MICROECONOMICS III – Information Economics and Contract Theory
Problem Set Solutions
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Problem Set 1

1 Adverse Selection and Monopoly screening in the Loan Market

Part I, Symmetric Information

a) Pareto-Optimum

To find the overall optimal allocation in the economy we maximize the total surplus.

Investor Utility: $U_i = p_i(\pi - r_i)$ –> accept contract iff $\pi \geq r_i$.
Bank profit: $\pi_B = \int_0^1 (p_i r_i - 1) I_L(i) di$,
where $I_L(i)$ is an indicator function for type i accepting a loan.

Total Surplus: $\pi_T = \int_0^1 [(p_i r_i - 1) I_L(i) + p_i(\pi - r_i)] di$
$\pi_T = \int_0^1 (p_i \pi - 1) I_L(i) di$.
$\max \pi_T$ –> loan iff $p_i \pi - 1 \geq 0 \iff p_i \in [\frac{1}{\pi}, 1]$

b) Equilibrium Allocation

Before starting any calculations, we already now from the First Welfare Theorem that with symmetric information and no other distortions the competitive equilibrium is pareto-optimal. To derive the market equilibrium we look at the decentralized decisions of the agents.

In the banking sector there is perfect competition. This implies free entry and thus zero profits.
bank decision:
$\pi_B = \int_0^1 (p_i r_i - 1) I_L(i) di = 0$
$\Leftrightarrow (p_i r_i - 1) = 0, \forall i$
$\Leftrightarrow r_i = \frac{1}{p_i}, \forall p_i \neq 0.$
investor decision: $U_i = p_i(\pi - r_i)$
$\Leftrightarrow U_i = p_i(\pi - \frac{1}{p_i}) = p_i \pi - 1 \geq 0$
–> Invest iff $p_i \geq \frac{1}{\pi}$.

Thus we get the pareto-optimal allocation of loans, i.e. loan iff $p_i \in [\frac{1}{\pi}, 1]$.

c) Single monopolistic bank (first degree price discrimination)

As there is symmetric information, we know that monopoly power will only lead to a redistribution of surplus from the investors to the bank. This is exactly what we get from the formal derivation.

$\max \pi_B = \int_0^1 (p_i r_i - 1) I_L(i) di$,
s.t.: participation constraint: $p_i(\pi - r_i) \geq 0$

Profit maximization implies PC binding:

proof by contradiction: Suppose not. Then monopolist could increase r.by ϵ This would increase profit, but constraint would still hold –> Contradiction!.q.e.d.

This implies:

$p_i(\pi - r_i) = 0 \Leftrightarrow \pi = r_i$.
$\max \pi_B = \int_0^1 (p_i \pi - 1) I_L(i) di$ –> $p_i \in [\frac{1}{\pi}, 1]$
Again we get the pareto-optimal allocation, i.e. loan iff $p_i \in [\frac{1}{\pi}, 1]$.

Part II, Asymmetric Information

d) loan equilibria under asymmetric information

Now banks cannot distinguish investors. Thus they can offer only one interest rate.

investor decision: $U_i = p_i(\pi - r)$ –> accept contract iff $\pi \geq r$.
i.e. either all invest or nobody.

$$E(p_i \mid r) = \left\{ \begin{array}{cc} 0, & \pi < r \\ E(p_i), & \pi \geq r \end{array} \right\} = \left\{ \begin{array}{cc} 0, & \pi < r \\ \frac{1}{2}, & \pi \geq r \end{array} \right\}$$

1st case: competitive banking sector
In the banking sector there is again perfect competition. This implies free entry and thus zero profits.

bank decision:
$\pi_B = \int_0^1 (p_i r - 1) di = 0$ (in case of $r \leq \pi$, i.e. if everybody takes a loan)

The integral can be simplified in the following way:
$\Leftrightarrow r \int_0^1 p_i di - 1 = 0$
$\Leftrightarrow r E(p_i) - 1 = 0$
$\Leftrightarrow r \frac{1}{2} - 1 = 0$

This results in the equilibrium interest rate of $r = 2$.

Equilibrium:
if $\pi < 2$: no loans
if $\pi \geq 2$: everybody takes a loan at interest rate $r = 2$.

2nd case: monopolistic bank
bank decision:
$\pi_B = \int_0^1 (p_i r - 1) di$

s.t.: participation constraint: $p_i(\pi - r) \geq 0 \Leftrightarrow \pi \geq r$

Profit maximization implies PC binding: (proof by contradiction)
This implies:
$\pi = r$.
$\pi_B = \int_0^1 (p_i\pi - 1)di = \frac{1}{2}\pi - 1$

The monopolistic bank makes positive profit iff $\pi \geq 2$.
$\pi_B \geq 0 \Leftrightarrow \pi \geq 2$.

Equilibrium:
if $\pi < 2$: no loans
if $\pi \geq 2$: everybody takes a loan at interest rate $r = \pi$.

As before, the equilibrium allocation of loans does not change when we move from competition to monopoly in the banking sector. The only thing that changes is the distribution of surplus.

Part III, Monopoly Screening

Now we allow for a second control for the monopolistic bank: a collateral. This allows for monopolistic screening. I.e. the bank can design its loan contracts in a way such that the two types of investors choose different contracts and thereby truthfully reveal their type.

e) participation constraint for an investor:
The participation constraint assures that the investor prefers a contract over the outside option.
$PC_i : U_I = p_i(\pi - r_i) - (1 - p_i)c_i \geq 0$.
The expected utility from taking a loan is profit minus interest in case of success and loss of the collateral in case of failure.

f) incentive compatibility contraint for an investor:
The second type of constraint is the incentive compatibility constraint for an investor.It is optimal for each investor to reveal truthfully his/her type by picking the contract designed for himself/herself.
$IC_{i,j} : p_i(\pi - r_i) - (1 - p_i)c_i \geq p_i(\pi - r_j) - (1 - p_i)c_j$
$\Leftrightarrow p_i(r_j - r_i) \geq (1 - p_i)(c_i - c_j)$

g) monopoly maximization problem:
In general there are three possible cases: bank offers loans to both types, only to type H or only to type L. We will look at all three cases and compare possible bank profits to find the profit maximizing equlibrium.

First Case: bank offers loans to both types
We can write the monopoly maximization problem using the four constraints:

$\max \pi_B = \alpha(p_H r_H + (1-p_H)c_H) + (1-\alpha)(p_L r_L + (1-p_L)c_L) - 1$
s.t.: PC_L, PC_H, IC_L, IC_H

Second Case: bank offers loans only to type H

$\max \pi_B = \alpha(p_H r_H + (1-p_H)c_H - 1)$
s.t.: PC_H
and type L does not participate –> PC_L not holding.

Third Case: bank offers loans only to type L

$\max \pi_B = (1-\alpha)(p_L r_L + (1-p_L)c_L - 1)$
s.t.: PC_L
and type H does not participate –> PC_H not holding.

h) profit maximization solution

First Case: bank offers loans to both types

To solve the problem we first analyze the structure of the constraints. For this we derive corresponding results to Lemmas 1-5 in the lecture.

1st proof: $PC_L + IC_H \Rightarrow PC_H$ (corresponds to Lemma 1 of lecture)
$PC_L : p_L(\pi - r_L) - (1-p_L)c_L \geq 0.$
–> $\pi \geq r_L$ for $c_L \geq 0$, i.e. for non-negative collateral.
$IC_H : p_H(\pi - r_H) - (1-p_H)c_H \geq p_H(\pi - r_L) - (1-p_H)c_L$

Add and substract PC_L on the right hand side of IC_H:
$p_H(\pi - r_H) - (1-p_H)c_H \geq$
$p_H(\pi - r_L) - (1-p_H)c_L + (p_L(\pi - r_L) - (1-p_L)c_L) - (p_L(\pi - r_L) - (1-p_L)c_L)$

Rearrange to get:
$p_H(\pi - r_H) - (1-p_H)c_H \geq$
$(p_L(\pi - r_L) - (1-p_L)c_L) + (p_H - p_L)(\pi - r_L) - (p_L - p_H)c_L$

$p_H(\pi - r_H) - (1-p_H)c_H \geq 0 + (p_H - p_L)(\pi - r_L + c_L) \geq 0.$ q.e.d.

2nd proof:$(c_H - r_H) \geq (c_L - r_L)$ (corresponds to Lemma 2 of lecture)

$IC_L : -p_L(r_L - r_H) + (1-p_L)(c_H - c_L) \geq 0$
$IC_H : p_H(r_L - r_H) - (1-p_H)(c_H - c_L) \geq 0$

Add the two constraints together:
–> $(p_H - p_L)(r_L - r_H + c_H - c_L) \geq 0$

$\Leftrightarrow (r_L - r_H + c_H - c_L) \geq 0$

$\Leftrightarrow (c_H - r_H) \geq (c_L - r_L)$.. q.e.d.

