Microeconomics 2^{nd} term Prof. P. Courty

- Book: Mas Colell, Whinston & Green, Microeconomic Analysis
- Handouts: Lecture notes, past exams, and problem sets
- Website: http://www.iue.it/Personal/Courty/courses.html
- Questions: karel.mertens@iue.it or pascal.courty@iue.it
- Evaluation: Exercises (30%) + 2 hour exam (open book) (70%)
- Office Hours: Thursday 16-18

This course will cover:

1. Competitive Markets and Welfare (MWG Ch 10 & 15)

2. Uncertainty, Insurance and Asset Markets (MWG Ch 19)

3. Externalities and Public Goods (MWG Ch 11)

4. Monopoly, Oligopoly and Monopolistic Competition (MWG Ch 12)

5. Incentives and Mechanism Design (MWG Ch 23)

1. Competitive Markets & Welfare

A. Competitive Equilibrium and Pareto Optimality (MWG Ch 10.B)

- Walrasian Equilibrium
- Pure Exchange Economy (Edgeworth Box) (MWG Ch 15.B)
- Closed Form Solutions
- First and Second Welfare Theorems

B. Partial Equilibrium Competitive Analysis (MWG Ch 10.C-F)

- The Quasilinear Model
- Welfare Analysis
- Comparative Statics

C. Extensions (MWG Ch 15.C-D)

- Wealth Effects
- Free Entry and Long Run CE
- One consumer - One Producer Economy
- The 2×2 Production Model

A. Competitive Equilibrium and Pareto Optimality

This section reviews some key concepts in economics in a very general setting. Consider the following world:

- endowment of $\omega_l \geq 0$ of commodity $l = 1, ..., L$
- consumer $i = 1, ..., I$ chooses bundles x_i in consumption set $X_i \subset \mathbb{R}^L$ and has preferences represented by $u_i(\cdot)$
- firm $j = 1, ..., J$ chooses production vector y_j in production set $Y_j \subset \mathbb{R}^L$

An **economic allocation** $(x_1, ..., x_I, y_1, ..., y_J)$ is **feasible** if

$$\sum_{i=1}^{I} x_{li} \leq \omega_l + \sum_{j=1}^{J} y_{lj}$$

for $l = 1, ..., L$.

A feasible allocation is (strongly) **Pareto optimal** if there is no other feasible allocation $(x'_1, ..., x'_I, y'_1, ..., y'_J)$ such that $u_i(x'_i) \geq u_i(x_i)$ for all i and $u_i(x'_i) > u_i(x_i)$ for some i.

The set of attainable utility levels is called the **utility possibility set** U.

$$U = \{(u_1, ..., u_I) \in \mathbb{R}^I : \text{there exists a feasible allocation}$$
$$(x_1, ..., x_I, y_1, ..., y_J) \text{ such that } u_i \leq u_i(x_i) \text{ for } i = 1, ..., I\}$$

Now suppose markets exist for each of the L goods and agents act as price takers.

A **competitive** or **Walrasian equilibrium** is an allocation $(x_1^*, .., x_I^*, y_1^*, ...y_J^*)$ of commodities and a vector p^* of L prices such that:

i. Firms maximize profits subject to technology constraints

ii. Consumers maximize utility subject to budget constraints

iii. All markets clear, i.e. $\sum_{i=1}^{I} x_{li}^* = \omega_l + \sum_{j=1}^{J} y_{lj}^*$ for $l = 1, ..., L$

This equilibrium concept is not uncontroversial.

- **Walras' law**

  If prices clear $L-1$ markets and preferences are locally non-satiated, then the L^{th} market must also clear.

  Proof. Adding up binding budget constraints and rearranging

$$\sum_{l\neq L} p_l(\sum_{i=1}^{I} x_{li} - \omega_l - \sum_{j=1}^{J} y_{lj}) = -p_L(\sum_{i=1}^{I} x_{Li} - \omega_L - \sum_{j=1}^{J} y_{Lj})$$

  By market clearing in the $L-1$ markets, the LHS equals 0. So the RHS must be 0 as well and the L^{th} market clears. □

- **Price Level Indeterminacy**

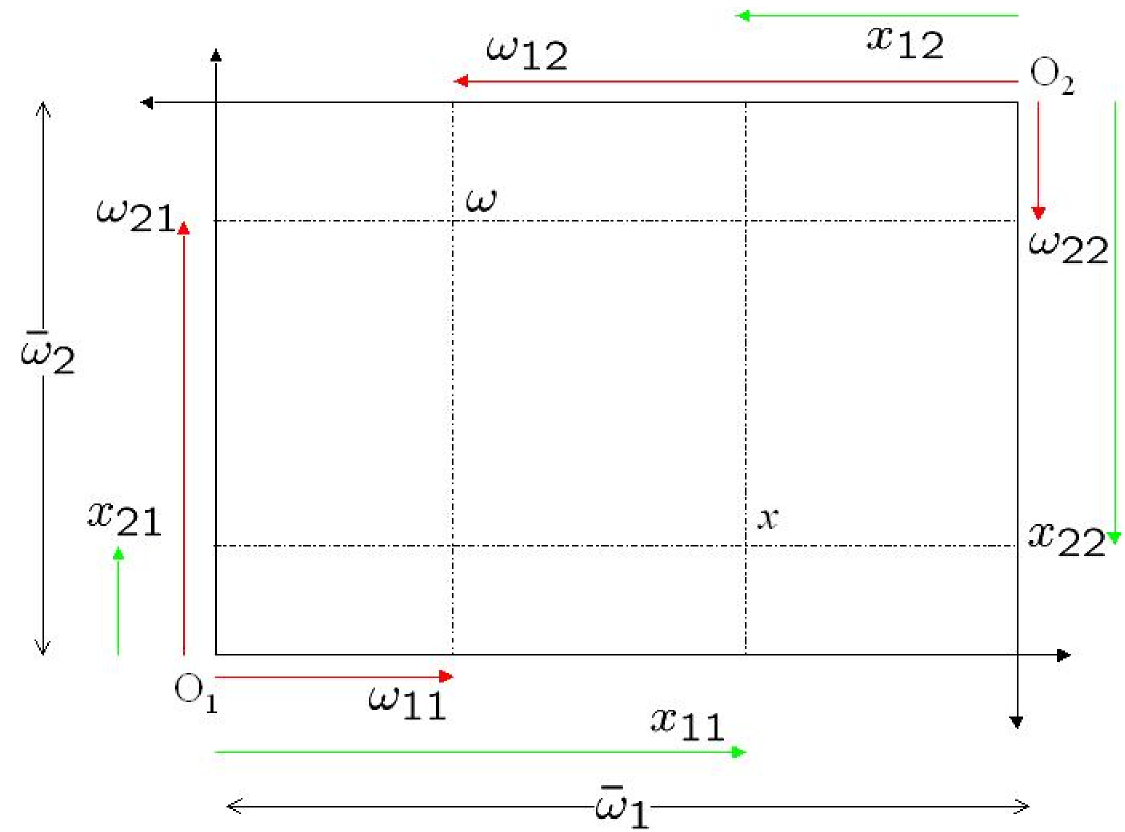
  If (x^*, y^*, p^*) is a CE, then $(x^*, y^*, \alpha p^*)$ as well, for any nonnegative scalar α. What is the **numeraire** ?

The Pure Exchange Economy

Consider the following world:

- two commodities, two agents
- total endowment of $\bar{\omega}_l \geq 0$, $l = 1, 2$ of which ω_{li} is owned by consumer i, $i = 1, 2$
- consumer $i = 1, 2$ chooses bundles $x_i = (x_{1i}, x_{2i})$ in consumption set $X_i \subset \mathbb{R}^2_+$ and has preferences represented by $u_i(\cdot)$
- There are no production opportunities

- An allocation is feasible if $x_{l1} + x_{l2} \leq \bar{\omega}_l$ for $l = 1, 2$.
- If the latter holds with equality, allocations are called **nonwasteful**.
- Nonwasteful allocations can be represented by an **Edgeworth Box**.



An Edgeworth Box

- For any prices $p = (p_1, p_2)$, consumer i's wealth is $p \cdot \omega_i = p_1 \omega_{1i} + p_2 \omega_{2i}$.

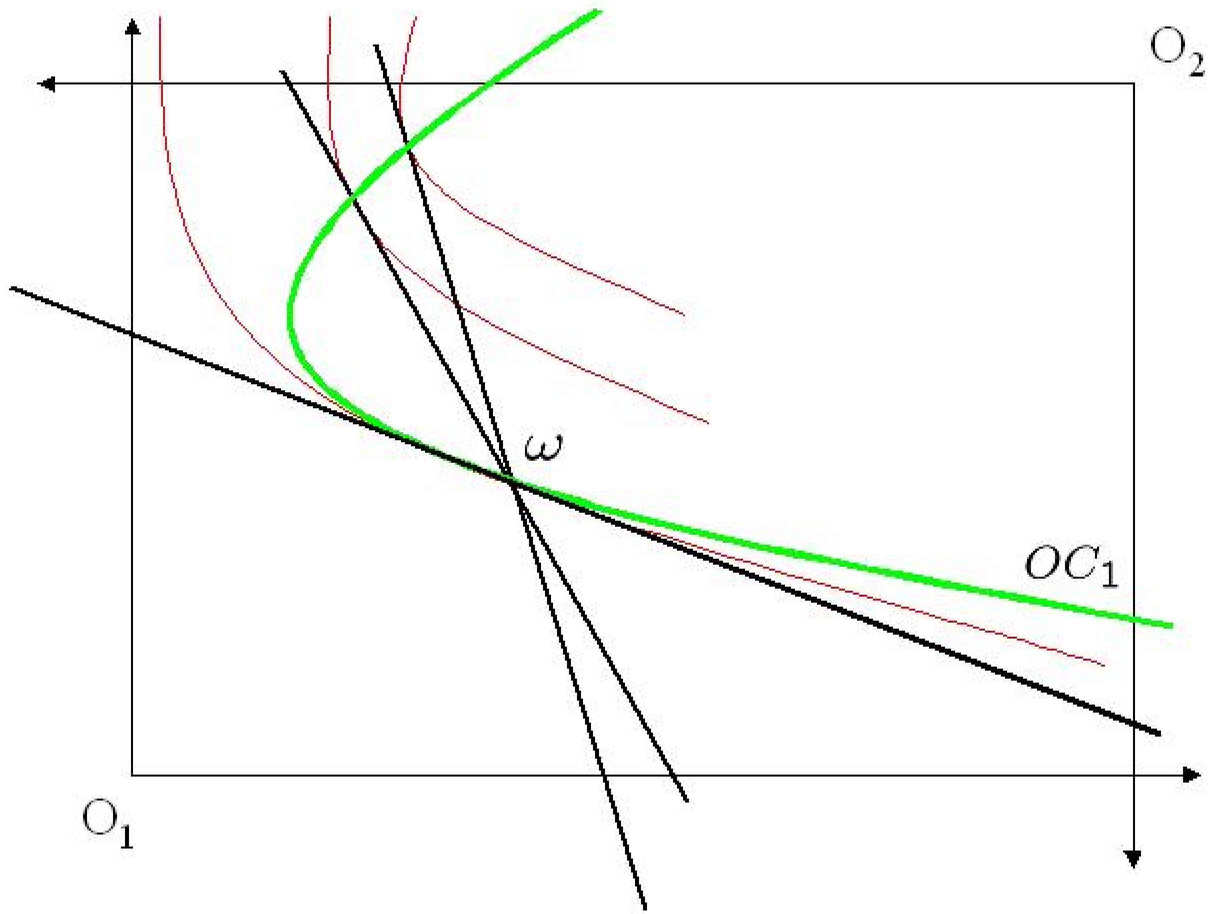
- We can draw the **budget line** through the **endowment point** ω with slope $-(p_1/p_2)$. (Why?)

- We can also represent preferences by **indifference curves** with slope $-\frac{\partial u_i}{\partial x_{1i}} / \frac{\partial u_i}{\partial x_{2i}}$. (Why?)

- Letting p vary, we can trace the **offer curve**, which passes through the endowment point.

- The consumer finds every point on his offer curve at least as good as his endowment point (why?).

- The offer curve is tangent to the budget line at the endowment point.



Consumer 1's Offer Curve

- Assume that preferences are strictly convex, continuous and strongly monotone.

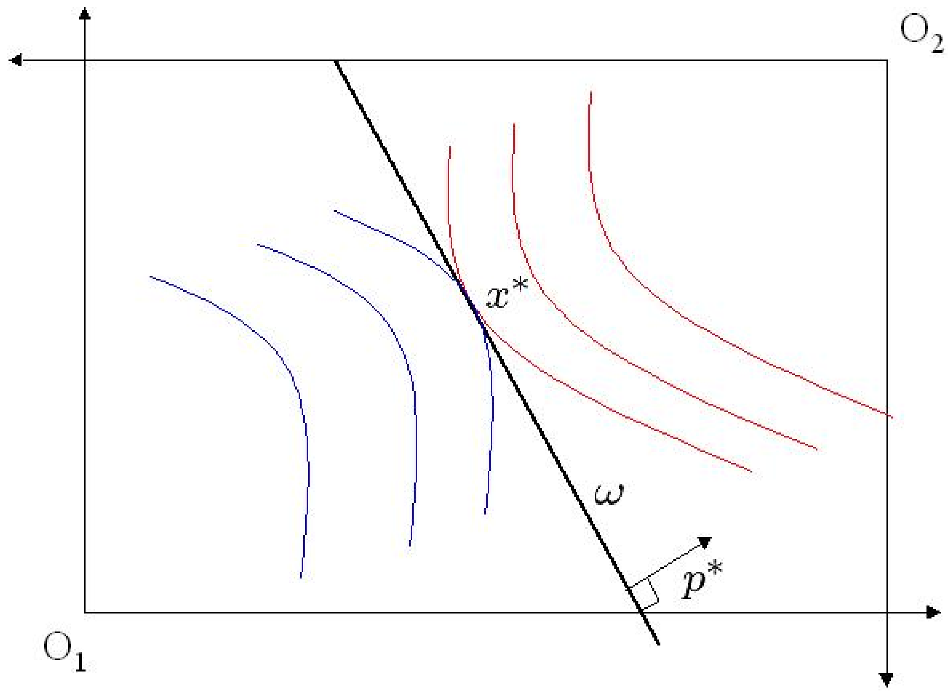
- A **competitive equilibrium for an Edgeworth box economy** is a price vector p^* and an allocation $x^* = (x_1^*, x_2^*)$ in the Edgeworth box such that for $i = 1, 2$,

$$u_i(x_i^*) \geq u_i(x_i')$$

for all x_i' in the budget set $B_i(p^*) = \{x_i \in \mathbb{R}_+^2 : x_i \leq p \cdot \omega_i\}$.

- In a CE, the consumers' offer curves intersect at the equilibrium allocation.

- Remark: Is price taking a sensible assumption with only two agents? See MWG Ch 18.C for an argument why it might be.



A competitive equilibrium in an Edgeworth Box Economy

It is easy to find examples of nonexistence or nonuniqueness of equilibria. Also, watch out for corner solutions!

When will the two agents trade?

