Microeconomics III	Instructor: Prof. Pascal Courty

Problem Set 1

Deadline: January 14, 2009	*TA session: Michal Lewandowski*

1 Expected Utility Theorem

Let X be a set of alternatives. The objects of choice are lotteries with **finite support**:

$$L = \left\{ P : X \to [0,1] \,\middle|\, \#\{x|P(x) > 0\} < \infty,\ \sum_{x\in X} P(x) = 1 \right\}$$

Notice that $\sum_{x\in X} P(x) = 1$ condition is well defined due to the finite support assumption. The observable choices are modeled by a binary relation on L, written $\succsim \subset L \times L$.

Axiom (Weak order). *$\succsim$ is complete and transitive*

Axiom (Continuity). *For every $P, Q, R \in L$,*

$$P \succ Q \succ R \ \Rightarrow\ \exists \alpha, \beta \in (0,1) :\ \alpha P + (1-\alpha)R \succ Q \succ \beta P + (1-\beta)R$$

Axiom (Independence). *For every $P, Q, R \in L$, and every $\alpha \in (0,1)$,*

$$P \succsim Q \quad \Longleftrightarrow \quad \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R$$

Theorem 1 (vNM). *$\succsim \subset L \times L$ satisfies Axioms 1-3 if and only if there exists $u : X \to \mathbb{R}$ such that, for every $P, Q \in L$*

$$P \succsim Q \quad \Longleftrightarrow \quad \sum_{x\in X} P(x)u(x) \geq \sum_{x\in X} Q(x)u(x)$$

Moreover, in this case u is unique up to a positive linear transformation.

In what follows, **you are asked to prove sufficiency of the first part of this theorem.** Necessity is easy to check and is omitted. Second part (uniqueness) is also omitted.
Please follow steps outlined in this problem set and do not change notation. To save your time you can use the following lemmas without giving proofs:

Lemma 1. *For every $P, Q, R \in L$, and every $\alpha \in (0,1)$,*

$$P \succ Q \quad \Longleftrightarrow \quad \alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R \tag{1}$$

$$P \sim Q \quad \Longleftrightarrow \quad \alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)R \tag{2}$$

Lemma 2. *For any $P, Q \in L$ and every $\alpha \in [0,1]$,*

$$P \succsim Q \Rightarrow P \succsim \alpha P + (1-\alpha)Q \succsim Q \tag{3}$$

The same holds for strict preference and for indifference.

PROVE THE FOLLOWING LEMMAS AND INTERMEDIARY RESULTS

Problem 1.1

Lemma 3. *For every* $P, Q \in L$ *and every* $\alpha, \beta \in [0, 1]$,

$$P \succ Q \Rightarrow (\alpha P + (1-\alpha)Q \succsim \beta P + (1-\beta)Q \iff \alpha \geq \beta) \tag{4}$$

Solution:

We assume $P \succ Q$ and want to prove:

$$\alpha P + (1-\alpha)Q \succsim \beta P + (1-\beta)Q \iff \alpha \geq \beta \tag{5}$$

Let's start with ($\Longleftarrow$).
It is easy to verify that in case any of the following occurs: $\alpha = 0$, $\alpha = 1$, $\beta = 0$, $\beta = 1$ or $\alpha = \beta$ the claim reduces to triviality or to condition (3) from the previous lemma. Assume then $1 > \alpha > \beta > 0$. Now define $\gamma = \frac{\beta}{\alpha}$. Note that $\gamma \in (0, 1)$. Now we can write the following:

$$\begin{aligned}
\beta P + (1-\beta)Q &= \gamma\alpha P + (1-\gamma\alpha)Q \\
&= \gamma(\alpha P + (1-\alpha)Q) - \gamma(1-\alpha)Q + (1-\gamma\alpha)Q \\
&= \gamma(\alpha P + (1-\alpha)Q) + (1-\gamma)Q
\end{aligned}$$

If we denote $R = \alpha P + (1-\alpha)Q$ then we can write:

$$\beta P + (1-\beta)Q = \gamma R + (1-\gamma)Q \tag{6}$$

From assumption that $P \succ Q$ and by lemma 2 we have:

$$P \succ Q \Rightarrow P \succsim Q \Rightarrow \alpha P + (1-\alpha)Q \succsim Q$$

Using our definition of R we can write $R \succsim Q$. Given that let's apply lemma 2 once more to obtain:

$$R \succsim Q \Rightarrow R \succsim \gamma R + (1-\gamma)Q$$

Substituting equation (6) for the RHS and $R = \alpha P + (1-\alpha)Q$ for the LHS we get:

$$\alpha P + (1-\alpha)Q \succsim \beta P + (1-\beta)Q$$

This finishes the proof of one direction. Let's now prove the converse direction, i.e. ($\Longrightarrow$) of (5)
The contrapositive of the statement to prove is:

$$\neg(\alpha \geq \beta) \Rightarrow \neg(\alpha P + (1-\alpha)Q \succsim \beta P + (1-\beta)Q)$$

This is equivalent to:

$$\alpha < \beta \Rightarrow \beta P + (1-\beta)Q \succ \alpha P + (1-\alpha Q$$

Notice that this is very similar to the statement which we already proved above. The only difference is the switched role of α and β and strict preference instead of weak preference. This is where we use the strict preference version of equation (3). This finishes the proof.

Problem 1.2

Step 1. *Assume that there are worst and best outcomes in X denoted x_* and x^*, respectively. For every $P \in L$ there exists a unique $\alpha_P \in [0,1]$ such that:*

$$P \sim \alpha_P x^* + (1-\alpha_P)x_*$$

where x^* and x_* are treated as degenerate lotteries.

Solution:

It is easy to verify that condition (4) from lemma 3 holds for strict inequality and strict preference. Suppose we have three lotteries such that $P \succ Q \succ R$. By continuity we know that there exist $\alpha, \beta \in (0,1)$ such that the following holds:

$$\alpha P + (1-\alpha)R \succ Q \succ \beta P + (1-\beta)R$$

. From lemma 3 and transitivity we know that it must be that $\alpha > \beta$. Again by continuity, transitivity and lemma 3 we know that there exists $\alpha_1, \beta_1 \in (0,1)$ such that $\alpha_1 < \alpha$, $\beta_1 > \beta$ and:

$$\alpha_1 P + (1-\alpha_1)R \succ Q \succ \beta_1 P + (1-\beta_1)R$$

If we continue in this way we can construct a decreasing sequence $(\alpha_i)_{i=1,\dots,n}$ and an increasing sequence $(\beta_i)_{i=1,\dots,n}$ such that:

$$P \succ \alpha_i P+(1-\alpha_i)R \succ \alpha_{i+1}P+(1-\alpha_{i+1})R \succ Q \succ \beta_{i+1}P+(1-\beta_{i+1})R \succ \beta_i P+(1-\beta_i)R \succ R$$

It is easy to see now that the limits of two constructed sequences must be the same:

$$\lim_{n\to\infty} \alpha_n = \alpha_Q = \lim_{n\to\infty} \beta_n$$

and $\alpha_Q P + (1-\alpha_Q)R \sim Q$. If we assume now that $x^* \succ Q \succ x_*$, the result follows immediately (just replace P with x_*, R with x_* and Q with P). If $x^* \sim Q \succ x_*$, then $\alpha_Q = 1$, if $x^* \succ Q \sim x_*$, then $\alpha_Q = 0$ and finally if $x^* \sim x_*$, then $\sim = L \times L$ and a constant u does the job. Uniqueness of α_Q follows from lemma 3.
Notice that this proof also shows that the sets:

$$\{\alpha \in [0,1] | \alpha P + (1-\alpha)R \succ Q\}$$
$$\{\alpha \in [0,1] | Q \succ \alpha P + (1-\alpha)R\}$$

are open or alternatively that the sets:

$$\{\alpha \in [0,1] | \alpha P + (1-\alpha)R \succsim Q\}$$
$$\{\alpha \in [0,1] | Q \succsim \alpha P + (1-\alpha)R\}$$

are closed which are alternative definitions of continuity.

Problem 1.3

Step 2. *Consider a lottery* $P \equiv (x_1, p_1; x_2, p_2; ...; x_n, p_n)$*, where* $P(x_i) \equiv p_i > 0$*. Index* α_P *such that* $P \sim \alpha_P x^* + (1 - \alpha_P)x_*$ *is linear in probabilities* p_i *i.e. it can e written as* $\alpha_P = \sum_{i=1}^n p_i \alpha_{x_i}$*, for some scalars* α_{x_i}*.*

Hint: Eliminate step by step each outcome of the lottery replacing it with the mixture of x^* and x_*.

Solution:

Construct a sequence of lotteries $P_i, \ i = 1, ..., n$ such that:

- $P_0 = P$
- $P_i \sim P_{i-1}$
- $supp(P_i) = \{x_*, x^*, x_{i+1}, ..., x_n\}$

Observe that

$$P \quad = \quad p_1 x_1 + (1 - p_1) \left(x_2, \frac{p_2}{1 - p_1}; ...; x_n, \frac{p_n}{1 - p_1} \right)$$

We can now replace x_1 by $\alpha_{x_1} x^* + (1 - \alpha_{x_1}) x_*$, where $x_1 \sim \alpha_{x_1} x^* + (1 - \alpha_{x_1}) x_*$. We obtain:

$$P_1 = p_1(\alpha_{x_1} x^* + (1 - \alpha_{x_1}) x_*) + p_2 x_2 + ... + p_n x_n$$

We can write is as:

$$P_1 \quad = \quad p_2 x_2 + (1 - p_2) \left(x^*, \frac{p_1 \alpha_{x_1}}{1 - p_2}; x_*, \frac{p_1(1 - \alpha_{x_1})}{1 - p_2}; x_3, \frac{p_3}{1 - p_2}; ...; x_n, \frac{p_n}{1 - p_1} \right)$$

Again we can replace x_2 by $\alpha_{x_2} x^* + (1 - \alpha_{x_2}) x_*$, where $x_2 \sim \alpha_{x_2} x^* + (1 - \alpha_{x_2}) x_*$. If we continue this way, at the end we will end up with the following lottery:

$$P_n = \left(x^*, \sum_{i=1}^n p_i \alpha_{x_i}, x_*, \sum_{i=1}^n p_i (1 - \alpha_{x_i}) \right)$$

But notice that $\sum_{i=1}^n p_i (1 - \alpha_{x_i}) = \sum_{i=1}^n p_i - \sum_{i=1}^n p_i \alpha_{x_i}) = 1 - \sum_{i=1}^n p_i \alpha_{x_i})$. So

$$P_n = \left(x^*, \sum_{i=1}^n p_i \alpha_{x_i}, x_*, 1 - \sum_{i=1}^n p_i \alpha_{x_i}) \right)$$

This shows that $\alpha_P = \sum_{i=1}^n p_i \alpha_{x_i}$ is indeed linear in probabilities.