3rd proof: IC_H binding and $(c_H - r_H) \geq (c_L - r_L) \Rightarrow IC_L$ (corresponds to Lemma 3 of lecture)

$IC_H : p_H(r_L - r_H) - (1 - p_H)(c_H - c_L) = 0$

Add and substract IC_L on the left hand side:.

$p_L(r_H - r_L) - (1 - p_L)(c_L - c_H)$

$p_H(r_L - r_H) - (1 - p_H)(c_H - c_L) + p_L(r_H - r_L) - (1 - p_L)(c_L - c_H) - [p_L(r_H - r_L) - (1 - p_L)(c_L - c_H)] = 0$

rewrite:$p_L(r_H - r_L) = (p_H - p_L)(r_L - r_H + c_H - c_L) + (1 - p_L)(c_L - c_H)$

$p_L(r_H - r_L) = (1 - p_L)(c_L - c_H) + (p_H - p_L)((c_H - r_H) - (c_L - r_L))$
$\geq (1 - p_L)(c_L - c_H)$ q.e.d.

4th proof:PC_L binds (corresponds to Lemma 4 of lecture)

Suppose not. Then monopolist could increase both r_L and r_H.by ϵ This would increase profits, but all constraints would still hold –> Contradiction!.q.e.d.

5th proof:IC_H binds (corresponds to Lemma 5 of lecture)

Suppose not. Then bank could increase c_H. by ϵ This would increase profits, but IC_H would still hold. Type L would have even less incentive to pretend to be type H.(IC_L still holds) Also the participation constraints would still hold. –> Contradiction! q.e.d.

Now we can use the information on the constraints to rewrite and solve the profit maximization of the monopolistic bank.

PC_L binds –> $p_L(\pi - r_L) - (1 - p_L)c_L = 0$

–> $r_L = \pi - \frac{(1-p_L)}{p_L}c_L$

Plug this back into the original profit maximization:

$\max \pi_B = \alpha(p_H r_H + (1 - p_H)c_H) + (1 - \alpha)(p_L r_L + (1 - p_L)c_L) - 1$

$\max \pi_B = \alpha(p_H r_H + (1 - p_H)c_H) + (1 - \alpha)(p_L(\pi - \frac{(1-p_L)}{p_L}c_L) + (1 - p_L)c_L) - 1$

$\max \pi_B = \alpha(p_H r_H + (1-p_H)c_H) + (1-\alpha)p_L\pi - 1$

IC_H binds –> $p_H(r_L - r_H) - (1-p_H)(c_H - c_L) = 0$

$\Leftrightarrow p_H r_H = p_H r_L - (1-p_H)(c_H - c_L)$

Plug this into profit maximization:

$\max \pi_B = \alpha(p_H r_L - (1-p_H)(c_H - c_L) + (1-p_H)c_H) + (1-\alpha)p_L\pi - 1$

$\max \pi_B = \alpha(p_H r_L + (1-p_H)c_L) + (1-\alpha)p_L\pi - 1$

$\Leftrightarrow \max \pi'_B = p_H r_L + (1-p_H)c_L$

Plug in information from PC_L: $r_L = \pi - \frac{(1-p_L)}{p_L}c_L$

$\max \pi'_B = p_H(\pi - \frac{(1-p_L)}{p_L}c_L) + (1-p_H)c_L$

$\max \pi'_B = p_H\pi + c_L(1 - p_H - \frac{p_H}{p_L}(1-p_L))$

$\max \pi'_B = p_H\pi + c_L(1 - \frac{p_H}{p_L})$

–> maximized for $c_L = 0$.

Plug this into the PC_L:
$p_L(\pi - r_L) = 0$

$\Leftrightarrow \pi = r_L$

Plug this into IC_H
$p_H(\pi - r_H) = (1-p_H)c_H$

$r_H = \pi - \frac{(1-p_H)}{p_H}c_H$

Bank profit: $\pi_B = \alpha(p_H r_L + (1-p_H)c_L) + (1-\alpha)p_L\pi - 1$

$\pi_B = \alpha p_H\pi + (1-\alpha)p_L\pi - 1$

We get an infinite number of solutions which satisfy the conditions PC_L and IC_H binding. They are characterized in the following way:

$c_L = 0$ and $\pi = r_L$ and $r_H = \pi - \frac{(1-p_H)}{p_H}c_H$ and $\pi_B = \alpha p_H\pi + (1-\alpha)p_L\pi - 1$

Finally we check if $(c_H - r_H) \geq (c_L - r_L)$ is fulfilled:

$\Leftrightarrow(c_H - (\pi - \frac{(1-p_H)}{p_H}c_H)) \geq -\pi$

$\Leftrightarrow c_H(1 + \frac{(1-p_H)}{p_H}) \geq 0$

$\Leftrightarrow \frac{c_H}{p_H} \geq 0$ always holds.

Second Case: bank offers loans only to type H

$\max \pi_B = \alpha(p_H r_H + (1-p_H)c_H - 1)$
s.t.: type L does not participate.

1st proof: PC_H binds:
Suppose not. Then bank could increase r_H by ϵ. This would increase profits and all constraints would still hold. –> Contradiction! q.e.d.

$p_H(\pi - r_H) = (1-p_H)c_H$

$r_H = \pi - \frac{(1-p_H)}{p_H}c_H$

–> $\pi_B = \alpha(p_H(\pi - \frac{(1-p_H)}{p_H}c_H) + (1-p_H)c_H - 1) = \alpha(p_H\pi - 1)$

Now type L should not be willing to participate:

$p_L(\pi - r_H) - (1-p_L)c_H < 0$

$\Leftrightarrow c_H > \frac{p_L}{(1-p_L)}(\pi - r_H) = \frac{p_L}{(1-p_L)}(\pi - r_H)$

$\Leftrightarrow c_H > \frac{p_L}{(1-p_L)}(\pi - \pi + \frac{(1-p_H)}{p_H}c_H)$

$\Leftrightarrow 1 > \frac{p_L}{p_H}\frac{(1-p_H)}{(1-p_L)}, \forall c_H > 0$ holding.

Again we get an infinite number of solutions that satisfy these conditions. The equlibria can be characterized in the following way:

$r_H = \pi - \frac{(1-p_H)}{p_H}c_H$ and $c_H > 0$ and $\pi_B = \alpha(p_H\pi - 1)$

Third Case: bank offers loans only to type L

$\max \pi_B = (1-\alpha)(p_L r_L + (1-p_L)c_L - 1)$
s.t.: type H does not participate.

We can prove that this case of separation is not possible as there does not exist a contract which will be accepted by type L, but not by type H.

Suppose PC_L binds:

–> $\pi \geq r_L$ for $c_L \geq 0$, i.e. for non-negative collateral.

$p_L(\pi - r_L) - (1 - p_L)c_L = 0$

and suppose PC_H does not hold:

$p_H(\pi - r_L) - (1 - p_H)c_L < 0$

$(p_H - p_L)(\pi - r_L) + (p_H - p_L)c_L < 0$

–> Contradiction! –> offering contract only to type L is not possible. q.e.d.

Last Step: Compare profits of Case 1 and Case 2

1st case: $\pi_B = \alpha p_H \pi + (1 - \alpha)p_L \pi - 1$
2nd case:$\pi_B = \alpha(p_H \pi - 1)$

$\Delta\pi_B = \alpha p_H \pi + (1 - \alpha)p_L \pi - 1 - \alpha(p_H \pi - 1)$

$\Delta\pi_B = (1 - \alpha)(p_L \pi - 1)$

$\Delta\pi_B \geq 0 \Leftrightarrow p_L \pi - 1 \geq 0 \Leftrightarrow p_L \geq \frac{1}{\pi}$

–> Offer contract to both types if $p_L \geq \frac{1}{\pi}$

–> Offer contract only to type Hif $p_H \geq \frac{1}{\pi} > p_L$

–> No contracts at all if$\frac{1}{\pi} > p_H > p_L$

Problem Set 2

1. Entrepreneurial incentives and taxation

a) budget constraint: $C \leq \alpha\delta_1 + (1-\alpha)\delta_2$
–> binds in optimum –> proof by contradiction
$\delta_1 = \frac{C}{\alpha} - \frac{(1-\alpha)}{\alpha}\delta_2$

Symmetric information:

define effort cost function: $K(e) = \frac{1}{2}e^2$

define total effort-corrected consumption: $X_i = \gamma_i e_i - \delta_i - K(e_i)$

$\max W = \alpha U(X_1) + (1-\alpha)U(X_2) = \alpha U[\gamma_1 e_1 - \delta_1 - K(e_1)] + (1-\alpha)U[\gamma_2 e_2 - \delta_2 - K(e_2)]$

use: $\delta_1 = \frac{C}{\alpha} - \frac{(1-\alpha)}{\alpha}\delta_2$

$\max W = \alpha U[\gamma_1 e_1 - \frac{C}{\alpha} + \frac{(1-\alpha)}{\alpha}\delta_2 - K(e_1)] + (1-\alpha)U[\gamma_2 e_2 - \delta_2 - K(e_2)]$

FOC 1: $\frac{\partial W}{\partial \delta_2} = \alpha \frac{\partial U(X_1)}{\partial X_1}\frac{\partial X_1}{\partial \delta_2} + (1-\alpha)\frac{\partial U(X_2)}{\partial X_2}\frac{\partial X_2}{\partial \delta_2} =$

$\alpha \frac{\partial U(X_1)}{\partial X_1}\frac{(1-\alpha)}{\alpha} + (1-\alpha)\frac{\partial U(X_2)}{\partial X_2}(-1) = 0$

$\Leftrightarrow \frac{\partial U(X_1)}{\partial X_1} = \frac{\partial U(X_2)}{\partial X_2}$

$\Leftrightarrow X_1 = X_2$

–> In the symmetric case optimum both types have the same consumption levels.