Example 1:

- Consumer 1: $u_1 = x_{11}x_{21}$, $(\omega_{11}, \omega_{21}) = (2, 1)$
- Consumer 2: $u_1 = x_{12}x_{22}$, $(\omega_{12}, \omega_{22}) = (1, 2)$

A competitive equilibrium is a price vector (p_1^*, p_2^*) and an allocation $(x_{11}^*, x_{21}^*, x_{12}^*, x_{22}^*)$ such that

i. $(x_{11}^*, x_{21}^*) = \left(\frac{p_1^*\omega_{11}+p_2^*\omega_{21}}{2p_1^*}, \frac{p_1^*\omega_{11}+p_2^*\omega_{21}}{2p_2^*}\right) = \left(\frac{2p_1^*+p_2^*}{2p_1^*}, \frac{2p_1^*+p_2^*}{2p_2^*}\right)$

ii. $(x_{12}^*, x_{22}^*) = \left(\frac{p_1^*\omega_{12}+p_2^*\omega_{22}}{2p_1^*}, \frac{p_1^*\omega_{12}+p_2^*\omega_{22}}{2p_2^*}\right) = \left(\frac{p_1^*+2p_2^*}{2p_1^*}, \frac{p_1^*+2p_2^*}{2p_2^*}\right)$

iii. $x_{11}^* + x_{12}^* = \bar{\omega}_1 = 3$, $x_{21}^* + x_{22}^* = \bar{\omega}_2 = 3$

Substituting the Marshallian demands into the market clearing conditions:

$$\frac{2p_1^*+p_2^*}{2p_1^*} + \frac{p_1^*+2p_2^*}{2p_1^*} = 3$$
$$\frac{2p_1^*+p_2^*}{2p_2^*} + \frac{p_1^*+2p_2^*}{2p_2^*} = 3$$

All we can say is $p_1^* = p_2^*$. To determine equilibrium prices we only need to find prices at which one market clears. The other market will clear necessarily. (Why?)

If we normalize $p_1^* = 1$, the competitive equilibrium is $p_2^* = 1$ and $x_{11}^* = x_{12}^* = x_{21}^* = x_{22}^* = 3/2$.

Example 2:

- Consumer 1: $u_1 = \min\{x_{11}, x_{21}\}$, $(\omega_{11}, \omega_{21}) = (2, 1)$
- Consumer 2: $u_1 = \min\{x_{12}, x_{22}\}$, $(\omega_{12}, \omega_{22}) = (2, 3)$

A competitive equilibrium is a price vector (p_1^*, p_2^*) and an allocation $(x_{11}^*, x_{21}^*, x_{12}^*, x_{22}^*)$ such that

i. $(x_{11}^*, x_{21}^*) = \left(\frac{p_1^*\omega_{11}+p_2^*\omega_{21}}{p_1^*+p_2^*}, \frac{p_1^*\omega_{11}+p_2^*\omega_{21}}{p_1^*+p_2^*}\right) = \left(\frac{2p_1^*+p_2^*}{p_1^*+p_2^*}, \frac{2p_1^*+p_2^*}{p_1^*+p_2^*}\right)$

ii. $(x_{12}^*, x_{22}^*) = \left(\frac{p_1^*\omega_{12}+p_2^*\omega_{22}}{p_1^*+p_2^*}, \frac{p_1^*\omega_{12}+p_2^*\omega_{22}}{p_1^*+p_2^*}\right) = \left(\frac{2p_1^*+3p_2^*}{p_1^*+p_2^*}, \frac{2p_1^*+3p_2^*}{p_1^*+p_2^*}\right)$

iii. $x_{11}^* + x_{12}^* = \bar{\omega}_1 = 4$, $x_{21}^* + x_{22}^* = \bar{\omega}_2 = 4$

Normalizing $p_1^* = 1$ and substituting the Marshallian demands into the market clearing conditions, we are left with the identity:

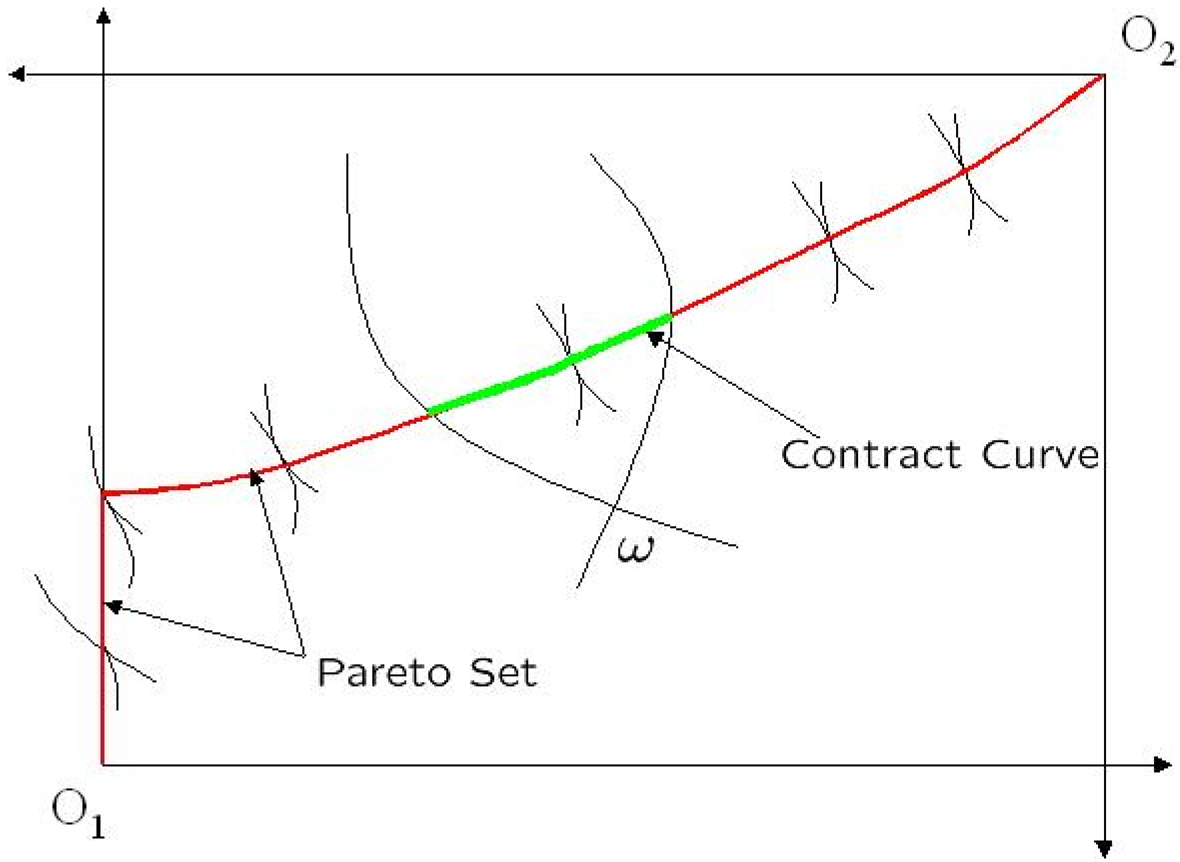
$$4 + 4p_2^* = 4 + 4p_2^*$$

Hence, any $p_2^* \geq 0$ clears the markets. This is an example of multiple competitive equilibria (in fact there are infinitely many equilibrium allocations).

- An allocation x in the Edgeworth box is **Pareto optimal** if there is no other allocation x' in the Edgeworth box with $u_i(x_i') \geq u_i(x_i)$ for $i = 1, 2$ and $u_i(x_i') > u_i(x_i)$ for some i.

- The set of all Pareto optimal allocations is called the **Pareto set**.

- The part of the Pareto set where both consumers do at least as well as their initial endowments is called the **contract curve**.



The Pareto set and the contract curve

First Fundamental Welfare Theorem

Assume that preferences are locally nonsatiated. If $\{p^*, x^*\} = \{(p_1^*, p_2^*), (x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^*)\}$ is a competitive equilibrium, then the equilibrium allocation x^* is Pareto optimal.

Proof. Suppose x^* is not Pareto optimal. Then there exists a feasible allocation x' such that $u_i(x_i') \geq u_i(x_i^*)$ for $i = 1, 2$ and $u_i(x_i') > u_i(x_i^*)$ for some i.
By local nonsatiation, it must be true that $p^* \cdot x_i \geq p^* \cdot \omega_i$ for $i = 1, 2$ and $p^* \cdot x_i > p^* \cdot \omega_i$ for some i. Summing across consumers we have that $p^* \cdot \sum_{i=1}^{2} x_i > p^* \cdot \sum_{i=1}^{2} \omega_i \Leftrightarrow \sum_{i=1}^{2} x_i > \sum_{i=1}^{2} \omega_i = \omega$ which contradicts feasibility. $\square$

Moreover, any Walrasian equilibrium lies on the contract curve of the Pareto set. (Why?)

Second Fundamental Welfare Theorem

If preferences are convex, locally nonsatiated (+ some other technical assumptions), then any Pareto optimal allocation can be implemented as a competitive equilibrium through lump-sum redistribution.

The proof is outside of the scope of these lectures.

See MWG Ch 16 for detailed proofs in more general context.

B. Partial Equilibrium Competitive Analysis

- Originates with Marshall (Principles of Economics, 1920).
- Partial Equilibrium Analysis focuses on one market only and ignores interaction with other markets.
- Interaction between markets is due to cross-price effects and wealth effects on demand.
- With some simplifying assumptions, these are absent even within a general equilibrium context.

The Quasilinear Model

- Consumer $i = 1, ..., I$ has utility function $u_i(m_i, x_i) = m_i + \phi_i(x_i)$ with $\phi_i'(x_i) > 0$ and $\phi_i''(x_i) < 0$ at all $x_i \geq 0$. Normalize $\phi_i(0) = 0$.

- m_i is a composite numeraire good, whereas $x_i \geq 0$ is consumption of a good l. (Note: For convenience, m_i can be negative)

- Firm $j = 1, ..., J$ can produce q_j of good l using the numeraire as input and has a cost function $c_j(q_j)$ with $c_j'(q_i) > 0$ and $c_j''(q_j) \geq 0$ at all $q_j \geq 0$.

- Consumer i is endowed with ω_{mi} of the numeraire good. Let $\omega_m = \sum_{i=1}^{I} \omega_{mi}$ be the total endowment of the numeraire good in the economy. There is no initial endowment of good l

- The composite good can be treated as expenditure on all other goods. The market for good l is sufficiently small.

- Normalize the price of the numeraire good to one and let p be the price of good l, then the competitive equilibrium conditions are:

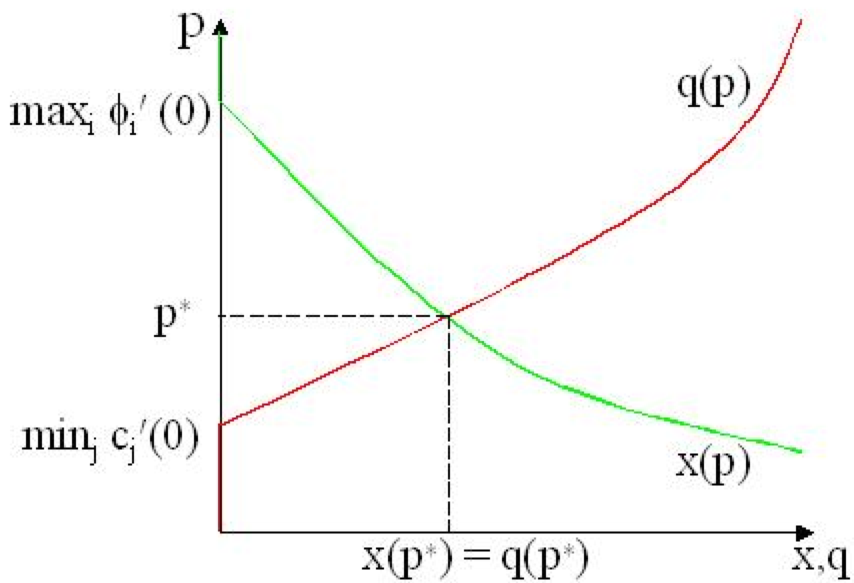
  i. profit maximization:
  $$p^* \leq c'_j(q_j^*) \text{ with equality if } q_j^* > 0,\ j = 1, ..., J$$
  ii. utility maximization:
  $$\phi'_i(x_i^*) \leq p^* \text{ with equality if } x_i^* > 0,\ i = 1, ..., I$$
  iii. market clearing:
  $$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

- This means $I + J + 1$ conditions, for as many unknowns.
- Recall that with quasilinear preferences, Marshallian demands depend only on p. There are no wealth effects.
- Because of Walras' law, we can focus on good l.
- As long as $\max_i \phi_i'(0) > \min_j c_j'(0)$, there is trade in good l.(Why?)

i. **Aggregate Demand** $x(p) = \sum_i x_i(p) = \sum_i \phi_i'^{-1}$, continuous and non-increasing at all $p > 0$. It is the horizontal summation of individual Walrasian demands.

ii. **Aggregate Supply** $q(p) = \sum_j q_j(p) = \sum_j c_j'^{-1}(p)$, continuous and nondecreasing at all $p > \min_j c_j'(0)$. It is the horizontal summation of individual firms' supply functions.

Note: Convexity is key to the existence of a competitive equilibrium. (Why?)



p
$\max_i \phi_i'(0)$
q(p)
p*
$\min_j c_j'(0)$
x(p)
$x(p^*) = q(p^*)$
x,q

- Quasilinear preferences allow unlimited unit-for-unit transfer of utility across consumers through transfers of the numeraire.

Proof. Fix the consumption and production level of good l at the feasible level $(\bar{x}, \bar{q}) = (\bar{x}_1, \ldots, \bar{x}_I, \bar{q}_1, \ldots, \bar{q}_J)$. Feasibility implies

$$\sum_{i=1}^{I} \bar{x}_i \leq \sum_{j=1}^{J} \bar{q}_j \text{ and } \sum_{i=1}^{I} m_i + \sum_{j=1}^{J} c_j(\bar{q}_j) \leq \omega_m$$

Summing utility across consumers yields $\sum_{i=1}^{I} u_i = \sum_{i=1}^{I} m_i + \sum_{i=1}^{I} \phi_i(\bar{x}_i)$, but then

$$\sum_{i=1}^{I} u_i \leq \sum_{i=1}^{I} \phi_i(\bar{x}_i) - \sum_{j=1}^{J} c_j(\bar{q}_j) + \omega_m$$

The boundary of the set $\{(u_1, \ldots, u_I) \in \mathbb{R}^I : \sum_{i=1}^{I} u_i \leq \sum_{i=1}^{I} \phi_i(\bar{x}_i) - \sum_{j=1}^{J} c_j(\bar{q}_j) + \omega_m\}$ is just a hyperplane with normal vector $(1, \ldots, 1)$. Clearly, how ω_m is distributed among consumers does not affect this boundary. There can be unlimited unit-for-unit transfer of utility across consumers through transfers of the numeraire. □

- The utility possibility set is

$$\{(u_1, ..., u_I) \in \mathbb{R}^I : \sum_{i=1}^{I} u_i \leq \sum_{i=1}^{I} \phi_i(x_i^*) - \sum_{j=1}^{J} c_j(q_j^*) + \omega_m\}$$

where

$$(x_i^*, q_j^*) = arg \max_{\substack{x_i \gg 0 \\ q_j \gg 0}} \{\sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j) + \omega_m \text{ s.t. } \sum_{i=1}^{I} x_i - \sum_{j=1}^{J} q_j = 0\}$$

This amounts to choosing (x_i, q_j) that shifts the boundary out as far as possible. By definition, (x_i^*, q_j^*) belongs to the Pareto set.

- **First Fundamental Welfare Theorem** : Verify that the conditions that determine the Pareto point (x_i^*, q_j^*) are formally identical to the competitive equilibrium conditions with the Lagrange multiplier on the feasibility constraint, or the **"shadow price"**, equal to the market price.

- $\sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$ is nothing more than Marshallian aggregate surplus. Thus, a Pareto allocation maximizes Marshallian surplus. Equivalently, in a competitive equilibrium, Marshallian surplus is maximized.

- Welfare analysis is easy since Marshallian aggregate surplus is a proper welfare metric for *any* social welfare function.