In principle the proof is finished since we can define $u(x_i) = \alpha_{x_i}$ and $U(P) = \alpha_P$ and then from the above step it follows that the utility from lottery P may be written as $U(P) = \sum_{i=1}^n p_i u(x_i)$. For the sake of completeness, below I give the last step, WHICH YOU DO NOT HAVE TO PROVE.

Step 3 (Final). *$\succsim \subset L \times L$ satisfies Axioms 1-3. Then there exists $u : X \to \mathbb{R}$ such that for any lotteries $P, Q \in L$:*

$$P \succsim Q \iff \sum_{x \in X} P(x)u(x) \geq \sum_{x \in X} Q(x)u(x)$$

Consider any two lotteries: $P \equiv (x_1, p_1; x_2, p_2; ...; x_n, p_n)$ and $Q \equiv (y_1, q_1; y_2, q_2; ...; y_m, q_m)$, such that $x^* \succ x_i, y_j \succ x_*,\ i = 1, ..., n\ j = 1, ..., m$. We know that $P \sim P_n$ and $Q \sim Q_m$, where P_n and Q_m are constructed as in the last step:

$$P_n = \left(x^*, \sum_{i=1}^{n} p_i \alpha_{x_i}, x_*, 1 - \sum_{i=1}^{n} p_i \alpha_{x_i}) \right)$$

$$Q_m = \left(x^*, \sum_{i=1}^{m} q_i \alpha_{y_i}, x_*, 1 - \sum_{i=1}^{m} q_i \alpha_{y_i}) \right)$$

Define $\alpha_x = u(x)$. By construction $P \succsim Q$ is equivalent to:

$$\left(x^*, \sum_{i=1}^{n} p_i u(x_i), x_*, \sum_{i=1}^{n} p_i (1 - u(x_i)) \right) \succsim \left(x^*, \sum_{i=1}^{m} q_i u(y_i), x_*, 1 - \sum_{i=1}^{m} q_i u(y_i)) \right)$$

By lemma 3 since $x^* \succ x_*$ the above is in turn equivalent to:

$$\sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{m} q_i u(y_i)$$

2 Risk attitudes

PROVE THE FOLLOWING CLAIMS AND PROPOSITIONS:

Problem 2.1

Proposition 2.1. *All CARA utility function with $U(0) = 0$ and $U'(0) = 1$ are of the following form: $U_a(x) = \frac{1}{a} - \frac{1}{a}e^{-ax},\ a > 0$. Show that risk neutrality occurs for $a \to 0^+$*

Solution:

We have to solve the differential equation: $-\frac{U''(x)}{U'(x)} = a > 0$, where $a > 0$ is a constant representing absolute risk aversion. We have two initial conditions which makes it a Cauchy problem. Integrate the equation once from 0 to x to obtain $\log U'(x) - \log U'(0) = -ax + 0$

or by using the fact that $U'(0) = 1$, $\log U'(x) = -ax$. Rewrite as $U'(x) = e^{-ax}$. Integrate it once more from 0 to x to obtain $U(x) - U(0) = -\frac{1}{a}e^{-ax} + \frac{1}{a}$ or by using another initial condition $U(0) = 0$, $U(x) = \frac{1}{a} - \frac{1}{a}e^{-ax}$
Now we calculate the limit as a goes to zero. Note that we can use L'Hospital rule:

$$\lim_{a \to 0^+} \frac{1 - e^{-ax}}{a} = \lim_{a \to 0^+} \frac{xe^{-ax}}{1} = x$$

Problem 2.2

Proposition 2.2. *Given a twice continuously differentiable utility function U, the following holds:*

$$ARA(W) = \lim_{h \to 0^+} \frac{4}{h}\left(p(W,h) - \frac{1}{2}\right) \tag{7}$$

where $ARA(W) = -\frac{U''(W)}{U'(W)}$ and $p(W,h)$ is a probability premium defined implicitly by:

$$p(W,h)U(W+h) + (1 - p(W,h))U(W-h) = U(W) \tag{8}$$

Moreover, for $h = \lambda W$, we have:

$$RRA(W) = \lim_{\lambda \to 0^+} \frac{4}{\lambda}\left(p(W,\lambda W) - \frac{1}{2}\right) \tag{9}$$

Hint: Use second order Taylor approximation of U.

Solution:

Rewriting equation (8) by using second order Taylor expansion of U around W, we obtain for small h:

$$\begin{aligned} &p(W,h)[U(W) + U'(W)h + \frac{1}{2}U''(W)h^2] \\ &+(1 - p(W,h))[U(W) - U'(W)h + \frac{1}{2}U''(W)h^2] \approx U(W) \end{aligned}$$

And after simplifying:

$$\frac{1}{2}U''(W)h + U'(W)(2p(W,h) - 1) \approx 0$$

Or

$$ARA(W) = \lim_{h \to 0^+} \frac{4}{h}\left(p(W,h) - \frac{1}{2}\right)$$

as was to be shown. If we set $h = \lambda W$, equation (9) immediately follows.

Problem 2.3

A gamble is a real-valued random variable with finite support. We can denote it by $\mathbf{x} \equiv (x_1, p_1; ...; x_n, p_n)$, where x_i denotes monetary values and p_i the corresponding probabilities, $i = 1, ..., n$. In what follows we will implicitly assume that the arguments of utility function belong to the domain. For instance if the domain of U is $(0, +\infty)$, then $W + x \in (0, +\infty)$. Assume that we are in the world of Expected Utility, i.e. there exists a vNM utility function U that represents the decision maker's preferences.

Definition 1. *A strategy whether to accept a gamble* $\mathbf{x}$ *or not is called wealth-invariant iff the following holds:*

$$\mathbf{E}U(W_1 + \mathbf{x}) \geq U(W_1) \iff \mathbf{E}U(W_2 + \mathbf{x}) \geq U(W_2), \quad \forall W_1, W_2 \tag{10}$$

Prove the following proposition:

Proposition 2.3. *If strategy is wealth-invariant then utility is CARA.*

Remark: It is easy to check that the converse direction is also true.

Solution:

Given utility function U, consider two lotteries $\mathbf{x_i} \equiv (h, p(W_i, h); -h, 1-p(W_i, h))$, where $W_i, h > 0,\ i \in \{1, 2\}$, such that:

$$\mathbf{E}U(W_i + \mathbf{x_i}) = U(W_i) \tag{11}$$

Contrary to what we are to show, assume that $ARA(W_1) > ARA(W_2)$. By proposition 2.2 equation (7), for h sufficiently small we know that $p(W_1, h) > p(W_2, h)$. Let q be between the two probability premiums: $p(W_1, h) > q > p(W_2, h)$. I define another lottery $\mathbf{y} \equiv (h, q; -h, 1 - q)$. By definition of a probability premium utility function U rejects lottery $\mathbf{y}$ at wealth W_1 and accepts it at wealth W_2, which contradicts wealth invariance.

Problem 2.4

Definition 2. *A strategy whether to accept a gamble* $\mathbf{x}$ *or not is called homogeneous (or scale-invariant) iff the following holds:*

$$\mathbf{E}U(W + \mathbf{x}) \geq U(W) \iff \mathbf{E}U(\lambda W + \lambda \mathbf{x}) \geq U(\lambda W), \quad \forall \lambda > 0 \tag{12}$$

Prove the following proposition:

Proposition 2.4. *If strategy is homogeneous then utility is CRRA.*

Remark: It is easy to check that the converse direction is also true.

Solution:

We will be using proposition 2.2, equation (9). In order to avoid notational problem I will rewrite this equation using symbol θ instead of λ, since λ is used in the definition of a homogeneous strategy (see equation (12)). So we will use the following equation:

$$RRA(W) = \lim_{\theta \to 0^+} \frac{4}{\theta} \left(p(W, \theta W) - \frac{1}{2} \right) \tag{13}$$

where $p(W, \theta W)U(W + \theta W) + (1 - p(W, \theta W))U(W - \theta W) = U(W)$.
Given utility function U, consider lottery $\mathbf{x} \equiv (\theta W, p(W, \theta W); -\theta W, 1 - p(W, \theta W))$, where $W, \theta > 0$, such that:

$$\mathbf{E}U(W + \mathbf{x}) = U(W) \tag{14}$$

Contrary to what we are to show, assume that $RRA(\lambda W) > RRA(W)$, for some $\lambda > 0$ and $\lambda \neq 1$. Define $\mathbf{x}' \equiv (\theta W, p(\lambda W, \lambda\theta W); -\theta W, 1 - p(\lambda W, \lambda\theta W))$, such that:

$$\mathbf{E}U(\lambda W + \lambda \mathbf{x}') = U(\lambda W) \tag{15}$$

For θ sufficiently small, by equation (13), we know that: $p(\lambda W, \lambda\theta W) > p(W, \theta W)$. Therefore, we can find q such that: $p(\lambda W, \lambda\theta W) > q > p(W, \theta W)$. Define $\mathbf{y} \equiv (\theta W, q; -\theta W, (1 - q))$.
By equation (15), since $p(\lambda W, \lambda\theta W) > q$:

$$\mathbf{E}U(\lambda W + \lambda \mathbf{y}) < U(\lambda W)$$

By equation (14), and since $q > p(W, \theta W)$, we have:

$$\mathbf{E}U(W + \mathbf{y}) > U(W)$$

which is a contradiction.

Micro III 2009
The Problem Sets

Oege Dijk

February 11, 2009

Problem set 2: Moral Hazard
Deadline: 21/01/09 11:00am (At the end of the lecture)
TA: Oege Dijk, office SP019

Question One: Presidential tastes.

Barack just became the new head of a decently sized Northern American Country. Barack likes to eat moose.

Moose is like the fruit of the arctic. You can barbecue it, boil it, broil it, bake it, saute it. There's moose-kabobs, moose creole, moose gumbo. Pan fried, deep fried, stir-fried. There's pineapple moose, lemon moose, coconut moose, pepper moose, moose soup, moose stew, moose salad, moose and potatoes, moose burger, moose sandwich. And that- that's about it.

But because he is a busy man, Barack has to hire someone else to do his moose hunting for him. The person he thinks is best for this job is Sarah. Sarah stems from Alaska and has a lot of experience with hunting large game. However there is a problem because Sarah likes shopping more than hunting for moose. In order to convince Sarah to still do his hunting for him, Barack has to come up with a way to give her some incentives.

The only thing Barack can observe after a hunting trip is the number of moose that Sarah brought back. However, Sarah might be either lucky or unlucky in her hunting and so her catch m depends both on her hunting effort e and some noise: $m = e + \varepsilon_m$ where $\varepsilon_m \sim N(0, \sigma_m^2)$.

Assume Sarah has a cost of effort $c(e) = \frac{ce^2}{2}$, utility function $u_S(e, w) = -\exp[-r(w - c(e))]$ and a reservation utility $\underline{u}$. Barack's utility is given by $u_B(m, w) = E(m - w)$, where w is the amount he pays to Sarah.