$\gamma_1^2 - \delta_1 - \frac{1}{2}\gamma_1^2 = \gamma_2^2 - \delta_2 - \frac{1}{2}\gamma_2^2$

$\Leftrightarrow \delta_2 = \frac{1}{2}(\gamma_2^2 - \gamma_1^2) + \delta_1$

plug into budget constraint to get:

$\delta_1 = \frac{C}{\alpha} - \frac{(1-\alpha)}{\alpha}(\frac{1}{2}(\gamma_2^2 - \gamma_1^2) + \delta_1)$

$\delta_1^* = C - \frac{(1-\alpha)}{2}(\gamma_2^2 - \gamma_1^2)$

$\delta_2^* = C + \frac{\alpha}{2}(\gamma_2^2 - \gamma_1^2)$

FOC 2: $\frac{\partial W}{\partial e_1} = \alpha \frac{\partial U(X_1)}{\partial X_1}(\gamma_1 - e_1^*) = 0$
$\Leftrightarrow e_1^* = \gamma_1$

FOC 3: $\frac{\partial W}{\partial e_2} = \alpha \frac{\partial U(X_2)}{\partial X_2}(\gamma_2 - e_2^*) = 0$
$\Leftrightarrow e_2^* = \gamma_2$

In the full information benchmark agents exert optimal efforts and redistribution is perfect.

Asymmetric information

Now we need incentive compatibility constraints assuring that agents chose the right contract:

we know that: $e_i\gamma_i = R_i \Leftrightarrow e_i = \frac{R_i}{\gamma_i} \Leftrightarrow e_i(R_j) = \frac{R_j}{\gamma_i} = \frac{\gamma_j}{\gamma_i}e_j$

incentive compatability constraint:

$IC_1 : \gamma_1 e_1 - \delta_1 - K(e_1) \geq \gamma_2 e_2 - \delta_2 - K(\frac{\gamma_2}{\gamma_1}e_2)$

$IC_2 : \gamma_2 e_2 - \delta_2 - K(e_2) \geq \gamma_1 e_1 - \delta_1 - K(\frac{\gamma_1}{\gamma_2}e_1)$

In the symmetric solution we had:

$X_1 = X_2$

$\Leftrightarrow \gamma_1 e_1 - \delta_1 - K(e_1) = \gamma_2 e_2 - \delta_2 - K(e_2)$

$IC_1 {-}{>} \gamma_1 e_1 - \delta_1 - K(e_1) = \gamma_2 e_2 - \delta_2 - K(e_2) \geq \gamma_2 e_2 - \delta_2 - K(\frac{\gamma_2}{\gamma_1}e_2)$

$\Leftrightarrow K(\frac{\gamma_2}{\gamma_1}e_2) \geq K(e_2)$ holds for all e_2.

$IC_2 {-}{>} \gamma_2 e_2 - \delta_2 - K(e_2) = \gamma_1 e_1 - \delta_1 - K(e_1) \geq \gamma_1 e_1 - \delta_1 - K(\frac{\gamma_1}{\gamma_2}e_1)$

$\Leftrightarrow K(\frac{\gamma_1}{\gamma_2}e_1) \geq K(e_1)$ does not hold for any e_2. –> Contradiction! –> IC_2 does not hold. Type 2 would prefer to take the Type 1 contract.

b) Proof that IC_2 is binding:

suppose not: $\gamma_2 e_2 - \delta_2 - K(e_2) > \gamma_1 e_1 - \delta_1 - K(\frac{\gamma_1}{\gamma_2}e_1)$

Suggest the following variation:

$\tilde{\delta}_1 = \delta_1 - \epsilon$

From budget constraint we get

$\delta_2 = \frac{C}{(1-\alpha)} - \frac{\alpha}{(1-\alpha)}\delta_1$

$\text{–>}\Delta\delta_2 = -\frac{\alpha}{(1-\alpha)}\Delta\delta_1$

$\tilde{\delta}_2 = \delta_2 + \frac{\alpha}{(1-\alpha)}\epsilon$

Plug this into the Welfare maximization to get:

$\max W = \alpha U[\gamma_1 e_1 - (\delta_1 - \epsilon) - K(e_1)] + (1-\alpha)U[\gamma_2 e_2 - \frac{C}{(1-\alpha)} + \frac{\alpha}{(1-\alpha)}(\delta_1 - \epsilon) - K(e_2)]$

$\frac{\partial W}{\partial \epsilon} = \alpha\frac{\partial U(X_1)}{\partial X_1} - (1-\alpha)\frac{\partial U(X_2)}{\partial X_2}\frac{\alpha}{(1-\alpha)} = \alpha(\frac{\partial U(X_1)}{\partial X_1} - \frac{\partial U(X_2)}{\partial X_2})$

Here we only have the direct effect of ϵ on W as the indirect effect through effort levels will be zero (envelope theorem).

We know that : $X_2 = \gamma_2 e_2 - \delta_2 - K(e_2) > \gamma_1 e_1 - \delta_1 - K(\frac{\gamma_1}{\gamma_2}e_1) \geq \gamma_1 e_1 - \delta_1 - K(e_1) = X_1$

$\Leftrightarrow X_2 > X_1 \Leftrightarrow \frac{\partial U(X_1)}{\partial X_1} > \frac{\partial U(X_2)}{\partial X_2}$ (from concavity of the utility function)

$\text{–>} \frac{\partial W}{\partial \epsilon} = \alpha(\frac{\partial U(X_1)}{\partial X_1} - \frac{\partial U(X_2)}{\partial X_2}) > 0$

Thus the variation would increase welfare. Furthermore IC_1 would hold even stronger. Thus IC_2 must be binding.

c) Maximization problem

with IC_2 binding the maximization problem becomes:

$BC : \delta_1 = \frac{C}{\alpha} - \frac{(1-\alpha)}{\alpha}\delta_2$

$IC_2 : \gamma_2 e_2 - \delta_2 - K(e_2) = \gamma_1 e_1 - \delta_1 - K(\frac{\gamma_1}{\gamma_2}e_1)$

$\Leftrightarrow \delta_2 = \gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2}e_1)) + \frac{C}{\alpha} - \frac{(1-\alpha)}{\alpha}\delta_2$

$\Leftrightarrow \delta_2 = C + \alpha(\gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2}e_1)))$

$\text{–>} \ \delta_1 = \frac{C}{\alpha} - \frac{(1-\alpha)}{\alpha}(C + \alpha(\gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2}e_1))))$

$\Leftrightarrow \delta_1 = C - (1-\alpha)(\gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2}e_1)))$

Plug into the maximization problem:

$\max W = \alpha U[\gamma_1 e_1 - \frac{C}{\alpha} + \frac{(1-\alpha)}{\alpha}\delta_2 - K(e_1)] + (1-\alpha)U[\gamma_2 e_2 - \delta_2 - K(e_2)]$

$\Leftrightarrow \max W = \alpha U[\gamma_1 e_1 - \frac{C}{\alpha} + \frac{(1-\alpha)}{\alpha}[C + \alpha(\gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2} e_1)))] - K(e_1)]$
$+(1-\alpha)U[\gamma_2 e_2 - [C + \alpha(\gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2} e_1)))] - K(e_2)]$

$\Leftrightarrow \max W = \alpha U[\gamma_1 e_1 - C + (1-\alpha)(\gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2} e_1))) - K(e_1)]$

$+(1-\alpha)U[\gamma_2 e_2 - C - \alpha(\gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2} e_1)))] - K(e_2)]$

Rearrange to get:

$\max W = \alpha U[\alpha\gamma_1 e_1 - K(e_1) - C + (1-\alpha)[\gamma_2 e_2 - K(e_2) + K(\frac{\gamma_1}{\gamma_2} e_1)]]$

$+(1-\alpha)U[(1-\alpha)(\gamma_2 e_2 - K(e_2)) - C + \alpha(\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2} e_1))]$

FOC:

$\frac{\partial W}{\partial e_2} = \alpha\frac{\partial U(X_1)}{\partial X_1}(1-\alpha)(\gamma_2 - K'(e_2)) + (1-\alpha)\frac{\partial U(X_2)}{\partial X_2}(1-\alpha)(\gamma_2 - K'(e_2)) = 0$

$\Leftrightarrow (\gamma_2 - K'(e_2))[\alpha\frac{\partial U(X_1)}{\partial X_1} + (1-\alpha)\frac{\partial U(X_2)}{\partial X_2}] = 0$ $(\alpha \neq 0)$

