_e \left[e(\pi + \theta) - \frac{1}{2}e^2 \right] \tag{183}$$

Notice that b, w cancel as this is joint payoff. The resulting first best effort level then is

$$e = \pi + \theta \tag{184}$$

and the expected joint surplus is

$$u^{P+A} = e^2 - \frac{1}{2}e^2 = \frac{1}{2}e^2 = \frac{1}{2}(\pi + \theta)^2 \tag{185}$$

We see that the contract (w, b) does not play any allocative role here.

7.2 Write P's optimal contracting problem when effort is not observable. (Hint: The incentive compatability constraint says that the equilibrium effort level maximizes the agent's utility given (b, w).)

Now we are in the second best case as effort is not observable anymore, hence there is hidden action (assymetric information). So P's optimal contracting problem implies to maximize her payoff subject to three constraints:

$$\max_{(b,w)} u^P = (\pi - b)e - w \tag{186}$$

subject to

$$LLC \; : \; w \geq \underline{w} \tag{187}$$

$$PC \; : \; u^A = e(b+\theta) + w - \frac{1}{2}e^2 \geq \overline{u} \tag{188}$$

$$ICC \; : \; e = \arg\max_e(u^A) = \arg\max(e(b+\theta) + w - \frac{1}{2}e^2) \tag{189}$$

We can rewrite the ICC as follows

$$e = b + \theta \tag{190}$$

and substitute for e in P's objective function, s.t. we are left with the following problem

$$\max_{(b,w)} u^P = (\pi - b)(b + \theta) - w \tag{191}$$

s.t.

$$LLC, PC \tag{192}$$

7.3 Show that under any optimal incentive contract the participation and/or the limited liability constraint bind.

Lemma 3 *We refer to this as Lemma 1.*

Proof. We proof by contradiction: Suppose NOT, i.e. Both the LLC and PC do not bind. Then since the PC does not bind, P can maximize her payoff with respect to b:

$$\max_b (u^P) \tag{193}$$

The FOC reads

$$\pi - 2b - \theta = 0 \tag{194}$$

$$\implies b = \max\left\{\frac{\pi - \theta}{2}, 0\right\} \tag{195}$$

It follows that

$$e = b + \theta = \max\left\{\frac{\pi + \theta}{2}, \theta\right\} \tag{196}$$

Furthermore we have

$$w > \underline{w} \implies w - \varepsilon \tag{197}$$

i.e. P can reduce the wage offered marginally without violating the LLC and PC. Furthermore e does not depend on w, hence there is no effect on effort chosen neither. On the other hand profits

$$\uparrow u^P = (\pi - b)(b + \theta) - w \downarrow \tag{198}$$

increase, so in equilibrium this cannot be $\implies$ contradiction. So at least one of the two, i.e. either the LLC or the PC must bind in equilibrium. ■

7.4 Show that the limited liability constraint binds.

Hint: Suppose the LLC does not bind, then show that

- e is at its first best level and
- the expected payoff of P is strictly negative.

So we can proove the equivalent statement, i.e.

Lemma 4 *If $e \neq \pi + \theta$ (first best), then the LLC must bind.*

Proof. We know that $\pi \geq b$ as P will never offer a contract implying negative expected profit u^P. We know from first best that

(199) $$e^{FB} = \pi + \theta \Longrightarrow e \leq e^{FB}$$

effort cannot exceed its first best level. So we are left to check that if $e < e^{FB}$, then the LLC must bind. Suppose NOT, i.e. LLC is not binding, then $\exists$ an optimal contract $\left(b^{\circ}, w^{\circ}\right)$ such that $b^{\circ} < \pi$ and $w^{\circ} > \underline{w}$ (as LLC is not binding). So we can change this contract in the following way:

(200) $$w^{\circ} - \varepsilon$$

i.e. we reduce the wage and

(201) $$b^{\circ} + \delta$$

increase the bonus such that the changes exactly offset each other, i.e. $\delta = \sqrt{2\varepsilon}$. This implies that the expected utility of A remains unchanged as