1

a) Under a specification almost the same to the above one can prove that a linear contract form will be efficient. Solve for the optimal symmetric information contract of the form $w = \beta_0 + \beta m$.(of

interest may be the fact that for $x \sim N(\mu, \sigma^2)$, $E[\exp(x)] = \exp[\mu + \frac{\sigma^2}{2}]$. The latter is called the certainty equivalent)

b) Now given asymmetric information, when Barack gives Sarah a contract of the form $w = \beta_0 + \beta m$, what is the amount of effort that Sarah will put in?

b) What is the optimal contract (β_0, β) under asymmetric informationthat Barack will offer to Sarah?

c) Interpret the results.

SOLUTION

1 a)

In the asymmetric optimum, Barack chooses the optimal level of effort and then choose a compensation schedule that will bind Sarah's Participation Constraint (PC). Proof: suppose otherwise, that a (β_0^*, β^*) maximizes Barack's objective function while leaving Sarah with some $u_S > \underline{u}$. Then Barack could lower either β_0 or β by an ε still ensuring participation by Sarah, while increasing his payoff. This is in contradiction with (β_0^*, β^*) being optimal. Thus in the optimal $u_S = \underline{u}$.Since the utility function depends on m and m is a stochastic variable, we need to look at expected utility. Here we make use of the standard result that for a normally distributed $x \sim N(\mu, \sigma^2)$, $E[\exp(x)] = \exp[\mu + \frac{\sigma^2}{2}]$. Thus Barack solves the optimization programme:

$$\max_{\{e, \beta_0, \beta\}} E[m - \beta_0 - \beta m]$$
s.t. $E[u_S(e, w)]$
$= E[-\exp[-r(\beta_0 + \beta e + \beta\varepsilon_m - \frac{ce^2}{2})]]$
$= -\exp[-r(\beta_0 + \beta e - \frac{ce^2}{2}) + \frac{r^2\beta^2\sigma_m^2}{2}] = \underline{u}$

We will make our live easier by taking the log of both sides of the PC. Now we can set up the lagrangian:

$L = e - \beta_0 - \beta e + \lambda[-r(\beta_0 + \beta e - \frac{ce^2}{2}) + \frac{r^2\beta^2\sigma_m^2}{2} - \log(-\underline{u})]$
FOC:
$\beta_0 : -1 - \lambda r = 0 => \lambda = -\frac{1}{r}$
$\beta : -e - \lambda re + r^2\beta\sigma_m^2 = 0$
$=> \beta = 0$
$e : 1 - \beta + \lambda(-r\beta + rce) = 0$
$=> 1 - \beta + \beta - ce = 0$
$=> e = \frac{1}{c}$

Hence Barack will pay Sarah a fixed wage β_0 and require an effort equal to $\frac{1}{c}$. This is the efficient outcome because whereas Barack is risk neutral, Sarah is risk averse. Thus Barack takes all the risk upon himself and fully insures Sarah who gets utility $-\exp[-r(\beta_0 - \frac{r}{2c})] = \underline{u}$. So finally calculating $\beta_0 = \frac{-\log(-\underline{u})}{r} + \frac{1}{2c}$.

b) Now given asymmetric information, when Barack gives Sarah a contract of the form $w = \beta_0 + \beta m$, what is the amount of effort that Sarah will put in?

This time we(Where we equals Barack. Yes We Can!) do not have symmetric information, and thus cannot verify the effort that Sarah is putting in. Hence we need to analyze what effort Sarah would put in given a contract of the form $w = \beta_0 + \beta m$. First we can try the first best $(\beta_0, \beta) = (\frac{-\log(-\underline{u})}{r} + \frac{r^2}{2c}, 0)$. Sarah will maximize:

$$\max_e E[u_S(e,w)] = -\exp[-r(\beta_0 + \beta e - \frac{ce^2}{2}) + \frac{r^2\beta^2\sigma_m^2}{2}] = -\exp[-r(\beta_0 - \frac{ce^2}{2})]$$

Because of the negative exponential this utility is maximized when the argument is minimized, so we can write the problem as:

$\max_e r(\beta_0 - \frac{ce^2}{2})$
FOC e: $-ce = 0$
$=> e = 0$.

Because Sarah is hunting-averse, and extra effort into hunting will not result in extra shopping money, she will put in zero effort. However clearly this is not optimal for moose-loving Barack. Setting a positive β would make Sarah partly dependent on the amount of moose she catches, and thus induce her to start hunting:
$\max_e E[u_S(e,w)] = -\exp[-r(\beta_0 + \beta e - \frac{ce^2}{2}) + \frac{r^2\beta^2\sigma_m^2}{2}]$

Which is again equivalent to:
$\max_e r(\beta_0 + \beta e - \frac{ce^2}{2}) - \frac{r^2\beta^2\sigma_m^2}{2}$

FOC: $r\beta - rce = 0$
$=> e = \frac{\beta}{c}$

The higher the per-moose-payoff the more effort Sarah puts in, and as effort becomes more costly she will put in less effort. Note though that the term $-\frac{r^2\beta^2\sigma_m^2}{2}$ induces a welfare cost to the increased risk that Sarah is taking on.

c) What is the optimal contract (β_0, β) that Barack will offer to Sarah?

An optimal contract achieves two goals: It induces Sarah to put in effort and minimizes the cost of doing so.

Barack solves an optimization problem including a Participation Constraint(PC) and an incentive compatability constraint (IC). The latter we already know from the previous exercise: $e = \frac{\beta}{c}$. Thus the problem becomes:

$\max E[m - (\beta_0 + \beta m)]$
s.t.
$- \exp[-r(\beta_0 + \beta e - \frac{ce^2}{2}) + \frac{r^2\beta^2\sigma_m^2}{2}] \geq \underline{u}$ (PC)
$e = \frac{\beta}{c}$(IC)

Again the PC binds. Now the trick is to find an expression for $E(\beta_0 + \beta m)$ from the PC and plug this into the objective function:

$$- \exp[-r(\beta_0 + \beta e - \frac{ce^2}{2}) + \frac{r^2\beta^2\sigma_m^2}{2}] = \underline{u}$$
$$-r(\beta_0 + \beta e - \frac{ce^2}{2}) + \frac{r^2\beta^2\sigma_m^2}{2} = \log(-\underline{u})$$
$$\beta_0 + \beta e = \frac{ce^2}{2} + \frac{r\beta^2\sigma_m^2}{2} + \log(-\underline{u}) = \frac{ce^2}{2} + \frac{r\beta^2\sigma_m^2}{2} + k$$

Now we can plug in $e = \frac{\beta}{c}$ and $\beta_0 + \beta e = \frac{ce^2}{2} + \frac{r\beta^2\sigma_m^2}{2} + k$ in the objective function to get:

$$\max_{\beta_0, \beta} E[\frac{\beta}{c} - (\frac{c(\frac{\beta}{c})^2}{2} + \frac{r\beta^2\sigma_m^2}{2} + k)] = \frac{\beta}{c} - \frac{\beta^2}{2c} - \frac{r\beta^2\sigma_m^2}{2} - k$$

FOC: $\frac{1}{c} - \frac{\beta}{c} - r\beta\sigma_m^2 = 0$
$=> \beta = \frac{1}{1+rc\sigma_m^2}$

Some interpretation:

$0 < \beta \leq 1$: You never pay more for the moose than what it's worth to you
$\frac{\partial \beta}{\partial r} < 0$: More risk averse agents face less powerful incentives
$\frac{\partial \beta}{\partial \sigma} < 0$:More risky environments give rise to less powerful incentives

$\frac{\partial \beta}{\partial c} < 0$: Less powerful incentives when greater disutility of effort.

Question 2: Moral Hazard in Banking

With the recent economic crisis, every time the Fed or the Treasury bailed out yet another big irresponsible bank, commentators could be heard shouting, "Butbut...Moral Hazard! Moral Hazard!"

2 a) Based on the moral hazard theory you have learned so far, write in a at most two paragraphs a more erudite critique of the Treasury's actions.

You could see the government as the principal who wants banks to provide credit to firms. The government wants the bank to provide credit to all firms who are credit-worthy, and to correctly price the riskiness of the loans. The moral hazard stems from the fact that only the bank knows the risk profile of its loans. Thus the government has to give incentives to provide loans, i.e. you get the keep the profit if it goes well, and to prevent the bank from overextending credit the bank has to lose everything when a loan goes bad. However, when you have a policy of bailing out banks when its loans go sour, there is no longer a downside risk to the bank, and thus it will overextend credit.

Heads I win, Tails you loose.

Thus the policy of bank rescues could exacerbate the Moral Hazard stemming from limited liability.

Question Three: Fisherman's friends

Imagine yourself to be a poor country. A very poor country. In fact one of the few things you have going for your country is that it has a beach. So you spend most of your days lying on your beach, being poor. But one day you suddenly have a jolt of insight: if I have a beach, I have a sea! And if I have a sea, then I have fish! And according to the Chinese as soon as you can catch fish you're not hungry anymore! Only problem is that you are too poor to build your own shipping fleet. But no worries! You just hire some foreign fisherman to do your fishing for you.

But then the Worldbank relocates one of their beaurocrats to your country. This man had bought a million umbrella's for Chad thinking this would cause rain there. . This man knew not much about rain, but he does about fish and he warns you that it is important not too overfish, otherwise your fishing stock will deplete and you will be poor again. After days hunched behind his laptop, crunching numbers in Stata, running GMM estimations in R, the beaurocrat finally makes a wonderful PowerPoint presentation filled with dazzling animations and this important factoid: the optimal amount of fishing effort is e^*.

Now you set to work designing an optimal contract for your foreign fisherman, let's call him Fred.

Since fishing is a very unpredictable business, even with such an experience fisherman as our Fred, there is some uncertainty about the catch. Let $P_j(e)$ =prob$(x_j|e)$, and $P_j(e) > 0$ for all j and $\sum_{j=1}^{n} P_j(e) = 1$. Where $x_n > x_{n-1} > ... > x_1$. Also, there is a discrete set of possible efforts $\{e^1, e^2, ..., e^m\}$. Again $e^m > e^{m-1} > ... > e^1$.

Fred's utility depends on how much fish x he catches, the market price for fish p, the sum he has to pay to our poor country for the fishing rights $T(x)$ and finally the amount of effort he puts into fishing e:

$$U_f(x, e) = u(px - T(x)) - v(e), \text{ with } u' > 0, u'' < 0, v' > 0, v'' > 0.$$

Fred is not at all concerned about fishing stocks, tends to overfish, and has a reserve utility $\underline{u}$.