$\Leftrightarrow \gamma_2 - K'(e_2) = 0 \Leftrightarrow \gamma_2 = K'(e_2)$ (first best effort)

$\frac{\partial W}{\partial e_1} = \alpha\frac{\partial U(X_1)}{\partial X_1}(\alpha\gamma_1 - K'(e_1) + (1-\alpha)K'(\frac{\gamma_1}{\gamma_2}e_1)\frac{\gamma_1}{\gamma_2}) + (1-\alpha)\frac{\partial U(X_2)}{\partial X_2}\alpha(\gamma_1 - K'(\frac{\gamma_1}{\gamma_2}e_1)\frac{\gamma_1}{\gamma_2}) = 0$

Use $K'(e) = e$

$-> \frac{\partial W}{\partial e_1} = \alpha\frac{\partial U(X_1)}{\partial X_1}(\alpha\gamma_1 - e_1 + (1-\alpha)\frac{\gamma_1}{\gamma_2}e_1\frac{\gamma_1}{\gamma_2}) + (1-\alpha)\frac{\partial U(X_2)}{\partial X_2}\alpha(\gamma_1 - \frac{\gamma_1}{\gamma_2}e_1\frac{\gamma_1}{\gamma_2}) = 0$

$\Leftrightarrow \alpha\gamma_1 - e_1 + (1-\alpha)e_1(\frac{\gamma_1}{\gamma_2})^2 + (1-\alpha)\frac{\partial U(X_2)/\partial X_2}{\partial U(X_1)/\partial X_1}(\gamma_1 - e_1(\frac{\gamma_1}{\gamma_2})^2) = 0$

Define: $\mu = \frac{\partial U(X_2)/\partial X_2}{\partial U(X_1)/\partial X_1} < 1$ (from IC_H binding)

$\Leftrightarrow \gamma_1 - e_1 - (1-\alpha)(\gamma_1 - e_1(\frac{\gamma_1}{\gamma_2})^2) + (1-\alpha)\mu(\gamma_1 - e_1(\frac{\gamma_1}{\gamma_2})^2) = 0$

$\Leftrightarrow e_1 = (1 - e_1\frac{\gamma_1}{\gamma_2^2})\gamma_1(1 - (1-\alpha)(1-\mu))$

$\Leftrightarrow e_1(1 + \frac{\gamma_1}{\gamma_2^2}\gamma_1(1 - (1-\alpha)(1-\mu))) = \gamma_1(1 - (1-\alpha)(1-\mu))$

$\Leftrightarrow e_1 = \frac{\gamma_1(1-(1-\alpha)(1-\mu))}{(1+\frac{\gamma_1}{\gamma_2^2}\gamma_1(1-(1-\alpha)(1-\mu)))}$

Define $x = (1 - (1-\alpha)(1-\mu)) < 1$

$\Leftrightarrow e_1 = \gamma_1 \frac{x}{1+\frac{\gamma_1^2}{\gamma_2^2}x} = \gamma_1 \frac{x\gamma_2^2}{x\gamma_1^2+\gamma_2^2} < \gamma_1$ as $x\gamma_2^2 < \gamma_2^2$ because we know that $x < 1$.

Under asymmetric information the effort level of type 1 is below the first best.

What about the fees delta:

Delta 1

$\delta_1 = C - (1-\alpha)(\gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2}e_1)))$

$\delta_1 = C - (1-\alpha)(\gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2}e_1)))$

$\delta_1 = C - (1-\alpha)(\frac{1}{2}\gamma_2^2 - \gamma_1 e_1 + \frac{1}{2}(\frac{\gamma_1}{\gamma_2}e_1)^2)$

Delta 2

$\delta_2 = C + \alpha(\gamma_2 e_2 - K(e_2) - (\gamma_1 e_1 - K(\frac{\gamma_1}{\gamma_2}e_1)))$

$\delta_2 = C + \alpha(\gamma_2^2 - \frac{1}{2}\gamma_2^2 - \gamma_1 e_1 + \frac{1}{2}(\frac{\gamma_1}{\gamma_2}e_1)^2) = C + \alpha(\frac{1}{2}\gamma_2^2 - \gamma_1 e_1 + \frac{1}{2}(\frac{\gamma_1}{\gamma_2}e_1)^2)$

we know that $e_1 < \gamma_1$replace e_1 by γ_1 on RHS to get

$\delta_2 = C + \alpha(\frac{1}{2}\gamma_2^2 - \gamma_1 e_1 + \frac{1}{2}(\frac{\gamma_1}{\gamma_2}e_1)^2) > C + \alpha(\frac{1}{2}\gamma_2^2 - \gamma_1^2 + \frac{1}{2}\frac{\gamma_1^4}{\gamma_2^2})$

Look at second term inside the brackets
$\frac{1}{2}\gamma_2^2 - \gamma_1^2 + \frac{1}{2}\frac{\gamma_1^4}{\gamma_2^2} = \frac{1}{2}(\gamma_2^2 - 2\gamma_1^2 + \frac{\gamma_1^4}{\gamma_2^2}) = \frac{1}{2}(\gamma_2 - \frac{\gamma_1^2}{\gamma_2})^2 = \frac{1}{2}\gamma_2^2(1 - \frac{\gamma_1^2}{\gamma_2^2})^2 > 0$

–>$\delta_2 > C$ –>$\delta_1 < C$

$\delta_1^* = C - \frac{(1-\alpha)}{2}(\gamma_2^2 - \gamma_1^2)$

$\delta_1 = C - \frac{(1-\alpha)}{2}(\gamma_2^2 - \gamma_1 e_1 + (\frac{\gamma_1}{\gamma_2}e_1)^2 - \gamma_1 e_1)$

$\delta_2^* = C + \frac{\alpha}{2}(\gamma_2^2 - \gamma_1^2)$

$\delta_2 = C + \frac{\alpha}{2}(\gamma_2^2 - \gamma_1 e_1 + (\frac{\gamma_1}{\gamma_2} e_1)^2 - \gamma_1 e_1)$

Type 2 will always pay a larger share of the costs for the EO.

Finally we note that from IC_2 (as shown above), the utility of type 2 is always larger than the utility of type 1. This is the case as type 2 could always pretend to be type 1 and the EO pays information rent to type 2 in order to make them truthfully reveal their type.

—> $U(X_2) > U(X_1)$

Thus we can characterize the following equilibrium results:

$e_2 = \gamma_2$(efficient effort for type 2)

$e_1 < \gamma_1$(below efficient effort of type 1)

$\delta_1 = C - \frac{(1-\alpha)}{2}(\gamma_2^2 - \gamma_1 e_1 + (\frac{\gamma_1}{\gamma_2} e_1)^2 - \gamma_1 e_1)$

$\delta_2 = C + \frac{\alpha}{2}(\gamma_2^2 - \gamma_1 e_1 + (\frac{\gamma_1}{\gamma_2} e_1)^2 - \gamma_1 e_1)$

$\delta_2 > \delta_1$

$U(X_2) > U(X_1)$

2. Licensing to a monopolist

a) Profit maximization of the monopolist

Define: x : cost of production

$\Pi^m(x) = (p^m(x) - x)Q(p^m(x)$

where $p^m(x) \in \arg\max_p (p - x)Q(p)$

Profit without taking the license:

$\Pi^m(c) = (p^m(c) - c)Q(p^m(c))$

Profit when taking the license:

$\Pi^m(c_1 + \alpha) =(p^m(c_1 + \alpha) - c_1 - \alpha)Q(p^m(c_1 + \alpha)) - L$

b) Profit maximization of the researcher

$\max_{L,\alpha} \Pi_R = L + \alpha Q(p^m(c_1 + \alpha)$

s.t.: $L \leq \Pi^m(c_1 + \alpha) - \Pi^m(c), \qquad \alpha \geq 0, \qquad L \geq 0$

PC_M has to bind –> proof by contradiction.

$\max_\alpha \Pi_R = \Pi^m(c_1 + \alpha) - \Pi^m(c) + \alpha Q(p^m(c_1 + \alpha)$

s.t.: $\alpha \geq 0, \qquad L \geq 0$

Assume constant elasticity of demand:

$\epsilon = -pQ'(p)/Q(p) > 1$

$p^m(x) \in \arg\max_p (p - x)Q(p)$

–> $Q'(p)(p - x) + Q(p) = 0$

$\Leftrightarrow p(x) = -\frac{Q(p)}{Q'(p)} + x \Leftrightarrow p(x) = \frac{P}{\epsilon} + x$

$\Leftrightarrow p(x) = \frac{\epsilon}{\epsilon - 1} x = \Gamma x$ with $\Gamma = \frac{\epsilon}{\epsilon - 1}$ markup over marginal costs.