Comparative Statics in the Partial CE Model

- Consider the effects of a **sales tax**. The equilibrium conditions imply, assuming an interior solution:

$$x(p^*(t)+t) = q(p^*(t))$$

- Assuming $x(\cdot)$ and $q(\cdot)$ are differentiable:

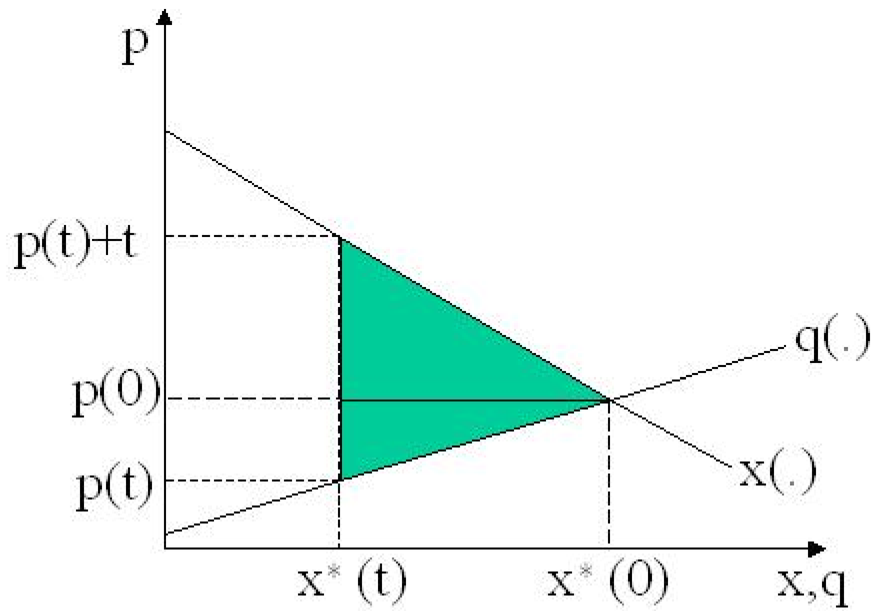
$$-1 \leq p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)-q'(p^*(t))} < 0$$

- **Tax incidence**

$$q'(p^*(t)) \to \infty \text{ or } x'(p^*(t)) = 0, p^{*\prime}(t) \to 0\text{: consumers pay}$$
$$q'(p^*(t)) = 0 \text{ or } x'(p^*(t)) \to \infty, p^{*\prime}(t) = -1\text{: producers pay}$$

i. The change in **Marshallian Aggregate Surplus**:
$\Delta MAS = \Delta CS + \Delta PS = -\int_{p^*(0)}^{p^*(t)+t} x(s)ds - \int_{p^*(t)}^{p^*(0)} q(s)ds$

ii. The total **Dead Weight Loss** of the sales tax:
$DWL = \mid \Delta MAS + tx^*(t) \mid$



C. Extension: Wealth Effects and Consumer Surplus

- Why does the area below the demand curve does not always capture the change in consumer surplus?

- Consumer i maximizes $u_i(x_i)$ subject to the constraint that x_i is in the budget set $B_i(p^*) = \{x_i \in \mathbb{R}^L_+ : x_i \leq p \cdot \omega_i\}$.

- F.O.C. (for interior solution) $\frac{d}{dx_{i,l}} u_i(x_i) = \lambda_i p_l$

- $du_i = \sum_l \frac{d}{dx_{i,l}} u_i(x_i) dx_{i,l} = \sum_l \lambda_i p_l dx_{i,l}$

- Holding p_l constant for $l > 1$ implies $\int du_i = \int \lambda_i p_l dx_{i,l}$

- But $\lambda_i = \lambda_i(p, \omega_i)$! (For quasi-linear preferences $\lambda_i = 1$)

C. Extension: Free Entry and Long Run CE

- In the long run, firms can enter or exit the market.
- Firms will enter as long as profits are positive at the margin.

Assume common technology and $c(0) = 0$ (Why?), a triple (p^*, q^*, J^*) is a **long run competitive equilibrium** if

i. q^* solves $\max\limits_{q \geq 0} p^* q - c(q)$

ii. $x(p^*) = J^* q^*$

iii. $p^* q^* - c(q^*) = 0$

J^* is the number of firms in equilibrium.

- The long run equilibrium price can be thought of as equating demand with long run supply.

- To generate existence of a long run equilibrium with a determinate number of firms, there must exist a strictly positive output level $\bar{q} = arg\min\{\frac{c(q)}{q}\}$.

- Define $\bar{c} = \frac{c(\bar{q})}{\bar{q}}$ and assume $x(\bar{c}) > 0$, then any long run equilibrium implies: $p^* = \bar{c}$.

Proof. Suppose $p^* > \bar{c}$, then $p^*\bar{q} > \bar{c}\bar{q}$ and $\pi(p^*) > 0$. Suppose $p^* < \bar{c}$, then $x(p^*) > 0$, but since $p^*q - c(q) = p^*q - (\frac{c(q)}{q})q \leq (p^* - \bar{c})q < 0$ for all $q > 0$, $\pi(p^*) < 0$. So $p^* = \bar{c}$ must be true. □

C. Extension: Examples of General Equilibrium Models

Before:

- **The pure exchange economy**: endowment economy, no production
- **The partial equilibrium model**: motivated from a GE perspective with many simplifying assumptions (Which?), no interaction between markets

Now: production and consumption with interacting markets.

- **One consumer - One Producer Economy**
- **The** 2×2 **Production Model**

One consumer - One Producer Economy

Introduce production in the simplest-possible setting: **The Robinson Crusoe Economy.**

- One price-taking firm uses labour input z to produce $f(z)$ units of the consumption good, where $f' > 0, f'' < 0$ and maximizes profits.

$$\max_{z \geq 0}(pf(z) - w(z))$$

Given prices (p, w), labour demand is $z(p, w)$, output $q(p, w)$ and profits $\pi(p, w)$.

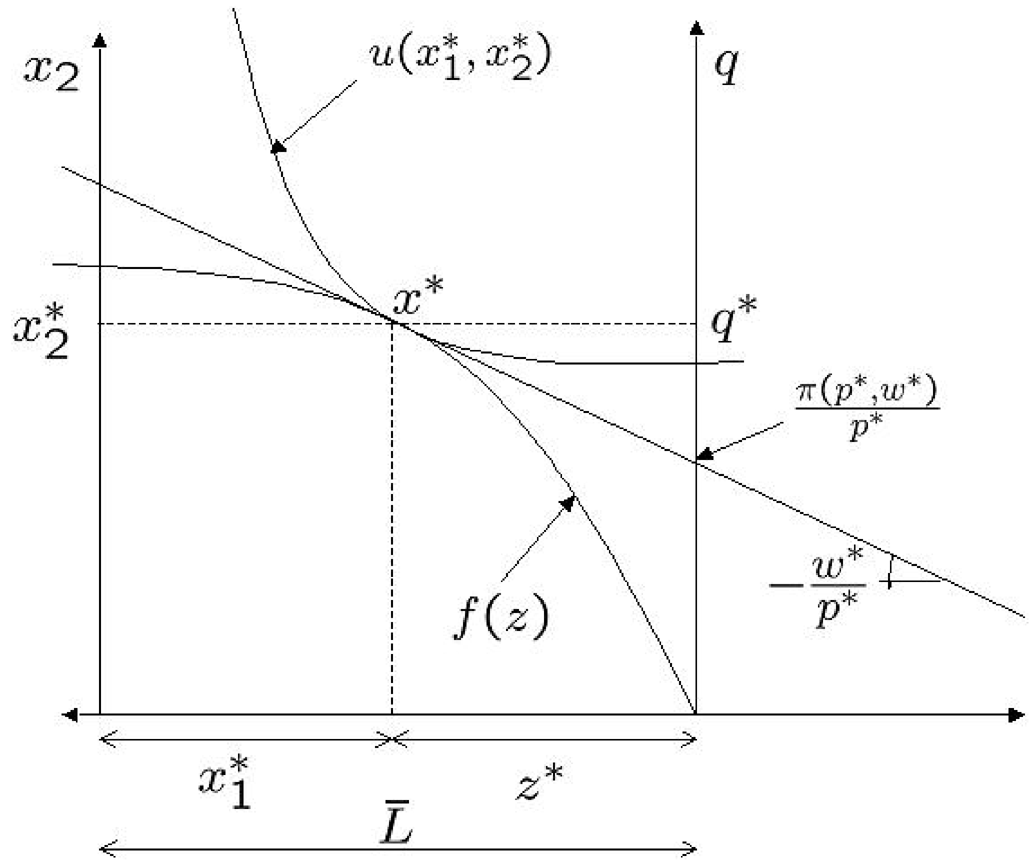
- One price-taking consumer chooses leisure x_1 and a consumption good x_2 to maximize $u(x_1, x_2)$ and has time endowment of $\bar{L}$. He also owns the firm.

$$\max_{x_1, x_2 \in \mathbb{R}^2_+} u(x_1, x_2) \text{ s.t. } px_2 \leq w(\bar{L} - x_1) + \pi(p, w)$$

  Given prices (p, w), demand is $(x_1(p, w), x_2(p, w))$.

- A competitive equilibrium is a price vector (p^*, w^*) at which consumption and labour markets clear.

$$x_2(p^*, w^*) = q(p^*, w^*)$$
$$z(p^*, w^*) = \bar{L} - x_1(p^*, w^*)$$



Competitive Equilibrium in a Robinson Crusoe Economy

- For any (p, w), the consumer's budget line is exactly the firm's isoprofit line and has slope $-\frac{w}{p}$.

- For any (p, w), the budget line must cross the vertical q-axis at height $\frac{\pi(p,w)}{p}$.

- x^* can arise in a competitive equilibrium if and only if it maximizes utility subject to technological and endowment constraints. (First Fundamental Welfare Theorem)

- Under certain assumptions, this model can be thought of as a representative agent model.

The 2×2 Production Model

- There are $J = 2$ firms that each produce one consumer good q_j using a vector of $L = 2$ production factors, $z_j = (z_{1j}, z_{2j}) \geq 0$. There are no intermediate goods.

- Firm j has a CRTS production function $f_j(z_j)$.

- The economy has total factor endowments $(\bar{z}_1, \bar{z}_2)$, initially owned by consumers.

- To focus on factor markets, assume consumption good prices are fixed at $p = (p_1, p_2)$ (cfr. small open economy).

A competitive equilibrium for the factor markets in this economy consists of an input price vector $w^* = (w_1^*, w_2^*) \gg 0$ and a factor allocation $z^* = (z_1^*, z_2^*) = ((z_{11}^*, z_{21}^*), (z_{12}^*, z_{22}^*))$ such that:

i. $p_j \frac{\partial f_j}{\partial z_{1j}} = w_1^*$ for $j = 1, 2$ (first FOC of firm j)

ii. $p_j \frac{\partial f_j}{\partial z_{2j}} = w_2^*$ for $j = 1, 2$ (second FOC of firm j)

iii. $z_{l1}^* + z_{l2}^* = \bar{z}_l$ for $l = 1, 2$ (market for factor l must clear)

$\Longrightarrow$ In general, we have a non-linear system of 6 equations to solve for the 6 unknown variables.

However, the equilibrium in this model is usually defined in another way.

- For every $w = (w_1, w_2)$, define $c_j(w)$ as the minimum cost of of producing one unit of good j.

- Also define $\nabla c_j(w) = (a_{1j}(w), a_{2j}(w))$. This vector describes the input combination for which this minimum cost is reached (by Shephard's lemma). This input combination is assumed to be unique.

- **The factor intensity assumption** :
  the production of good 1 is relatively more intensive in factor 1 than the production of good 2, i.e.

$$\frac{a_{11}(w)}{a_{21}(w)} > \frac{a_{12}(w)}{a_{22}(w)}$$

  at all factor prices w.

- Because production is CRTS, $c_j(w)$ is homogenous of degree one. By Euler's theorem we can state

$$c_j(w) = a_{1j}w_1 + a_{2j}w_2 \text{ for } j = 1,2$$

- Now we can define a competitive equilibrium (w^*, y_1^*, y_2^*) by

  i. $c_j(w^*) = a_{1j}w_1^* + a_{2j}w_2^* = p_j$ for $j = 1,2$
  ii. $a_{l1}y_1^* + a_{l2}y_2^* = \bar{z}_l$ for $l = 1,2$

- Now only 4 equations in 4 unknowns!

- We can also solve i) independently for w^*!

- Because of the factor intensity assumption, there is at most a single w^* that solves i) that can arise as an equilibrium factor price vector of an interior equilibrium.

- The 2×2 model has proven quite popular and has been used in:
  - international trade: the Heckscher-Ohlin model
  - public finance: the Harberger model of tax incidence
  - neoclassical growth theory
- see Exercises for more on the 2×2 model

A. Uncertainty in an Exchange Economy

Describing an Uncertain World

- Consider a simple exchange economy with L physical commodities and I consumers.

- Contrary to the last section, endowments and/or preferences are uncertain and depend on the state of the world.

- The **state of the world** is a complete description of a possible outcome of uncertainty, sufficiently fine for any two distinct states of the world to be mutually exclusive. The finite set S is the set of possible states with element $s = 1, .., S$.

- For every physical commodity $l = 1, ..., L$ and state $s = 1, ..., S$, a unit of **state contingent commodity** ls (or an **Arrow-Debreu security**) is a title to receive a unit of the physical good l if and only if state s occurs.

- A state contingent commodity vector is

$$x = (x_{11}, ..., x_{L1}, ..., x_{1S}, ..., x_{LS}) \in \mathbb{R}^{LS}$$

It is a collection of L random variables, the l^{th} random variable being $(x_{l1}, ..., x_{lS})$.

- Let the endowments of consumer $i = 1, ..., I$ be a contingent commodity vector

$$\omega_i = (\omega_{11i}, ..., \omega_{L1i}, ..., \omega_{1Si}, ..., \omega_{LSi})$$

- Consumer preferences are also state-dependent and are specified by a rational preference relation $\succsim_i$ on a consumption set $X_i \subset \mathbb{R}^{LS}$.

- Moreover, it is assumed that consumer i assigns to a state s a probability π_{si} and then evaluates physical commodity vectors at state s according to a **Bernouilli state-dependent utility function** $u_{si}(\cdot)$, i.e. for any $x_i, x_i' \in X_i \subset \mathbb{R}^{LS}$

$$x_i \succsim_i x_i' \text{ if and only if } U_i(x_i) > U_i(x_i')$$

where $U_i(x_i) = \sum_s \pi_{si} u_{si}(x_{1si}, ..., x_{Lsi})$ is the **(von-Neumann-Morgenstern) expected utility function**. (recap MWG Ch 6)

Arrow-Debreu equilibrium

- Assume that the state of the world is observable and contractable by every agent (**symmetric information**).

- Assume a market exists for every contingent commodity ls. In other words, there is a **full set of Arrow-Debreu (AD) securities**.

- Markets for AD-securities open *before* the resolution of uncertainty (at date zero).

- The price for every contingent commodity/AD security is p_{ls}.

- Note that consumers trade in commitments to receive or deliver amounts of physical good l contingent on state s. Payments, however, are *not* state-contingent.

- An **Arrow-Debreu equilibrium** is an allocation $(x_1^*, ..., x_I^*) \in X_1 \times ... \times X_I \in \mathbb{R}^{LSI}$ and a system of prices for the contingent commodities $p = (p_{11}, ..., p_{LS}) \in \mathbb{R}^{LS}$ such that

   i. For every i, x_i^* maximizes expected utility $U_i(x_i)$ and is in the budget set $\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i\}$

   ii. All markets clear, i.e. $\sum_i x_i^* = \sum_i \omega_i$

- This is just the standard definition of a competitive equilibrium in an exchange economy, so the welfare theorems apply. An AD-equilibrium is Pareto optimal.

- Efficiency is achieved because we assume consumers can trade all state contingent commodities, that is, we have **complete markets**.

Economic Issues

- How is risk allocated in the economy?
- How do state contingent prices depend on the parameters of the economy (state utility, subjective probabilities, endowments)?
- Is it necessary to trade all contingent claims to achieve the efficient outcome?
- What happens when markets are not complete?
- How to price (new) assets?

AD equilibrium in the Edgeworth box

- Suppose $I = S = 2$ and $L = 1$, i.e. there are two consumers, two states, and a single good.