Our poor country is risk neutral: $U_p(x) = E[T(x)]$

3 a) Write down the maximization problem when the country wants to implement optimal effort $e^* \leq e^m$ and **effort is observable. Solve for the optimal contract and comment.**

SOLUTION:

Since the optimal level of effort is exogenously given by $e^* \leq e^m$ we can just plug this into the objective function and the PC:

$$\max_{\{T(x_j)\}} \sum_{j=1}^{n} P_j(e^*)[T(x_j)]$$

$$\text{s.t.} \sum_{j=1}^{n} P_j(e^*)u(px_j - T(x_j)) - v(e^*) \geq \underline{u}.\text{(PC)}$$

Now we simply set up the Lagrangian:

$$L = \sum_{j=1}^{n} P_j(e^*)[T(x_j)] + \lambda[\sum_{j=1}^{n} P_j(e^*)u(px_j - T(x_j)) - v(e^*) - \underline{u}]$$

Taking FOC with respect to $T(x_j)$:

$$P_j(e^*) - \lambda P_j(e^*)u'(px_j - T(x_j)) = 0 \ \forall j$$

Or

$$\frac{1}{u'(px_j - T(x_j))} = \lambda \text{ for } \forall j$$

Since λ is a constant, $px_j - T(x_j)$ has to be constant as well, $px_j - T(x_j) = k$. Hence $T(x_j) = px_j - k$.

In other words the Poor Country makes a constant payment k to Fred, and receives the market price of his catch. This follows from Fred's risk aversion and Poor Country's risk neutrality. Since Fred is risk averse, the Poor Country will fully insure Fred against variance in his catch.

b) Now write down the incentive compatibility constraint(s) needed to get Fred to fish at effort e^*.

SOLUTION:

The IC has to be such that Fred prefers effort e^* to all other effort levels $e^i \neq e^*$.Thus:

$$\sum_{j=1}^{n} P_j(e^*)u(px_j - T(x_j)) - v(e^*) \geq \sum_{j=1}^{n} P_j(e^i)u(px_j - T(x_j)) - v(e^i) \text{ for } \forall e^i \neq e^*.$$

c) Characterize the system of equations governing the optimal contract with asymmetric information.

$$\max_{\{T(x_j)\}} \sum_{j=1}^{n} P_j(e^*)[T(x_j)]$$

s.t. $\sum_{j=1}^{n} P_j(e^*)u(px_j - T(x_j)) - v(e^*) \geq \underline{u}$(PC)

$\sum_{j=1}^{n} P_j(e^*)u(px_j - T(x_j)) - v(e^*) \geq \sum_{j=1}^{n} P_j(e^i)u(px_j - T(x_j)) - v(e^i)$ for $\forall e^i \neq e^*$.(IC$_i$)

The IC in this case is a whole collection of IC's.

We set up the lagrangian:

$$L = \sum_{j=1}^{n} P_j(e^*)[T(x_j)] + \lambda[\sum_{j=1}^{n} P_j(e^*)u(px_j - T(x_j)) - v(e^*) - \underline{u}]$$
$$+\sum_{i=1}^{m} \mu^i [\sum_{j=1}^{n} P_j(e^*)u(px_j - T(x_j)) - v(e^*) - \sum_{j=1}^{n} P_j(e^i)u(px_j - T(x_j)) - v(e^i)]$$

Taking FOC with respect to $T(x_j)$:

$$P_j(e^*) - \lambda P_j(e^*)u'(px_j - T(x_j)) + \sum_{i=1}^{m} \mu^i[-P_j(e^*)u'(px_j - T(x_j)) + P_j(e^i)u'(px_j - T(x_j))] = 0$$

Rewriting:

$$\frac{1}{u'(px_j - T(x_j))} = \lambda + \sum_{i=1}^{m} \mu^i [\frac{P_j(e^*) - P_j(e^i)}{P_j(e^*)}]$$

Hence:

$$T(x_j) = (u')^{-1}(\frac{1}{\lambda + \sum_{i=1}^{m} \mu^i [\frac{P_j(e^*) - P_j(e^i)}{P_j(e^*)}]})$$

d) Assume that left to itself Fred has a tendency to overfish. That is, if Fred would keep the whole of his catch $T(x) = px$ then Fred's optimal effort would be higher than e^*. Now consider two possible catches $x_2 > x_1$. Assume that with greater effort it becomes more

likely to catch a large catch and less likely to catch a small catch. Hence if $e^i > e^*$ then $P_2(e^*) - P_2(e^i) < 0$ and $P_1(e^*) - P_1(e^i) > 0$ for . Now compare the payments Fred has to make on both catches with his market revenues. Does this make sense? Would it work?

SOLUTION:

Since only IC_i for effort levels higher than e^* would be violated, we can ignore all ICC's for $i \leq *$.

Since $P_2(e^*) - P_2(e^i) < 0$ and $P_1(e^*) - P_1(e^i) > 0$

$$\frac{1}{u'(px_1 - T(x_1))} = \lambda + \sum_{i=*+1}^{m} \mu^i [\frac{P_1(e^*) - P_1(e^i)}{P_1(e^*)}] > \lambda + \sum_{i=*+1}^{m} \mu^i [\frac{P_2(e^*) - P_2(e^i)}{P_2(e^*)}] = \frac{1}{u'(px_2 - T(x_2))}$$

Hence $u'(px_1 - T(x_1)) < u'(px_2 - T(x_2))$ and by concavity it follows that $px_1 - T(x_1) > px_2 - T(x_2)$.

Rearranging: $T(x_2) - T(x_1) > px_2 - px_1$.

Thus for an extra catch of $x_2 - x_1$ Fred has to make more payments to Poor Country than he will make in the marketplace! This makes as an incentive not to overfish, but would probably not work in practice, as Fred would just lump excess fish overboard...

In other words, to make the whole plan feasible we should add some Implementability Constraint $T(x_j) \leq px_j$. This means that the country cannot simply fine Fred for overfishing, because he can lump fish overboard and under-report his catch. Ofcourse we could say that the country could send a monitor along with Fred who perfectly observes x_j but this same monitor should then also be able to observe e, in which case we would be back to the symmetric information case as before.

Question Four: Moral Hazard and Monotonicity

Consider the two outcomes and two efforts moral hazard model with risk aversion. The agent's utility is $u(w) - e$ where $u()$ is strictly concave and $e \in \{e_l, e_h\}$ with $e_l < e_h$. The agent has outside option $\underline{u}$. Under high/low effort, the probability of success is $p_h/\ p_l$. The pay-off to the principal when the outcome is $j = H, L$ is $x_j - w_j$. Assume $x_H > x_L$ and $p_h > p_l$.

4 **a) Assume it is optimal to implement the high action. Characterize the optimal contract.**

b) Show that the incentive compatibility constraint implies that the payment has to increase with the level of outcome.

c) Assume there are 3 outcomes and let p_{ei} denote the probability of outcome $i \in \{L, M, H\}$ under effort $e \in \{l, h\}$. Assume the distibution under high effort first order stochastic dominante the distribution under low effort. Is it possible to use the same argument as in the case with 2 outcomes to show that the payment has to increase with the level of outcome?

SOLUTION

a) Using the first order condition to the principal problem, one can show as we did in class that the participation constraint and incentive compatibility constraint bind. The optimal contract (w_L, w_H) solves the system of two equations in two unknowns,

$p_h u(w_H) + (1 - p_h)u(w_L) - e_h = \underline{u}$
$p_h u(w_H) + (1 - p_h)u(w_L) - e_h = p_l u(w_H) + (1 - p_l)u(w_L) - e_l$

b) Rearranging terms in the incentive compatibility constraint imply $(p_h - p_l)(u(w_H) - u(w_L)) = e_h - e_l > 0$ which implies $u(w_H) - u(w_L) > 0$ and since u is increasing we have $w_H > w_L$.

c)

Now the IC becomes:

$p_{hH} u(w_H) + p_{hL} u(w_L) + (1 - p_{hH} - p_{hL})u(w_M) - e_h = p_{lH} u(w_H) + p_{lL} u(w_L) + (1 - p_{lH} - p_{lL})u(w_M) - e_l$

The proof does not follow with three outcomes. Incentive compatibility requires $(p_{hH} - p_{lH})(u(w_H) - u(w_M)) + (p_{lL} - p_{hL})(u(w_M) - u(w_L)) = e_h - e_l > 0$. First order stochastic dominance implies that $p_{hH} - p_{lH} \geq 0$ and $p_{lL} - p_{hL} \geq 0$ but this implies only that it is not possible that both $w_H < w_M$ and $w_M < w_L$. Stated differently, the above inequality can be rewritten as $\alpha(u(w_H) - u(w_M)) + (1 - \alpha)(u(w_M) - u(w_L)) > 0$ for $\alpha \in (0, 1)$ which only says that the payment has to increase in an average sense. This is not surprising since we already saw that the wage does not have to increase with outcome in the continuous outcome case.

Micro III: Problem set 3
SOLUTIONS

Rationing and Price Discrimination

Question one.

What's up dawgz?? Listen up, y'all, gott 'a little mission for you. As y 'all know I've been hustling the streets, busting caps up anyone's ass that dare intruding on our turf. It's a thug's life, and somebody's gotta do it, but this thug is strung out from all this gangstering. G-for-life and all that, but right now it's G-to-bed. Anyway, I'm writing y 'all cuz I know you're all phat playa's with the number 'n shit. So better break your balls on this problem. As y 'all know we caught some dope from the Gangster Disciples, but the quality of the shit is fucking all over the place. No worries though, we just gotta figure out how to divvy up the snow among our respected clientele, if you know what i'm saying. Our main man P to the C came with this hizzledeshizzle plan to turn this outfit of ours into a snort lottery. Sounds to my like he's been huffing a bit too much on the Bhang, the Marijuana, the Reefer, the Wahupta , Dope, the Ganja, the Smoke, the Weed, the Herb, the Doobie, the Sensemilla, the Green, the Stash, the Mexican, the Pounds, the Dirtweed, the Shake, the Ganjah, the Guaza, the Shit, the Hops, the Mary Jane, the Boo, the Grass, the Tea, the

Bush, the Buzz, the Diggy, the Blunt, the Ragweed, the Broccoli, if you know what I'm saying. Anyway solve the following and keep it real!

Peace out!

Our friendly neighbourhood addicts form a unit mass with type distribution $F(t)$, $t \in [t_L, t_H]$. We as the local monopolist have to price an exogenously given continuum of drugs of quality distributed according to $G(q)$, $q \in [q_L, q_H]$. Addict t gets utility $U^t(q) - p$ from buying a good of quality q at price p, where $U_q^t > 0$, $U_{qq}^t < 0$, and $U_{qt}^t > 0$.

a) Derive the efficient allocation rule $t^e(q)$ (it assigns quality q to type $t^e(q)$).

b) Assume the monopolist price discriminates and sells all products. Derive the optimal monopoly allocation $t^{pd}(q)$, the optimal price $p(q)$ for quality q, and the monopoly profits.

c) Assume the monopolist sells all products at the same price. Under uniform pricing, a consumer buys a lottery over qualities. Consumer t receives quality q with probability $\pi^t(q)$ and associated distribution $\Pi^t(q)$. The probability distribution is given. We assume that $\int \pi^t(q)U^t(q)dq \geq \int \pi^{t_L}(q)U^{t_L}(q)dq$ for all $t > t_L$. Derive the monopoly profits under uniform pricing.

d) When does uniform pricing dominate price discrimination? Demonstrate that a sufficient condition is $\frac{1-\Pi^L(q)}{1-G(q)} \geq \frac{U_q^{t^e(q)}(q)}{U_q^t(q)}$ for $\forall q$. Interpret.

e) Compare total welfare under price discrimination and uniform pricing. Discuss.

