European University Institute
Economics Department

MICROECONOMICS II
Course Materials – Part II

Final Exams
with Answer Keys

by
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with assistance of
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Florence, Italy
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MOCK FINAL
Microeconomics II
Pascal Courty/Liliane Karlinger
EUI, Florence, November 2003

ANSWER ALL QUESTIONS. TIME: TWO HOURS.
TOTAL POINTS: 100

Exercise 1 (20 pts)

Indicate whether each of the following is TRUE or FALSE. Justify your answer. No credit given without justification.

NOTE: These questions are mainly open questions, i.e. they do not have a single admissible answer; rather, you are supposed to show that you can reason consistently, referring to the material that was covered in the lecture.

(a) (4 pts) In a competitive equilibrium, firms earn zero profits if and only if there is free entry and firms have a constant-returns-to-scale technology.

(b) (4 pts) It would be Pareto-suboptimal to force night clubs to charge the same price to all patrons (ie no free nights for females).

(c) (4 pts) At the optimal monopoly price, the price elasticity of demand is always strictly greater than one.

(d) (4 pts) If identical production technologies are available in all countries, then free trade in outputs implies that input prices will be equalized.

(e) (4 pts) Arrow-Debreu contingent claims remove all uncertainty from the economy so that consumption does not vary across states of the world.

Exercise 2 (25 pts)

Consider a situation in which there is a monopolist in a market with inverse demand function $p(q)$. The monopolist makes two choices: How much to invest in cost reduction, I, and how much to sell, q. If the monopolist invests I in cost reduction, his (constant) per-unit cost of production is $c(I)$. Assume that $c'(I) < 0$ and that $c''(I) > 0$. Assume throughout that the monopolist's objective function is concave in q and I.

(a) (5 pts) Derive the first-order conditions for the monopolist's choices.

(b) (10 pts) Compare the monopolist's choices with those of a benevolent social planner who can control both q and I (a "first-best" comparison).

(c) (10 pts) Compare the monopolist's choices with those of a benevolent social planner who can control I but not q (a "second-best" comparison). Suppose that the planner chooses I and then the monopolist chooses q.

Exercise 3 (20 pts)

Lojack is a car retrieval system, o¤ered by a private …rm to car owners in several major US cities. With Lojack, a small radio transmitter is hidden in one of many possible locations within a car. When the car is reported stolen, the police remotely activate the transmitter, allowing specially equipped police cars and helicopters to track the precise location and movement of the stolen vehicle. Thus, Lojack-equipped cars have higher retrieval rates and lower theft damages once a vehicle is stolen.

For legal reasons (which are of no relevance for our discussion here), car owners are not allowed to display anywhere on their Lojack-equipped vehicle that Lojack is installed. Thus, Lojack is an example of unobservable self-protection against crime.

(a) (5 pts) Compare the private bene…ts of Lojack (ie the bene…ts to the car owner who decides to install Lojack) with the private bene…ts of a highly visible precaution system like, for instance, a car alarm (ignore the possibility of fake car alarms for the purpose of this exercise).

(b) (5 pts) Discuss possible externalities generated by Lojack-equipped cars on other car owners, and compare these to the externalities generated by vehicles equipped with a visible car alarm on other car owners.

(c) (10 pts) Set up a very simple model to argue which precaution system (visible or invisible) is likely to be oversupplied, and which one will be undersupplied (ie compare the individual car owner's decision to buy the system with the socially optimal level).

Exercise 4 (35 pts)

Consider a 2-consumer pure exchange economy and assume quasi-linear preferences $u_i(m_i, x_i) = m_i + \phi_i(x_i)$ for $i = 1, 2$, where m_i 2 < and x_i 2 $<^+$ and ϕ_i is an unbounded concave function. Consumer i is endowed with $w_i = (w_{mi}, w_{xi})$.

(a) (7 pts) Characterize the interior competitive equilibrium for this economy (ie the equilibrium where $x_1 > 0$ and $x_2 > 0$). Does the consumption of good x_i depend on initial endowments w_i? Under what condition will the equilibrium consumption of x be an interior solution? Assume for the rest of this exercise that this is always the case.

(b) (7 pts) Generalize the concept of Edgeworth box to this economy and show with the use of a …gure that there are no wealth e¤ects in this economy.

(c) (7 pts) For the rest of this exercise, assume that m_i 2 $<^+$. Under what condition will the consumption of good x_i not depend on initial endowments w_i?

(d) (7 pts) Using an Edgeworth box, draw the Pareto Set. Distinguish three sets of Pareto optimal allocations corresponding to $\phi_1^0(x_1) > \phi_2^0(x_2)$, $\phi_1^0(x_1) < \phi_2^0(x_2)$, and $\phi_1^0(x_1) = \phi_2^0(x_2)$.

(e) (Extra Credit) Does the second welfare theorem apply? That is, can all Pareto optimal allocations be supported as a competitive equilibrium?

MOCK FINAL - ANSWER KEY
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Exercise 1 (20 pts)

Indicate whether each of the following is TRUE or FALSE. Justify your answer. No credit given without justification.

NOTE: These questions are mainly open questions, i.e. they do not have a single admissible answer; rather, you are supposed to show that you can reason consistently, referring to the material that was covered in the lecture.

(a) (4 pts) In a competitive equilibrium, firms earn zero profits if and only if there is free entry and firms have a constant-returns-to-scale technology.

FALSE - There are two possible ways to disprove this statement: (1) Even if the number of firms is fixed (ie no free entry), but CRS holds, profits will be zero in a competitive equilibrium. (2) If the firms' technology is not CRS (but instead e.g. decreasing-returns to scale with fixed costs), and free entry holds, we can still have zero profits in a competitive equilibrium.

(b) (4 pts) It would be Pareto-suboptimal to force night clubs to charge the same price to all patrons (ie no free nights for females).

TRUE - Night clubs are a competitive industry, so that price discrimination here does not raise the same welfare concerns as under monopoly. Female patrons generate a demand externality (attracting male patrons), so that price discrimination between these two groups allows night clubs to internalize this externality and to maintain the right balance of partons.

(c) (4 pts) At the optimal monopoly price, the price elasticity of demand is always strictly greater than one.

FALSE. If marginal cost is zero ($c'(q^M) = 0$), then price elasticity of demand will be exactly 1 at the monopoly price (recall: $\left(p^M - c'(q^M)\right)/p^M = 1/\varepsilon_{p^M}$)

(d) (4 pts) If identical production technologies are available in all countries, then free trade in outputs implies that input prices will be equalized.

FALSE - This is true only under constant-returns-to-scale, no factor-intensity reversal, and no specialization.

(e) (4 pts) Arrow-Debreu contingent claims remove all uncertainty from the economy so that consumption does not vary across states of the world.

FALSE - true only if state contingent prices are equal and this will hold when aggregate endowment does not vary across states of the world.

Exercise 2 (25 pts)

Consider a situation in which there is a monopolist in a market with inverse demand function $p(q)$. The monopolist makes two choices: How much to invest in cost reduction, I, and how much to sell, q. If the monopolist invests I in cost reduction, his (constant) per-unit cost of production is $c(I)$. Assume that $c'(I) < 0$ and that $c''(I) > 0$. Assume throughout that the monopolist's objective function is concave in q and I.

(a) (5 pts) Derive the first-order conditions for the monopolist's choices.

The monopolist will solve

$$\max_{q,I \geq 0} p(q)q - c(I)q - I$$

which yields the first-order conditions

$$\begin{aligned} \text{(i) } p'(q^m)q^m + p(q^m) &= c(I^m) \\ \text{(ii) } -c'(I^m)q^m &= 1 \end{aligned}$$

(b) (10 pts) Compare the monopolist's choices with those of a benevolent social planner who can control both q and I (a "first-best" comparison).

The social planner will maximize total surplus

$$\max_{q,I \geq 0} \underbrace{\int_0^q P(x)dx}_{\text{gross consumer surplus}} \underbrace{-c(I)q - I}_{\text{total cost}}$$

and the first-order conditions are

$$\begin{aligned} \text{(iii) } p(q^*) &= c(I^*) \\ \text{(iv) } -c'(I^*)q^* &= 1 \end{aligned}$$

The monopolist produces less output than is socially optimal, $q^m < q^*$, and price is above marginal cost. Given this, equations (ii) and (iv) imply that $-c'(I^*) < -c'(I^m)$, which in turn implies that $I^* > I^m$. Note, however, that given the output level of the monopolist, he chooses the optimal level of investment.

(c) (10 pts) Compare the monopolist's choices with those of a benevolent social planner who can control I but not q (a "second-best" comparison). Suppose that the planner chooses I and then the monopolist chooses q.

Given a level $\hat{I}$ set by the government, the monopolist will set q to maximize its profits, i.e., it will set q to equate marginal revenue with marginal cost. Therefore, the governments problem is to maximize social surplus subject to the monopolist's behavior. That is,

$$\begin{aligned} \max_{q,I\geq 0} S(q,I) &= \int_0^q P(x)dx - c(I)q - I \\ \text{subject to } p'(q)\,q + p(q) &= c(I) \end{aligned}$$

The Lagrangian is

$$L = \int_0^q P(x)dx - c(I)q - I - \lambda\left[p'(q)\,q + p(q) - c(I)\right]$$

which yields first-order conditions

$$\begin{aligned} \text{(v)}\ p(\hat{q}) - c(\hat{I}) - \lambda\left[p''(\hat{q})\,\hat{q} + 2p'(\hat{q})\right] &= 0 \\ \text{(vi)}\ -c'(\hat{I})\,(\hat{q} - \lambda) &= 1 \end{aligned}$$

Note that (vi) implies that the chosen investment level $\hat{I}$ is not optimal given the quantity $\hat{q}$, and since $\lambda > 0$ we will have $\hat{I}$ greater than optimal given $\hat{q}$. The intuition is that in this second-best environment, the social planner chooses an investment level larger than optimal for the given level of output in order to induce the monopolist to produce more.

Exercise 3 (20 pts)

Lojack is a car retrieval system, offered by a private firm to car owners in several major US cities. With Lojack, a small radio transmitter is hidden in one of many possible locations within a car. When the car is reported stolen, the police remotely activate the transmitter, allowing specially equipped police cars and helicopters to track the precise location and movement of the stolen vehicle. Thus, Lojack-equipped cars have higher retrieval rates and lower theft damages once a vehicle is stolen.