(202) $$du^A = edb + dw = 0$$

s.t. $dw < 0$ and $db > 0$. As the agent is choosing effort such that he maximizes his expected utility, which remains unchanged overall, we apply the envelope theorem (which just tells us how the optimal value of the objective function, here u^A changes as one or more of the parameters change see Simon/Blume page 454) and hence ignore the effects of changing wage and bonus although they affect the choice of e effort. However, on the side of P expected payoff is changing such that

(203) $$du^P = de\underbrace{(\pi - b)}_{>0} - \underbrace{(edb + dw)}_{=0}$$

as $de = db + \theta$ and $db > 0$. There is an incentive to deviate as P can be better off $\Longrightarrow$ contradiction. ■

Lemma 5 *If LLC does not bind, then* $u^P < 0$.

Proof. From Lemma 2 we know that if LLC does not bind then $e = e^{FB} = \pi + \theta$.

From the ICC we know that $e = b + \pi$ so it follows that

(204) $$b = \pi$$

Since the LLC reads $w > \underline{w}$ and $\underline{w} \geq 0$ we see that

$$u^P = e(\pi - b) - \underset{=0}{w} = -w < 0 \quad (205)$$

P's payoff is strictly negative. ■

So overall we have shown that the LLC must always bind in equilibrium.

Now let

$$\overline{v} = \frac{1}{2}\left(\frac{\theta + \pi + \sqrt{(\theta+\pi)^2 - 4\underline{w}}}{2}\right)^2 + \underline{w} \quad (206)$$

be the reservation payoff of A such that P makes 0−Expected Profits under an optimal contract. Assume that $\pi^2 - 4\underline{w} > 0$. Denote $\underline{v}$ as the reservation payoff of A such that for $\overline{u} \geq \underline{v}$ A's participation constraint is binding. They are both positive real numbers and $\overline{v} > \underline{v}$ holds. It follows that there are three cases to consider when characterizing the optimal contract:

7.5 Case 1: LLC and PC are binding. Characterize the optimal contract and show that it is feasible iff $\overline{u} \in [\underline{v}, \overline{v}]$.

Both constraints are binding and hence uniquely pin down the two choice variables for P, w and b. First it follows directly that

$$w^* = \underline{w} \quad (207)$$

Furthermore the optimal b follows from the binding PC

$$e(b+\theta) + w - \frac{1}{2}e^2 = \overline{u} \quad (208)$$

$$\Longrightarrow \frac{1}{2}(b+\theta)^2 + \underline{w} = \overline{u} \quad (209)$$

as $e = b + \theta$. So it follows that

$$b^* = \sqrt{2(\overline{u} - \underline{w})} - \theta \quad (210)$$

Plugging these values into the ICC (which is determining effort in equilibrium), we get

$$e^* = b^* + \theta = \sqrt{2(\overline{u} - \underline{w})} \quad (211)$$

Now we have to check whether this contract is feasible:

ANALYSIS: We have shown above already that $b \leq \pi \implies e^* = \sqrt{2(\overline{u} - \underline{w})} \leq \pi + \theta = e^{FB}$. So it follows that

$$\overline{u} - \underline{w} \leq \frac{1}{2}(\pi + \theta)^2 \tag{212}$$

and $b^* > 0$, i.e.

$$\overline{u} - \underline{w} > \frac{1}{2}\theta^2 \tag{213}$$

which must be true as otherwise the PC would not bind anymore:

$$\frac{1}{2}(b + \theta)^2 = \overline{u} - \underline{w} \tag{214}$$

The payoffs to A and P are

$$u^A = \overline{u} \tag{215}$$

and

$$u^P = \underbrace{\sqrt{2(\overline{u} - \underline{w})}}_{e}(\pi - \underbrace{\sqrt{2(\overline{u} - \underline{w})}}_{b} + \theta) - \underline{w} \tag{216}$$

We know now that if $u^A = \overline{u} = \overline{v} = \frac{1}{2}\left(\frac{\theta+\pi+\sqrt{(\theta+\pi)^2-4\underline{w}}}{2}\right)^2 + \underline{w} \implies u^P = 0$.