SOLUTION

a) Since higher types have a higher marginal utility of quality than lower types, the efficient allocation rule $t^e(q)$ is such that higher qualities get assigned to higher types.

Thus we get that $t^e(q) = F^{-1}(G(q))$

b) When the monopolist price discriminates and tries to extract surplus from the high types he binds their Incentive Compatability constraints. Low types get extracted all their surplus. The monopolist implements the efficient allocation in order to maximize extractable surplus. (We can prove this by totally differentiating the FOC of consumers)

Given a price schedule $p(q)$ a consumer t maximizes $U^t(q) - p(q)$.

$$max \int_{q_L}^{q_H} p(q)g(q)dq$$

s.t.

$$U^{t^e(q)}(q) - p(q) \geq 0 \quad \forall q \text{ (PC's)}$$

$$q = \arg\max_{\widetilde{q}} U^{t^e(q)}(\widetilde{q}) - p(\widetilde{q}) \forall q \text{(IC's)}$$

If we have a PCL $U^{t_L}(q_L) - p(q_L) \geq 0$ and given that the IC 's garantee that all other $t > t_L$ will prefer their quality $q = (t^e)^{-1}(t)$ to q_L. And so all other PC's will hold as well. PCL binds as well. (by contradiction).

$$max \int_{q_L}^{q_H} p(q)g(q)dq$$

s.t.

$U^{t_L}(q_L) - p(q_L) = 0$ (PC)

$q = \arg\max_{\widetilde{q}} U^{t^e(q)}(\widetilde{q}) - p(\widetilde{q}) \forall q$(IC's)

Now we need to take prices such that the IC holds, so we take the first order condition of the IC: $U_q^{t^e(q)}(\widetilde{q}) - p_q(\widetilde{q}) = 0$ for $\widetilde{q} = q$.

Integrating both sides $\int_{q_L}^{q} p_q(q)dq = \int_{q_L}^{q} U_q^{t^e(q)}(q)dq$ we can now solve for $p(q)$. Since the lowest type get all his surplus extracted $p(q_L) = U^{t_L}(q_L)$. So now we can write:

$$p(q) = U^L(q_L) + \int_{q_L}^{q} U_q^{t^e(q)}(q)dq$$

To calculate total profits we just integrate the prices over all qualities:

$$R_{pd} = \int_{q_L}^{q_H} [U^L(q_L) + \int_{q_L}^{q} U_q^{t^e(x)}(x)dx]g(q)dq$$

$$= U^L(q_L) + [\int_{q_L}^{q} U_q^{t^e(x)}(x)dx)G(q)]_{q_L}^{q_H} - \int_{q_L}^{q_H} U_q^{t^e(x)}(x)G(q)dq$$

$$= U^L(q_L) + \int_{q_L}^{q_H} U_q^{t^e(x)}(x)(1 - G(q))dq$$

c) Since the monopolist wants to cover the whole market, and if the lowest type participates, everybody participates, the highest price such that everybody particates is equal to the expected utility of the lowest type:

$R_{uniform} = p = \int \pi^L(q)U^{t_L}(q)dq$

Full market coverage implies $\int_{t_L}^{t_H} \pi^t(q)dt = g(q)$.

Again integrating by parts:

$$R_{uniform} = \int_{q_L}^{q_H} \pi^L(q)U^{t_L}(q)dq$$

$$= [U^L(q)\Pi^L(q)]_{q_L}^{q_H} - \int_{q_L}^{q_H} \Pi^L(q)U_q^{t_L}(q)dq$$

$$= U^L(q_H) - \int_{q_L}^{q_H} \Pi^L(q)U_q^{t_L}(q)dq$$

$$= U^L(q_H) - \int_{q_L}^{q_H} \Pi^L(q)U_q^{t_L}(q)dq + \int_{q_L}^{q_H} U_q^L(q)dq - U^{t_L}(q_H) + U^{t_L}(q_L)$$

$$= U^L(q_L) + \int_{q_L}^{q_H} (1 - \Pi^L(q))U_q^{t_L}(q)dq$$

d)
$R_{uniform} > R_{pd}$
$U^L(q_L) + \int_{q_L}^{q_H} (1 - \Pi^L(q))U_q^{t_L}(q)dq > U^L(q_L) + \int_{q_L}^{q_H} U_q^{t^e(q)}(q)(1 - G(q))dq$

$\int_{q_L}^{q_H} [(1 - \Pi^L(q))U_q^{t_L}(q) - U_q^{t^e(q)}(q)(1 - G(q)]dq \geq 0$

$(1 - \Pi^L(q))U_q^L(q) - U_q^{t^e(x)}(x)(1 - G(q) \geq 0$
$\dfrac{1 - \Pi^L(q)}{1 - G(q)} \geq \dfrac{U_q^{t^e(q)}(q)}{U_q^{t_L}(q)}$

So uniform pricing works when there are not too many high types, when $\Pi^L(q)$ is more reverse monotone (low types have high probability of getting high quality) and when high types are not willing to pay much more than low types.

e) Since under uniform pricing firms do not select the optimal allocation, overall welfare is lower. And since firm profits are higher (otherwise the firm would not charge uniform prices in the first place), we can conclude that consumer welfare must be lower as well.

Consumers would be better of with price discrimination.

Regulation

A regulator wants to minimize the price charged by a natural monopolist who is privately informed about its cost. The monopolist produces a good at cost $c = \theta - e$ which is publicly observed but where $\theta \in \{\theta_H, \theta_L\}$ is a fixed type privately observed by the monopolist and $\Delta\theta = \theta_H - \theta_L > 0$ and $e \geq 0$ represents a level of effort chosen by the monopolist, and also not observed by the regulator, to reduce cost. The cost of exerting effort e is $\frac{1}{2}e^2$. The monopolist is low type with probability β. The regulator minimizes the payment $P = s + c$ that is given to the monopolist in exchange for the good where s is a fixed subsidy. The monopolist maximizes profits $P - c - \frac{1}{2}e^2$.

a) Assume the regulator observes θ. What is the optimal level of effort?

b) Assume the regulator observes only c and not θ nor e. Let (s_i, c_i) denote the contract taken in equilibrium by type $i = L, H$. Under contract i, the monopolist has to supply the good at cost c_i and then gets compensated s_i. Denote $e_{ii'}$ the level of effort that type i has to supply in equilibrium if she selects contract i' where $i, i' \in \{L, H\}^2$. We assume throughout that these levels of efforts are all positive. Show that $e_{LH} = e_{HH} - \Delta\theta$ and $e_{HL} = e_{LL} + \Delta\theta$.

c) Write down the participation constraints (PC_i) and incentive compatibility constraints ($IC_{ii'}$) that type i does not take contract i' using only the variables s_i and e_{ii}, $i = L, H$.

d) Show that PC_L is strict and that PC_H and IC_L bind.

e) Solve for the optimal effort level e_{LL} and e_{HH}. assuming that IC_{HL} holds.

f) Derive the optimal contracts.

SOLUTION

a) The regulator minimizes

$\min s + c$
s.t. $s + c - c - \frac{1}{2}e^2 \geq 0$ (PC)

$\min s + \theta - e$
s.t. $s - \frac{1}{2}e^2 = 0$ (PC)

$\min \frac{1}{2}e^2 + \theta - e$

FOC: $e - 1 = 0$
$=> e = 1$
$s = 1/2$.

b) Writing down the different effort levels:
$e_{LL} = \theta_L - c_L$
$e_{HH} = \theta_H - c_H$
$e_{LH} = \theta_L - c_H$
$e_{HL} = \theta_H - c_L$.

After replacement, we get

$e_{LH} = \theta_L - \theta_H + e_{HH} = e_{HH} - \Delta\theta$
$e_{HL} = \theta_H - \theta_L + e_{LL} = e_{LL} + \Delta\theta$.

c)

$s_L - \frac{1}{2}e_{LL}^2 \geq 0$ (PCL)
$s_H - \frac{1}{2}e_{HH}^2 \geq 0$ (PCH)
$s_L - \frac{1}{2}e_{LL}^2 \geq s_H - \frac{1}{2}(e_{HH} - \Delta\theta)^2$ (ICL)
$s_H - \frac{1}{2}e_{HH}^2 \geq s_L - \frac{1}{2}(e_{LL} + \Delta\theta)^2$ (ICH)

d) Put IC_L and PC_H together:

$s_L - \frac{1}{2}e_{LL}^2 \geq s_H - \frac{1}{2}(e_{HH} - \Delta\theta)^2 > s_H - \frac{1}{2}e_{HH}^2 \geq 0$

Hence:
$s_L - \frac{1}{2}e_{LL}^2 > 0$ (PCL)

PC_L is strict.

The other two claims are proved by contradiction.

If PC_H does not bind then decrease both s_H by a small amount.

If IC_L does not bind decrease s_H and s_L by a small amount.

e) Since PC$_H$ binds:
$s_H = \frac{1}{2}e_{HH}^2$

Since ICL binds:

$s_L = \frac{1}{2}(e_{HH}^2 + e_{LL}^2) - \frac{1}{2}(e_{HH} - \Delta\theta)^2$

The maximisation problem is:

min $\beta(s_L + \theta_L - e_{LL}) + (1-\beta)(s_H + \theta_H - e_{HH})$
s.t.

$s_L - \frac{1}{2}e_{LL}^2 \geq 0$ (PCL)
$s_H - \frac{1}{2}e_{HH}^2 \geq 0$ (PCH)
$s_L - \frac{1}{2}e_{LL}^2 \geq s_H - \frac{1}{2}(e_{HH} - \Delta\theta)^2$ (ICL)
$s_H - \frac{1}{2}e_{HH}^2 \geq s_L - \frac{1}{2}(e_{LL} + \Delta\theta)^2$ (ICH)

We assume that ICH holds (and check afterwards!) and plug in the values for s_H and s_L.

min $\beta(\frac{1}{2}(e_{HH}^2 + e_{LL}^2) - \frac{1}{2}(e_{HH} - \Delta\theta)^2 + \theta_L - e_{LL}) + (1-\beta)(\frac{1}{2}e_{HH}^2 + \theta_H - e_{HH})$

FOCs:
$e_{LL} : \beta(e_{LL} - 1) = 0$
$=> e_{LL} = 1$

$e_{HH} : \beta(e_{HH} - e_{HH} + \Delta\theta) + (1-\beta)(e_{HH} - 1) = 0$
$\beta\Delta\theta + (1-\beta)(e_{HH} - 1) = 0$

$=> e_{HH} = 1 - \frac{\beta}{1-\beta}\Delta\theta.$

f) Check that IC_H is satisfied.