For legal reasons (which are of no relevance for our discussion here), car owners are not allowed to display anywhere on their Lojack-equipped vehicle that Lojack is installed. Thus, Lojack is an example of unobservable self-protection against crime.

(a) (5 pts) Compare the private benefits of Lojack (ie the benefits to the car owner who decides to install Lojack) with the private benefits of a highly visible precaution system like, for instance, a car alarm (ignore the possibility of fake car alarms for the purpose of this exercise).

Private benefits of Lojack: An individual car owner's decision to install Lojack only marginally affects the likelihood of his or her own vehicle being stolen since thieves base their theft decisions on average Lojack installation rates. The only private benefits of Lojack are thus the higher retrieval rates and lower theft damages once a vehicle is stolen (see text of question).

Private benefits of visible car alarm: A visible car alarm has an immediate impact on the probability of theft. A thief screening a street for a possible theft target will likely avoid the car that has a visible car alarm installed, and will instead go for a different car (of similar value) without visible precaution system.

(b) (5 pts) Discuss possible externalities generated by Lojack-equipped cars on other car owners, and compare these to the externalities generated by vehicles equipped with a visible car alarm on other car owners.

Externalities of Lojack: Auto thieves presumably have an idea about the average Lojack installation rate, but they can never tell if a specific car has Lojack installed or not. Thus, the presence of Lojack makes auto theft riskier and less profitable, leading to a reduction in the number of such crimes. We can conclude that Lojack generates **positive** externalities for all other car owners (in particular for those who do not have Lojack installed).

Externalities of visible car alarms: This form of victim precaution will primarily redistribute crime across victims rather than reduce crime. Consequently, those who engage in observable self-protection will impose a cost on those who do not. Thus, visible car alarms generate mainly **negative** externalities for all those car owners who do not have such a device installed.

(c) (10 pts) Set up a very simple model to argue which precaution system (visible or invisible) is likely to be oversupplied, and which one will be undersupplied (ie compare the individual car owner's decision to buy the system with the socially optimal level).

Note: This is an open problem, and the following suggestions for modelling the issue probably overshoot (ie within the time limit you have at the exam, your model is likely to be less elaborate, but you will still get a high grade...)

Case 1: Supply of Lojack

Let our "city" consist of 2 car owners, $i = 1, 2$; each of them owns a car of value V. Assume for simplicity that only consumer 1 can buy Lojack, while consumer 2 cannot buy it. Assume also that once a car is stolen, it will never be retrieved unless it has Lojack installed. In that case, the car is retrieved with probability $\alpha > 0$. Denote by $h \in \{0, 1\}$ consumer 1's decision of whether to buy or not to buy the precaution device. Let $c > 0$ be the cost of installing Lojack, and let $\pi(h)$ be the (average) probability of car theft (where $\pi(1) < \pi(0)$). Consumer 1's utility is defined as follows:

$$u_1(h) = \begin{cases} \pi(1)\alpha V + (1 - \pi(1))V - c & \text{if } h = 1 \\ \pi(0)0 + (1 - \pi(0))V & \text{if } h = 0 \end{cases}$$

Thus, consumer 1 will install the device iff

$$\pi(1)\alpha V + (1 - \pi(1))V - c \geq \pi(0)0 + (1 - \pi(0))V$$

or equivalently, iff

$$c \leq \pi(1)\alpha V + (1 - \pi(1))V - (1 - \pi(0))V$$

Now, consumer 2 will be affected by 1's decision to buy or not to buy Lojack, because the presence of a Lojack-equipped car lowers the average probability of car theft; in particular, 2's utility is

$$u_2(h) = \begin{cases} \pi(1)0 + (1 - \pi(1))V & \text{if } h = 1 \\ \pi(0)0 + (1 - \pi(0))V & \text{if } h = 0 \end{cases}$$

Thus, consumer 2 gets to enjoy some of the benefits of Lojack (recall that $\pi(1) < \pi(0)$) without contributing to its cost.

Now, let's assume that car thieves are "specialized" in this "industry" and will not easily switch to other kinds of property offences (like burglary or pick-pocketing), ie the decrease in car thefts induced by Lojack ($\pi(0) - \pi(1)$) is not matched by an increase in crime rates elsewhere in society. Then, the social planner's problem of reducing car theft can be analyzed in isolation from other areas of crime.

In particular, a social planner would choose to have consumer 1 buy Lojack iff

$$[\pi(1)\alpha V + 2(1 - \pi(1))V] - c \geq 2[\pi(0)0 + (1 - \pi(0))V]$$

Comparing this to the individually optimal behavior, we see that the social planner will buy the device whenever the cost of Lojack is below the socially optimal threshold c^o

$$c \leq c^o = \pi(1)\alpha V - 2\pi(1)V + 2\pi(0)V$$

while the individual car owner will buy Lojack iff

$$c \leq c^* = \pi(1)\alpha V - \pi(1)V + \pi(0)V$$

Since $\pi(1) < \pi(0)$, we have $c^* < c^o$. Thus, Lojack is likely to be undersupplied if the decision is left to an individual car owner.

Case 2: Supply of visible car alarm

Using a similar setup: Let our "city" consist of 2 car owners, $i = 1, 2$; each of them owns a car of value V. Assume for simplicity that only consumer 1 can buy the car alarm, while consumer 2 cannot buy it. Assume also that once a car is stolen, it will never be retrieved. Denote by $h \in \{0, 1\}$ consumer 1's decision of whether to buy or not to buy the precaution device. Let $c > 0$ be the cost of installing the alarm, and let $\pi(0) = \frac{1}{2}$ be the (average) probability of car theft if no car has a car alarm. If 1's car has the car alarm, we have: $\pi_1(1) = 0$, and $\pi_2(1) = 1$, ie 1's car will never get stolen, and 2's car will get stolen for sure (note that we assume a pure redistribution of crime; on average, the theft rate is still $\frac{1}{2}$, ie the car alarm has no overall deterrence effect). Now, let 1's utility be defined as follows:

$$u_1(h) = \begin{cases} V - c & \text{if } h = 1 \\ \frac{1}{2}0 + \frac{1}{2}V & \text{if } h = 0 \end{cases}$$

Thus, consumer 1 will install the device iff

$$V - c \geq \frac{1}{2}0 + \frac{1}{2}V$$

Now, consumer 2's utility is analogously:

$$u_2(h) = \begin{cases} 0 & \text{if } h = 1 \\ \frac{1}{2}0 + \frac{1}{2}V & \text{if } h = 0 \end{cases}$$

Now, we see immediately that a social planner would never buy the visible car alarm, because the social probability of car theft is not reduced by it, but crime is just redistributed from consumer 1 to 2. In other words, since we assumed $c > 0$, we have

$$u_1(1) + u_2(1) = V - c < u_1(0) + u_2(0) = V$$

Thus, if the device is costly and "steeling car 2" is a perfect substitute for "steeling car 1" (no aggregate deterrence effect), then the visible car alarm is socially inefficient. Still, the private agent would buy the device whenever $c \leq \frac{1}{2}V$, and so visible car alarms are likely to be oversupplied by private agents.

Exercise 4 (35 pts)

Consider a 2-consumer pure exchange economy and assume quasi-linear preferences $u_i(m_i, x_i) = m_i + \phi_i(x_i)$ for $i = 1, 2$, where $m_i \in \Re$ and $x_i \in \Re^+$ and ϕ_i is an unbounded concave function. Consumer i is endowed with $w_i = (w_{mi}, w_{xi})$.

(a)-(7 pts) Characterize the interior competitive equilibrium for this economy (ie the equilibrium where $x_1 > 0$ and $x_2 > 0$). Does the consumption of good x_i depend on initial endowments w_i? Under what condition will the equilibrium consumption of x be an interior solution? Assume for the rest of this exercise that this is always the case.

The interior competitive equilibrium is characterized by a price p and an allocation $(x_1^*, x_2^*, m_1^*, m_2^*)$ such that

$$\begin{aligned} \phi_1'(x_1^*) &= \phi_2'(x_2^*) = p \\ x_1^* + x_2^* &= w_{x1} + w_{x2} \\ m_1^* &= w_{m1} + p(w_{x1} - x_1^*) \\ m_2^* &= w_{m2} + p(w_{x2} - x_2^*) \end{aligned}$$

The consumption of good x_i is independent of initial endowments w_i: Since $m_i \in \Re$, ie m_i^* can take any value from $-\infty$ to $+\infty$, the equilibrium consumption levels (x_1^*, x_2^*) and equilibrium price p will be uniquely determined by the condition $\phi_1'(x_1^*) = \phi_2'(x_2^*) = p$. For each change in endowment point from (w_1, w_2) to $(\tilde{w}_1, \tilde{w}_2)$, only consumption of the numeraire will change. In particular, the corresponding $(\tilde{m}_1^*, \tilde{m}_2^*)$ will be obtained as $\tilde{m}_i^* = \tilde{w}_{mi} + \phi_i'(x_i^*)(\tilde{w}_{xi} - x_i^*)$, while (x_1^*, x_2^*) remain the same.

We will have an interior solution if $\phi_1'(0) > \phi_2'(w_{x1} + w_{x2})$ and $\phi_2'(0) > \phi_1'(w_{x1} + w_{x2})$, ie if each agent's valuation of the first unit is strictly larger than the other agent's marginal utility of consuming the total endowment of x.

(b)-(7 pts) Generalize the concept of Edgeworth box to this economy and show with the use of a figure that there are no wealth effects in this economy.

As discussed in (a), $m_i \in \Re$, ie m_i^* can take any value from $-\infty$ to $+\infty$; thus, the Edgeworth "box" is open on two sides, looking more like a "band" than a "box". The equilibrium consumption levels (x_1^*, x_2^*) and equilibrium price p will be uniquely determined by the condition $\phi_1'(x_1^*) = \phi_2'(x_2^*) = p$. For each endowment point (w_1, w_2), the corresponding (m_1^*, m_2^*) will be obtained as $m_i^* = w_{mi} + \phi_i'(x_i^*)(w_{xi} - x_i^*)$, so that a redistribution of endowments corresponds to a vertical parallel shift of indifference curves along the "Edgeworth band", leaving consumption levels (x_1^*, x_2^*) unchanged. Hence, there are no wealth effects in this economy.