Proof. $\implies$: Hence as long as $\overline{u} \leq \overline{v} \implies u^P \geq 0$ or if $\overline{u} > \overline{v} \implies u^P < 0$ we get a contradiction as this contract would never be offered. On the other hand if $\overline{u} < \underline{v}$, the PC is not binding anymore and our optimal b would change (see next two cases). This somehow follows by definition of reservation utilities.

$\impliedby$: So if $\overline{u} \in [\underline{v}, \overline{v}]$ it follows that PC is binding (again by definition) as $\overline{v} > \underline{v}$ and the contract takes the form above (is feasible) as b and w are determined directly by the PC and LLC respectively. ∎

Short EXCURSION on how to get $\overline{v}$: By Lemma 2 we have that the LLC must bind. In the optimal contracting problem

$$\max_{(b)}(u^P) = (\pi - b)(b + \theta) - \underline{w} \tag{217}$$

the only choice variable left for P is b. Assume P's expected payoff is zero, i.e.

$$(\pi - b)(b + \theta) - \underline{w} = 0 \tag{218}$$

This is a quadratic functoion in b:

$$b^2 - (\pi - \theta)b + \underline{w} - \pi\theta = 0 \tag{219}$$

We take the higher root, as it is the relevant one due to the fact that u^A is increasing in b, so

$$b = \frac{\pi - \theta + \sqrt{(\theta - \pi)^2 - 4(\underline{w} - \pi\theta)}}{2} \tag{220}$$

and then substituting into u^A yields

$$\overline{v} = \frac{1}{2}\left(\frac{\theta + \pi + \sqrt{(\theta + \pi)^2 - 4\underline{w}}}{2}\right)^2 + \underline{w} \tag{221}$$

and is strictly positive as $\pi, \theta > 0$ and $\pi^2 - 4\underline{w} > 0$. (Notice the change of sign in the brackets under the square root which cancels the $4\pi\theta$ term.)

Lemma 6 *Further we can show that $\underline{v} \in (0, \overline{v})$.*

Proof. From Lemma 1 we know that LLC binds and $b = \max(0, \frac{\pi-\theta}{2})$. It follows that

$$u^A = \frac{1}{2}(\theta + \max(0, \frac{\pi - \theta}{2}))^2 + \underline{w} = \frac{1}{8}(\theta + \max\{\pi, \theta\})^2 + \underline{w} \geq 0 \tag{222}$$

So we see that there are two possible cases: First $\pi > \theta$ implies that

$$\underline{v} = \frac{1}{8}(\theta+\pi)^2+\underline{w} < \overline{v} = \frac{1}{2}\left(\frac{\theta + \pi + \sqrt{(\theta + \pi)^2 - 4\underline{w}}}{2}\right)^2 + \underline{w} = \frac{1}{8}(\theta+\pi+\sqrt{(\theta + \pi)^2 - 4\underline{w}})^2 \tag{223}$$

by Assumption $\pi^2 - 4\underline{w} > 0$. The second case is $\theta \geq \pi$: So we need to show that

$$\underbrace{\frac{\theta + \pi + \sqrt{(\theta + \pi)^2 - 4\underline{w}}}{2}}_{\overline{v}} > \underbrace{\theta}_{\underline{v}} \tag{224}$$

which is true as simplified we get

$$\frac{1}{2}(\pi^2 + \pi\theta) - \underline{w} > 0 \tag{225}$$

which holds by assumption and as $\pi\theta > \pi^2 \implies \frac{1}{2}(\pi^2 + \pi\theta) - \underline{w} > \pi^2 - 4\underline{w} > 0$. ■

7.6 Case 2: PC is not binding: We will distinguish two sub-cases:

- Characterize the optimal contract when $\pi > \theta$ and show that it is feasible iff $\overline{u} \in [0, \underline{v}]$.