- We will consider in turn an Edgeworth Box economy:

  Case A without aggregate uncertainty and with identical probabilities
  Case B without aggregate uncertainty and with different probabilities
  Case C with aggregate uncertainty and with identical probabilities
  Case D with aggregate uncertainty and with different probabilities

Case A: No Aggregate Uncertainty and Identical Probabilities

- Let $\omega_1 = (1, 0)$ and $\omega_2 = (0, 1)$ such that $\omega_1 + \omega_2 = (1, 1)$ and there is no aggregate uncertainty. The state determines only which consumer gets the endowment.

- Probability assessments are the same for the two consumers, i.e. $(\pi_{11}, \pi_{21}) = (\pi_{12}, \pi_{22}) = (\pi_1, \pi_2)$.

- Assume concave Bernouilli utility functions that are state-independent such that utility of $i = 1, 2$ is given by $U_i = \pi_1 u_i(x_{1i}) + \pi_2 u_i(x_{2i})$.

- Recall that the concavity assumption (from convex preferences) has the interpretation of **risk aversion**.

- The MRS_i of consumer i is given by $\frac{\pi_1 u_i'(x_{1i})}{\pi_2 u_i'(x_{2i})}$ (Why?)

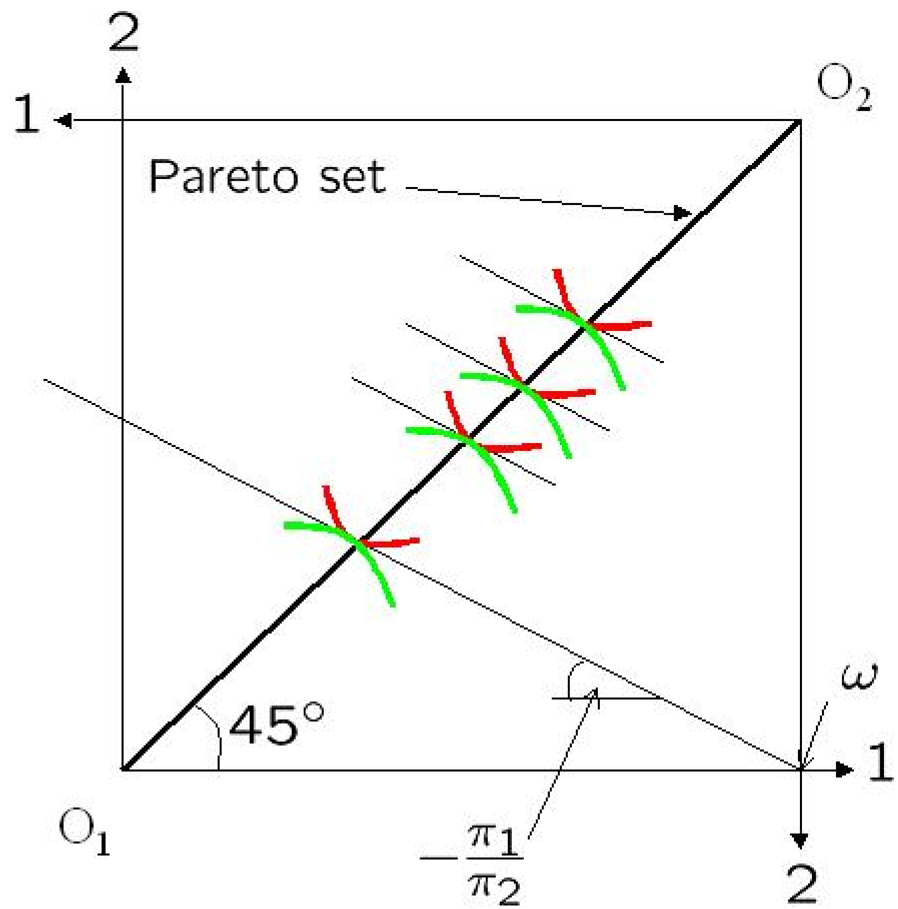
- Denote the AD equilibrium by a pair of prices (p_1^*, p_2^*) and by the allocation $(x_{11}^*, x_{21}^*, x_{12}^*, x_{22}^*)$.

- In an interior equilibrium, we must have

$$\frac{p_1^*}{p_2^*} = MRS_1 = MRS_2 \Leftrightarrow \frac{\pi_1 u_1'(x_{11}^*)}{\pi_2 u_1'(x_{21}^*)} = \frac{\pi_1 u_2'(x_{12}^*)}{\pi_2 u_2'(x_{22}^*)}$$

  which requires $x_{11}^* = x_{21}^*$ and $x_{12}^* = x_{22}^*$ such that $\frac{p_1^*}{p_2^*} = \frac{\pi_1}{\pi_2}$.

- There is **full insurance**: equilibrium consumption does not vary across states. The price of the AD security whose state is deemed more likely is higher.



Equilibrium in Case A

Case B: No Aggregate Uncertainty and Different Probabilities

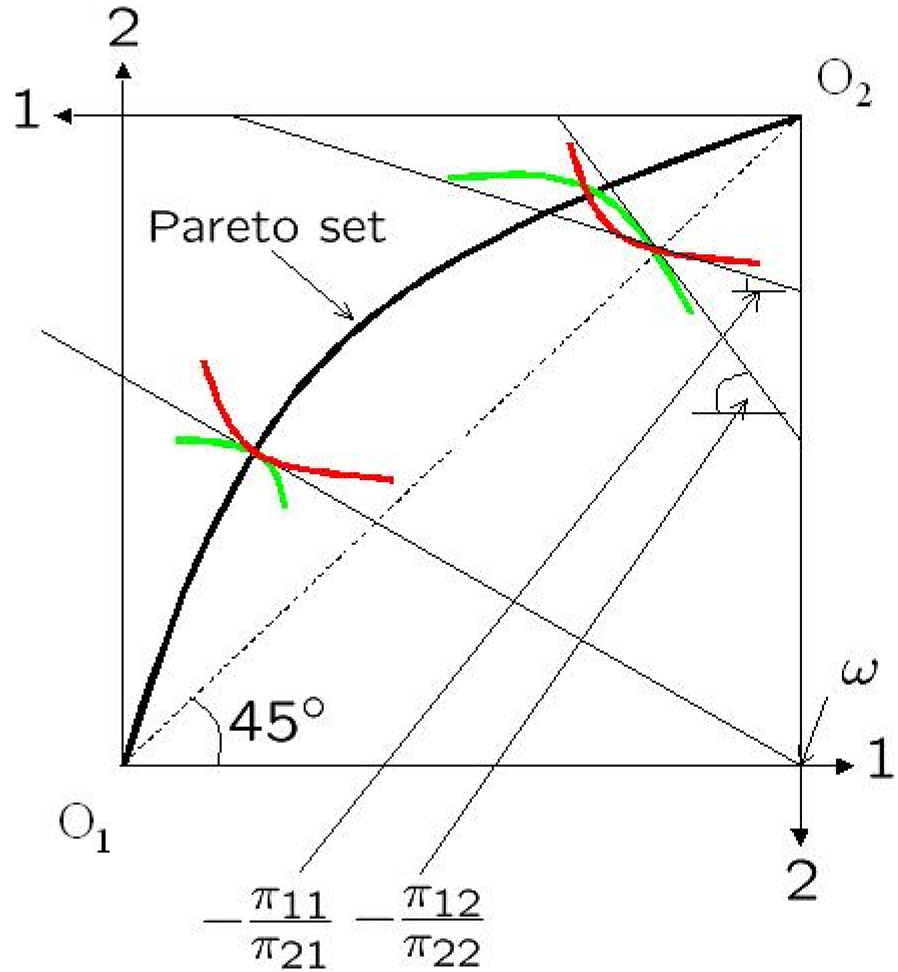
- Suppose that the consumers have different subjective beliefs about the likeliness of the states, i.e. $(\pi_{11}, \pi_{21}) \neq (\pi_{12}, \pi_{22})$.

- In particular, suppose $\pi_{11} < \pi_{12}$, i.e. consumer 2 assigns a higher probability to state 1.

- In an interior equilibrium, we must have

  $\frac{p_1^*}{p_2^*} = \frac{\pi_{11} u_1'(x_{11}^*)}{\pi_{21} u_1'(x_{21}^*)} = \frac{\pi_{12} u_2'(x_{12}^*)}{\pi_{22} u_2'(x_{22}^*)}$ such that $x_{11}^* < x_{21}^*$, $x_{12}^* > x_{22}^*$.

- Each consumer's equilibrium consumption is higher in the state he thinks more probable relative to the other consumer's beliefs.



Equilibrium in Case B

Insurance: Issues

- Under what condition does trade take place?
- Computation of fair insurance prices?
- Equilibrium when one consumer is risk neutral and has objective probabilities

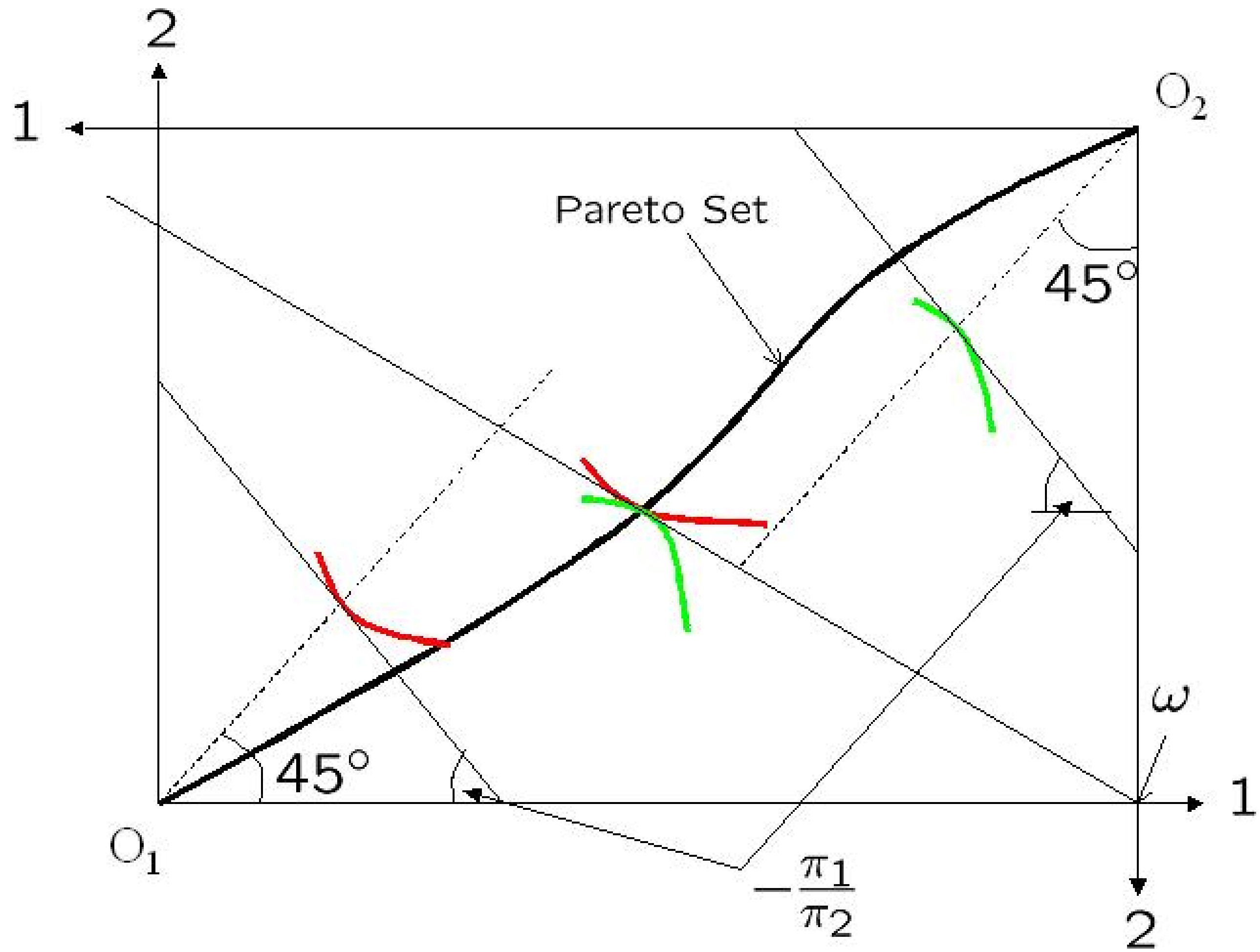
Case C: Aggregate Uncertainty and Identical Probabilities

- Suppose there is aggregate risk, i.e. $\omega_1 + \omega_2 = (2,0) + (0,1) = (2,1)$.

- The consumers' probability assessments are again identical: (π_1, π_2).

- Again, we must have $\frac{p_1^*}{p_2^*} = \frac{\pi_1 u_1'(x_{11}^*)}{\pi_2 u_1'(x_{21}^*)} = \frac{\pi_1 u_2'(x_{12}^*)}{\pi_2 u_2'(x_{22}^*)}$. At any point of the Pareto set, the common MRS must be smaller then the ratio of probabilities such that $\frac{p_1}{p_2} < \frac{\pi_1}{\pi_2}$.

- Suppose $\pi_s = \frac{1}{2}$, then $p_1 < p_2$ and the price is larger for the state in which the good is scarcer. This illustrates a fundamental result in finance: an asset (here an AD security) whose pay-off is negatively correlated with the market return (here the endowment) is more valuable.



Equilibrium in Case C

Case D: Aggregate Uncertainty and Different Probabilities

State Dependent Utility

State dependent utility: utility depends on state of nature if do not value money as much in some states

Application: Army draft

- $v_A(I)$ and $v_C(I)$ are the indirect utility of income in the army and civilian
- v_A and v_C are positive, increasing and concave
- $v_A(I) < v_C(I)$ $v'_A(I) < v'_C(I)$

Equilibrium with and without transfer

- Equilibrium under professional army $V_A(I_A) = V_C(I_C)$
- $I_A > I_C$ compensating differential principle
- Can a social planner do better by reallocating income?
- Assume fraction p of population engages in the army
- Consider transfer: draftees give Δ_A and non-draftees receive Δ_C such that $p\Delta_A = (1-p)\Delta_C$
- Alternative interpretation: Eligible candidates are drafted with probability p and can buy lotteries that are contingent on the draft's outcome

- Social planner maximizes $pV_A(I_A - \Delta_A) + (1-p)V_C(I_C - \Delta_C)$ subject to $p\Delta_A = (1-p)\Delta_C$

- F.O.C. $V_A'(I_A - \Delta_A) = V_C'(I_C - \Delta_C) \Rightarrow \Delta_A > 0$

- Consume less in the army!

- No perfect insurance although objective probabilities and fair prices

- Intuition: Since marginal utility of income is low in the army, it is optimal from an ex-ante point of view to transfer income from army status to civilian status

B. Asset Markets

- Sequential trade
- Option pricing with two states of the world
- Arbitrage free condition and asset pricing
- Interpretation in terms of probabilities and utilities

Sequential Trade

- Arrow-Debreu equilibrium assumes that it is possible to trade contingent claims for all goods in all state of the world
- In a T-period model, this assumes $LxSxT$ markets
- **Radner Equilibrium**: Sufficient to trade one good in all states and let spot markets allocate this fixed endowment across goods
- Two step decision (a) how much wealth to carry to next period? (b) how to allocate today's wealth across goods?
- In two period model, only need to make $L + S$ trades at date 0 and L trades at date one instead of $(L + 1)S$ trades

Asset Markets

- An **asset**, or **security**, is a title to receive either physical goods (real assets) or money (financial assets) at some point in the future. The pay-off is known as the **return** and may depend on the state of the world.

- Suppose there are two dates ($t = 0, 1$). All information about the state $s \in S$ is revealed at $t = 1$.

- All payoffs are in terms of good 1, the numeraire.

- There is only consumption at $t = 1$.

- Then the return of an asset at $t = 1$ is characterized by its return vector $r = (r_1, ..., r_S) \in \mathbb{R}^S$.