Note: To get full credit on this part of the exercise, you need to derive the indifference curves analytically, then argue that they are vertical parallel shifts of one another, and that this graphical feature implies that there are no wealth effects. (see Figure 1)

(c)-(7 pts) For the rest of this exercise, assume that $m_i \in \Re^+$. Under what condition will the consumption of good x_i not depend on initial endowments w_i?

Consumption of good x_i will not depend on initial endowments w_i as long as none of the two consumers are budget constrained, ie if $w_{mi} \geq -\phi_i'(x_i^*)(w_{xi} - x_i^*)$ is satisfied for both $i = 1, 2$. In terms of the Edgeworth box: as long as we don't hit it's upper or lower boundary.

(d)-(7 pts) Using an Edgeworth box, draw the Pareto Set. Distinguish three sets of Pareto optimal allocations corresponding to $\phi_1'(x_1) > \phi_2'(x_2)$, $\phi_1'(x_1) < \phi_2'(x_2)$, and $\phi_1'(x_1) = \phi_2'(x_2)$.

see Figure 2: Lower left corner: consumer 1 is budget-constrained and $\phi_1'(x_1) > \phi_2'(x_2)$; interior section: no consumer is budget-constrained, ie $m_i^* > 0$ for both $i = 1, 2$ and $\phi_1'(x_1) = \phi_2'(x_2)$. Upper right corner: consumer 2 is budget-constrained and $\phi_2'(x_2) > \phi_1'(x_1)$.

(e)-(Extra Credit) Does the second welfare theorem apply? That is, can all Pareto optimal allocations be supported as a competitive equilibrium?

Case 1: allocations where $m_i^* > 0$ for both $i = 1, 2$ and $\phi_1'(x_1^*) = \phi_2'(x_2^*)$ can always be sustained by price vector $p^* = \phi_1'(x_1^*) = \phi_2'(x_2^*)$

Case 2: allocations where one of the consumers is budget-constrained, ie $m_i^* = w_{mi} + w_{mj}$ and $m_j^* = 0$, can be sustained by a price vector $p^* = \phi_i'(x_i^*)$ and $\phi_i'(x_i^*) < \phi_j'(x_j^*)$. By agent i's budget constraint, we have $x_i^* = \frac{1}{p^*}(w_{mi} - m_i^*) + w_{xi}$, and by j's budget constraint, we have $x_j^* = \frac{1}{p^*}w_{mj} + w_{xj}$.

FIGURE 1

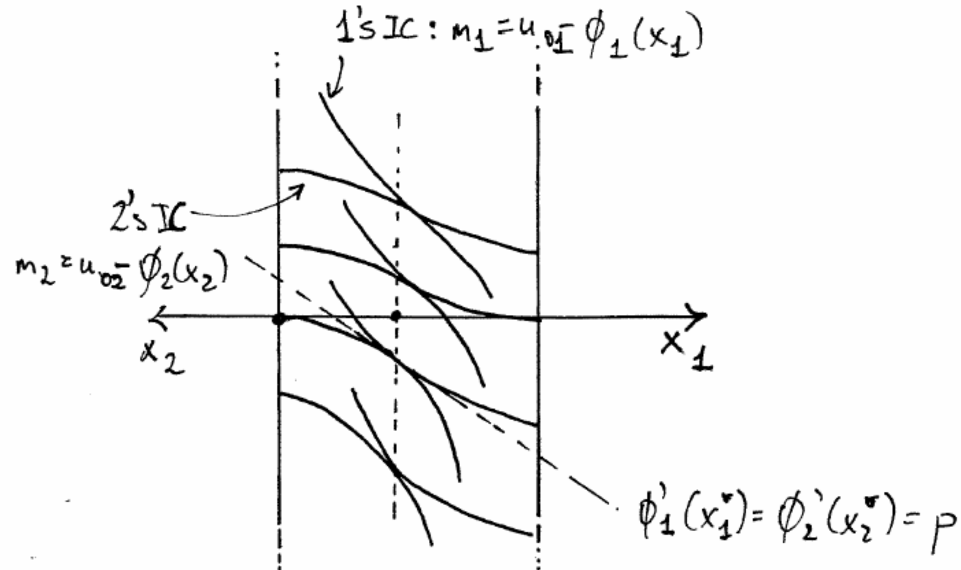

Figure 1:

FIGURE 2

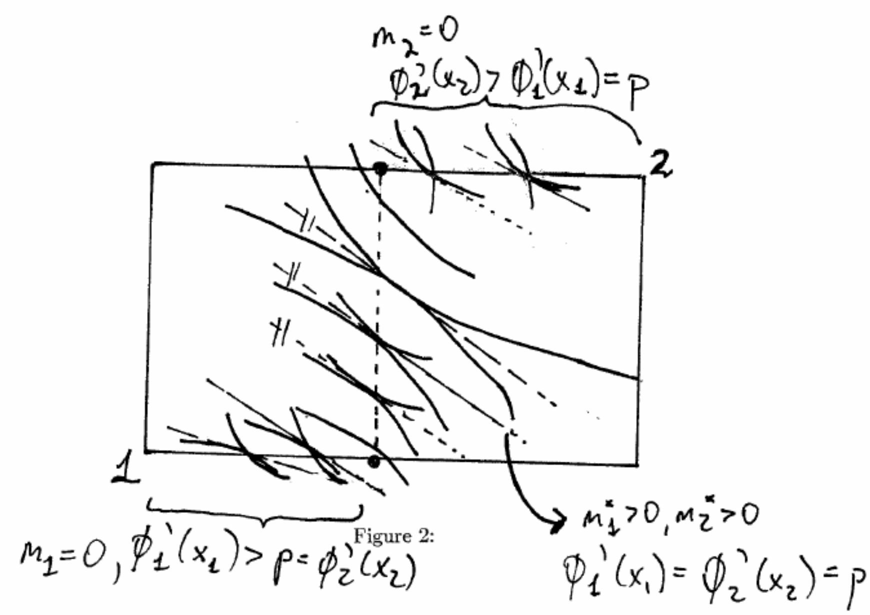

Figure 2:

SECOND MID-TERM EXAM
Microeconomics
Pascal Courty
EUI, Florence, December 12, 2003

ANSWER ALL QUESTIONS. TIME: TWO HOURS.
TOTAL POINTS: 100

Exercise 1 (20 pts)

Indicate whether each of the following is TRUE or FALSE. Justify your answer. No credit given without justi…cation.

NOTE: These questions are mainly open questions, i.e. they do not have a single admissible answer; rather, you are supposed to show that you can reason consistently, referring to the material that was covered in the lecture.

(a) (4 pts) In an economy with quasilinear preferences, a tax is not ine¢cient if and only if the elasticity of demand is zero or the elasticity of supply is zero.

(b) (4 pts) In a team with N members where total output is a function of the sum of all the members' e¤orts, the e¢cient e¤ort contribution is achieved if each member receives a bonus that is equal to the team's output.

(c) (4 pts) Under the assumption of linear demands and CRS technology, a law mandating …rms to set a uniform price in all markets always increases aggregate welfare.

(d) (4 pts) Consider a 2x2 Edgeworth box exchange economy such that trade takes place in equilibrium. Both consumers strictly prefer to trade at the equilibrium price than to receive a take it or leave it o¤er from the other consumer.

(e) (4 pts) In an economy with a single consumption good, in which households receive an endowment in each period that depends on both idiosyncratic and aggregate shocks, and in which di¤erent households have access to di¤erent storage technologies (i.e. di¤erent waste rates), there would be unanimity about whether or not to allow the economy to trade with the rest of the world.

Exercise 2 (25 pts)

Consider the following two-period model: A …rm is a monopolist in a market with an inverse demand function (in each period) of $p(q) = a ¡ bq$. The cost per unit in period 1 is c_1. In period 2, however, the monopolist has "learned by doing", and so its constant cost per unit of output is $c_2 = c_1 ¡ mq_1$, where q_1 is the monopolist's period 1 output level. Assume $a > c_1$ and $b > m$. Also assume that the monopolist does not discount future earnings.

(a) (5 pts) What is the monopolist's level of output in each of the periods?

(b) (10 pts) What outcome would be implemented by a benevolent social planner who fully controlled the monopolist? Is there any sense in which the planner's period 1 output is selected so that "price equals marginal cost"?

(c) (10 pts) Given that the monopolist will be selecting the period 2 output level, would the planner like the monopolist to slightly increase the level of period 1 output above that identi…ed in (a)? Can you give any intuition for this?

Exercise 3 (20 pts)

When VCRs were …rst introduced, two main competing standards were offered, VHS and shortly after (so we will assume for the purpose of this exercise) Betamax. Although some experts argued that Betamax was a superior standard, it quickly lost ground against VHS and eventually disappeared. Interestingly, other standards that were introduced around that period have also disappeared to leave the entire market to VHS.

(a) (5 pts) Does a consumer's decision to adopt a given standard depend on the number of people who have adopted that standard? Why?

(b) (5 pts) Does the decision to produce complementary products (i.e. videotapes) for a speci…c standard depend on the number of people who have adopted that standard? Why?

(c) (10 pts) Identify the central features of markets with competing standards that would play a key role if you were to write a model to explain the dynamic of market share in these markets. You may want to introduce some notations to clarify your points.

Exercise 4 (35 pts)

Consider a 2x2 exchange economy where consumer i's utility is $u_i(x_{1,i}, x_{2,i}) = \alpha_{1,i} \ln x_{1,i} + \alpha_{2,i} \ln x_{2,i}$ with $\alpha_{i,j} > 0$ for $i, j = 1, 2$. Consumer i's endowment of good j is $w_{j,i}$. Throughout the exercise we will normalize the price of good 1 to 1 and denote the price of good 2 by p.

(a) (9 pts) De…ne consumer i's Marginal Elasticity of Substitution (MES) similarly as the Marginal Rate of Substitution but with an additional normalization: $MES_i = x_{1,i}dx_{2,i}/x_{2,i}dx_{1,i}$. Show that MES_i is constant. Assume for now that consumer i has income I. Derive consumer i's demand and express this demand as a function of MES_i and I.

(b) (9 pts) Assume that the competitive equilibrium is achieved at an interior point. Derive the competitive equilibrium price p and express it as a function of MES_i and $w_{j,i}$ for $i, j = 1, 2$.