Now we have found already that then

$$b^* = \max\left\{0, \frac{\pi-\theta}{2}\right\} = \frac{\pi-\theta}{2} \tag{226}$$

and again

$$w^* = \underline{w} \tag{227}$$

and

$$e^* = \frac{\pi+\theta}{2} \tag{228}$$

A's payoff is

$$u^A = \frac{1}{8}(\pi+\theta)^2 + \underline{w} > 0 \tag{229}$$

by assumption. Since PC does not bind it must be true that

$$u^A = \frac{1}{8}(\pi+\theta)^2 > \overline{u} - \underline{w} \tag{230}$$

As $\overline{u} \leq \underline{v}$, we have that $\underline{v} = \frac{1}{8}(\pi+\theta)^2 + \underline{w} - \varepsilon$ as then PC starts to bind and we switch to the above case with the respective b^*. On the other hand, if $\overline{u} < 0$, P could reduce b further as A's outside option gets worse and the PC even stricter.

1. • Characterize the optimal contract when $\pi < \theta$ and show that it is feasible iff $\overline{u} \in [0, \underline{v}]$.

Here we get

$$b^* = \max\left\{0, \frac{\pi-\theta}{2}\right\} = 0 \tag{231}$$

$$e^* = \theta \tag{232}$$

A's utility is

$$u^A = \frac{1}{2}\theta^2 > \overline{u} - \underline{w} \tag{233}$$

and

$$u^P = \theta\pi - \underline{w} \tag{234}$$

This is positive by assumption as $\pi\theta > \pi^2$ and $\pi^2 - 4\underline{w} > 0$.

So we see that in both cases of CASE 2 A receives strictly positive expected payoff even though $\overline{u} = 0$. On the other hand, If $u^P = 0$, A's payoff is $\overline{v} > 0$ strictly positive. For all $\overline{u} > \underline{v}$ the PC binds and u^P is a continuous and decreasing function of A's reservation utility $\overline{u}$, which prooves existence for all $\overline{u} \in [0, \underline{v}]$.

To sum up our result we write Proposition 1.

Proposition 1 *Suppose $\pi+\theta < 1$ and $\pi^2 - 4\underline{w} > 0$ hold. An optimal contract (b^*, w^*) between a principal and an agent given a reservation utility $\overline{u} \in [0, \overline{v}]$ exists and has the following features:*

1. *The fixed wage is set at the subsistence level, i.e. $w^* = \underline{w}$;*
2. *The bonus payment (monetary) is characterized by*

$$b^* = \begin{cases} \max\left\{0, \frac{\pi-\theta}{2}\right\} & \text{if } \overline{u} \in [0, \underline{v}] \\ \sqrt{2(\overline{u} - \underline{w})} - \theta & \text{if } \overline{u} \in [\underline{v}, \overline{v}] \end{cases} \tag{235}$$

3. *The optimal effort level solves*

$$e^* = b^* + \theta \tag{236}$$

7.7 Discuss how the optimal contract depends on the role of θ.

Here we assume that θ is a feature of all agents, i.e. we can see it as motivation or intrinsic utility. It enters the optimal contract in two ways:

1. It reduces b, i.e. the monetary incentive.
2. If agent's motivation is large enough, more precisely $\theta > \pi$, then there is no need for any monetary incentive at all. We note that A always receives a minimum level of consumption per period, i.e. $\underline{w}$ that assures his survival. The final effort level then depends on the two ways of incentivizing, extrinsic b and intrinsic θ.

For further and more detailed discussion and extension of this model towards matching and heterogenous agents and principals see "Competition and Incentives with Motivated Agents" (Besley, Ghatak) in *The American Economic Review* (June 2005) Vol.95no3.