Examples

- If $r = (1, ..., 1)$, we have a **commodity future**. It promises a non-contingent pay-off of 1. Is r a **riskless** asset?

- If $r = (0, ..., 0, 1, 0, ..., 0)$, we have an **Arrow Debreu security** as seen before. AD securities are largely theoretical constructs.

- **Options** are **derivative assets** rather than **real assets**, i.e., their returns are derived from the returns of other assets:
  - A **call option** provides the holder with the right to purchase the primary asset (stock) at a specified strike price K and specified expiration date.
  - A **put option** provides the holder with the right to *sell*...

Option Pricing

Study how the option price depend on (a) the asset price, (b) asset distribution of return, (c) strike price, and (d) expiration date

- Two period model (date 0 and 1) and two states of the world, up or down, that occur with respective probability q and $1-q$

- Two primary assets: the stock is worth S at date 0 and gives return uS (dS) at date 1, $r(Stock) = (uS, dS)$, and the riskfree asset, a bond, which is worth 1 at date 0 and gives return r at date 1, $r(Bond) = (r, r)$

- A call option with strike price K and expiration date 1 is worth C at date 0

- Can we express C as a function of (u, d, S, K, q, r)?

- A call option with strike price K will only be exercised if K is less than the stock value

- Hence the return vector of the call option is

$$r(Option) = (C_u, C_d) = (\max(0, uS - K)), \max(0, dS - K))$$

- Consider a portfolio composed of Δ units of stocks and B units of bonds

- This portfolio has return $r(Portfolio) = (\Delta uS + B, \Delta dS + B)$

- Set Δ and B to equalize this return with the call return $(C_u, C_d) = (\Delta uS + B, \Delta dS + B)$

No Arbitrage Condition

- $\Delta = \frac{C_u - C_d}{(u-d)S}$ and $B = \frac{uC_d - dC_u}{(u-d)r}$
- Under **no arbitrage** no portfolio can have both non-negative and non-zero return at date one and non-negative value at date 0
- No arbitrage implies that the price of the call option has to be equal to the price of the portfolio that gives identical returns
- $C = \frac{1}{r}\left[\frac{r-d}{u-d}C_u + \frac{u-r}{(u-d)}C_d\right]$ but C is independent of q!
- Interpretation in terms of risk free probabilities:

$$p = \frac{r-d}{u-d};\ 1-p = \frac{u-r}{u-d}$$

Asset Pricing

- Two dates, I consumers, K assets, L goods, S states of the world

- Consumer i: U_i utility, $\omega_i \in \mathbb{R}^L$ date one endowment, $z_i \in \mathbb{R}^K$ date zero portfolio, and $x_i \in \mathbb{R}^{LS}$ date one consumption (no consumption at date zero)

- Assets: $r_k \in \mathbb{R}^S$ return of asset k non-negative non-zero, R is the SxK matrix of asset returns

- Prices: $q_k \in \mathbb{R}^K$ date zero asset prices, and $p_s \in \mathbb{R}^L$ date one spot-prices in state s

- A **Radner equilibrium** is an allocation $(x_1^*, ..., x_I^*) \in X_1 \times ... \times X_I \in \mathbb{R}^{LSI}$, a set of portfolios $(z_1^*, ..., z_I^*) \in \mathbb{R}^{IK}$, and a system of prices for the contingent commodities $p = (p_{11}, ..., p_{LS}) \in \mathbb{R}^{LS}$ and for the assets $q = (q_1, ..., q_K) \in \mathbb{R}^K$ such that

  i. For every i, (x_i^*, z_i^*) maximizes expected utility $U_i(x_i)$ subject to the constraints:
     (a) $p_s \cdot x_{s,i} \leq p_s \cdot \omega_{s,i} + \sum_k p_{1,s} z_{k,i} r_{s,k}$ for $s = 1..S$
     (b) $\sum_k q_k \cdot z_{k,i} \leq 0$

  ii. All markets clear, i.e.
     (a) $\sum_i x_{s,i}^* = \sum_i \omega_{s,i}$ for $s = 1..S$
     (b) $\sum_i z_{k,i}^* \leq 0$ for $k = 1..K$

- We can set $p_{1,s} = 1$ for $s = 1..S$ without loss of generality (Why?)

Proposition: $\exists \mu \in \mathbb{R}^S$, $\mu \geq 0$, such that $q_k = \sum_s \mu_s r_{s,k}$ for $k = 1..K$, or $q = R'\mu'$.

- Asset prices are linear functions of asset return

- Can μ_s be interpreted as the shadow price of one unit of good 1 in state s?

- Two different ways to prove this result: (a) Derivation from no-arbitrage condition, (b) Derivation from consumer's first order condition

No Arbitrage Condition

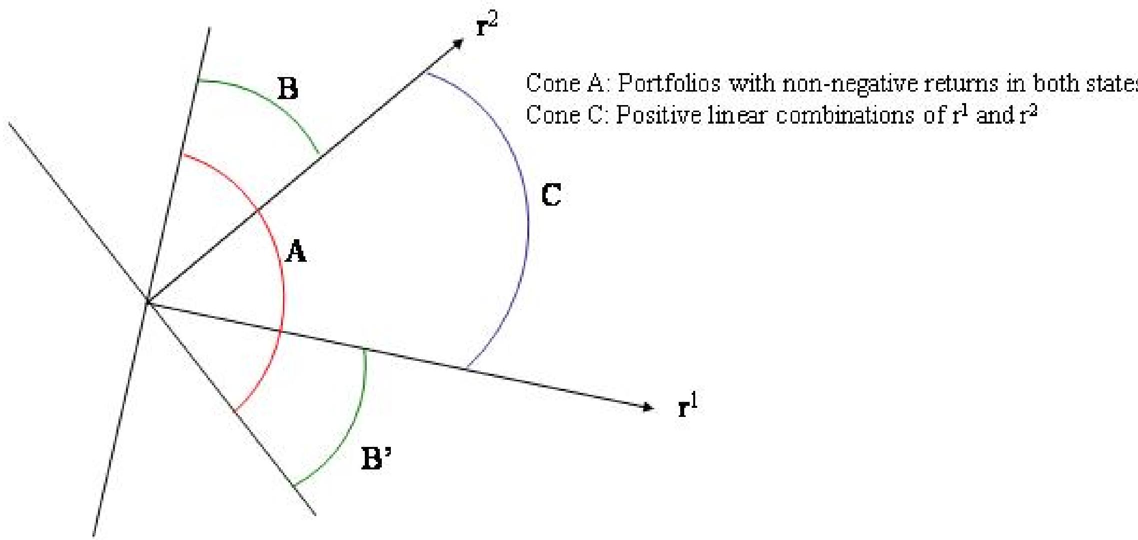
Definition: q is **arbitrage free** if there does not exist a portfolio z such that (a) $q'z \leq 0$ and (b) $Rz \geq 0$ and $Rz \neq 0$

- What condition on consumer preferences implies arbitrage free prices?
- The proposition follows from convexity theory

Lemma: If q is arbitrage free, then there exists a $\mu \in \mathbb{R}^S$ such that $q' = \mu' R$.

2 states of the world, K assets, $R = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix}$

Can q belong to cone B? No=>By contradiction
(There exists a portfolio vector z in B' such that q'z<0 and Rz≥0)
By elimination, q has to belong to cone C
q=R'μ'



Graphical Intuition for Lemma 1

Consumer F.O.C

- Assume consumers have Bernoulli utility representation
- Consumer i has indirect utility $v_{s,i}(p_s, w_{s,i})$ where

$$w_{s,i} = p_s \cdot \omega_{s,i} + \sum_k z_{k,i} r_{s,k}$$

- Consumer i maximizes $\sum_s \pi_{s,i} v_{s,i}(p_s, p_s \cdot \omega_{s,i} + \sum_k z_{k,i} r_{s,k})$ subject to $\sum_k q_k \cdot z_{k,i} \leq 0$
- Let λ_i be the Lagrange multiplier on consumer i's date 0 budget constraint (economic interpretation of λ_i?)

- Consumer's F.O.C with respect to $z_{k,i}$ gives

$$\lambda_i q_k = \sum_s \pi_{s,i} \frac{\partial}{\partial w_{k,i}} v_{s,i}(p_s, w_{s,i}) r_{s,k}$$

- Let $\mu_s = \pi_{s,i} \frac{1}{\lambda_i} \frac{\partial}{\partial w_{k,i}} v_{s,i}(p_s, w_{s,i}) \geq 0$

- $q_k = \sum_s \mu_s r_{s,k}$

- Interpretation of μ_s?

Discussion

- Do asset prices depend on objective probabilities?
- Interpretation in terms of risk neutral probabilities
- Assume $r_1 = (1, .., 1)$. What is the interpretation of q_1?
- Is μ_s unique? When?
- What is the price of a new asset $r \in \mathbb{R}^K$?

Overview of this section:

A. Externalities (MWG Ch 11.B)

- Example 1: Two-consumer partial equilibrium
- Example 2: R&D rivalry
- Property Rights and The Coase Theorem

B. Public Goods (MWG Ch 11.C)

C. Multilateral Externalities (MWG Ch 11.D)

A. Externalities

An **externality** is present whenever the well-being of a consumer or the production possibilities of a firm are directly affected by the actions of another agent in the economy.

$\Longrightarrow$ *directly* means excluding any effects of prices (**pecuniary** externalities)

- positive externality: e.g. R&D, vaccination, nice front garden,...

- negative externality: e.g. pollution, ugliness, cigarette smoke,...

A Two-Consumer Partial Equilibrium Model

- consumer $i = 1, 2$ chooses a vector of L traded goods, $x_i \in X_i \subset \mathbb{R}^L_+$.

- consumer $i = 1, 2$ has preferences over x_i, but also over an action $h \in \mathbb{R}_+$ taken by consumer 1. The utility function of i is $u_i(x_i, h)$.

- If $\frac{\partial u_2}{\partial h} > 0$, positive externality. If $\frac{\partial u_2}{\partial h} < 0$, negative externality.

- If utility of consumer i is quasi-linear, we can define (Why?) a derived utility function $v_i(p, w_i, h) = \phi_i(p, h) + w_i$ where $p \in \mathbb{R}^L$ is the price vector for the L goods and w_i is i's wealth.

- Assuming prices are unaffected by changes in h, we can even further simplify to $\phi_i(h)$. Also assume $\phi_i''(\cdot) < 0$.

Suppose that we are at a competitive equilibrium. Consumer 1 maximizes utility, which implies

$$\phi_1'(h^*) \leq 0 \text{ with equality if } h^* > 0$$

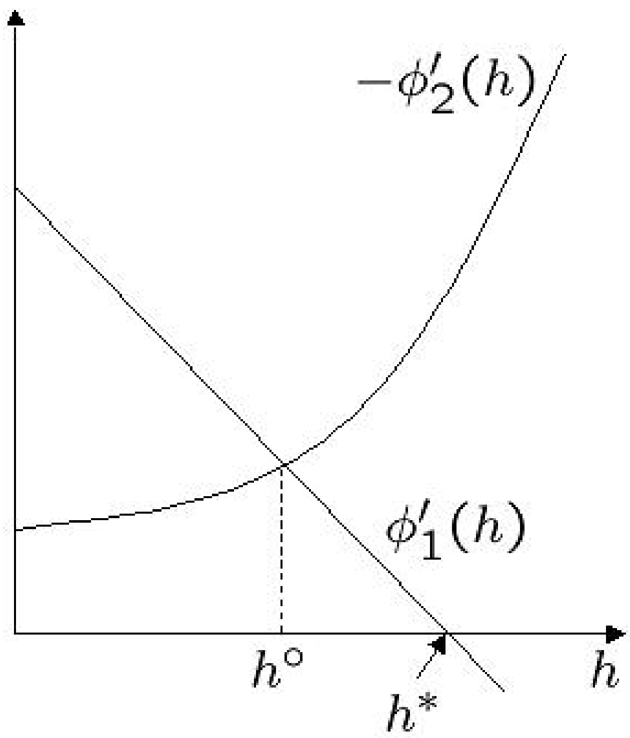
The Pareto optimal level of h is given by $h^\circ = arg \max_{h \geq 0}(\phi_1(h) + \phi_2(h))$. It is thus the solution to

$$\phi_1'(h^\circ) \leq -\phi_2'(h^\circ) \text{ with equality if } h^* > 0$$

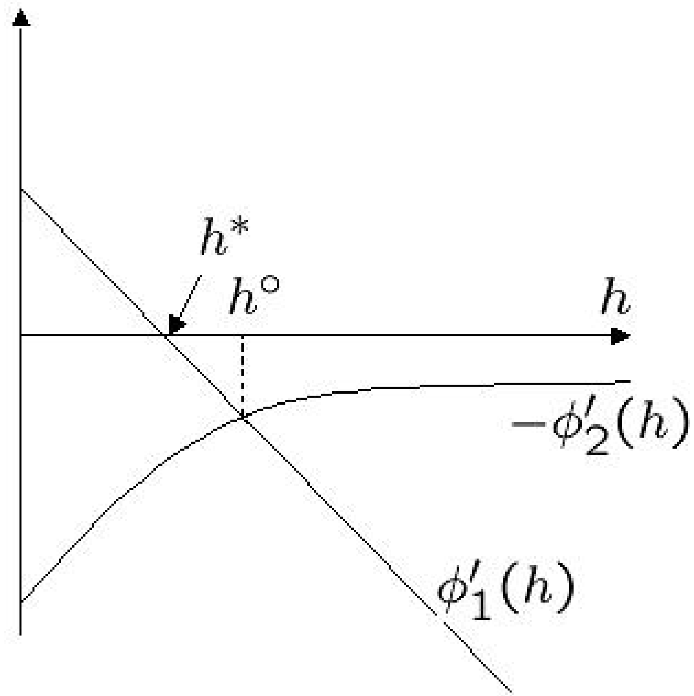
At an interior competitive solution, $\phi_1'(h^*) = 0$, but when h has external effects such that $\phi_2'(h^*) \neq 0$, the Pareto condition cannot hold at h^* unless $h^* = h^\circ = 0$.

The problem is that consumer 1 does not take into account the impact of his actions on consumer 2's well-being.