$\frac{1}{2}(1 - \frac{\beta}{1-\beta}\Delta\theta)^2 - \frac{1}{2}(1 - \frac{\beta}{1-\beta}\Delta\theta)^2 \geq \frac{1}{2}((1 - \frac{\beta}{1-\beta}\Delta\theta)^2 + 1) - \frac{1}{2}(1 - \frac{\beta}{1-\beta}\Delta\theta - \Delta\theta)^2 - (1 + \Delta\theta)^2$
$0 \geq \frac{1}{2}((1 - \frac{\beta}{1-\beta}\Delta\theta)^2 + 1) - \frac{1}{2}(1 - \frac{\beta}{1-\beta}\Delta\theta - \Delta\theta)^2 - (1 + \Delta\theta)^2$

$\frac{1}{2}(1-\frac{\beta}{1-\beta}\Delta\theta-\Delta\theta)^2+(1+\Delta\theta)^2\geq\frac{1}{2}((1-\frac{\beta}{1-\beta}\Delta\theta)^2+1)$

$((1-\frac{\beta}{1-\beta}\Delta\theta)-\Delta\theta)^2+2(1+\Delta\theta)^2\geq(1-\frac{\beta}{1-\beta}\Delta\theta)^2+1$

$(1-\frac{\beta}{1-\beta}\Delta\theta)^2+\Delta\theta^2-2\Delta\theta(1-\frac{\beta}{1-\beta}\Delta\theta)+2(1+\Delta\theta)^2\geq(1-\frac{\beta}{1-\beta}\Delta\theta)^2+1$
$\Delta\theta^2-2\Delta\theta(1-\frac{\beta}{1-\beta}\Delta\theta)+2(1+\Delta\theta)^2\geq1$
$\Delta\theta^2-2\Delta\theta(1-\frac{\beta}{1-\beta}\Delta\theta)+2(1+\Delta\theta^2+2\Delta\theta)\geq1$
$\Delta\theta^2-2\Delta\theta+2\frac{\beta}{1-\beta}\Delta\theta^2+2+2\Delta\theta^2+4\Delta\theta\geq1$
$2\frac{\beta}{1-\beta}\Delta\theta^2+3\Delta\theta^2+2\Delta\theta\geq-1$

This always holds. Yippie!
The optimal contracts are

$c_L=\theta_L-e_{LL}=\theta_L-1$
$s_L=\frac{1}{2}(e_{HH}^2+e_{LL}^2)-\frac{1}{2}(e_{HH}-\Delta\theta)^2$
$=\frac{1}{2}e_{LL}^2+e_{HH}\Delta\theta-\frac{1}{2}\Delta\theta^2$
$=\frac{1}{2}+\Delta\theta\left(1-\frac{\beta}{1-\beta}\Delta\theta\right)--\frac{1}{2}\Delta\theta^2$
$=\frac{1}{2}\left(1-\Delta\theta^2\right)+\Delta\theta\left(1-\frac{\beta}{1-\beta}\Delta\theta\right)$

$c_H=\theta_H-1+\frac{\beta}{1-\beta}\Delta\theta$
$s_H=\frac{1}{2}\left(1-\frac{\beta}{1-\beta}\Delta\theta\right)^2$
$=\frac{1}{2}\left(1+\left(\frac{\beta}{1-\beta}\right)^2\Delta\theta^2-2\frac{\beta}{1-\beta}\Delta\theta\right)$

As expected, the high cost firm produces at a higher cost (both because it has higher initial cost and because it supplies less effort). The relative size of the subsidy depends on β and $\Delta\theta$:

$\frac{1}{2}\left(1+\left(\frac{\beta}{1-\beta}\right)^2\Delta\theta^2-2\frac{\beta}{1-\beta}\Delta\theta\right)\leq\frac{1}{2}\left(1-\Delta\theta^2\right)+\Delta\theta\left(1-\frac{\beta}{1-\beta}\Delta\theta\right)$
$\frac{1}{2}+\frac{1}{2}\left(\frac{\beta}{1-\beta}\right)^2\Delta\theta^2-\frac{\beta}{1-\beta}\Delta\theta\leq\frac{1}{2}-\frac{1}{2}\Delta\theta^2+\Delta\theta-\frac{\beta}{1-\beta}\Delta\theta^2$
$+\frac{1}{2}\left(\frac{\beta}{1-\beta}\right)^2\Delta\theta^2-\frac{\beta}{1-\beta}\Delta\theta\leq-\frac{1}{2}\Delta\theta^2+\Delta\theta-\frac{\beta}{1-\beta}\Delta\theta^2$
$(\frac{1}{2}+\frac{\beta}{1-\beta}+\frac{1}{2}\left(\frac{\beta}{1-\beta}\right)^2)\Delta\theta^2\leq(1+\frac{\beta}{1-\beta})\Delta\theta$
$(\frac{1}{2}+\frac{\beta}{1-\beta}+\frac{1}{2}\left(\frac{\beta}{1-\beta}\right)^2)\Delta\theta\leq(1+\frac{\beta}{1-\beta})$
$(\frac{1}{2}+\frac{\beta}{1-\beta}+\frac{1}{2}\left(\frac{\beta}{1-\beta}\right)^2)\Delta\theta\leq\frac{1}{1-\beta}$
$(\frac{1}{2}(1-\beta)+\beta+\frac{1}{2}\frac{\beta^2}{1-\beta})\Delta\theta\leq1$

Micro III: Problem set 4
Deadline: Wednesday Feb 4 11:00

Reefer Madness

Remember our main man P to the C? Well, turns out that he's not the only one hooked on the herb. Loads of people are. In fact, there a whole bloody friggin ' army of Ganjah Geezers out there. All looking for the same thing: their daily grass. Unfortunately, there are heavy side effects to this green hobby. Not only do you come up with strange ideas about distributing valuable drugs by lottery, you wouldn't even remember the next day that you did! Reefer Madness can seriously affect your memory and long term planning ability, kids!

However these terrible long lasting medical effects and daunting moral questions are ofcourse of no concern to us: We are economists. So let us put our moral and medical qualms aside, and start computing pricing equilibria!

Given an economy with identical smokers buying an identical quality of Bhang, one would expect the Law of One Price to hold. However, this is not necessarily the case, as you are about to show. The crux is the interplay of search and storage costs.

Assume an economy with L smokers and n hashbars which sell an identical quality of weed. Smokers get utility u from a day supply of blunt. Our smokers only remember and plan for two days at a time. When buying on day one, the huffer has two choices. She can either decide to buy two days of supply , and store another one of them at cost δ. Or she can buy only one day of supply and search again the next day but with an added search cost of c. After the second day the doobie has affected the brain so much that the cycle starts all over.

1

a) Suppose that consumers know the exact distribution of prices $F(p)$ across hash bars and draw an i.i.d. price each period that they search. For what price are (risk neutral) smokers indifferent between buying either one or two days of supply in period one?

b) Suppose that only two prices are charged in equilibrium, by fraction λ, respectively $(1-\lambda)$ of firms. Solve for prices p_H and p_L.

c) Show that no firm will deviate with a price $p_3 \notin \{p_L, p_H\}$

c) Derive the demands each individual firm faces and solve for (p_L, p_H, λ) of the Two Price Equilibrium (TPE) in terms of u, δ and c.

d) When transaction costs are zero and prices are positive, show that a neccesary and sufficient condition for the existence of a TPE is that $\frac{u+3\delta}{2} < u$.

e) Show that a Single Price equilibrium $p = u$ is robust as long as

$\delta \geq \frac{u}{3}$.

f) Show that smoker welfare W as given by average price, storage and transaction costs is: $W = u - 3\delta + 2c$. Why would increasing second period search costs increase welfare?

SOLUTION

a)

When you know the full distribution of prices, you also know the average price $\overline{p}$. The price then for which consumers are indifferent is the price for which, taking into account storage cost, the price for the second good is equal to the expected price to be paid in the second period including transaction costs:

$\widehat{p} + \delta = \overline{p} + c$

$\widehat{p} = \overline{p} + c - \delta$

b)

The mechanism that causes the price dispersion the different prices that cause people to either buy on product or two. Thus the high price will be equal to the maximum price people are willing to pay for one unit: $p_H = u$.

And the low price will be equal to the maximum price people are willing to pay to buy two units(in the first period ofcourse):$p_L = \widehat{p}$.

$p_L = \widehat{p} = \overline{p} + c - \delta = \lambda p_H + (1-\lambda)p_L + c - \delta = \lambda u + (1-\lambda)p_L - (\delta - c)$

$p_L = u - \frac{\delta - c}{\lambda}$

c)

Now we derive demands. In each period each firm attracts $\lambda \frac{L}{n}$ old consumers who faced a high price in the first period, and only bought one good. Each firms also attrachts $\frac{L}{n}$ young consumers. Since the high priced stores only sell one good, and the low price good sell two goods to young consumers the demands for both firms look like this:

$x_H = (1+\lambda)\frac{L}{n}$
$x_L = (2+\lambda)\frac{L}{n}$

In equilibrium profits have to be equal. This helps us to calculate λ.