(c) (9 pts) Show that the equilibrium price p derived in (b): (i) increases with $w_{1,1}$ and $w_{1,2}$ and decreases with $w_{2,1}$ and $w_{2,2}$, (ii) is constant if all endowments $w_{j,i}$ increase by the same proportional amount, (iii) increases if and only if $MES_2 > MES_1$ when $w_{1,1}$ increases while $w_{1,2}$ decreases by an equal amount. Interpret these results.

(d) (8 pts) Assume a social planner wants to maximize $u_1(x_{1,1}, x_{2,1}) + u_2(x_{1,2}, x_{2,2})$. Derive the optimal allocation assuming that it is achieved at an interior point.

(e) (Extra Credit) In the allocation you derived in (d), under what condition does consumer 1 consume (i) the same amount of good 1 as consumer 2, (i) a …xed fraction of each good's total endowment.

SECOND MID-TERM EXAM - ANSWER KEY
Microeconomics II
Pascal Courty
EUI, Florence, December 12, 2003

ANSWER ALL QUESTIONS. TIME: TWO HOURS.
TOTAL POINTS: 100

Exercise 1 (20 pts)

Indicate whether each of the following is TRUE or FALSE. Justify your answer. No credit given without justi…cation.

NOTE: These questions are mainly open questions, i.e. they do not have a single admissible answer; rather, you are supposed to show that you can reason consistently, referring to the material that was covered in the lecture

(a) (4 pts) In an economy with quasilinear preferences, a tax is not ine¢ cient if and only if the elasticity of demand is zero or the elasticity of supply is zero.

TRUE - a tax is ine¢ cient if it reduces equilibrium quantities. This will not happen if consumers or …rms do not respond to prices.

(b) (4 pts) In a team with N members where total output is a function of the sum of all the members' e¤orts, the e¢ cient e¤ort contribution is achieved if each member receives a bonus that is equal to the team's output.

TRUE - each member chooses e¤ort to equalize marginal cost of e¤ort with marginal return of e¤ort. If the bonus is equal to the team's output, then the marginal return of e¤ort corresponds to the teams' marginal return of e¤ort and each member chooses the …rst best e¤ort contribution.

(c) (4 pts) Under the assumption of linear demands and CRS technology, a law mandating …rms to set a uniform price in all markets always increases aggregate welfare.

FALSE - the monopolist may instead decide to shut o¤ one (or more) markets, and still charge the (discriminatory) monopoly price to the high-demand market, thus reducing aggregate welfare.

(d) (4 pts) Consider a 2x2 Edgeworth box exchange economy such that trade takes place in equilibrium. Both consumers strictly prefer to trade at the equilibrium price than to receive a take it or leave it o¤er from the other consumer.

FALSE - Actually it is true as long as consumers have strictly convex preferences but not necessarily true if this is not the case. Consider the case where a consumer has a linear segment in her indi¤erence curve and assume that both the initial endowment and equilibrium allocation are located on that segment. The other consumer strictly benefits from trade (assuming that her indifference curves are convex) but this consumer does not. S/he therefore is indifferent between the competitive equilibrium and the take it or leave it offer.

(e) (4 pts) In an economy with a single consumption good, in which households receive an endowment in each period that depends on both idiosyncratic and aggregate shocks, and in which different households have access to different storage technologies (i.e. different waste rate), there would be unanimity about whether or not to allow the economy to trade with the rest of the world.

FALSE - Under autarchy the households with access to storage can insure themselves and also others through trading. An equilibrium emerges where there is a trade off between the cost of storage and the gains from inter-temporal insurance. Trading with the rest of the world Pareto dominates autarchy but those households with superior storage technology may loose the rents they receive from providing insurance to their neighbors and may therefore oppose trading.

Exercise 2 (25 pts)

Consider the following two-period model: A firm is a monopolist in a market with an inverse demand function (in each period) of $p(q) = a - bq$. The cost per unit in period 1 is c_1. In period 2, however, the monopolist has "learned by doing", and so its constant cost per unit of output is $c_2 = c_1 - mq_1$, where q_1 is the monopolist's period 1 output level. Assume $a > c_1$ and $b > m$. Also assume that the monopolist does not discount future earnings.

(a) (5 pts) What is the monopolist's level of output in each of the periods?

The monopolist's intertemporal problem is

$$\max_{q_1,q_2} (a - bq_1 - c_1)\, q_1 + (a - bq_2 - (c_1 - mq_1)\, q_2)$$

and the FOCs yield $q_1^m = q_2^m = \frac{a - c_1}{2b - m} > 0$ (by the exercise's assumptions).

(b) (10 pts) What outcome would be implemented by a benevolent social planner who fully controlled the monopolist? Is there any sense in which the planner's period 1 output is selected so that "price equals marginal cost"?

A benevolent social planner maximizes total surplus (assuming no discounting of the future, we just add up both periods' consumer surplus, to both periods' firm's profits, and subtract both periods' costs),

$$\begin{aligned}\max_{q_1,q_2} SW &= \int_0^{q_1} p(x)\,dx + \int_0^{q_2} p(x)\,dx - c_1 q_1 - (c_1 - mq_1)\, q_2 \\ &= a(q_1 + q_2) - \frac{1}{2} b \left(q_1^2 + q_2^2\right) - c_1 q_1 - (c_1 - mq_1)\, q_2\end{aligned}$$

and the FOCs are,

$$\text{(i)}\ (a - bq_1) + mq_2 = c_1$$
$$\text{(ii)}\ (a - bq_2) = c_1 - mq_1$$

which yields $q_1^* = q_2^* = \frac{a - c_1}{b - m} > 0$. Comparing this with the monopoly quantities, we see that $q_i^m < q_i^*$ for $i = 1, 2$. The way we wrote down the FOCs show that in fact there is a sense of "price equals marginal cost". Recall that price is marginal surplus from each period's good: In the first period, aside from marginal consumer surplus, given q_2, any additional unit of q_1 reduces marginal cost next period by mq_2. The right hand side is the effective marginal cost in each period.

(c) (10 pts) Given that the monopolist will be selecting the period 2 output level, would the planner like the monopolist to slightly increase the level of period 1 output above that identified in (a)? Can you give any intuition for this?

As we have seen, the social planner would want to produce more in every period. By increasing the output in the first period above q_1^m, welfare in the first period will be higher, and this will lead to a lower second period marginal cost. This lower second period marginal cost will induce the monopolist to produce more in the second period and will therefore further increase welfare.

Exercise 3 (20 pts)

When VCRs were first introduced, two main competing standards were offered, VHS and (we will assume for the purpose of this exercise) briefly later Betamax. Although some experts argued that Betamax was a superior standard, it quickly lost ground against VHS and eventually disappeared. Interestingly, other standards that were introduced around that period have also disappeared to leave the entire market to VHS.

(a) (5 pts) Does a consumer's decision to adopt a given standard depend on the number of people who have adopted that standard? Why?

(b) (5 pts) Does the decision to produce complementary products (i.e. videotapes) for a specific standard depend on the number of people who have adopted that standard? Why?

(c) (10 pts) Identify the central features of markets with competing standards that would play a key role if you were to write a model to explain the dynamic of market share in these markets. You may want to introduce some notations to clarify your points.

Exercise 4 (35 pts)

Consider a 2x2 exchange economy where consumer i's utility is $u_i(x_{1,i}, x_{2,i}) = \alpha_{1,i} \ln x_{1,i} + \alpha_{2,i} \ln x_{2,i}$ with $\alpha_{i,j} > 0$ for $i, j = 1, 2$. Consumer i's endowment of good j is $w_{j,i}$. Throughout the exercise we will normalize the price of good 1 to 1 and denote the price of good 2, p.

(a) (9 pts) Define consumer i's Marginal Elasticity of Substitution (MES) similarly as the Marginal Rate of Substitution but with an additional normalization $MES_i = x_{1,i}dx_{2,i}/x_{2,i}dx_{1,i}$. Show that MES_i is constant. Assume for now that consumer i has income I. Derive consumer i's demand and express this demand as a function of MES_i and I.

We derive MES_i as

$$\begin{aligned}\frac{x_{1,i}dx_{2,i}}{x_{2,i}dx_{1,i}} &= \frac{x_{1,i}\partial u_i\left(x_{1,i},x_{2,i}\right)/\partial x_{1,i}}{x_{2,i}\partial u_i\left(x_{1,i},x_{2,i}\right)/\partial x_{2,i}} \\ &= \frac{x_{1,i}\alpha_{1i}\frac{1}{x_{1,i}}}{x_{2,i}\alpha_{2i}\frac{1}{x_2}} = \frac{\alpha_{1i}}{\alpha_{2i}}\end{aligned}$$

which is a ratio of two constants and is thus indeed a constant itself.

Given consumer i has income I, she needs to solve the following UMP:

$$\begin{aligned}\max_{x_{1,i},x_{2,i}} u_i\left(x_{1,i},x_{2,i}\right) &= \alpha_{1,i}\ln x_{1,i} + \alpha_{2,i}\ln x_{2,i} \\ \text{s.t. } p_1x_{1,i} + p_2x_{2,i} &\le I\end{aligned}$$

The FOCs with respect to $(x_{1,i},x_{2,i})$ yield the following optimality condition:

$$\frac{\alpha_{1i}}{\alpha_{2i}} = \frac{p_1x_{1,i}}{p_2x_{2,i}}$$

which we can insert into the budget constraint to obtain the following demand functions:

$$x_{1,i}\left(p_1,I\right) = \frac{\alpha_{1i}}{\alpha_{2i}}\frac{I}{p_1\left(\frac{\alpha_{1i}}{\alpha_{2i}}+1\right)}$$

and

$$x_{2,i}\left(p_2,I\right) = \frac{I}{p_2\left(\frac{\alpha_{1i}}{\alpha_{2i}}+1\right)}$$

(b) (9 pts) Assume that the competitive equilibrium is achieved at an interior point. Derive the competitive equilibrium price p^* and express it as a function of MES_i and $w_{j,i}$ for $i,j = 1,2$.