Negative externality $h^* > h^\circ$



Positive externality $h^* < h^\circ$

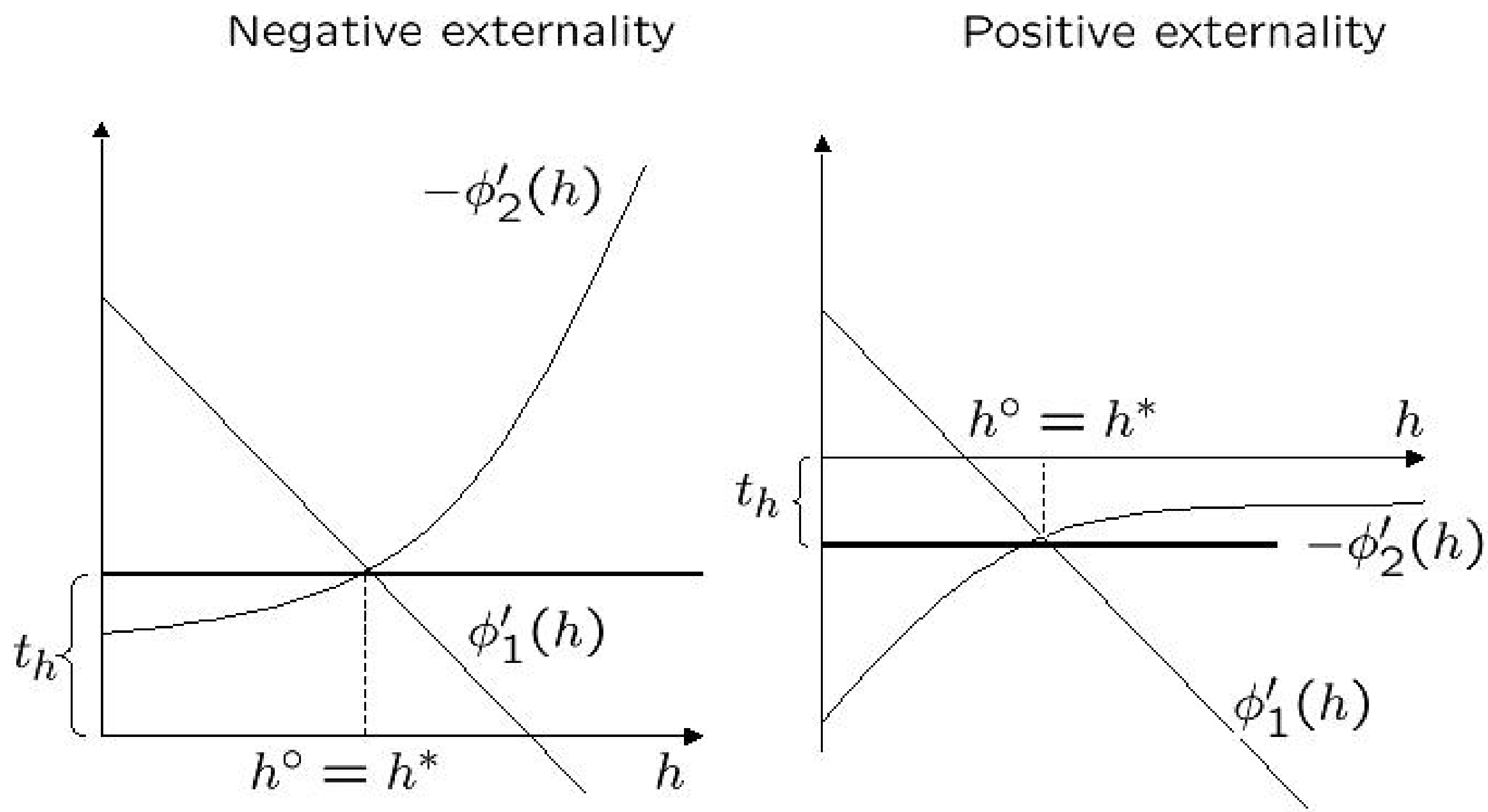


Solutions to the externality problem through direct government intervention:

i. a **quotum**: a rule that states that $h \leq h^\circ$ if negative externality and $h \geq h^\circ$ if positive externality.

ii. **Pigouvian taxation**: imposing a tax on the externality generating activity.

- Suppose consumer 1 is made to pay t_h per unit of h. The idea is to force consumer 1 to internalize the externality.
- The optimal tax equals the marginal externality at the optimal solution $t_h = -\phi_2'(h^\circ)$. Verify that this restores Pareto optimality.
- Negative externality means taxation $t_h > 0$, positive externality means subsidization $t_h < 0$.



Pigouvian taxation

- Taxing a negative externality is equivalent to subsidizing its reduction. Suppose government pays $s_h = -\phi_2'(h^\circ) > 0$ for every unit as long as $h \leq h^\circ$. Consumer chooses $h^* = arg \max_{h \geq 0}(\phi_1(h) + s_h(h^\circ - h)) = h^\circ$.

- It is essential to tax the externality generating activity directly. Consider the case of car pollution. Taxing cars upon purchase will generally not lead to optimal levels of pollution. Taxing emission (or fuel usage) will.

Quota and taxation are equally effective in achieving optimal outcomes for society. The problem is whether the information on costs and benefits of the externality are available to the government.

R&D Rivalry

- Two firms (1 and 2) compete in R&D, denoted by x_1 and x_2 respectively.

- Sales revenue for each firm is $R(x_1 + x_2)$ with $R'' < 0$ such that revenue depends on the sum of R&D spending of both firms.

- Each firm faces an identical cost function $c(x) = \alpha x$ with $\alpha > 0$.

- A **Nash equilibrium** (x_1^*, x_2^*) is such that x_i^* maximizes profits given x_{-i}^* for $i = 1, 2$.

- Define x^* as the solution to $R'(x^*) = \alpha$. Any pair (x_1, x_2) such that $x_1 + x_2 = x^*$ is a Nash equilibrium.

- Industry profits are given by $2R(x_1+x_2)-\alpha(x_1+x_2)$ and are maximized at a level x° of total R&D which is the solution to $R'(x^\circ)=\frac{\alpha}{2}$.

- Clearly $x^\circ > x^*$ and the level of R&D spending in the industry is suboptimal.

- Firms do not take into account the positive spill-over effects of their R&D on other firms.

- Possible solutions to the externality problem are:
  - Firms write a contract in which they commit to choose $\frac{x^\circ}{2}$.
  - A per unit subsidy of $\frac{\alpha}{2}$.

Property Rights and The Coase Theorem

Generally, a less intrusive form of intervention based on the assignment of enforceable property rights is possible. Consider the case of a negative externality (e.g. loud nightclub in a quiet neighbourhood). The property rights can be assigned in two ways:

- **Case 1** Agents who are at the root of the externality can be assigned the right to generate that externality (club can play music at any number of decibels).

- **Case 2** Agents who suffer from the externality can be assigned the right to an externality-free environment (neighbours have the right to absolute silence).

The **Coase Theorem** states that if trade of the externality can occur, competitive markets will deliver Pareto optimality, no matter how property rights were assigned initially.

The externality problem is thus treated as a problem of missing markets.

In the example: suppose the club owner and the neighbours can trade in the number of decibels. In Case 1, the club owner will pay the neighbours to produce some decibels. In Case 2, the neighbours will pay the club owner to turn down the volume. Under competitive conditions, the number of decibels is Pareto optimal.

The information requirement for the regulator is much lower. Optimality is restored through the price mechanism.

Consider the two-consumer partial equilibrium model .

- Assign the right to an externality free environment to consumer 2. Bargaining takes a form in which consumer 2 makes consumer 1 a take-it-or-leave-it offer, demanding a payment T in return for allowing consumer 1 to generate h.

- Consumer 1 agrees if she is at least as well off as she would by rejecting it, i.e. if $\phi_1(h) - T \geq \phi_1(0)$

- Consumer 2 chooses (h^*, T^*) to solve

$$\max_{h \geq 0, T} \phi_2(h) + T \text{ s.t. } \phi_1(h) - T \geq \phi_1(0)).$$

- Since the constraint is always binding, this is equivalent to $\max_{h \geq 0} \phi_1(h) + \phi_2(h) - \phi_1(0)$ which yields exactly the Pareto optimal level h°.

- Verify that assigning the right to generate the externality to consumer 1 will also yield h°, but now, $T < 0$ such that consumer 2 needs to pay consumer 1.

- Also verify that h° will also be the outcome if it is consumer 1 that makes the take-it-or-leave-it offer, regardless of the initial allocation of rights.

Thus, the Coase theorem holds in this example, provided that there are no bargaining inefficiencies and there are no wealth effects. Note that the consumers need to know each other's preferences, but the government need not.

B. Public Goods

A **public good** is a commodity that is **non-depletable** or **non-rivalrous**, i.e. for which use of a unit of the good by one agent does not preclude its use by other agents. e.g. knowledge, public defense, non-congested road, air quality... .

Public goods can be **excludable** (knowledge through patents) or **non-excludable** (public defense, air quality), depending on whether exclusion of an individual from the benefits of a public good is possible.

Consider a partial equilibrium setting with I consumers, L traded private goods and one public good.

- As before, define for each consumer i a derived utility function $\phi_i(x)$ over the level of the public good x. Let $\phi_i''(x) < 0$ at all $x \geq 0$.

- The cost of supplying q units of the public good is $c(q)$, with $c''(q) > 0$ at all $q \geq 0$.

- If $\phi_i' > 0$ and $c' > 0$, we have a public good whose production is costly. If $\phi_i' < 0$ and $c' < 0$, we have a public bad whose reduction is costly.

- The Pareto optimal level q° of the public good must solve

$$\max_{q \geq 0} \sum_{i=1}^{I} \phi_i(q) - c(q)$$

- The optimality condition is:

$$\sum_{i=1}^{I} \phi_i'(q^\circ) \leq c'(q^\circ) \text{ with equality if } q^\circ > 0$$

- At an interior solution, the optimal level of the public good is such that the sum of the consumers' marginal benefit equals the marginal cost.

- A competitive equilibrium is a price p^* and an allocation (x^*, q^*) such that

i. $x_i^* = arg \max_{x_i \geq 0} \left(\phi_i \left(x_i + \sum_{k=-i} x_k^* \right) - p^* x_i^* \right)$ for $i = 1, ..., I$

ii. $q^* = arg \max_{q \geq 0} (p^* q - c(q))$

iii. $x^* = \sum_{i=1}^{I} x_i^* = q^*.$

- Notice the concept of **Nash equilibrium**. Each consumer i maximizes over x_i taking x_{-i} as given.

- Whenever there is an i for which there is an interior solution, i.e. $\phi_i'(q^*) = c'(q^*)$, we have

$$\sum_{i=1}^{I} \phi_i'(q^*) > c'(q^*)$$

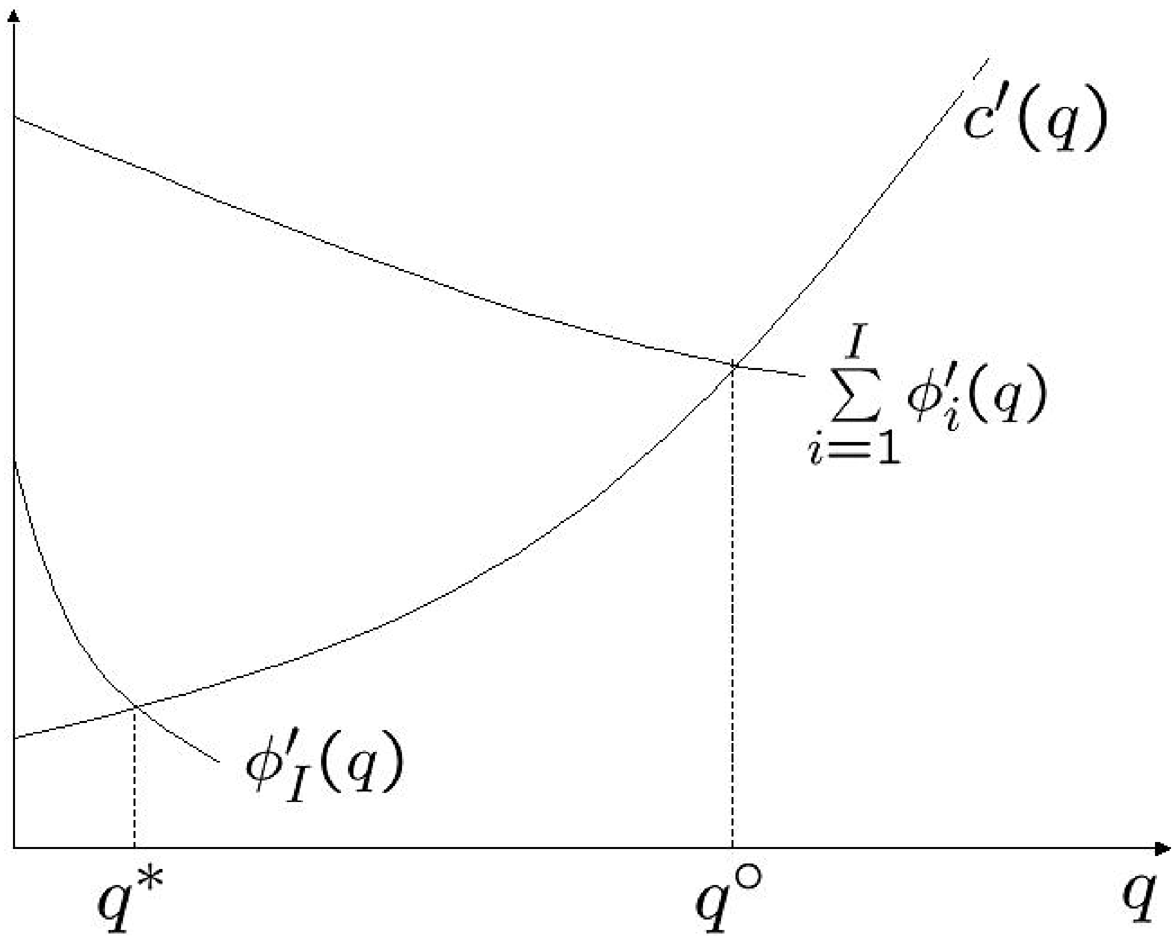
  and the Pareto condition is violated.

- At a competitive equilibrium, $q^* < q^\circ$ such that the provision of the public good is too low, (or too high, in the case of a public bad such that $q^* > q^\circ$).

- The failure of each consumer to consider the benefits for others of her public good provision is known as **the free-rider problem**.

- In the current example: Assume we can order $\phi'_1(x) < ... < \phi'_I(x)$ at all $x \geq 0$. The first order condition to consumer i's problem, $\phi'_i(x^*) \leq p^*$, can only hold with equality for one consumer. So only one consumer will choose $x_i > 0$. Moreover this must be consumer I.

- Therefore, only the consumer who derives the largest (marginal) benefit from the public good will provide it.



Private and Pareto optimal provision of a public good

C. Multilateral Externalities

- Often, externalities affect numerous agents. In that case, we can distinguish

  depletable externalities: experience of the externality by one agent reduces the amount that is felt by other agents, e.g. garbage thrown in one garden cannot be thrown anymore in another.

  nondepletable externalities: externalities with the characteristics of public goods or bads, e.g. air pollution.

- With depletable externalities, the establishment of a market for the externality may be expected to lead to the Pareto optimal outcome because of the first fundamental theorem of welfare.

- With nondepletable externalities, Pareto optimality may not be restored with the creation of markets.

Private Information

- Ususally, how an agent is affected by externalities or public goods is only known to her.

- This **asymmetric information** can be a serious obstacle to both centralized and market based attempts to restore optimality.

- The study of ways of dealing with this problem is called **mechanism design** and will be studied in the last section.

Economic Issues

- Understand market outcomes in concentrated industries
- Model strategic interactions (interdependent decision making)
- Evaluate welfare properties of industry competition
- This field of research is known as 'Industrial Organization' or 'Industrial Economics'
- Applications: Competition policy, regulation, international trade
- Tools: Game theory

A. Monopoly Pricing

Consider a **monopolist**, i.e. a firm that is the only producer of a good.

- The monopolist faces a continuous demand curve $x(p)$, with $x(\cdot) > 0$, $x'(\cdot) < 0$ and inverse $p(q) \equiv x^{-1}(q)$ and a continuous cost function $c(q)$ with $c'(\cdot) > 0$ where q is the output level.

- For convenience, assume there exists $\bar{p} < \infty$ such that $x(p) = 0$ for all $p > \bar{p}$ and that $p(0) > c'(0)$.