$p_H x_H = p_L x_L$
$u(1+\lambda)\frac{L}{n} = (u - \frac{\delta-c}{\lambda})(2+\lambda)\frac{L}{n}$
$u = 2u - 2\frac{\delta-c}{\lambda} - (\delta - c)$
$\lambda = \frac{2(\delta-c)}{u-(\delta-c)}$

Now we can calculate p_H(trivially) and p_L:
$p_L = u - \frac{\delta-c}{\lambda} = u - (\delta-c)\frac{u-(\delta-c)}{2(\delta-c)} = \frac{u+(\delta-c)}{2}$

$p_H = u.$

d)

For the TPE to exist both prices must exist with positive probability, so $0 < \lambda < 1$. If $c = 0$, then

$\lambda < 1$
which is
$\frac{2(\delta-c)}{u-(\delta-c)} < 1$
which then is equal to

$\frac{2\delta}{u-\delta} < 1$

This gives a condition on δ :
$\delta < \frac{u}{3}$
For the lower bound:
$0 < \frac{2\delta}{u-\delta}$

Hence:
$\delta < u$

Now writing $p_L + \delta = \frac{u+(\delta-c)}{2} + \delta = \frac{u+3\delta}{2}$

Now if we plug the condition $\delta < \frac{u}{3}$ into the above we get $\frac{u+3\delta}{2} < u$. Since p_L is positive the lower bound condition follows automatically.

e)

In a single price equilibrium the single price is $p = u$. When $c = 0$, firms would deviate to $p = u - \delta$ if this would be profitable. Under SPE firms would sell one unit to both young and old customers. When charging $p = u - \delta$ a single deviating firm would sell two units to young customer and one unit to old customers. So a firm would deviate if

$$(2+1)(u-\delta)\frac{L}{n} \geq 2u\frac{L}{n}$$
$$(2+1)(u-\delta) \geq 2u$$
$$\delta \geq \frac{1}{3}u$$

Thus when storage costs are low enough Two Price equilibrium emerges, but when they are prohibitively high then only one price will be charged.

f)

First we write down average price, storage and search costs. If a consumer finds a high priced store the first period the average prices she pays are $p_H + \overline{p}$ and she incurs an addes transaction cost c in the second period. If she stumbles on a low price store the first period she pays prices $2p_l$ and incurs a storage cost δ.

$$W = 2u - \lambda(u + \overline{p} + c) - (1-\lambda)(2p_l + \delta)$$

Using the fact that $p_l = \overline{p} + c - \delta$ we write:
$W = 2u - \lambda(u + \overline{p} + c) - (1-\lambda)(p_l + \overline{p} + c)$

Since we know that $(1-\lambda)p_l = \overline{p} - \lambda p_h = \overline{p} - \lambda u$ we can write:

$$W = 2u - 2\overline{p} - c$$

Now since we know that $\overline{p} = \widehat{p} + \delta - c = p_l + \delta - c = \frac{u+(\delta-c)}{2} + \delta - c$

$$W = 2u - u - (\delta - c) - 2(\delta - c) - c$$
$$= u - 3\delta + 2c$$

As transaction costs go up there is more price dispersion, and this lowers the average price thus increasing welfare.

MICROIII:Problemset5:Signalling

MICROIII:Problemset5:Signalling

Deadline: February 11 2009, 11:00

Meet John. John wakes up every morning in a dreary house in a rainy suburb, that looks just like every other dreary house in that rainy suburb.

While rubbing his still sleepy eyes, a thought suddenly shellacks through his brain that causes an immediate and frightening drop in his mood. A sudden pale somberness comes over him, that will not leave him for the rest of the day. Deeply depressed he goes to work and through his daily routine, without joy, without even a trace of a smile on his lips. His colleagues all have the same desolate grimace, often staring numb out in the distance for minutes at a time. A deep dark brown melancholy permeates the scene, as everyone just slogs the hours away, knowing that any hope of a sprinkling of cheer is a pointless exercise leading to inevitable disapointment.

Contrary to what you might think John's first thought was not "I voluntarily signed up for an economics PhD ", rather it was "I work for an insurance company ".

Don 't ever work for an insurance company folks! CHOOSE LIFE!

But luckily none of us made that terrible mistake, so instead our days are filled with cheer and song! Every day a new outpouring of raw delight and gaiety. Laughter and jubilation shared by all, a pure delirious ecstacy of mindcrushingly interesting thoughts from the giants of the Worldy Philosophers. What better way to put a smile on childs face then to quote Friedman? What more elegant way to win a a pub argument then to take a first order condition? Truly we lead a blessed life!

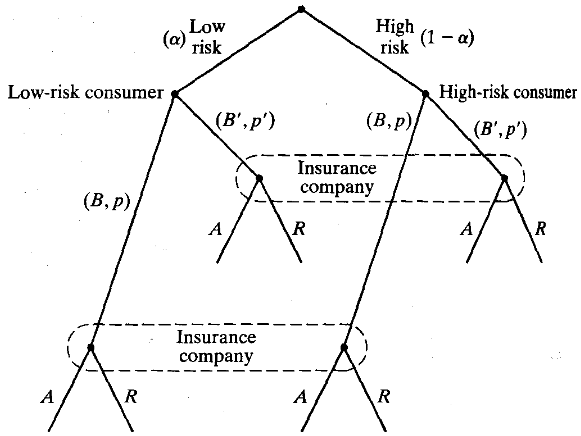
And with that in mind, consider the following Insurance Signalling Game:

We have two types of consumers (h,l) with a high ($\overline{\pi}$) respectively low ($\underline{\pi}$) probability of accident. We have a fraction α low-risk types and $(1-\alpha)$ high risk types. Both types have wealth w and suffer a loss L in the case of an accident.

The consumers can sign (B,p)-contracts with an insurance company, in which case the consumer pays a premium p in all states of the world, but gets a claim B in case of an accident. The game works such that first Nature decides on the type of consumer, then the consumer makes a proposal (B,p) to the insurance company, who then either Accepts or Rejects.

1 Seperating equilibria

a) Draw the extensive form of this game for the case where consumers can only propose contracts (B,p) or (B',p').



b) Let $\psi_l(B_l, p_l)$ and $\psi_h(B_h, p_h)$ denote the pure strategies of both types. Let $\beta(B,p) \in (0,1)$ denote the belief of the insurer about the probability that a consumer proposing (B,p) is of low risk. And finally let $\sigma(B,p) \in \{A, R\}$ denote the insurer's strategy. Characterize the Weak Perfect Bayesian Equilibrium

SOLUTION:
$\{\psi_l(B_l, p_l), \psi_h(B_h, p_h), \beta(B,p), \sigma(B,p)\}$ *is a PBE if*

1. Given the strategy $\sigma(B,p)$, $\psi_l(B_l, p_l)$ is the optimal strategy for low-risk types and $\psi_h(B_h, p_h)$ is the optimal strategy for high-risk types.
2. The insurer's beliefs satisfy Bayes' rule, that is
(a) $\beta(\psi) \in [0,1]$
(b) if $\psi_l \neq \psi_h$ then $\beta(\psi_l) = 1$ and $\beta(\psi_h) = 0$
(c) if $\psi_l = \psi_h$ then $\beta(\psi_l) = \beta(\psi_h) = \alpha$
3. Strategy $\sigma(B,p)$ maximizes the firms profits for every ψ given the firm's beliefs $\beta(B,p)$.

c) Write down the expected utility for both types (in terms of B and p), and show that the single crossing property holds.

SOLUTION:
$u_l(B,p) = \underline{\pi} u(w - L + B - p) + (1 - \underline{\pi}) u(w - p)$
$u_h(B,p) = \overline{\pi} u(w - L + B - p) + (1 - \overline{\pi}) u(w - p)$

$$MRS_l = -\frac{\partial u_l / \partial B}{\partial u_l / \partial p} = -\frac{\underline{\pi} u'(w-L+B-p)}{-\underline{\pi} u'(w-L+B-p) - (1-\underline{\pi}) u'(w-p)} =$$

$$MRS_h = -\frac{\partial u_h / \partial B}{\partial u_h / \partial p} = -\frac{\overline{\pi} u'(w-L+B-p)}{-\overline{\pi} u'(w-L+B-p) - (1-\overline{\pi}) u'(w-p)}$$

Takingderivatives of MRS w.r.t. π :

0.1 $\frac{\partial MRS}{\partial \pi} = \frac{-u'(w-L+B-p)[-\pi u'(w-L+B-p)-(1-\pi)u'(w-p)]+\pi u'(w-L+B-p)[-u'(w-L+B-p)+u'(w-p)]}{[-\pi u'(w-L+B-p)-(1-\pi)u'(w-p)]^2}$

$$= \frac{\pi u'(w-L+B-p)^2+(1-\pi)u'(w-p)u'(w-L+B-p)-\pi u'(w-L+B-p)^2+\pi u'(w-L+B-p)u'(w-p)}{[-\pi u'(w-L+B-p)-(1-\pi)u'(w-p)]^2}$$

$$= \frac{u'(w-L+B-p)u'(w-p)}{[-\pi u'(w-L+B-p)-(1-\pi)u'(w-p)]^2} > 0$$

Hence $MRS_h > MRS_l$. Thus proving single crossing.

d) If the insurer knows the type, what minimal condition on (B,p) would it set for both types?

SOLUTION: Since the insurer wants to at least break even, for the low risk types the insurer will accept any $p > \underline{\pi}B$ *and for the high risk types it will accept any* $p > \overline{\pi}B$.

e) Write down the conditions on strategies, prices and utilities for a separating equilibrium to exist.

1. Strategies of both types should differ: $\psi_l \neq \psi_h$

2. The firm only accepts profitable offers.

$p_l \geq \underline{\pi}B_l$

$p_h \geq \overline{\pi}B_h$

3. The low risk type should do better under separation than it would be able to under pooling:

$u_l(\psi_l) \geq \widehat{u}_l = \max_{\{B,p\}} u_l(B,p)$ s.t. $p = \overline{\pi}B$

4. The high risk types prefers it's own contract to the low risk contract:

$u_h(\psi_h) \geq u_h(\psi_l)$

f) Sketch a (p,B) quadrant with p along the vertical axis containing:

(1) Both the high risk and low risk zero profit lines for the firm

(2) The optimal offers with indifference curve for both types if the firm believes them to be of high risk type.

(3) The region in the graph in which (B,p) offers are (a) preferred by low-risk types over the offer in (2), (b) not preffered by high risk types over the offer in (2), (c) accepted by the firm.

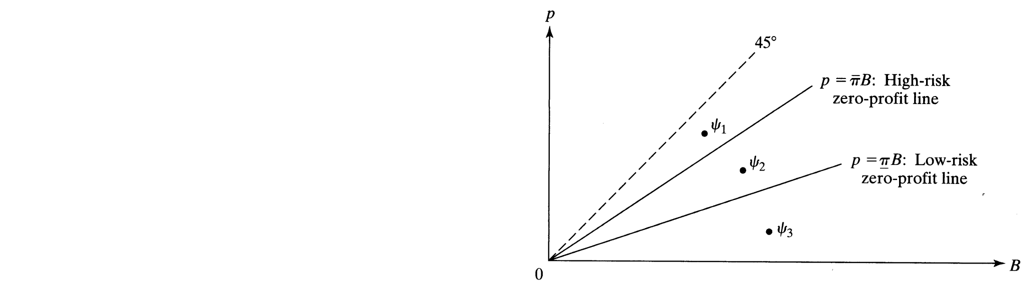


Figure 8.3. Zero-profit lines. Policy ψ_1 earns positive profits on both consumer types; ψ_2 earns positive profits on the low-risk consumer and negative profits on the high-risk consumer; ψ_3 earns negative profits on both consumer types.

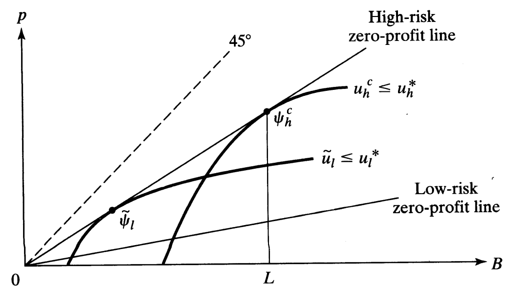


Figure 8.5. Lower bounds. Because all policies (B, p) above the high-risk zero-profit line are accepted by the insurance company in equilibrium, the low-risk consumer must obtain utility no smaller than $\tilde{u}_l = u_l(\tilde{\psi}_l)$ and the high-risk consumer utility no smaller than $u_h^c = u(\psi_h^c)$. Note that although in the figure $\tilde{\psi}_l \neq (0, 0)$, it is possible that $\tilde{\psi}_l = (0, 0)$.