–>$\frac{\partial p}{\partial x} = \Gamma > 1$

FOC:

$\frac{\partial \Pi_R}{\partial \alpha} = \frac{\partial \Pi^m(c_1+\alpha)}{\partial \alpha} + Q(p^m(c_1 + \alpha) + \alpha \frac{\partial Q(p^m(c_1+\alpha)}{\partial p^m(c_1+\alpha)} \frac{\partial p^m(c_1+\alpha)}{\partial \alpha}$

$\frac{\partial \Pi^m(c_1+\alpha)}{\partial \alpha} = (\frac{\partial p^m(c_1+\alpha)}{\partial \alpha} - 1)Q(p^m(c_1+\alpha) + (p^m(c_1+\alpha) - c_1 - \alpha) \frac{\partial Q(p^m(c_1+\alpha)}{\partial p^m(c_1+\alpha)} \frac{\partial p^m(c_1+\alpha)}{\partial \alpha}$

$\frac{\partial \Pi_R}{\partial \alpha} = (\frac{\partial p^m(c_1+\alpha)}{\partial \alpha} - 1)Q(p^m(c_1+\alpha) + (p^m(c_1+\alpha) - c_1 - \alpha) \frac{\partial Q(p^m(c_1+\alpha)}{\partial p^m(c_1+\alpha)} \frac{\partial p^m(c_1+\alpha)}{\partial \alpha} + Q(p^m(c_1 + \alpha) + \alpha \frac{\partial Q(p^m(c_1+\alpha)}{\partial p^m(c_1+\alpha)} \frac{\partial p^m(c_1+\alpha)}{\partial \alpha}$

$\frac{\partial \Pi_R}{\partial \alpha} = (\frac{\partial p^m(c_1+\alpha)}{\partial \alpha})(Q(p^m(c_1 + \alpha) + ((p^m(c_1 + \alpha) - c_1 - \alpha) \frac{\partial Q(p^m(c_1+\alpha)}{\partial p^m(c_1+\alpha)}) + \alpha \frac{\partial Q(p^m(c_1+\alpha)}{\partial p^m(c_1+\alpha)})$

$\frac{\partial \Pi_R}{\partial \alpha} = \Gamma(Q(\Gamma(c_1 + \alpha)) + (\Gamma(c_1 + \alpha) - c_1) \frac{\partial Q(p^m(c_1+\alpha)}{\partial p^m(c_1+\alpha)})$

$\frac{\partial \Pi_R}{\partial \alpha} = \Gamma(Q(\Gamma(c_1 + \alpha)) + (\Gamma - 1)c_1) \frac{\partial Q(p^m(c_1+\alpha)}{\partial p^m(c_1+\alpha)}) < 0$

In symmetric case: $\alpha = 0$

d) relevant constraints

$IC_G : \Pi^m(c_G + \alpha_G) - L_G \geq \Pi^m(c_G + \alpha_B) - L_B$

$IC_B : \Pi^m(c_B + \alpha_B) - L_B \geq \Pi^m(c_B + \alpha_G) - L_G$

$PC_G : \Pi^m(c_G + \alpha_G) - L_G - \Pi^m(c) \geq 0$

$PC_B : \Pi^m(c_B + \alpha_B) - L_B - \Pi^m(c) \geq 0$

e) which constraints are binding

Lemma 1: IC_G **+** PC_B –> PC_G

$\Pi^m(c_G + \alpha_G) - L_G \geq \Pi^m(c_G + \alpha_B) - L_B > \Pi^m(c_B + \alpha_B) - L_B \geq \Pi^m(c)$

Lemma 2: IC_G + IC_B –> $\Pi^m(c_G + \alpha_G) - \Pi^m(c_G + \alpha_B) \geq \Pi^m(c_B + \alpha_B) - \Pi^m(c_B + \alpha_G)$

$\Pi^m(c_G + \alpha_G) - L_G - \Pi^m(c_G + \alpha_B) + L_B + \Pi^m(c_B + \alpha_B) - L_B - \Pi^m(c_B + \alpha_G) + L_G \geq 0$

$\Leftrightarrow \Pi^m(c_G + \alpha_G) - \Pi^m(c_G + \alpha_B) \geq \Pi^m(c_B + \alpha_B) - \Pi^m(c_B + \alpha_G)$ L2
(single crossing condition: good type profits relatively more from better contract)

Lemma 3: L2 + IC_G binds –> IC_B

$IC_G : \Pi^m(c_G + \alpha_G) - L_G - \Pi^m(c_G + \alpha_B) + L_B = 0$

add IC_B ob both sides:

$\Pi^m(c_G + \alpha_G) - L_G - \Pi^m(c_G + \alpha_B) + L_B + \Pi^m(c_B + \alpha_B) - L_B - \Pi^m(c_B + \alpha_G) + L_G = \Pi^m(c_B + \alpha_B) - L_B - \Pi^m(c_B + \alpha_G) + L_G$

$\Leftrightarrow \Pi^m(c_B + \alpha_B) - L_B - \Pi^m(c_B + \alpha_G) + L_G = \Pi^m(c_G + \alpha_G) - \Pi^m(c_G + \alpha_B) + \Pi^m(c_B + \alpha_B) - \Pi^m(c_B + \alpha_G)$

by L2: ≥ 0 q.e.d.

Lemma 4: PC_B binds:

Suppose not. $\Pi^m(c_B + \alpha_B) - L_B > \Pi^m(c)$

variation: $\tilde{L}_B = L_B + \epsilon$ and $\tilde{L}_G = L_G + \epsilon$ –>higher profit for researcher

–> PC_B still holds.IC_G does not change. IC_B and PC_G hold by Lemma 3 and Lemma 1 respectively.

–>$\exists$ deviation with higher profit –>PC_B cannot be holding –>PC_B binds q.e.d

Lemma 5:IC_G binds:

Suppose not. $\Pi^m(c_G + \alpha_G) - L_G - \Pi^m(c_G + \alpha_B) + L_B > 0$

variation: $\tilde{L}_G = L_G + \epsilon$ –>higher profit for researcher

–>PC_B unaffected,.IC_B gets even stronger, PC_G holds by Lemma 1.

–>$\exists$ deviation with higher profit –>IC_G cannot be holding –>IC_G binds q.e.d

Maximization problem

q: probability of invention being good

$\max q(L_G + \alpha_G Q(p^m(c_G + \alpha_G))) + (1 - q)(L_B + \alpha_B Q(p^m(c_B + \alpha_B)))$

s.t.: $\Pi^m(c_G + \alpha_G) - L_G - \Pi^m(c_G + \alpha_B) + L_B = 0$

$\Pi^m(c_B + \alpha_B) - L_B = \Pi^m(c)$

$\Pi^m(c_G + \alpha_G) - \Pi^m(c_G + \alpha_B) - \Pi^m(c_B + \alpha_B) + \Pi^m(c_B + \alpha_G) \geq 0$

Using constraints to get:

–> $L_B = \Pi^m(c_B + \alpha_B) - \Pi^m(c)$

–> $L_G = \Pi^m(c_G + \alpha_G) - \Pi^m(c_G + \alpha_B) + \Pi^m(c_B + \alpha_B) - \Pi^m(c)$

$\max_{\alpha_G,\alpha_B} \Pi_R = q(\Pi^m(c_G + \alpha_G) - \Pi^m(c_G + \alpha_B) + \Pi^m(c_B + \alpha_B) - \Pi^m(c) + \alpha_G Q(p^m(c_G + \alpha_G)) + (1-q)(\Pi^m(c_B + \alpha_B) - \Pi^m(c) + \alpha_B Q(p^m(c_B + \alpha_B))$

write $Q(p^m(x)) = Q^m(x)$

$\frac{\partial \Pi_R}{\alpha_G} = q(-Q^m(c_G + \alpha_G) + Q^m(c_G + \alpha_G) + \alpha_G \frac{\partial Q^m(c_G+\alpha_G)}{\partial \alpha_G}) = 0$

$\Leftrightarrow q\alpha_G \frac{Q(c_G+\alpha_G)}{\partial \alpha_G} = q\alpha_G \frac{\partial Q(c_G+\alpha_G)}{\partial p} \frac{\partial p}{\partial \alpha_G} = \alpha_G q \Gamma \frac{\partial Q(c_G+\alpha_G)}{\partial p} \Leftrightarrow \alpha_G = 0$

$\frac{\partial \Pi_R}{\alpha_B} = q(Q^m(c_G + \alpha_B) - Q^m(c_B + \alpha_B)) + (1-q)(-Q^m(c_B + \alpha_B) + Q^m(c_B + \alpha_B) + \alpha_B \frac{\partial Q^m(c_B+\alpha_B)}{\partial \alpha_B}) = 0$

$\frac{\partial \Pi_R}{\alpha_B} = q(Q^m(c_G + \alpha_B) - Q^m(c_B + \alpha_B)) + (1-q)\alpha_B \Gamma \frac{\partial Q(c_G+\alpha_G)}{\partial p} = 0$

$\alpha_B \neq 0$ as $Q^m(c_G + \alpha_B) - Q^m(c_B + \alpha_B) > 0$

–>$\alpha_B > 0$

We can use this information on the optimal per unit payments with the binding constraints to find the properties of the fixed payments L.