- The monopolist's decision problem can be to

  i. choose the price p that solves $\max_p \left(px(p) - c(x(p))\right)$

  ii. choose the output level q that solves $\max_q \left(p(q)q - c(q)\right)$

- Assuming q is the decision variable, the optimal q^m must equate marginal revenue $MR(q)$ with marginal cost:

$$MR(q^m) = p(q^m) + p'(q^m)q^m = c'(q^m)$$

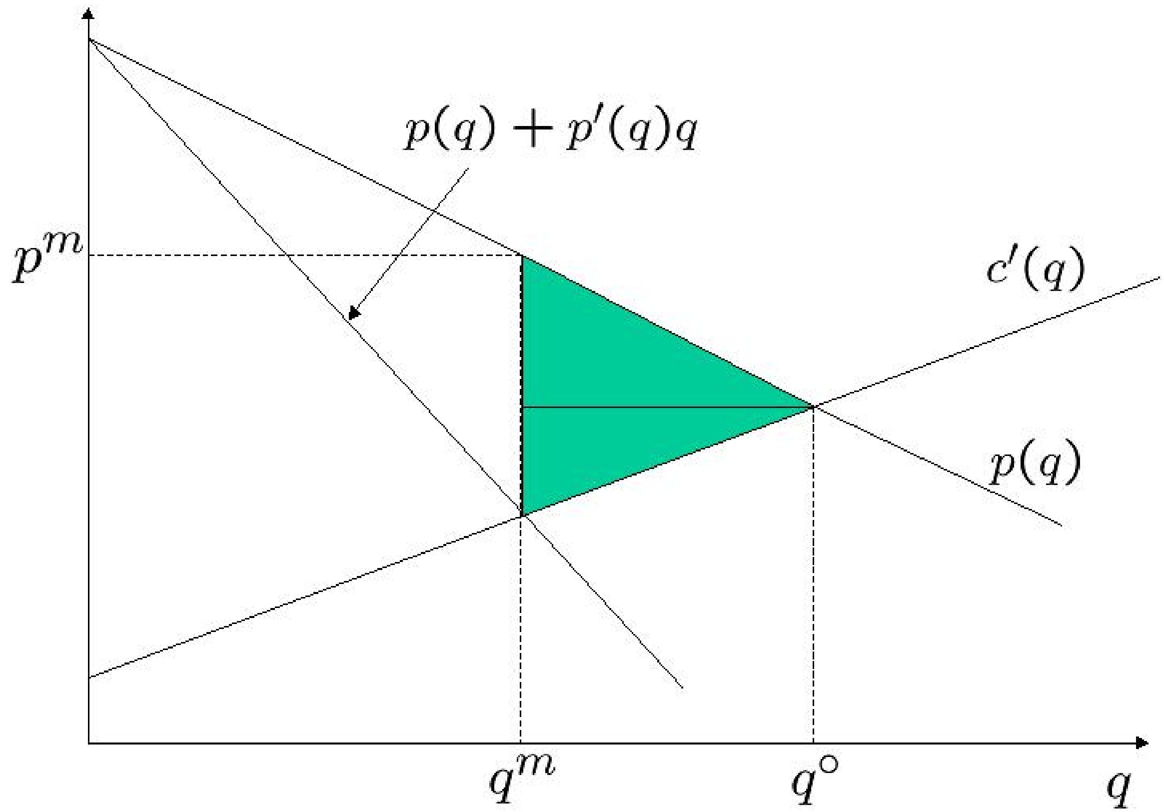
or equivalently

$$p(q^m)\left[1 + \frac{1}{\epsilon}\right] = c'(q^m)$$

where $\epsilon < 0$ is the price elasticity of demand.

- In the basic monopoly model, whether p or q is the decision variable does not matter for the outcome (Check!). This is not generally true (see oligopoly: Bertrand vs. Cournot).

- Comparing with the Pareto optimal q° that solves $p(q^\circ) = c'(q^\circ)$, $\frac{\epsilon}{1+\epsilon}$ is the monopolist's mark-up over marginal cost that gives rise to a dead weight loss.



- Since $c'(\cdot) > 0$, it must be that $\epsilon < -1$ and $\frac{\epsilon}{1+\epsilon} > 1$ at the optimal solution, such that $p^m > p^\circ$ is always true. The monopolist supplies on the elastic part of the demand curve.

- $\frac{p^m - c'(q^m)}{p^m} = -\frac{1}{\epsilon}$ is the **Lerner index** and is an empirical measure of market power.

- The requirement for a global maximum (second-order condition) is $MR'(q^m) < c''(q^m)$. Unlike the competitive case, this can be satisfied even if $c''(q^m) < 0$, i.e. with increasing returns.

- The supply curve is not defined for a monopolist.(Why?)

Example:

- Suppose $c(q)$ is CRTS with $c'(\cdot) = 0$ and $p(q) = a - bq$.
- The monopolist's problem is $\max\limits_{q}((a - bq)q)$.
- The FOC is $a - 2bq^m = 0 \Leftrightarrow q^m = \frac{a}{2b} < q^\circ = \frac{a}{b}$
- The monopolist price is $p^m = a - b\frac{a}{2b} = \frac{a}{2}$
- The monopolist profits are $\pi^m = \frac{a^2}{4b}$

B. Oligopoly

Consider a situation in which more than one, but still not many, firms compete in a market: an **oligopoly**.

- Competition among firms is characterized by strategic interaction.
- Game theory is the appropriate tool for analysis.
- If firms choose prices simultaneously we have **Bertrand competition**.
- If firms choose quantities simultaneously we have **Cournot competition**.

The Bertrand Model

- Consider a duopoly, i.e. two maximizing firms 1 and 2 in a market with demand function $x(p)$ which has the same properties as in the monopoly model.

- Production is CRTS with per unit cost c, identical for both firms.

- Firm 1 and 2 simultaneously choose p_1 and p_2

- Sales for firm $j = 1, 2$ are given by

$$x(p_j, p_{-j}) = \begin{cases} x(p_j) & \text{if } p_j < p_{-j} \\ \frac{1}{2}x(p_j) & \text{if } p_j = p_{-j} \\ 0 & \text{if } p_j > p_{-j} \end{cases}$$

- The unique **Nash Equilibrium** is $p_1^* = p_2^* = c$

  Proof. If $p_1^* = p_2^* = c$ both firms earn zero profits. Given $p_{-j}^* = c$, suppose firm j sets $p_j > c$, it would sell nothing and still earn zero profits. If firm j sets $p_j < c$ it would make a loss. There is no gain from deviating. Thus $p_1^* = p_2^* = c$ is a NE. If $p_{-j}^* > c$, firm j always asks an infinitesimal lower price and captures the entire market. If $p_{-j}^* < c$, firm j asks a higher price to avoid losses. Hence the NE is unique. □

- With two (or more) firms, Bertrand competition yields the competitive, Pareto optimal outcome!

- This strong result fails under

  i. quantity or Cournot competition
  ii. product differentiation (e.g. monopolistic competition)
  iii. decreasing returns to scale
  iv. repeated interaction (with infinite horizon)

The Cournot Model

- Suppose now the two firms simultaneously decide on q_1 and q_2.

- Given a pair (q_1, q_2), the price adjusts such that it clears the market, i.e. $x(p) = q = q_1 + q_2$ or using the inverse demand, $p = p(q)$.

- To obtain a Nash equilibrium, we must find firm j's **best-response function** $b_j(\bar{q}_{-j})$ (or best-response correspondence if it is not single-valued). That is, the optimal quantity q_j, given that firm $-j$ produces $\bar{q}_{-j}$.

- A pair (q_1^*, q_2^*) is a Nash equilibrium if and only if $q_j^* \in b_j(q_{-j}^*)$ for $j = 1, 2$.

- The best-response function $b_j(\bar{q}_{-j})$ is found as the solution to

$$\max_{q_j \geq 0} (p(q_j + \bar{q}_{-j})q_j - cq_j)$$

with first-order condition

$$p'(q_j + \bar{q}_{-j})q_j + p(q_j + \bar{q}_{-j}) \leq c \text{ with equality if } q_j > 0$$

- A Nash equilibrium is a pair $(q_1^*, q_2^*) \gg 0$ that is a solution to

$$\begin{aligned} p'(q^*)q_1^* + p(q^*) = c \\ p'(q^*)q_2^* + p(q^*) = c \end{aligned}$$

where $q^* = q_1^* + q_2^*$.
Note that if c is equal for both firms $(q_1^*, q_2^*) \gg 0$. (Why?)

- In any Nash equilibrium, the market price is greater than the competitive price c and smaller than the monopoly price p^m.

Proof. Adding the two equilibrium conditions, we get $p'(q^*)\frac{q^*}{2} + p(q^*) = c$. Since $p'(q^*)\frac{q^*}{2} < 0$, $p(q^*) > c$.
Now suppose $p(q^*) \geq p^m$, which implies $q^* \leq q^m$. If firm j chooses $\hat{q}_j = q^m - q^*_{-j}$, instead of $q^*_j = q^* - q^*_{-j}$, then the aggregate profits of the firms increase, since at q^m profits are maximal. However, since that means that firm j increases output, the market price must decrease, making firm $-j$ strictly worse off. Such a deviation would thus be optimal for firm j and q^* could not have been a NE. Moreover $q^* = q^m$ cannot be true since $p'(q^*)q^* + p(q^*) = c$ and the equation above cannot hold simultaneously. Hence $p(q^*) < p^m$.

□

- Thus, the presence of two firms is not sufficient to restore a competitive outcome. At the same time, the increase in competition lowers the price below the monopolistic level.

in the Example:

- firm j solves $\max_{q_j}((a - b(q_j + \bar{q}_{-j}))q_j)$

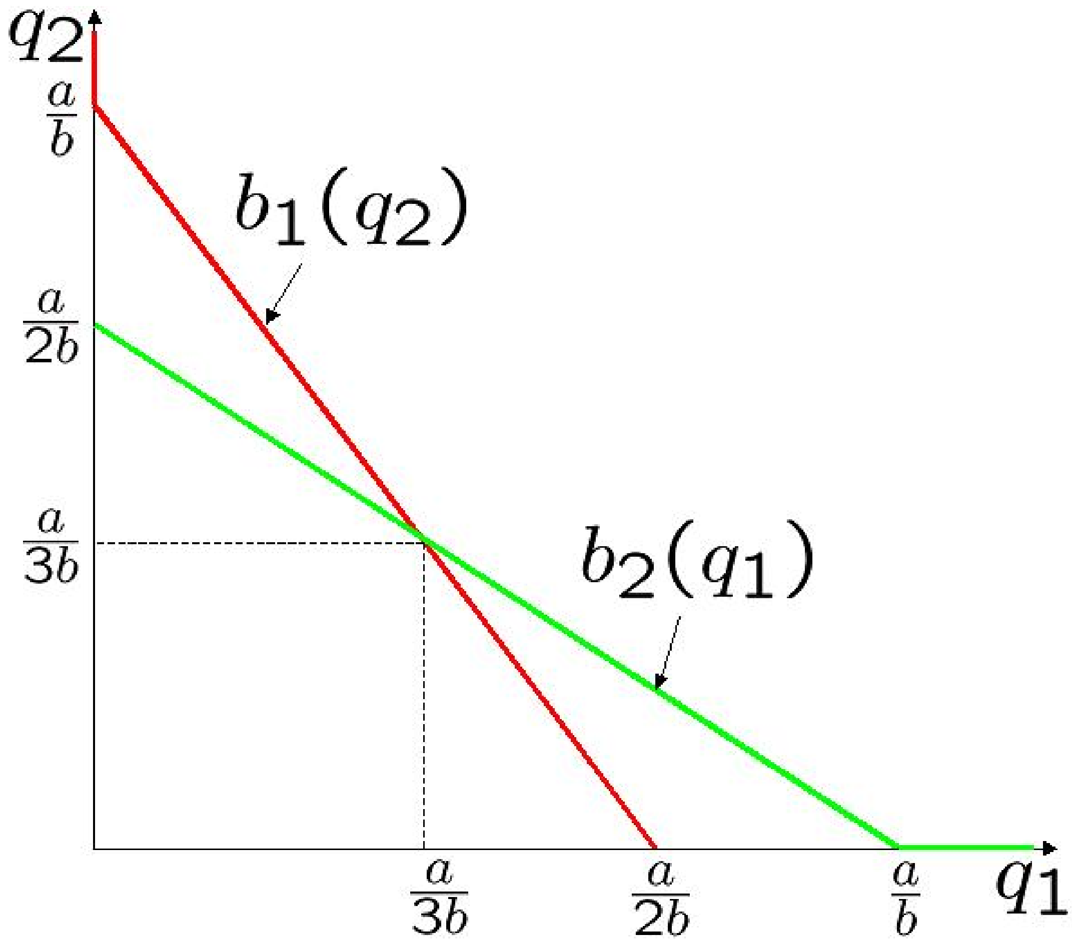
- The FOC is $a - b(2q_j + \bar{q}_{-j}) = 0$ and the best response is $q_j = \frac{a - b\bar{q}_{-j}}{2b}$

- In a symmetric equilibrium $q_j^* = \bar{q}_{-j}$ for $j = 1, 2$ such that $q_1^* = q_2^* = \frac{a}{3b}$

- Industry output is $q^* = q_1^* + q_2^* = \frac{2a}{3b}$. Note $q^m = \frac{a}{2b} < q^* < q^\circ = \frac{a}{b}$.

- The market price is $p^* = \frac{a}{3} < p^m = \frac{a}{2}$

- Firm j's profits are $\pi_j^* = \frac{a^2}{9b}$ such that $\pi_1^* + \pi_2^* < \pi^m = \frac{a^2}{4b}$



Cournot duopoly equilibrium in the example

The Competitive Limit

- The extension of the Cournot model to $J > 2$ identical firms is straightforward.

- Let Q_J^* denote the resulting aggregate output level in a symmetric Nash equilibrium, given by

$$p'(Q_J^*)\frac{Q_J^*}{J} + p(Q_J^*) = c$$

- When $J = 1$, this condition coincides with the first-order condition of a monopolist.

- On the other hand, if $J \to \infty$, $p'(Q_J^*)\frac{Q_J^*}{J} \to 0$, and the price approaches marginal cost. This is the **competitive limit** result.

Equilibrium Entry in The Cournot Model

- Until now the number of active firms was fixed exogenously.
- Suppose instead there is a infinite number of potential entrants, each of which can enter and produce if it is profitable.
- There is however a fixed set-up cost $K > 0$ in entering the industry.
- Consider this simple **dynamic game** setting:

  stage 1 All firms decide "in" or "out" simultaneously.
  stage 2 All firms engage in Cournot competition.

- In any **subgame perfect Nash equilibrium**, no firm wants to change its entry decision, given the entry decision of all the other firms.

- Define an equilibrium with J^* firms choosing to enter the market if and only if $\pi_{J^*} \geq K$ and $\pi_{J^*+1} < K$

- The equilibrium is found by **backwards induction**: First solve the Cournot game for a given J, then determine J^* given the second-stage profits.

Example:

With linear demand, zero constant marginal cost and J firms, the outcome of the 2^{nd} stage Cournot game is:

$$-b\frac{Q_J^*}{J} + a - bQ_J^* = 0 \Leftrightarrow Q_J^* = \frac{a}{b}\left(\frac{J}{J+1}\right)$$

- Each firm $j = 1, ..., J$ chooses to produce $q_J^* = \frac{Q_J^*}{J} = \frac{a}{b}\left(\frac{1}{J+1}\right)$ and earns a profit $\pi_J = \frac{1}{b}\left(\frac{a}{J+1}\right)^2$.
- The market price p_J^* is $\frac{a}{J+1}$.
- Note that as $J \to \infty$, $\pi_J \to 0$, $p_J^* \to 0$ and $Q_J^* \to \frac{a}{b}$, which is the competitive outcome.

Now we can solve for the real number $\tilde{J} \in \mathbb{R}$ at which $\pi_{\tilde{J}} = K$:

$$\tilde{J} = \frac{a}{\sqrt{bK}} - 1$$

The equilibrium number of entrants J^* is the largest integer that is less than or equal to $\tilde{J}$.

Is the equilibrium number of entrants J^* **socially optimal**?
A social optimal number of firms J° must maximize Marshallian aggregate surplus $W(J)$, i.e.

$$J^\circ = arg \max_{J \in \mathbb{Z}_+} W(J)$$

where $W(J) = \int_0^{Q^*_J} p(s)ds - JK = a\left[\frac{a}{b}\left(\frac{J}{J+1}\right)\right] - \frac{b}{2}\left[\frac{a}{b}\left(\frac{J}{J+1}\right)\right]^2 - JK$

- Let $\bar{J} \in \mathbb{R}$ be the real number such that $W'(\bar{J}) = 0$. Then

$$\bar{J} = \frac{a}{\sqrt[3]{bK}} - 1$$

- Hence, $\tilde{J} > \bar{J}$ such that the optimal number of firms is always smaller or equal than the number of firms that actually enter.

- The reason is **business stealing**: when an additional firm enters the market, the sales of the existing firms decline. However, the new entrant does not take this into account.