At the competitive equilibrium price vector $(p_1^* = 1, p_2^* = p^*)$, consumer i's income is given by the value of her endowments: $I_i = 1w_{1,i} + p^*w_{2,i}$. Market clearing for good 2 means:

$$x_{2,1}\left(p^*, w_{1,1} + p^*w_{2,1}\right) + x_{2,2}\left(p^*, w_{1,2} + p^*w_{2,2}\right) = w_{2,1} + w_{2,2} \qquad (1)$$

We can solve this equation for p^* to obtain

$$p^* = \frac{w_{1,1}\left(\frac{\alpha_{12}}{\alpha_{22}}+1\right) + w_{1,2}\left(\frac{\alpha_{11}}{\alpha_{21}}+1\right)}{w_{2,1}\left(\frac{\alpha_{12}}{\alpha_{22}}+1\right)\frac{\alpha_{11}}{\alpha_{21}} + w_{2,2}\left(\frac{\alpha_{11}}{\alpha_{21}}+1\right)\frac{\alpha_{12}}{\alpha_{22}}}$$

Note that by Walras' Law, $(p_1^* = 1, p_2^* = p^*)$ will clear the market for good 1 as well.

(c) (9 pts) Show that the equilibrium price p^* derived in (b): (i) increases with $w_{1,1}$ and $w_{1,2}$ and decreases with $w_{2,1}$ and $w_{2,2}$, (ii) is constant if all endowments $w_{j,i}$ increase by the same proportional amount, (iii) increases if and only if $MES_2 > MES_1$ when $w_{1,1}$ increases while $w_{1,2}$ decreases by an equal amount. Interpret these results.

(i)

$$\frac{\partial p^*}{\partial w_{1,1}} = \frac{\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right)}{w_{2,1}\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right)\frac{\alpha_{11}}{\alpha_{21}} + w_{2,2}\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)\frac{\alpha_{12}}{\alpha_{22}}} > 0$$

$$\frac{\partial p^*}{\partial w_{1,2}} = \frac{\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)}{w_{2,1}\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right)\frac{\alpha_{11}}{\alpha_{21}} + w_{2,2}\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)\frac{\alpha_{12}}{\alpha_{22}}} > 0$$

$$\frac{\partial p^*}{\partial w_{2,1}} = -\frac{\left[w_{1,1}\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right) + w_{1,2}\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)\right]\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right)\frac{\alpha_{11}}{\alpha_{21}}}{\left[w_{2,1}\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right)\frac{\alpha_{11}}{\alpha_{21}} + w_{2,2}\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)\frac{\alpha_{12}}{\alpha_{22}}\right]^2} < 0$$

$$\frac{\partial p^*}{\partial w_{2,2}} = -\frac{\left[w_{1,1}\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right) + w_{1,2}\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)\right]\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)\frac{\alpha_{12}}{\alpha_{22}}}{\left[w_{2,1}\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right)\frac{\alpha_{11}}{\alpha_{21}} + w_{2,2}\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)\frac{\alpha_{12}}{\alpha_{22}}\right]^2} < 0$$

(ii) Let $k > 1$ be some constant. Then,

$$\begin{aligned}
p^*(kw_{1,1}, kw_{1,2}, kw_{2,1}, kw_{2,2}) &= \frac{(kw_{1,1})\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right) + (kw_{1,2})\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)}{(kw_{2,1})\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right)\frac{\alpha_{11}}{\alpha_{21}} + (kw_{2,2})\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)\frac{\alpha_{12}}{\alpha_{22}}} \\
&= \frac{k}{k}\frac{w_{1,1}\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right) + w_{1,2}\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)}{w_{2,1}\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right)\frac{\alpha_{11}}{\alpha_{21}} + w_{2,2}\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)\frac{\alpha_{12}}{\alpha_{22}}} \\
&= p^*(w_{1,1}, w_{1,2}, w_{2,1}, w_{2,2})
\end{aligned}$$

(ii) Let $0 < dw_{1,1} = -dw_{1,2}$. Then,

$$dp^* = \frac{\partial p^*}{\partial w_{1,1}} dw_{1,1} + \frac{\partial p^*}{\partial w_{1,2}} dw_{1,2}$$

and so

$$\begin{aligned}
\frac{dp^*}{dw_{1,1}} &= \frac{\partial p^*}{\partial w_{1,1}} - \frac{\partial p^*}{\partial w_{1,2}} \\
&= \frac{\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right) - \left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)}{w_{2,1}\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right)\frac{\alpha_{11}}{\alpha_{21}} + w_{2,2}\left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right)\frac{\alpha_{12}}{\alpha_{22}}}
\end{aligned}$$

Thus, $\frac{dp^*}{dw_{1,1}} > 0$ iff $\left(\frac{\alpha_{12}}{\alpha_{22}} + 1\right) - \left(\frac{\alpha_{11}}{\alpha_{21}} + 1\right) > 0$, ie $MES_2 = \frac{\alpha_{12}}{\alpha_{22}} > MES_1 = \frac{\alpha_{11}}{\alpha_{21}}$.

Interpretation:

(i) If total endowment of good 1 increases, ie good 1 becomes relatively more abundant, then relative price of good 1 decreases (which is equivalent to saying that the relative price of good 2 increases). Conversely, if good 2 becomes relatively more abundant, relative price of good 2 decreases.

(ii) As we saw in (a), the loglinear utility function has the very special feature that for any given pair of prices (p_1, p_2), consumers will always spend a fixed fraction of their income on one and the other good, ie demands deriving from loglinear utility functions are homothetic in wealth. Thus, if we increase each consumer's endowments of both goods by the same proportional amount k (which means that consumer's wealth has increased by a factor k, given prices), both consumers will demand both goods in the exact same proportions as they did before the increase in endowments. Thus, the market-clearing condition (1) is multiplied by the same constant on both sides, and so the equilibrium price (the one that solves equation (1)) remains unchanged.

(iii) Suppose that instead of increasing total endowment of good 1, we just transfer some of it from consumer 2 to consumer 1, keeping the total amount of good 1 fixed. Then, consumer 1 perceives an increase in wealth (and will therefore demand strictly more of both goods) at the "old" equilibrium price. However, at these prices, consumer 1's increase in demands will in general not match consumer 2's decrease in demands. In particular, if $MES_2 > MES_1$, this means that consumer 2 values good 1 relatively more (good 2 relatively less) than consumer 1. Thus, at "old" equilibrium prices, consumer 1's increase in demand for good 2 will be larger than consumer 2's decrease in demand for good 2 (inspect demand functions above to convince yourself that $\Delta x_{2i}/\Delta I_i = 1/\left[p_2\left(\frac{\alpha_{1i}}{\alpha_{2i}} + 1\right)\right]$, so if $\Delta I_1 = p_1 dw_{1,1}$ and $\Delta I_2 = -p_1 dw_{1,1}$, then $\Delta x_{21} + \Delta x_{22} > 0$ iff $MES_2 > MES_1$); hence, the equilibrium price of good 2 has to increase to clear markets.

(d) (8 pts) Assume a social planner wants to maximize $u_1(x_{1,1}, x_{2,1}) + u_2(x_{1,2}, x_{2,2})$. Derive the optimal allocation assuming that it is achieved at an interior point.

The social planner has to solve

$$\max_{\{x_{1,i}, x_{2,i}\}_{i=1,2}} \sum_{i=1} u_i(x_{1,i}, x_{2,i})$$

$$\text{s.t.} \quad x_{1,1} + x_{1,2} = w_1$$

$$x_{2,1} + x_{2,2} = w_2$$

At an interior PO allocation, the marginal utilities of consumption must be equalized across agents; hence, we obtain for each good $j = 1, 2$

$$\frac{\alpha_{j,1}}{\alpha_{j,2}} = \frac{x_{j,1}^{PO}}{x_{j,2}^{PO}}$$

Using the economy's feasibility constraints, we can solve for $\left(x_{j,1}^{PO}, x_{j,2}^{PO}\right)$, $j = 1, 2$, to obtain the unique interior PO allocation as

$$x_{j,1}^{PO} = w_j \frac{\frac{\alpha_{j,1}}{\alpha_{j,2}}}{\frac{\alpha_{j,1}}{\alpha_{j,2}} + 1} \text{ and } x_{j,2}^{PO} = w_j \frac{1}{\frac{\alpha_{j,1}}{\alpha_{j,2}} + 1}$$

(e) (Extra Credit) In the allocation you derived in (d), under what condition does consumer 1 consume (i) the same amount of good 1 as consumer 2, (i) a fixed fraction of each good's total endowment.

(i) We have $x_{1,1}^{PO} = x_{1,2}^{PO}$ iff $\alpha_{1,1} = \alpha_{1,2}$

(ii) We have $x_{1,1}^{PO} = \theta w_1$ and $x_{2,1}^{PO} = \theta w_2$ (where $\theta \in [0, 1]$ is some constant) iff $\frac{\frac{\alpha_{1,1}}{\alpha_{1,2}}}{\frac{\alpha_{1,1}}{\alpha_{1,2}}+1} = \frac{\frac{\alpha_{2,1}}{\alpha_{2,2}}}{\frac{\alpha_{2,1}}{\alpha_{2,2}}+1}$ which can be reduced to $\frac{\alpha_{1,1}}{\alpha_{1,2}} = \frac{\alpha_{2,1}}{\alpha_{2,2}}$.

MICROECONOMICS FALL EXAM
Microeconomics II—Pascal Courty
EUI, Florence, January 1st, 2004

PART II
ANSWER ALL QUESTIONS.
TOTAL POINTS: 90

Exercise 1 (25 pts)

Indicate whether each of the following is TRUE or FALSE. Justify your answer. No credit given without justi…cation.

NOTE: These questions are mainly open questions, i.e. they do not have a single admissible answer; rather, you are supposed to show that you can reason consistently, referring to the material that was covered in the lecture.

(a) (5 pts) It is still possible to observe large price variations in a competitive industry where there are only small demand shocks and no supply shocks.

(b) (5 pts) If both …rms in a 2-…rm R&D joint-venture are credited the bene…t that the other …rm receives from their actions, then the e¢cient R&D investments will be achieved.

(c) (5 pts) If both …rms in a Cournot duopoly have to pay to the other …rm an amount which is equal to half the monopoly pro…ts minus that other …rm's realized pro…ts, then the duopoly total output will be equal to the monopoly output. (Firm A has to pay …rm B half of monopoly pro…ts minus …rm B's pro…ts and vice versa.)

(d) (5 pts) An allocation is part of a competitive equilibrium if and only if it is Pareto e¢cient.

(e) (5 pts) In a 2-individual economy with 2-states of the world, where individuals are risk averse and have Bernoulli utilities, each individual's consumption varies across states of the world if and only if aggregate endowment does.