$\Pi^m(c_G + \alpha_G) - L_G - \Pi^m(c_G + \alpha_B) + L_B = 0$

$\Leftrightarrow L_G - L_B = \Pi^m(c_G) - \Pi^m(c_G + \alpha_B) > 0$

$\Leftrightarrow L_G > L_B$

$L_G^* = \Pi^m(c_G) - \Pi^m(c)$

$L_G = \Pi^m(c_G) - \Pi^m(c_G + \alpha_B) + \Pi^m(c_B + \alpha_B) + \Pi^m(c)$

$L_G^* - L_G = \Pi^m(c_G) - \Pi^m(c_G) + \Pi^m(c_G + \alpha_B) - \Pi^m(c_B + \alpha_B)$

$L_G^* - L_G = \Pi^m(c_G + \alpha_B) - \Pi^m(c_B + \alpha_B) > 0$

$\Leftrightarrow L_G^* > L_G$

Problem Set 3

1. Moral hazard in farming

Symmetric Information

a) Symmetric Benchmark

Farmer: $U(w) = \sqrt{w} - \frac{1}{2}e^2 \geq 0$ (PC_F)

Participation constraint of farmer binds in optimum –> Proof by contradiction.

–> $\sqrt{w} - \frac{1}{2}e^2 = 0 \Leftrightarrow w = \frac{1}{4}e^4$

Landowner: $\max_e V = ey_G + (1-e)y_B - w$

s.t.: $w = \frac{1}{4}e^4$

$\Leftrightarrow \max_e ey_G + (1-e)y_B - \frac{1}{4}e^4$

FOC: $\frac{\partial V}{\partial e} = y_G - y_B - (e^*)^3 = 0$

$\Leftrightarrow e^* = (y_G - y_B)^{\frac{1}{3}}$

–> $w = \frac{1}{4}e^4 = \frac{1}{4}(y_G - y_B)^{\frac{4}{3}}$

–> $U = [\frac{1}{4}(y_G - y_B)^{\frac{4}{3}}]^{\frac{1}{2}} - \frac{1}{2}[(y_G - y_B)^{\frac{1}{3}}]^2 = 0$

–> $V = e(y_G - y_B) - \frac{1}{4}e^4 + y_B$

$\Leftrightarrow V = (y_G - y_B)^{\frac{1}{3}}(y_G - y_B) - \frac{1}{4}[(y_G - y_B)^{\frac{1}{3}}]^4 + y_B$

$\Leftrightarrow V = y_B + \frac{3}{4}(y_G - y_B)^{\frac{4}{3}}$

1st case: interior solution: $(y_G - y_B)^{\frac{1}{3}} < 1$
$e^* = (y_G - y_B)^{\frac{1}{3}}$, $w = \frac{1}{4}(y_G - y_B)^{\frac{4}{3}}$ $U = 0$, $V = y_B + \frac{3}{4}(y_G - y_B)^{\frac{4}{3}}$

2nd case: corner solution: $(y_G - y_B)^{\frac{1}{3}} \geq 1$

$e^* = 1$, $w = \frac{1}{4}$, $U = 0$, $V = y_G - y_B$

Symmetric Benchmark with alternative utility function from part (iv):

$U(w) = w - \frac{1}{2}e^2 \geq 0$ (PC_F)

$w = \frac{1}{2}e^2$

$\max_e ey_G + (1-e)y_B - \frac{1}{2}e^2$

FOC: $\frac{\partial V}{\partial e} = y_G - y_B - e^* = 0$

$\Leftrightarrow e^*_{(iv)} = y_G - y_B$

Asymmetric Information

b) case (i) take-it or leave-it wage employment contract

There is only one wage rate, as the landowner in case (i) cannot pay conditional on effort. As effort causes disutility the farmer will minimize effort to zero.

$U(w) = \sqrt{w} - \frac{1}{2}e^2 \geq 0$

$\frac{\partial U}{\partial e} = -e < 0 -> e^{(1)} = 0$

As the landowner wants to maximize net income, participation constraint of farmer binds. –> proof by contradiction.

$-> \sqrt{w} - \frac{1}{2}(e^{(1)})^2 = 0 \Leftrightarrow \sqrt{w} = 0 \Leftrightarrow w = 0$

The equilibrium has the following properties: $e^{(1)} = w = U = 0, \qquad V = y_B$

–> This basically replicates the complete market breakdown with asymmetric information.

case (ii) take-it or leave-it rental contract

$U(e, R) = e\sqrt{y_G - R} + (1-e)\sqrt{y_B - R} - \frac{1}{2}e^2 \geq 0$

FOC: $\frac{\partial U}{\partial e} = \sqrt{y_G - R} - \sqrt{y_B - R} - e^{(ii)} = 0$

$\Leftrightarrow e^{(ii)} = \sqrt{y_G - R} - \sqrt{y_B - R}$

–> limited liability constraint –>$y_G - R > y_B - R \geq 0$

$\Leftrightarrow y_B \geq R$

LLC has to bind in optimum –> proof by contradiction. Suppose not, then landowner can increase net income by raising R and all constraints would still hold. –> contradiction –> LLC has to bind.

–>$y_B = R$

–>$e^{(ii)} = \sqrt{y_G - y_B}$

$U(R) = \sqrt{y_G - y_B}\sqrt{y_G - y_B} - \frac{1}{2}(y_G - y_B) = \frac{1}{2}(y_G - y_B)$

$V = R = y_B$

1st case: interior solution: $\sqrt{y_G - y_B} < 1$
$e^{(ii)} = \sqrt{y_G - y_B}, \qquad R = y_B \qquad U = \frac{1}{2}(y_G - y_B), \qquad V = y_B$

2nd case: corner solution: $\sqrt{y_G - y_B} \geq 1$

$e^{(ii)} = 1, \qquad R = y_B, \qquad U = \sqrt{y_G - y_B} - \frac{1}{2}, \qquad V = y_B$

case (iii) take-it or leave-it sharecropping contract

The farmer keeps share β of the crop after harvest:

$U(e, R) = e\sqrt{1-\beta}(\sqrt{y_G} - \sqrt{y_B}) + \sqrt{1-\beta}\sqrt{y_B} - \frac{1}{2}e^2 \geq 0$

$\frac{\partial U}{\partial e} = \sqrt{1-\beta}(\sqrt{y_G} - \sqrt{y_B}) - e^{(iii)} = 0$

$\Leftrightarrow e^{(iii)} = \sqrt{1-\beta}(\sqrt{y_G} - \sqrt{y_B})$

define $A = \sqrt{y_G} - \sqrt{y_B}$ –>$e^{(iii)} = A\sqrt{1-\beta}$

and $B = (y_G - y_B)$

maximization of the landowner:

$V = e\beta(y_G - y_B) + \beta y_B$.

$V = \beta(\sqrt{1-\beta}AB + y_B)$

FOC: $\frac{\partial V}{\partial \beta} = (\sqrt{1-\beta}AB + y_B) + \beta\frac{1}{2}(1-\beta)^{-\frac{1}{2}}AB(-1) = 0$

$\Leftrightarrow \sqrt{1-\beta}AB + y_B = \frac{\beta AB}{2\sqrt{1-\beta}}$

Look at three possible solutions:

First case: $\beta = 0$

–> $e = 0, \qquad U = 0, \qquad V_{(i)} = y_B$

Second case: $\beta = 1$

–> $e = \min\{(\sqrt{y_G} - \sqrt{y_B}), \qquad 1\}$
$U = \min\{\frac{1}{2}(\sqrt{y_G} - \sqrt{y_B})^2 + \sqrt{y_B}, \qquad \sqrt{(1-\beta)y_G} - \frac{1}{2}\} \qquad V_{(ii)} = 0$

Second case is dominated by first case –> can exclude it.

Third case: $\beta \in (0, 1)$

–> $e^{(iii)} = \min\{\sqrt{1-\beta}(\sqrt{y_G} - \sqrt{y_B}), \qquad 1\}$

$U = e\sqrt{1-\beta}(\sqrt{y_G} - \sqrt{y_B}) + \sqrt{(1-\beta)y_B} - \frac{1}{2}e^2$

$\Leftrightarrow U = \min\{\frac{1}{2}(1-\beta)(\sqrt{y_G} - \sqrt{y_B})^2 + \sqrt{(1-\beta)y_B}, \qquad \sqrt{(1-\beta)}(\sqrt{y_G}) - \frac{1}{2}\}$

$V_{(iii)} = \min\{\sqrt{1-\beta}\beta(\sqrt{y_G} - \sqrt{y_B})(y_G - y_B) + \beta y_B, \qquad \beta y_G\}$

Solution for β: See Appendix

$1 > \beta > \frac{2}{3}$

$e^{(iii)} = \frac{\sqrt{y_B^2 + 3A^2B^2} - y_B}{3B} > 0$

Comparison of first case with third case: see Appendix

case (iv) take-it or leave-it offer (w; b) wage with extra payment employment contract.