- In more general settings, the entry bias due to business stealing is also present, but it is also possible to have too few entrants. (see MWG 12.E).

C. Monopolistic Competition

- So far we only considered markets for homogenous goods.
- Suppose now that each firm can differentiate its own variety of the good, such that consumers cannot substitute perfectly between varieties.
- One important model that captures product differentiation is the **Dixit-Stiglitz model**.
- The model is built on the idea of monopolistic competition, introduced by Chamberlin and Robinson in the 30s.

- There are J firms, each producing q_j of their own variety j according to a linear technology $q_j = \frac{1}{b}(l_j - a)$ with $a, b > 0$.
- There are I symmetric consumers that choose a vector x of the J differentiated goods and that have preferences represented by the utility function $u(x) = \sum_{j=1}^{J} x_j^{1-\frac{1}{\sigma}}$ with $\sigma > 1$.
- Each consumer also supplies one homogeneous unit of labour inelastically in return for a wage w. Thus the total labour supply is I.
- An equilibrium is a price vector $(p_1^*, ..., p_J^*, w^*)$ and an allocation $(x_1^*, ..., x_J^*, q_1^*, ..., q_J^*, l_1^*, ..., l_J^*)$ such that:
  i. Consumers maximize utility subject to their budget constraint
  ii. Firm maximize profits subject their technology constraint and the demand for their good
  iii. The market for labour and for every variety j clears, i.e. $\sum_{j=1}^{J} l_j^* = I$ and $I x_j^* = q_j^*$ for $j = 1, ..., J$.

Solution of the model:

- The representative consumer problem is:

$$\max_{x \geq 0} \sum_{j=1}^{J} x_j^{1-\frac{1}{\sigma}} \text{ s.t } \sum_{j=1}^{J} p_j x_j \leq w$$

and has as FOCs:

$$\frac{\sigma-1}{\sigma} x_j^{-\frac{1}{\sigma}} = \lambda p_j \text{ for } j = 1, ..., J$$
$$\sum_{j=1}^{J} p_j x_j = w$$

- Solving for the Lagrange multiplier yields $\lambda^{-\sigma} = w \left[\left(\frac{\sigma}{\sigma-1} \right)^{-\sigma} \sum_{j=1}^{J} p_j^{1-\sigma} \right]^{-1}$

- Subsituting this expression into the FOC for x_j yields the demand function:

$$x_j = \left(\frac{p_j}{P}\right)^{-\sigma} \frac{w}{P}$$

where $P = \left(\sum_{j=1}^{J} p_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$

- So $-\sigma$ is the (constant) price elasticity of demand for good x_j.

- Firm j's problem can now be written as $\max_{p_j \geq 0}(p_j I x_j - w(a + bIx_j)$, which has as a FOC

$$x_j + p_j \frac{\partial x_j}{\partial p_j} - bw \frac{\partial x_j}{\partial p_j} = 0$$

- Plugging the value of good j's price elasticity,

$$\Leftrightarrow p_j = \frac{\sigma}{\sigma - 1} bw$$

- Hence, each firm j behaves as a monopolist setting its price as a mark-up over marginal cost.

- Despite the fact that the number of firms J can be arbitrarily large, each firm has market power because of product differentiation.

Suppose there is free entry, what is the equilibrium number of firms?

- Profits are zero in any free entry equilibrium or equivalently price equals average cost $p^* = w^*(\frac{a}{q^*} + b)$.
- Substituting the monopolist price into this expression yields the production level $q^* = (\sigma - 1)\frac{a}{b}$.
- From labour market clearing $I = Jl^* = J(a + bq^*) = Ja\sigma$.
- So the number of firms that will enter is the largest integer smaller or equal than $J = \frac{I}{a\sigma}$.
- Note that consumers value diversity and utility is strictly increasing in J.
- Question: Consider two countries with I consumers each. What happens to consumption, production and product diversity in both countries after a move from autarky to free trade?

The monopolistic competition model knows many applications:

international trade: a theory of intra-industry trade (Krugman)

industrial organization: optimal product diversity (the subject of the original Dixit-Stiglitz (1977) paper)

macroeconomics: introducing price setting into general equilibrium models (Blanchard-Kiyotaki)

Economic Issues

- Symmetric versus asymmetric information
- Preferences are typically privately known (does this matter in G.E. analysis?)
- Evaluate welfare properties of institutions used allocate goods (e.g. first price sealed bid auction)
- Applications: Public choice (e.g. enviromental standards), monopoly pricing, auctions
- Tools: Mechanism design

Mechanism Design Problem

- $i = 1..I$ agents
- $x \in X$ alternative
- $\theta_i \in \Theta_i$ type of agent i,privately observed by i
- $u_i(x, \theta_i)$ utility of agent i when alternative x is chosen
- $\Phi(\theta)$ probability density on $\Theta = \Theta_1 x..x\Theta_I$
- $\{u_i, \Phi, X\}$ is common knowledge

Questions:

1. Efficient choice of alternative x conditional on realization of θ

2. Implementation through agreed-upon mechanism (institution)
   - Which choice functions are 'implementable'?
   - Efficiency properties of commonly used mechanisms

3. Is it possible to get agents to reveal truthfully their types?

Clarke-Groves Mechanism (1/3)

- Example: Allocation of right to play music with privately known preferences

  1. Music lover (ML) gets benefit b from playing music
  2. Neighbour (N) suffers c from music annoyance
  3. b and c are privately known and distributed $g(b)$ and $f(c)$

- What is the effficient allocation?

Take-it-or-leave-it Offer (2/3)

- Assume N has the right to a quite environment and makes a take-it-or-leave-it offer of the type "you can play if you pay me t"

  1. ML accepts offer t with probability $1 - G(t)$
  2. N chooses t to maximize $(1 - G(t))(t - c)$
  3. FOC implies $g(t)(t - c) = 1 - G(t)$
  4. The optimal offer is such that $t > c$

- The equilibrium allocation is not always efficient (Coase theorem fails)!

Gloves-Clarke Mechanism (3/3)

- Consider the following game
- Both ML and N first announce their private preference $\widehat{b}$ and $\widehat{c}$ to a third party
- The third party then choose the following allocation

  1. ML is allowed to play iff $\widehat{b} > \widehat{c}$
  2. If $\widehat{b} > \widehat{c}$, ML pays $\widehat{c}$ and N receives $\widehat{b}$

- Truth telling is wealky dominant startegy (why?)
- The equilibrium allocation is efficient
- Problem: not budget balanced!

Main Concepts

- Definition: A social choice function (SCF) is a mapping from types to alternative $f(\theta) \in X$ (example, $f(b,c) = \{y_b, y_c; t_b, t_c\}$, with $y = 0, 1$ and $t \in \mathbb{R}$)

- Definition: f is ex-post efficient if there does not exist (x, θ_i) such that $u_i(x, \theta_i) \geq u_i(f(\theta), \theta_i)$ for $i = 1..I$ with at least one inequality strict

- Definition: A mechanism $\Gamma = (S_1, .., S_I, g())$ is a collection of I strategy sets and an outcome function which maps each strategy vector into an alternative $g(s) \in X$ where $S = (s_1, .., s_I) \in S_1 x..x S_I$ (example: Clark-Groves mechanism)

- Formally, $(\Gamma, \Theta, u_i, \Phi)$ defines a Bayesian game of incomplete information (still need to define equilibrium concept)

- Definition: Mechanism Γ implement SCF f if there exists a strategy $(s_1^*(), .., s_I^*())$ of the game induced by Γ such that $g(s_1^*(\theta), .., s_I^*(\theta)) = f(\theta)$ for any $\theta \in \Theta$ (for example, the Clark-Groves mechanism implements the first-best allocation)

- Definition: A direct revealation mechanism is a mechanism in which $S_i = \Theta_i$ (for example, an English auction, as defined in later slide, is not a direct mechanism)

- Definition: SCF f is truthfully implementable (or incentive compatible IC) if the direct revealation mechanism $\Gamma = (\Theta_1, .., \Theta_I; f(\theta))$ has equilibrium $(s_1^*(), .., s_I^*())$ such that $s_1^*(\theta_i) = \theta_i$, that is, truthtelling is an equilibrium of Γ (for example, the first best allocation with transfer $t = \frac{b+c}{2}$ is not IC, why?)

- Questions: Can we restrict to truthfully implementable SCF?

Bayesian Implementation

- Let $\theta_{-i} = (\theta_1, .., \theta_{i-1}, \theta_{i+1}, .., \theta_I)$, $s_{-i} = (s_1, .., s_{i-1}, s_{i+1}, .., s_I)$, $s = (s_i, s_{-i})$ is a strategy, and $s() = (s_i(), s_{-i}())$ is a strategy profile

- Definition: $s^*(\theta) = (s_1^*(\theta), .., s_I^*(\theta))$ is a Bayesian Nash equilibrium (BNE) of Γ if $\forall i, \forall \theta$

  $$E_{\theta_{-i}}\left[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i\right] \geq E_{\theta_{-i}}\left[u_i(g(\widehat{s}_i, s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i\right]$$

  for any $\widehat{s}_i \in S_i$

- Definition: Γ implements SCF f in BNE if there exists a BNE of, Γ, such that $g(s^*(\theta)) = f\ (\theta)$ for any $\theta \in \Theta$

- Definition: SCF f is truthfully implementable in BNE (or IC) if $\forall i, \forall \theta_i$

  $E_{\theta_{-i}}\left[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i\right] \geq E_{\theta_{-i}}\left[u_i(f(\widehat{\theta}_i, \theta_{-i}), \theta_i)|\theta_i\right]$ for any $\widehat{\theta}_i \in \Theta$

Revelation Principle

- Proposition: If Γ implements SCF f in BNE then f is truthfully implementable in BNE.

- Intuition: Under Γ, θ_i finds it optimal to reveal $s_i^*(\theta_i)$. Assume θ_i reveals its own type, θ_i, to an automaton who then plays $s_i^*(\theta_i)$ on her behalf. It is optimal to reveal θ_i to the automaton.

- Conclusion: Can we restrict without loss of generality to truthfully implementable SCF.

Application to Auctions

- $x = \{y_1, .., y_I; t_1, .., t_I\}$ where $y_i = 0, 1$ and $\sum_i t_i \leq 0$
- $u_i(x, \theta_i) = \theta_i y_i + t_i$, where θ_i is type i's valuation
- A SCF is ex-post efficient iff $y_i(\theta_i)(\theta_i - Max_i \theta_i) = 0$ and $\sum_i t_i(\theta_i) = 0$
- $i = 2$ corresponds to bilateral trade.
- Auctions is a special case of unit-good allocation problems: assuming that agent 0 is the auctioneer and receives $\sum_i t_i$
- We assume that the θ_i are independently distributed with density $f(\theta)$ and distribution $F(\theta)$ with support $[\underline{\theta}, \overline{\theta}]$

General Direct Mechanism

- Denote a direct mechanism $\{y(\theta), t(\theta)\}$
- Consider the following SCF: $y_i(\theta_i)(\theta_i - Max_i\theta_i) = 0$ and $y_i(\theta_i)t_i(\theta_i) = \theta_i$
- It it incentive compatible?

- **Indirect Mechanism 1: Second Price Sealed Bid Auction (SPSBA)**

- Institutional Design

  1. All bidders make sealed bid
  2. The highest bid wins the good and pays an amount corresponding to the second highest bid

- Equilibrium and direct implementation

  1. What is a (weakly dominant strategy) equilibrium of the bidding game? Is the allocation efficient?
  2. What SCF is (indirectly implemented) by the SPSBA?
  3. What direct revealation mechanism implements this SCF?

- **Indirect Mechanism 2: First Price Sealed Bid Auction (FPSBA)**

- Institutional Design

  1. All bidders make sealed bid
  2. The highest bid wins the good and pays an amount corresponding to her bid

- Equilibrium and direct implementation (Assume f is uniform on $[0, 1]$)

  1. What is the Bayesian Nash equilibrium of the bidding game? Is the allocation efficient?
  2. What SCF is (indirectly implemented) by the FPSBA?
  3. What direct revealation mechanism implements this SCF?

More notation...

- $\bar{t}_i(\widehat{\theta}_i) = E_{\theta_{-i}} t_i(\widehat{\theta}_i, \theta_{-i})$ expected payment under $\widehat{\theta}_i$

- $\overline{y}_i(\widehat{\theta}_i) = E_{\theta_{-i}} y_i(\widehat{\theta}_i, \theta_{-i})$ expected probability of receiving the good under $\widehat{\theta}_i$

- $u_i(\theta_i) = \theta_i \overline{y}_i(\theta_i) - \bar{t}_i(\theta_i)$ equilibrium expected utility

- $\widetilde{u}_i(\theta_i, \widetilde{\theta}_i) = \theta_i \overline{y}_i(\widetilde{\theta}_i) - \bar{t}_i(\widetilde{\theta}_i)$ is the expected utility of type θ_i if she claims to be type $\widetilde{\theta}_i$

Implication of IC

Proposition: Assume $\{y(\theta), t(\theta)\}$ is IC. (a) y_i is non-decreasing, (b) $u_i(\theta_i) = u_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{y}_i(s)ds$ for $i = 1..I$.

- Proof (sketch): Type θ_i maximizes $\widetilde{u}_i(\theta_i, x)$ over x. FOC implies $\theta_i \overline{y}_i'(\theta_i) - \overline{t}_i'(\theta_i) = 0$. Taking full derivative of $\widetilde{u}_i(\theta_i, \theta_i)$ with respect to θ_i and plugging in FOC implies $u_i'(\theta_i) = \overline{y}_i(\theta_i)$.

- Intuition: Higher types are more likely to get the good. Expected utility is only a function of the utility of the lowest type and the allocation probability.

Revenue Equivalence Theorem

- Proposition: (Revenue Equivalence Theorem) Any auction mechanism in which (a) the good is always allocated to the highest bidder, and (b) any bidder with the lowest valuation $\underline{\theta}$ gets 0 expected surplus, yields the same expected revenue for the seller, and results in a buyer with valuation θ making the same expected payment.

- Intuition: $u_i(\theta_i)$ depends only on $\overline{y}_i(\theta_i)$ and $u_i(\underline{\theta}_i)$. Therefore expected payment of θ_i depends only on $\overline{y}_i(\theta_i)$ and $u_i(\underline{\theta}_i)$.

- Interpretation: Consider the following four auction mechanisms

  1. First price sealed bid auction
  2. Second price sealed bid auction
  3. English auction (the price increases by unit increment from $\underline{\theta}$ to $\overline{\theta}$ until one bidder bids)
  4. Dutch auction (the price decreases by unit increment from $\overline{\theta}$ to $\underline{\theta}$ until one bidder bids)

- If in equilibrium the bidder with the highest realized valuation always gets the good and a bidder with valuation $\underline{\theta}$ gets zero surplus, then the seller gets is indifferent (ex-ante) between these 4 schemes

Application to Uniform Case

- Assume the valuations are uniformly distributed between $\underline{\theta}$ and $\overline{\theta}$
- $F(\theta) = \frac{1}{\overline{\theta}-\underline{\theta}}$ and $F(\theta) = \frac{\theta-\underline{\theta}}{\overline{\theta}-\underline{\theta}}$
- The k^{th} highest expected value drawn among n independently drawn values is $\underline{\theta} + \frac{n+1-k}{n+1}(\overline{\theta} - \underline{\theta})$
- The expected revenue of any auction that achieves efficiency and leave no surplus to the lowest valuation is $\underline{\theta} + \frac{n-1}{n+1}(\overline{\theta} - \underline{\theta})$
- Can compute bidding strategies using revenue equivalence theorem
  1. Under FPSBA, $u_i(\theta_i) = \theta_i \overline{y}_i(\theta_i) - \bar{t}_i(\theta_i) = (\theta_i - b_i)\overline{y}_i(\theta_i)$
  2. This implies $b_i = \frac{\bar{t}_i(\theta_i)}{\overline{y}_i(\theta_i)} = \underline{\theta} + \frac{n-1}{n}(\theta_i - \underline{\theta})$