Exercise 2 (25 pts)

Suppose that the government can tax or subsidize a monopolist who faces inverse demand function $p(q)$ and has cost function $c(q)$ (assume that $p(q)q$ ¡ $c(q)$ is concave in q).

(a) (12.5 pts) What tax or subsidy per unit of output would lead the monopolist to act e¢ciently?

(b) (12,5 pts) Consider now a government regulating two Cournot duopolists who have identical convex cost functions $c(q)$. What tax or subsidy per unit of output would lead each duopolist to act e¢ciently?

Exercise 3 (40 pts)

Consider a 2x2 exchange economy where consumer i's utility is $u_i(x_{1,i}, x_{2,i}) = \alpha_{1,i}\sqrt{x_{1,i}} + \alpha_{2,i}\sqrt{x_{2,i}}$ with $\alpha_{i,j} > 0$ for $i, j = 1, 2$. Consumer i's endowment of good j is $w_{j,i}$. Throughout the exercise we will normalize the price of good 1 to 1 and denote the price of good 2 by p.

(a) (10 pts) Assume for now that consumer i has income I. Derive consumer i's demands for good 1 and 2 and express these demands as a function of $\alpha_{j,i}$ and I.

(b) (10 pts) Assume that the competitive equilibrium is achieved at an interior point and denote p^* the equilibrium price. Show that p^* can be expressed as an implicit function of $\alpha_{j,i}$ and $w_{j,i}$. Use this implicit function to show that the equilibrium price increases with $w_{1,1}$ and $w_{1,2}$ and decreases with $w_{2,1}$ and $w_{2,2}$. (You are not required to solve for the equilibrium price).

(c) (10 pts) Assume that $\alpha_{j,i} = 1$ for $i, j = 1, 2$. Derive an explicit expression for p^*. Show that p^* is constant when the total endowments of goods 1 and 2 change by an equal percentage amount.

(d) (10 pts) Assume a social planner wants to maximize $u_1(x_{1,1}, x_{2,1}) + u_2(x_{1,2}, x_{2,2})$. Derive the optimal allocation assuming that it is achieved at an interior point.

MICROECONOMICS FALL EXAM
ANSWER KEY
Microeconomics II—Pascal Courty
EUI, Florence, January 1st, 2004

PART II
ANSWER ALL QUESTIONS.
TOTAL POINTS: 90

Exercise 1 (25 pts)

Indicate whether each of the following is TRUE or FALSE. Justify your answer. No credit given without justi…cation.

NOTE: These questions are mainly open questions, i.e. they do not have a single admissible answer; rather, you are supposed to show that you can reason consistently, referring to the material that was covered in the lecture.

(a) (5 pts) It is still possible to observe large price variations in a competitive industry where there are only small demand shocks and no supply shocks.

TRUE - under very special conditions: if supply is very price-inelastic (say the industry operates at full capacity), then even a few additional buyers may have a signi…cant impact on equilibrium price. This is so because in the short-run, the industry cannot adjust aggregate supply through entry (or exit), so that any demand ‡uctuation will fully translate into price variations. Now, if these few additional buyers have a very high willingness to pay (e.g. for a last-minute plane ticket on a speci…c day with a speci…c destination), these buyers can push up the price quite dramatically.

(b) (5 pts) If both …rms in a 2-…rm R&D joint-venture are credited the bene…t that the other …rm receives from their actions, then the e¢cient R&D investments will be achieved.

TRUE - In an R&D joint-venture, each …rm generates positive externalities on the other when making a research investment, e.g. because the patents resulting from the research will be owned jointly, no matter who contributed more to …nancing the lab, the researchers etc. To achieve the …rst-best outcome, the marginal cost of each …rms investment has to equal the marginal social bene…t (i.e. joint bene…t of the two …rms), not the marginal private bene…t (i.e. bene…t of the …rm which makes the investment). In the case of positive externalities, the private bene…ts are generally lower than the social bene…ts, resulting in under-provision of the good/activity that generates the externality. Now, by internalizing the externality (e.g. through appropriately designed transfers from one …rm to the other), the …rst-best outcome will be achieved.

(c) (5 pts) If both …rms in a Cournot duopoly have to pay to the other …rm an amount which is equal to half the monopoly pro…ts minus that other …rm's realized profits, then the duopoly total output will be equal to the monopoly output. (Firm A has to pay firm B half of monopoly profits minus firm B's profits and vice versa.)

FALSE - Denote firm A's Cournot profits by $\pi_A^C(q_A, q_B)$, and monopoly profits by π^m. Assume for simplicity that inverse demand is strictly decreasing in aggregate output, i.e. $P'(Q) < 0$ for all $Q \geq 0$. Now, under the transfer system described above, firm A has to solve the following problem:

$$\max_{q_A} \underbrace{\pi_A^C(q_A, q_B)}_{\text{own profit}} - \underbrace{\left[\frac{1}{2}\pi^m - \pi_B^C(q_A, q_B)\right]}_{\text{transfer paid to rival}} + \underbrace{\left[\frac{1}{2}\pi^m - \pi_A^C(q_A, q_B)\right]}_{\text{transfer received from rival}}$$

But this expression reduces to

$$\max_{q_A} \pi_B^C(q_A, q_B)$$

In other words, firm A will choose its own quantity such that its rival's profits are maximized. Now, what is firm A's optimal output then?

First, suppose that firm B produces a strictly positive amount, $q_B > 0$. Then, firm B's profits (which are the only payoff that firm A cares about) are strictly decreasing in A's output (since, by assumption, $P'(Q) < 0$ for all $Q \geq 0$), and so firm A's best reply is to produce strictly zero.

Note that this is true even when joint profits, $\pi_A^C + \pi_B^C$, are increasing in q_A (this will be the case whenever $q_B < q^m$), so that it would be desirable to have firm A produce a strictly positive quantity: the only one to benefit from this increase in joint profits is firm B (who will receive firm A's profits through the transfers), while firm A's payoff will decrease.

Second, suppose that firm B's output is exactly zero, i.e. $q_B = 0$. In this case, firm B's profits (and hence firm A's payoff) are zero anyway, and so firm A is indifferent between all possible output levels $q_A \geq 0$. Thus, there is no guarantee that firm A will choose the "right" (i.e. joint-profit maximizing) output level, namely $q_A = q^m$.

Thus, the transfer system does not provide the right incentives for maximizing joint profits, and so we should not expect it to generate the monopoly outcome.

Additional Remark:

It may still be possible to design a transfer system that will yield the monopoly outcome. Suppose firm A agrees to transfer a share $\sigma \in (0, 1)$ of its profits to firm B, against receiving a share σ of B's profits. Then, firm A's problem can be written as:

$$\max_{q_A} (1 - \sigma)[P(q_A + q_B)q_A - C(q_A)] + \sigma[P(q_A + q_B)q_B - C(q_B)]$$

The first-order condition for this problem is

$$(1 - \sigma)[P'q_A + P - C'(q_A)] + \sigma[P'q_B] = 0$$

and analogously for firm B:

$$(1 - \sigma)[P' q_B + P - C'(q_B)] + \sigma [P' q_A] = 0$$

Adding up the two first-order conditions yields

$$P'(q_A + q_B) + (1 - \sigma) 2P - (1 - \sigma)[C'(q_A) + C'(q_B)] = 0$$

Now, if $\sigma = \frac{1}{2}$ and marginal costs are constant, i.e. $C'(q_A) = C'(q_B) = c$ for all $q_A, q_B \geq 0$, then this equation reduces to

$$P'(q_A + q_B) + P - c = 0$$

which corresponds exactly to the monopolist's first-order condition. Thus, under such a "profit-swapping" agreement, total industry output will be equal to monopoly output, and any pair (q_A, q_B) such that $q_A + q_B = q^m$ will be an equilibrium. Note that even though each firm retains full control over its own output level, they will behave as if they were fully integrated, i.e. as if they had delegated the quantity choice to a common management.

(d) (5 pts) An allocation is part of a competitive equilibrium if and only if it is Pareto efficient.

FALSE - There is no equivalence relationship between "competitive equilibrium" and "Pareto efficiency". Provided that preferences are non-satiated, every competitive equilibrium is Pareto efficient (First Welfare Theorem), but for a Pareto-efficient allocation to be sustainable as a competitive equilibrium, more restrictive assumptions have to be made (Second Welfare Theorem).

(e) (5 pts) In a 2-individual economy with 2-states of the world, where individuals are risk averse and have Bernoulli utilities, each individual's consumption varies across states of the world if and only if aggregate endowment does.

FALSE - for the two agents to insure completely, the absence of aggregate risk is not sufficient: they also have to have the same subjective probability assessments of the two states of the world. Otherwise, each agent's equilibrium consumption will be higher in the state that she thinks is more likely (see MWG, p. 692 f.)

Exercise 2 (25 pts)

Suppose that the government can tax or subsidize a monopolist who faces inverse demand function $p(q)$ and has cost function $c(q)$ (assume that $p(q)q - c(q)$ is concave in q).

(a) (12.5 pts) What tax or subsidy per unit of output would lead the monopolist to act efficiently?

Suppose the monopolist receives a subsidy (or has to pay a tax) of τ for each unit of output he sells. Then, his profit maximization problem would read:

$$\max_Q p(Q) Q + \tau Q - c(Q)$$

which has first-order condition:

$$p'(Q)\,Q + p(Q) + \tau - c'(Q) = 0$$

Now, the efficient output level, call it Q^{PO}, is characterized by

$$p\left(Q^{PO}\right) = c'(Q^{PO})$$

Thus, for the Q that solves the monopolist's problem to coincide with the efficient output level, the government must set

$$\tau = -p'\left(Q^{PO}\right)Q^{PO} > 0$$

Note that $\tau > 0$, i.e. the government has to subsidize (not tax!) the monopolist's output in order to make him produce the efficient quantity.

(b) (12,5 pts) Consider now a government regulating two Cournot duopolists who have identical convex cost functions $c(q)$. What tax or subsidy per unit of output would lead each duopolist to act efficiently?