Utility of farmer: $U = e(w + \gamma y_G) + (1-e)w = w + e\gamma y_G - \frac{1}{2}e^2$

$\frac{\partial U}{\partial e} = \gamma y_G - e = 0 \Leftrightarrow \gamma = \frac{e}{y_G}$

Maximization of the landowner:

$V = \max_{w,\gamma} e(y_G - w - \gamma y_G) + (1-e)(y_B - w)$

s.t. LLC_F, PC_F

$\Leftrightarrow V = \max_{w,e} e(1 - \frac{e}{y_G})y_G + (1-e)y_B - w$

$\Leftrightarrow V = \max_{w,e} ey_G - e^2 + (1-e)y_B - w$

FOC:

$\frac{\partial V}{\partial w} = -1 < 0$ –> set $LLC_F : w \geq 0$ –> set $w = 0$

$\frac{\partial V}{\partial e} = y_G - 2e^{(iv)} - y_B = 0$

$\Leftrightarrow e^{(iv)} = \frac{y_G - y_B}{2}$

$\gamma = \frac{e}{y_G} = \frac{y_G - y_B}{2y_G}$

Does this gamma fulfill the particpation constraint?

$U = e\gamma y_G - \frac{1}{2}e^2 \geq 0$

$\Leftrightarrow U = e\frac{e}{y_G}y_G - \frac{1}{2}e^2 = \frac{1}{2}e^2 = \frac{1}{2}(\frac{y_G - y_B}{2})^2 = \frac{1}{8}(y_G - y_B)^2 > 0$

$V = ey_G - e^2 + (1-e)y_B - w$

$\Leftrightarrow V = e(y_G - y_B) - e^2 + y_B$

$\Leftrightarrow V = \frac{y_G - y_B}{2}(y_G - y_B) - \frac{(y_G - y_B)^2}{4} + y_B$

$\Leftrightarrow V = \frac{1}{4}(y_G - y_B)^2 + y_B$

1st case: interior solution: $\frac{y_G - y_B}{2} < 1$

$e^{(iv)} = \frac{y_G - y_B}{2}, \qquad w = 0 \qquad \gamma = \frac{y_G - y_B}{2y_G},$

$U = \frac{1}{8}(y_G - y_B)^2, \qquad V = \frac{1}{4}(y_G - y_B)^2 + y_B$

–> effort below symmetric case optimum, because LLC sets limit to incentive payment scheme.

2nd case: corner solution: $\frac{y_G - y_B}{2} \geq 1$

$e^{(iv)} = 1, \qquad w = 0, \qquad U = \frac{1}{2}, \qquad V = y_G - 1$

c) Welfare comparison

Only look at interior solutions: $(y_G - y_B) < 1$

Symmetric Information: $e^* = (y_G - y_B)^{\frac{1}{3}}$, $\quad w = \frac{1}{4}(y_G - y_B)^{\frac{4}{3}}$ $\quad U = 0$, $\quad V = y_B + \frac{3}{4}(y_G - y_B)^{\frac{4}{3}}$

Asymmetric Information:

wage employment: complete market breakdown

(i) $e^{(1)} = w = U = 0$, $\quad V = y_B$

rental contract with LLC:

(ii) $e^{(ii)} = \sqrt{y_G - y_B}$, $\quad R = y_B$ $\quad U = \frac{1}{2}(y_G - y_B)$, $\quad V = y_B$

sharecropping agreement

(iii) $e^{(iii)} = \sqrt{\beta}(\sqrt{y_G} - \sqrt{y_B})$, $\quad \frac{2}{3} < \beta < 1$,

$U = \frac{1}{2}\beta(\sqrt{y_G} - \sqrt{y_B})^2 + \sqrt{\beta y_B}$

$V = \sqrt{\beta}(1-\beta)(\sqrt{y_G} - \sqrt{y_B})(y_G - y_B) + (1-\beta)y_B$

basic wage and extra payment

(iv) $e^{(iv)} = \frac{y_G - y_B}{2}$, $\quad w = 0$ $\quad \gamma = \frac{y_G - y_B}{2y_G}$,

$U = \frac{1}{8}(y_G - y_B)^2$, $\quad V = \frac{1}{4}(y_G - y_B)^2 + y_B$

Welfare:

Assume utalitarian welfare function with same weight for farmer and landowner –> $W = U + V$

$W^{sym} = y_B + \frac{3}{4}(y_G - y_B)^{\frac{4}{3}}$

$W^{(i)} = y_B$

$W^{(ii)} = \frac{1}{2}(y_G - y_B) + y_B$

$W^{(iii)} = \frac{1}{2}\beta(\sqrt{y_G} - \sqrt{y_B})^2 + \sqrt{\beta y_B} + \sqrt{\beta}(1-\beta)(\sqrt{y_G} - \sqrt{y_B})(y_G - y_B) + (1-\beta)y_B$

$\tilde{W}^{(iv)} = \frac{3}{8}(y_G - y_B)^2 + y_B$

We can see that $W^{(ii)} > W^{(i)}$

Further $W^{sym} > W^{(ii)} \Leftrightarrow (y_G - y_B) > \frac{8}{27}$

The other comparisons are more difficult. See the theoretical discussion.

Efforts:

$e^* = (y_G - y_B)^{\frac{1}{3}}$

$e^{(i)} = 0$

$e^{(ii)} = (y_G - y_B)^{\frac{1}{2}}$

$e^{(iii)} = \sqrt{\beta}(\sqrt{y_G} - \sqrt{y_B})$

$e^{(iv)} = \frac{(y_G - y_B)}{2} < e^*_{(iv)}$

We can see that $e^* > e^{(ii)} > e^{(i)}$

Some comments:

1. In the symmetric case all risk is taken by the landowner. Furthermore as effort is observable, the farmer has to make the optimal effort in order to receive the wage. This is the efficient benchmark.

2. In case (i) we have the complete market breakdown. Clearly this is the worst possible outcome.

3. In case (ii) the landlord gets the risk-free income, whereas the farmer gets the full risk. Even though effort is below optimal we can get a higher welfare here. This is due to the fact that farmers get a surplus now and they value income between 0 and 1 more than the landlord.

4. In case (iii) the risky and the risk-free income are shared equally between the landlord and the farmer. We expect the farmer to exert less effort than under the rental contract as now only a share of the additional revenue goes to him/her. On the other hand utility could still be higher for the farmer as uncertainty of income decreases.

5. In case (iv) we have an extra payment for the good outcome. Without LLC this should lead to the efficient effort.Yet the LLC limits the incentives the landowner can give to the farmer. Thus the farmer exerts less effort than would be efficient. In terms of welfare without LLC this should lead to the highest outcome under asymmetric information as the landowner takes the trade-off between incentives and risk-aversion of the farmer into account. Yet as the LLC limits the range of incentives it is possible that utility under the sharecropping agreement is higher

Appendix:

Solution for β:

substitute: $\sqrt{1-\beta} = x$

$\Leftrightarrow \beta = 1 - x^2$

$\Leftrightarrow 2x(xAB + y_B) = (1 - x^2)AB$

$3x^2AB = -2xy_B + AB$

$x^2 = -\frac{2}{3}\frac{y_B}{AB}x + \frac{1}{3}$

$x = -\frac{y_B}{3AB} + - \sqrt{(\frac{y_B}{3AB})^2 + \frac{1}{3}}$

$x = -\frac{y_B}{3AB} + - \frac{\sqrt{y_B^2 + 3A^2B^2}}{3AB}$

$x_1 = \frac{\sqrt{y_B^2 + 3A^2B^2} - y_B}{3AB}, \qquad x_2 < 0$

check sign: $\sqrt{y_B^2 + 3A^2B^2} - y_B > 0$?

$\Leftrightarrow \sqrt{y_B^2 - 3A^2B^2} > y_B$

$\Leftrightarrow 3A^2B^2 > 0$

check if $\sqrt{y_B^2 + 3A^2B^2} - y_B < 3AB$

$\Leftrightarrow y_B^2 + 3A^2B^2 < (3AB + y_B)^2$

$\Leftrightarrow 0 < 6A^2B^2 + 6ABy_B$

–> $x_1 \in (0, 1)$

Now $\beta = 1 - x^2$

–>$\beta \in (0, 1)$

$x_1 = \frac{\sqrt{y_B^2 + 3A^2B^2} - y_B}{3AB}$

$x_1^2 = \frac{2y_B^2 + 3A^2B^2 - 2y_B\sqrt{y_B^2 + 3A^2B^2}}{9A^2B^2}$

$\beta = 1 - x_1^2 = \frac{6A^2B^2 - 2y_B^2 + 2y_B\sqrt{y_B^2 + 3A^2B^2}}{9A^2B^2} = \frac{2}{3} + \frac{2y_B(\sqrt{y_B^2 + 3A^2B^2} - y_B)}{9A^2B^2} > \frac{2}{3}$

$e^{(iii)} = A\sqrt{1-\beta} = Ax_1 = \frac{\sqrt{y_B^2+3A^2B^2}-y_B}{3B} > 0$

Comparison of first case with third case:

$V_{(iii)} - V_{(i)} = \sqrt{1-\beta}\beta AB + \beta y_B - y_B > 0?$

$\Leftrightarrow \sqrt{1-\beta}\beta AB > (1-\beta)y_B$

$\Leftrightarrow \beta AB > \sqrt{1-\beta}y_B$

$AB(\frac{6A^2B^2+2y_B(\sqrt{y_B^2+3A^2B^2}-y_B)}{9A^2B^2}) > y_B x_1$

$\Leftrightarrow \frac{6A^2B^2+2y_B(\sqrt{y_B^2+3A^2B^2}-y_B)}{9AB} > y_B\frac{\sqrt{y_B^2+3A^2B^2}-y_B}{3AB}$

$\Leftrightarrow 6A^2B^2 + 2y_B(\sqrt{y_B^2+3A^2B^2} - y_B) > 3y_B\sqrt{y_B^2+3A^2B^2} - y_B^2$

$\Leftrightarrow 6A^2B^2 - y_B^2 > y_B\sqrt{y_B^2+3A^2B^2}$

$\Leftrightarrow 36A^4B^4 + y_B^4 + 12A^2B^2y_B^2 > y_B^2(y_B^2+3A^2B^2)$

$\Leftrightarrow 36A^4B^4 + 12A^2B^2y_B^2 > 3A^2B^2y_B^2$

$\Leftrightarrow 4A^2B^2 + y_B^2 > 0$

–>$V_{(iii)} > V_{(i)}$

Problem Set 4

1 Advertisement and Signaling

a) Definition of Perfect Bayesian Equilibrium

1. The firms ' pricing strategy is optimal given consumer's demand

$p_{i1}, p_{i2}(b_1(p_{i1}))$ –> maximizes firms profits

2. The consumer's belief that the good is high quality is calculated by Bayes ' Rule whenever possible

$\mu_1(p)$ –> Bayes Rule

3. The consumer maximizes utility given the pricing strategy of the firm and their believes.

$b_1(p_1), b_2(p_1, p_2)$

b) Perfect Bayesian Equilibria

Lemma 1: $b_1(p_1) = \left\{ \begin{array}{ccc} 1, & if & \mu_1(p)\theta_H + (1-\mu_1(p))\theta_L \geq p_1 \\ & 0, & otherwise \end{array} \right\}$ is the only NE in stage 3.

Lemma 2: Separating equilibrium. Conditional on price, consumer knows product quality.

$\mu_1(p_H) = 1, \mu_1(p_L) = 0$

Pooling: $\mu_1(p_1) = \alpha$, α: probability of new product being of good quality
1) Separating equilibrium:

$p_L \leq \theta_L < c_L \Leftrightarrow \Pi_L = p_L - c_L < 0$

–> separating equilibrium does not exist.

2) Pooling equilibrium:

$\mu_1(p_1) = \alpha, \qquad E(\theta) = \alpha\theta_H + (1-\alpha)\theta_L$

1) $E(\theta) \geq c_L \qquad (PC_L)$

2) $E(\theta) + \delta\theta_H \geq c_H(1+\delta) \qquad (PC_H)$

–> assume that indifferent consumer buys the product.

$\Leftrightarrow E(\theta) \geq c_H(1+\delta) - \delta\theta_H$

$p_1 \in [\min\{c_L, c_H(1+\delta) - \delta\theta_H\}, E(\theta)]$

$\Pi_H^P = E(\theta) + \delta\theta_H - c_H(1+\delta)$

3) Excluding equilibrium:

only low quality: not possible as $p_L - c_L < 0$

only high quality: $p_1 \leq c_L, \qquad EC \qquad$ 'Exclusion constraint '

–> assume that indifferent low quality firm does not participate.

$\Pi_H^E = p_1 + \delta\theta_H - c_H(1+\delta)$

Compare profits for high quality of the two candidates:

$\Pi_H^P - \Pi_H^E = E(\theta) + \delta\theta_H - c_H(1+\delta) - (p_1 + \delta\theta_H - c_H(1+\delta))$

$\Pi_H^P - \Pi_H^E = E(\theta) - p_1 = \alpha\theta_H + (1-\alpha)\theta_L - p_1$

$\Leftrightarrow \Pi_H^P - \Pi_H^E > \theta_L - p_1 > 0$

–> The only PBE are the pooling equilibria.

c) Perfect Bayesian Equilibria with signaling

Advertising allows for a new excluding equilibrium.

1) exclude low quality firm: $\Pi_L = p_1 - c_L - A \leq 0$

$\Leftrightarrow p_1 \leq c_L + A$

2) $PC_C : p_1 \leq \theta_H$

3) non-negative profit for high quality firm:

$\Pi_H = p_1 + \delta\theta_H - c_H(1+\delta) - A \geq 0$

$\Leftrightarrow p_1 \geq c_H(1+\delta) + A - \delta\theta_H$

–> $p_1 \in [c_H(1+\delta) + A - \delta\theta_H, \min\{\theta_H, c_L + A\}]$

1 Cheap talk in business

a) babbling equilibrium

Juniors: $U_i(a_i, e_i) = a_i e_i - \frac{1}{2}e_i^2$

FOC: $\frac{\partial U_i}{\partial e_i} = a_i - e_i \Leftrightarrow e_i^* = a_i$

Manager: $W(e_1, e_2, a_1, a_2) = a_1 e_1 + a_2 e_2$

$\frac{\partial W}{\partial e_i} = a_i > 0$ –> corner solution –> manager always wants effort level 1.

Juniors believe that their ability is 1/2 independently of the manager's announcement. They always take effort 1/2. The manager earns $1/2(a_1 + a_2)$. This is a PBE.

b) no information transmission, one junior

Assume the manager communicates some information about a and the junior believes it. The manager will say that the junior is of the highest ability possible ($a = 1$ under full transmission) irrespective of the junior's ability. The junior sees through the manager's incentive and does not use the manager's signal to update her belief.

c) no full information transmission, two juniors

Suppose the juniors believe in full information transmission. Then principal would have an incentive to deviate and always tell $a_1 = a_2 = 1$. This contradicts original believes. –> Full information transmission no equilibrium.

d) if juniors believe the manager

Here we use order statistics. From statistics we know that

$F(a < a') = 2(1 - F(a))F(a) + F(a)^2$

here $F(a) = y$

–> $F(a < a') = 2(1 - y)y + y^2 = 2y - y^2$

–> $f(a < a') = 2 - 2y$

–> $E(a \mid a < a') = \int_0^1 y(2 - 2y)dy = [y^2 - \frac{2}{3}y^3]_0^1 = \frac{1}{3}$

Similarily: $F(a > a') = F(a)^2$

here $F(a) = y$

–> $F(a > a') = y^2$

–> $f(a > a') = 2y$

–> $E(a \mid a > a') = \int\limits_0^1 y(2y)dy = [\frac{2}{3}y^3]_0^1 = \frac{2}{3}$

Thus the high agent believes that her ability is $E(a \mid a > a') = 2/3$ and the low one $E(a \mid a < a') = 1/3$. Under a tie, both agents believe that their ability is $E(a \mid a = a') = 1/2$. The high agent takes action $e^H = 2/3$, the low one $e^L = 1/3$, and equally good agents take actions $e^T = 1/2$.

e) equilibrium with truth revelation

Consider the case such that $a > a'$. Under truth telling the principal earns $e^H a + e^L a'$. Under deviation where the principal inverses the order, she earns $e^H a' + e^L a$. Under deviation where she claims a tie she earns $e^T(a + a')$. Complementarity between effort and action implies that the first deviation is dominated

$$e^H a + e^L a' = (e^H - e^L)(a + a') + e^H a' + e^L a.$$

The second deviation is dominated because

$$e^H a + e^L a' = e^T(a+a') + (e^H - e^T)a + (e^L - e^T)a' = e^T(a+a') + (e^H - e^T)(a - a') > e^T(a+a').$$

Under a tie, the principal earns $2e^T a$. By deviating the principal would earn $(e^H + e^L)a$ and she is indifferent.

f) General Proof

Under truth-telling, the high agent takes action $e^H = E(a \mid a > a')$. We have

$$EMax(a, a') = E(a \mid a > a')\Pr(a > a') + E(a' \mid a < a')\Pr(a < a')$$

But $\Pr(a > a') = 1/2$ and $EMax(a, a') = \int_0^1 a2F(a)f(a)da$ implies

$$e^H = E(a \mid a > a') = \int_0^1 a2F(a)f(a)da.$$

Using the same argument, the low agent takes action $e^L = \int_0^1 a2(1-F(a))f(a)da$, and the equally talented agents $e^M = \int_0^1 af(a)da$. We have $e^H - e^M = e^M - e^L$ independently of f. This identidy is all we need to prove equilibrium existence in 4.

Appendix Derivation of general ordered statistic for maximum.

$F(a > a') = F(a)^2$

–> $f(a > a') = 2F(a)f(a)$

–> $E(a \mid a > a') = \int_0^1 a2F(a)f(a)da$