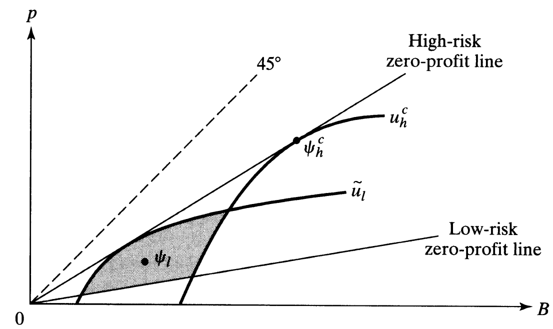


Figure 8.6. Potential separating equilibria. In a separating equilibrium in which both consumer types propose acceptable policies, the high-risk policy must be ψ_h^c and the low-risk policy, ψ_l, must be in the shaded region. Here, $MRS_l(0,0) > \bar{\pi}$. A similar figure arises in the alternative case, noting that $MRS_l(0,0) > \underline{\pi}$ always holds.

2) Pooling equilibria

There exists another possible equilibrium where both types make the same offer $\psi_l(B_l, p_l) = \psi_h(B_h, p_h) = \psi'$.

a) Characterize the positive expected profit condition for firms when both consumers offer ψ'.

SOLUTION:

When both types make the same (B,p) offer, the firm learns nothing about the identity of its customers. Therefore the expected accident rate is equal to

$\widehat{\pi} = \alpha\underline{\pi} + (1-\alpha)\overline{\pi}$

Thus the firm will accept any offer with $p \geq \widehat{\pi}B$.

b)Again sketch a (p,B) quadrant with p along the vertical axis containing:

(1) The high risk zero profit line, the low risk zero profit line, and the pooling expected zero profit line.

(2) Both types' indifference curves for when the firm believes them to be of a high-risk type

(3) The region of possible possible pooling equilibra.

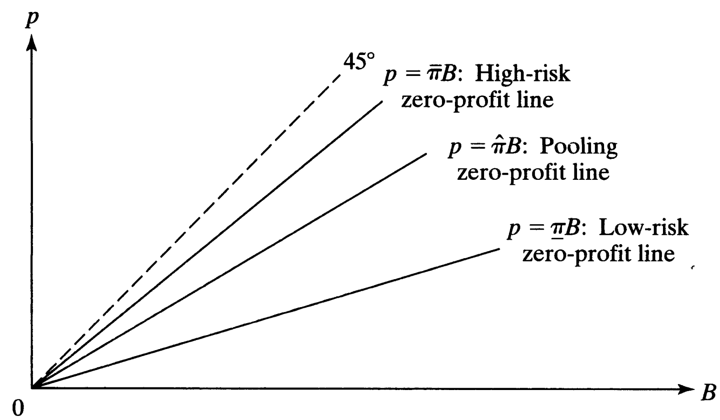


Figure 8.8. Pooling zero-profit line.

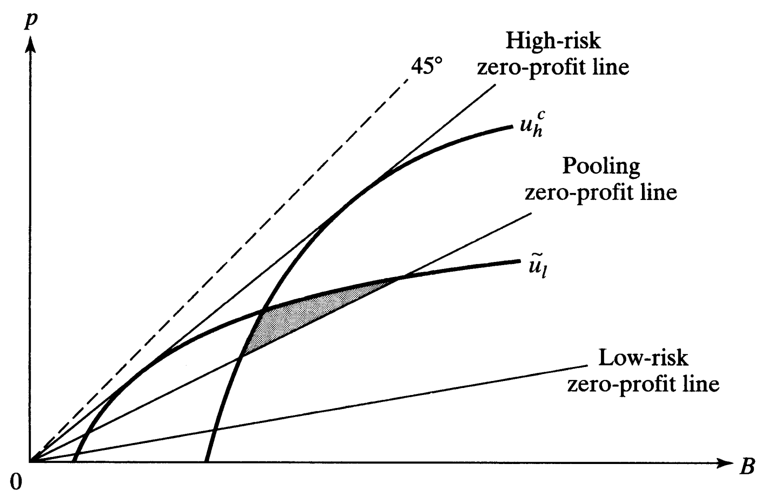


Figure 8.9. Pooling equilibria. The shaded region depicts the set of policies that can arise as pooling equilibria.

c) Characterize a belief $\beta(B,p)$ and a strategy $\sigma(B,p)$ that would result in a Bayesian Pooling Equilibrium. (Hint: think about who wants to deviate)

To characterize an equilibrium, we must find a (bayesian) belief and strategy that would be profit maximizing for the firm, and prefered by both types of consumers.

SOLUTION:

Because only the high risk type wants to deviate, the correct beliefs are that if both offer ψ' then $\beta(B,p) = a$, and otherwise it's a lowtype hence $\beta(B,p) = 0$. Given these beliefs, the correct strategy is accept if either the proposal is ψ' or otherwise when $p > \overline{\pi}B$.

d) What happens when the fraction of high risk types increases?

SOLUTION: As a decreases, the pooling expected zero profit line shifts upwards, reducing the are in which pooling is possible, until it disappears.

e) **Show graphically that in some cases the pooling equilibrium might be preffered by the low risk types to the seperating equilibrium.**

SOLUTION: As the population of high types increases, the pooling expected zero profit line shifts downwards. Drawing an indifference curve through the seperation equilibrium optimal ψ_l there can be an pooling equilibrium area under this curve. See page 349.

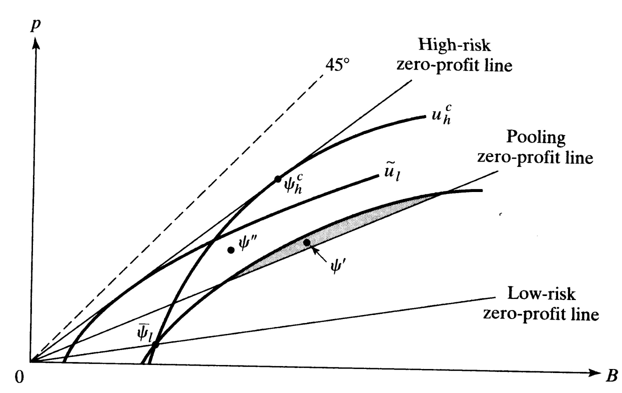


Figure 8.10. Pooling may dominate separation. The best separating equilibrium for consumers yields policies $\psi_l = \bar{\psi}_l$ and $\psi_h = \psi_h^c$. The pooling equilibrium outcome $\psi_l = \psi_h = \psi'$ in the shaded region is strictly preferred by both risk types. Other pooling equilibrium outcomes, such as $\psi_l = \psi_h = \psi''$, are not.

Question Two:

An expert E observes the realization of the state of the word $\omega \in [-\infty, \infty]$. E sends a message m to the decision maker DM which consists in a closed interval $[u, v] \subset \mathbb{R}$ with the only constraint that $\omega \in [u, v]$. DM takes a decision $d \in [-\infty, \infty]$. When the state of the world is ω,.E would like DM to take a decision as close as possible to $\omega + \beta$ where β is privately observed by E. DM believes that β is distributed uniformly over $[b - \Delta, b + \Delta]$ with $\Delta > 0$ and that ω is uniformly distributed on $[-\infty, \infty]$ (strictly speaking, this is not possible but assume that DM's prior is such that $\Pr(\omega \in [u, v]) = \varepsilon(v - u)$ where ε is a small positive number arbitrarly close to 0). DM chooses an action that corresponds to her posterior belief on the true state of nature $d = E[\omega \mid m = [u, v]]$. Denote $d(\omega, \beta)$ the decision taken in a Perfect Bayesian Equilibrium when the state is ω and E's type is β. A full revelation equilibrium is such that $d(\omega, \beta) = \omega$ while a fully manipulative equilibrium is such that $d(\omega, \beta) = \beta + \omega$.

1-Show that if $\Delta < b$ there exists a full revelation equilibrium.

Since $\Delta > 0$ we know that also $b > 0$. And since $\Delta < b$ we know that $\beta > 0$. Thus the DM knows that E has an upward bias.

We know that E always has to include the truth in his message. And he only wants to give favorable information. So suppose the strategy for E is:

$m(\omega, \beta) = [\omega, \omega + k]$ for some $k > 0$

Given this strategy the belief of the DM is

$\mu(u, v) = \mu(m(\omega, \beta)) = u$

And his decision $d(u, v) = u = \omega$

Given these beliefs the E can not improve upon the strategy $m(\omega, \beta) = [\omega, \omega + k]$:

- He cannot increase u otherwise he would no longer communicate the truth
- If he lowers u then he increases the distance between the DM's decision and his preferred outcome: $d(u, v) - (\omega + \beta)$
- Whether he increases or lowers v does not change the decision

Hence $\{m(\omega, \beta) = [\omega, \omega + k], \mu(u, v) = u, d(u, v) = u\}$ is a full revelation PBE.

2-Show that if $b = 0$ there exists a fully manipulative equilibrium.

Suppose $[u, v] = m(\omega, \beta) = [\omega + \beta - \Delta, \omega + \beta + \Delta]$ is the optimal strategy for the experts.

Given this strategy, what would be the belief $\mu(u, v)$ of the DM?

$u = \omega + \beta - \Delta$
$v = \omega + \beta + \Delta$

=>

$u = \omega + \beta + \omega + \beta - v$

$2\omega = -2\beta + u + v$
$\omega = \frac{u+v}{2} - \beta$

Now we take expectations. And since $b = 0$, $E(\beta) = 0$:

$\mu(u, v) = E(\omega) = \frac{u+v}{2}$

Given these beliefs, if the expert plays $m(\omega, \beta)$ as outlined above he will obtain full manipulation:

$d([u, v]) = d(m(\omega, \beta)) = \frac{u+v}{2} = \frac{\omega+\beta-\Delta+\omega+\beta+\Delta}{2} = \omega + \beta$

Thus $\{m(\omega, \beta) = [\omega + \beta - \Delta, \omega + \beta + \Delta]$,$\mu(u, v) = \frac{u+v}{2}, D(\omega, \beta) = \omega + \beta\}$ is a fully manipulative PBE.

3 Is the fully manipulative equilibrium you dervied in 2) also an equilibrium when $\Delta < b$?

When $\Delta < b$ then $\beta \in [b - \Delta, b + \Delta]$ which means that in some cases $\beta > \Delta$. This means that the signal $m(\omega, \beta) = [\omega + \beta - \Delta, \omega + \beta + \Delta]$ would in some cases not include the truth because $u = \omega + \beta - \Delta > \omega$. Thus when $\Delta < b$ the previous equilibrium does not work.