First, recall that, by assumption, $c(q)$ is convex in q, i.e. marginal cost increases with output, and so it is more efficient to have both firms produce half of total industry output rather than one firm producing all of it and the other one not producing at all. Thus, let us solve the duopolist's problem when both firms receive the same subsidy/have to pay the same tax:

$$\max_{q_1} p(q_1 + q_2)\,q_1 + \tau q_1 - c(q_1)$$

has first-order condition

$$p'(q_1 + q_2)\,q_1 + p(q_1 + q_2) + \tau - c'(q_1) = 0$$

and analogously for firm 2:

$$p'(q_1 + q_2)\,q_2 + p(q_1 + q_2) + \tau - c'(q_2) = 0$$

Consider a symmetric equilibrium where both firms produce the same quantity, i.e. $q_1 = q_2 = q$. Then, $c'(q_1) = c'(q_2) = c'(q)$, and the Pareto-efficient output level is characterized by

$$p\left(2q^{PO}\right) = c'(q^{PO})$$

i.e. given that total output is $Q^{PO} = 2q^{PO}$, the consumers' willingness to pay for the marginal unit should equal the marginal cost of producing it (no matter whether it was produced by firm 1 or 2). Then, for each duopolist to produce exactly q^{PO}, the government has to set

$$\tau = -p'\left(2q^{PO}\right)q^{PO} > 0$$

Note that, as in the monopoly case, the government will have to subsidize output (rather than tax it) to induce the duopolists to produce the socially optimal level. This is not surprising, since market power (i.e. the ability to deviate from competitive behavior, as in the case of monopolies or Cournot oligopolies) generally implies that output falls short of the efficient level.

Yet, since the problem is less severe under a Cournot duopoly than under monopoly (total Cournot output is generally higher than monopoly output), the subsidy that induces the efficient output level is only half of what it has to be under monopoly (denoting by Q^{PO} the Pareto-efficient total industry output, $Q^{PO} = 2q^{PO}$):

$$\tau^{Cournot} = -p'\left(Q^{PO}\right)q^{PO} < \tau^{monopoly} = -p'\left(Q^{PO}\right)Q^{PO}$$

Exercise 3 (40 pts)

Consider a 2x2 exchange economy where consumer i's utility is $u_i(x_{1,i}, x_{2,i}) = \alpha_{1,i}\sqrt{x_{1,i}} + \alpha_{2,i}\sqrt{x_{2,i}}$ with $\alpha_{i,j} > 0$ for $i, j = 1, 2$. Consumer i's endowment of good j is $w_{j,i}$. Throughout the exercise we will normalize the price of good 1 to 1 and denote the price of good 2 by p.

(a) (10 pts) Assume for now that consumer i has income I. Derive consumer i's demands for goods 1 and 2 and express these demands as a function of $\alpha_{j,i}$ and I.

Taking goods prices $(1, p)$ and her income I as given, consumer i will choose consumption levels $(x_{1,i}, x_{2,i})$ in order to maximize her utility, $u_i(x_{1,i}, x_{2,i})$:

$$\begin{aligned} \max_{\{x_{1,i}, x_{2,i} \geq 0\}} u_i(x_{1,i}, x_{2,i}) &= \alpha_{1,i}\sqrt{x_{1,i}} + \alpha_{2,i}\sqrt{x_{2,i}} \\ \text{subject to } 1x_{1,i} + px_{2,i} &= I \end{aligned}$$

Denoting by λ the Lagrangian multiplier, the solution to this maximization problem is characterized by the following first-order conditions:

$$\begin{aligned} \alpha_{1,i}\frac{1}{2}\frac{1}{\sqrt{x_{1,i}}} - \lambda &= 0 \\ \alpha_{2,i}\frac{1}{2}\frac{1}{\sqrt{x_{2,i}}} - \lambda p &= 0 \\ x_{1,i} + px_{2,i} &= I \end{aligned}$$

The first two of these equations imply that the consumer will consume goods 1 and 2 in fixed proportions:

$$x_{1,i} = \left(\frac{p\alpha_{1,i}}{\alpha_{2,i}}\right)^2 x_{2,i}$$

Inserting this expression into the budget constraint, $x_{1,i} + px_{2,i} = I$, and solving for $x_{2,i}$, we obtain consumer i's demand for good 2 as a function of price p, income I, and preference parameters $\alpha_{1,i}$, $\alpha_{2,i}$:

$$x_{2,i}(p, I) = \frac{I}{\left(\frac{p\alpha_{1,i}}{\alpha_{2,i}}\right)^2 + p}$$

To derive consumer i's demand for good 1, simply insert this expression and rearrange to obtain:

$$x_{1,i}(p,I) = \left(\frac{p\alpha_{1,i}}{\alpha_{2,i}}\right)^2 x_{2,i}(p,I) = \frac{pI}{\left(\frac{\alpha_{2,i}}{\alpha_{1,i}}\right)^2 + p}$$

(b) (10 pts) Assume that the competitive equilibrium is achieved at an interior point and denote p^* the equilibrium price. Show that p^* can be expressed as an implicit function of $\alpha_{j,i}$ and $w_{j,i}$. Use this implicit function to show that the equilibrium price increases with $w_{1,1}$ and $w_{1,2}$ and decreases with $w_{2,1}$ and $w_{2,2}$. (You are not required to solve for the equilibrium price).

First, note that consumers' incomes are given by the value of their endowments:

$$I_i = 1w_{1,i} + pw_{2,i} \text{ for } i = 1,2$$

Inserting this expression into the demand functions derived in (a), we see that they are functions of price p, endowments $w_{1,i}$, $w_{2,i}$, and preference parameters $\alpha_{1,i}$, $\alpha_{2,i}$.

Now, at an interior equilibrium, the equilibrium price p^* is defined as the price that achieves market clearing, i.e. the price at which aggregate demand for good j (sum of individual demands for good j) exactly coincides with the economy's total endowment of good j (sum of individual endowments of good j):

$$\begin{aligned} x_{1,1}(p^*, w_{1,1}, w_{2,1}, \alpha_{1,1}, \alpha_{2,1}) + x_{1,2}(p^*, w_{1,2}, w_{2,2}, \alpha_{1,2}, \alpha_{2,2}) &= w_{1,1} + w_{1,2} \\ x_{2,1}(p^*, w_{1,1}, w_{2,1}, \alpha_{1,1}, \alpha_{2,1}) + x_{2,2}(p^*, w_{1,2}, w_{2,2}, \alpha_{1,2}, \alpha_{2,2}) &= w_{2,1} + w_{2,2} \end{aligned}$$

The two market-clearing conditions above define a system of two equations in one unknown, p, and a vector of eight parameters,

$$\theta \equiv (w_{1,1}, w_{2,1}, w_{1,2}, w_{2,2}, \alpha_{1,1}, \alpha_{2,1}, \alpha_{1,2}, \alpha_{2,2})$$

Assume that there exists an interior solution to this equation system, and denote its solution by p^*.

By Walras' Law, we know that if p^* solves one of these equations, the other one will hold as well. Thus, either of the two equations defines p^* as an implicit function of all eight parameters. Consider the excess demand function for good 2 at equilibrium price p^*:

$$F(p^*, \theta) \equiv x_{2,1}(p^*, \ldots) + x_{2,2}(p^*, \ldots) - w_{2,1} - w_{2,2} = 0$$

The Implicit Function Theorem (see MWG, p. 941 f.) states that if $F(p^*, \theta)$ is continuously differentiable with respect to p and all elements of θ (which is indeed the case), and if $\frac{\partial F(p^*,\theta)}{\partial p} \neq 0$ at solution values (which will be shown below), then the system can be locally solved at (p^*, θ) by an implicitly defined function $p(\theta)$ which is continuously differentiable.

Moreover, the Implicit Function Theorem allows to perform comparative statics on this implicitly defined function $p(\theta)$. Let θ_i be an element of θ, i.e. one of the endowments or preference parameters of interest. By total differentiation of $F(p^*, \theta_i)$, we obtain

$$\frac{\partial F(p^*, \theta)}{\partial p}\frac{dp}{d\theta_i} + \frac{\partial F(p^*, \theta)}{\partial \theta_i} = 0$$

which can be rearranged to have

$$\frac{dp}{d\theta_i} = -\frac{\partial F(p^*, \theta)/\partial \theta_i}{\partial F(p^*, \theta)/\partial p}$$

The big advantage of using this method is that we do not need to know the closed-form solution of $p(\theta)$ to perform comparative statics on the equilibrium price. In fact, such a closed-form solution may not even exist; yet, as long as there is an equilibrium condition which implicitly defines this price, and the partial derivatives of this equilibrium condition with respect to price and parameters exist, we can still say something about the properties of $p(\theta)$.

Now, in order to determine how the equilibrium price changes with respect to endowments, we need to evaluate the signs of the partial derivative of excess-demand function 2 with respect to the endowment and to price.

In particular, we have for $i = 1, 2$:

$$\begin{aligned}\frac{\partial F(p^*, \theta)}{\partial w_{1,i}} &= \frac{\partial x_{2,1}(\cdot)}{\partial w_{1,i}} + \frac{\partial x_{2,2}(\cdot)}{\partial w_{1,i}} \\ &= \underbrace{\frac{\partial I_i(\cdot)}{\partial w_{1,i}}}_{=1} \underbrace{\left[\left(\frac{p\alpha_{1,i}}{\alpha_{2,i}}\right)^2 + p\right]^{-1}}_{>0} > 0\end{aligned}$$

$$\begin{aligned}\frac{\partial F(p^*, \theta)}{\partial w_{2,i}} &= \frac{\partial x_{2,1}(\cdot)}{\partial w_{2,i}} + \frac{\partial x_{2,2}(\cdot)}{\partial w_{2,i}} - 1 \\ &= \underbrace{\underbrace{\frac{\partial I_i(\cdot)}{\partial w_{2,i}}}_{=p} \underbrace{\left[\left(\frac{p\alpha_{1,i}}{\alpha_{2,i}}\right)^2 + p\right]^{-1}}_{>p}}_{<1} - 1 < 0\end{aligned}$$

$$\frac{\partial F(p^*,\theta)}{\partial p} = \frac{\partial x_{2,1}(\cdot)}{\partial p} + \frac{\partial x_{2,2}(\cdot)}{\partial p}$$

$$= \sum_{i=1}^{2} \left\{ \frac{w_{2,i}}{\left(\frac{p\alpha_{1,i}}{\alpha_{2,i}}\right)^2 + p} + \frac{I_i(-1)\left(2p\left(\frac{\alpha_{1,i}}{\alpha_{2,i}}\right)^2 + 1\right)}{\left(\left(\frac{p\alpha_{1,i}}{\alpha_{2,i}}\right)^2 + p\right)^2} \right\}$$

$$= \sum_{i=1}^{2} \left\{ \frac{-w_{2,i}\left(\frac{p\alpha_{1,i}}{\alpha_{2,i}}\right)^2 - w_{1,i}\left(2p\left(\frac{\alpha_{1,i}}{\alpha_{2,i}}\right)^2 + 1\right)}{\left(\left(\frac{p\alpha_{1,i}}{\alpha_{2,i}}\right)^2 + p\right)^2} \right\} < 0$$

Now, putting these results together, we have for $i = 1, 2$:

$$\frac{dp}{dw_{1,i}} = -\frac{\overbrace{\partial F(p^*,\theta)/\partial w_{1,i}}^{>0}}{\underbrace{\partial F(p^*,\theta)/\partial p}_{<0}} > 0$$

and

$$\frac{dp}{dw_{2,i}} = -\frac{\overbrace{\partial F(p^*,\theta)/\partial w_{2,i}}^{<0}}{\underbrace{\partial F(p^*,\theta)/\partial p}_{<0}} < 0$$

which confirms the claim: The equilibrium price p^* increases with $w_{1,1}$ and $w_{1,2}$ and decreases with $w_{2,1}$ and $w_{2,2}$. In other words, if the endowment of good 1 increases (no matter if the extra quantity goes to consumer 1 or 2), then good 2 will become relatively more expensive; if instead the endowment of good 2 increases, then good 2 will become relatively cheaper (again no matter which consumer owns the extra endowment).

(c) (10 pts) Assume that $\alpha_{j,i} = 1$ for $i, j = 1, 2$. Derive an explicit expression for p^*. Show that p^* is constant when the total endowments of goods 1 and 2 change by an equal percentage amount.

When setting all $\alpha_{j,i}$ to 1, our market-clearing condition for good 2 reduces to:

$$\frac{w_{1,1} + pw_{2,1}}{p^2 + p} + \frac{w_{1,2} + pw_{2,2}}{p^2 + p} = w_{2,1} + w_{2,2}$$

which can be solved for p to obtain the equilibrium price

$$p^* = \sqrt{\frac{w_{1,1} + w_{1,2}}{w_{2,1} + w_{2,2}}}$$

Now, let both total endowment of good 1, $w_{1,1} + w_{1,2}$, and total endowment of good 2, $w_{2,1} + w_{2,2}$, change by 4 percent. Then, the new equilibrium price is

$$p^{*}(4) = \sqrt{\frac{(1+4)(w_{1,1}+w_{1,2})}{(1+4)(w_{2,1}+w_{2,2})}} = \sqrt{\frac{w_{1,1}+w_{1,2}}{w_{2,1}+w_{2,2}}} = p^{*}$$

i.e. the equilibrium price will indeed remain the same.

(d) (10 pts) Assume a social planner wants to maximize $u_1(x_{1,1}, x_{2,1}) + u_2(x_{1,2}, x_{2,2})$. Derive the optimal allocation assuming that it is achieved at an interior point.

The social planner has to determine consumption levels of good 1 and 2 for both consumers so as to maximize "social welfare" given the economy's resource constraints:

$$\begin{aligned}
\max_{\{x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2} \geq 0\}} \sum_{i=1} u_i(x_{1,i}, x_{2,i}) &= \sum_{i=1} \left\{ \alpha_{1,i}\sqrt{x_{1,i}} + \alpha_{2,i}\sqrt{x_{2,i}} \right\} \\
\text{subject to } x_{1,1} + x_{1,2} &= w_{1,1} + w_{1,2} \\
\text{and } x_{2,1} + x_{2,2} &= w_{2,1} + w_{2,2}
\end{aligned}$$

We can simplify the problem by reducing it to a choice of consumption levels for one consumer, with the resource constraints implying that the other consumer will just get what is "left over". Hence, we have

$$\begin{aligned}
x_{1,2} &= w_{1,1} + w_{1,2} - x_{1,1} \\
\text{and } x_{2,2} &= w_{2,1} + w_{2,2} - x_{2,1}
\end{aligned}$$

and so the social planner's problem can be rewritten as

$$\begin{aligned}
&\max_{\{x_{1,1}, x_{2,1} \geq 0\}} \alpha_{1,1}\sqrt{x_{1,1}} + \alpha_{2,1}\sqrt{x_{2,1}} + \\
&+\alpha_{1,2}\sqrt{w_{1,1} + w_{1,2} - x_{1,1}} + \alpha_{2,2}\sqrt{w_{2,1} + w_{2,2} - x_{2,1}}
\end{aligned}$$

This problem has the following first-order conditions for goods $j = 1, 2$:

$$\alpha_{j,1}\frac{1}{2}\frac{1}{\sqrt{x_{j,1}}} = \alpha_{j,2}\frac{1}{2}\frac{1}{\sqrt{w_{j,1} + w_{j,2} - x_{j,1}}}$$

These first-order conditions simply state that at the optimum, the marginal utility of consuming good j must be equalized across consumers.

Rearrange terms to solve for the social planner's allocation in terms of endowments and preference parameters. Consumer 1 will consume goods $j = 1, 2$ as follows:

$$x_{j,1}^{SP} = \frac{w_{j,1} + w_{j,2}}{\left(\frac{\alpha_{j,2}}{\alpha_{j,1}}\right)^2 + 1}$$

while the social planner's consumption schedule for consumer 2 is (again for $j = 1, 2$)

$$x^{SP}_{j,2} = \frac{\frac{w_{j,1} + w_{j,2}}{2}}{\frac{\alpha_{j,1}}{\alpha_{j,2}} + 1}$$

Additional Remark:

Note that, by the First Welfare Theorem, the allocation induced by competitive markets (i.e. the competitive equilibrium discussed in (b)) must coincide with the allocation we derived in (d) as solution to the social planner's problem.[1]

Thus, even though we were unable to explicitly solve the system of excess-demand functions in (b), we do know the exact closed-form solutions of the allocation that would arise from free trade among the two consumers at equilibrium prices, namely

$$x_{j,1}\left(p^*, w_{1,1}, w_{2,1}, \alpha_{1,1}, \alpha_{2,1}\right) = x^{SP}_{j,1}$$
$$\text{and } x_{j,2}\left(p^*, w_{1,2}, w_{2,2}, \alpha_{1,2}, \alpha_{2,2}\right) = x^{SP}_{j,2}$$

[1] To be precise: the First Welfare Theorem states that every competitive equilibrium is Pareto e¢cient. Now, the solution to the social planner's problem in (d) is unique, and so if there exists a competitive equilibrium (which we simply assumed in (b) without proving it), then the allocation it implies has to be the same as the (unique) Pareto-optimal allocation.

MICROECONOMICS FALL EXAM
Microeconomics II—Pascal Courty
EUI, Florence, June 9th, 2004

PART II
ANSWER ALL QUESTIONS.
TOTAL POINTS: 90

Exercise 1 (50 pts)

Consider a competitive industry composed of $J > 1$ price taker …rms. The industry demand is $x(p)$. All …rms face the same cost function. The industry is characterized by learning by doing in that each …rm's total cost is declining in the level of industry output. A …rm's cost function, $c(q, Q)$, is increasing in it's own output and decreasing in industry output $Q = \sum_{j=1..J} q_j$.

Assume that $c_q + Jc_Q > 0$, $\frac{1}{n}c_{qq} + \frac{n+1}{n}c_{qQ} + c_{QQ} > 0$ for $n \in \{1, J\}$, and $c_{qq} + 2Jc_{qQ} + J^2 c_{QQ} > 0$.

(a) (12.5 pts) Derive the competitive equilibrium output level.

(b) (12.5 pts) Assume the partial equilibrium welfare approach applies. Derive the Pareto optimal output level. Is the competitive equilibrium Pareto optimal? Explain.

(c) (12.5 pts) What tax or subsidy restores e¢ciency?

(d) (12.5 pts) Let J be an even number. Suppose that each …rm j merges with one other …rm in the industry, so that after the mergers a total of $J/2$ …rms will be operating in the industry, each being the merged entity of two previously independent …rms. Will such mergers increase welfare?

Exercise 2 (40 pts)

Consider a 2-consumers 2-…rms economy. The consumers have utility $u(x, l) = \ln(x) + \ln(l)$ where x is good's consumption and l is leisure consumption. Consumers are endowed with one unit of leisure, one unit of good, and also own half of each …rm. The production technology is $y = f(z) = z^{1/2}$ where z is labor input and y is good output. Let w and p be the price of labor and good respectively.

(a) (10 pts) Assume consumers and …rms are price takers. Write down the consumer's utility maximization problem and the …rm's pro…t maximization problem.

(b) (10 pts) Characterize the conditions that a competitive equilibrium must satisfy and interpret these conditions. Show that one can set $p = 1$ without loss of generality. Compute the equilibrium.

(c) (10 pts) Assume one consumer owns both …rms and can make a take-it-or-leave-it o¤er to the other consumer of the kind "exchange l units of labor against y units of good". What is the optimal o¤er?

(d) (10 pts) Derive the Pareto optimal allocation. Are the outcomes in (b) and/or (c) e¢cient? Explain

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Exercise 1 (50 pts)

Consider a competitive industry composed of $J > 1$ price taker firms. The industry demand is $x(p)$. All firms face the same cost function. The industry is characterized by learning by doing in that each firm's total cost is declining in the level of industry output. A firm's cost function, $c(q, Q)$, is increasing in it's own output and decreasing in industry output $Q = \sum_{j=1..J} q_j$.

Assume that $c_q + Jc_Q > 0$, $\frac{1}{n}c_{qq} + \frac{n+1}{n}c_{qQ} + c_{QQ} > 0$ for $n \in \{1, J\}$, and $c_{qq} + 2Jc_{qQ} + J^2 c_{QQ} > 0$.

(a) (12.5 pts) Derive the competitive equilibrium output level.

The competitive equilibrium is defined as an allocation $(x^*, q_1^*, \ldots, q_J^*)$ and a price p^* such that

(i) Firms maximize profits: $q_j^* = \arg\max\ p^* q_j - c(q_j, Q)$ for all $j = 1, \ldots, J$

(ii) Consumers maximize utility (granted here since the industry demand $x(p)$ derives from the individual consumer's utility maximization problem)

(iii) Market clears: $x^* = \sum_{j=1} q_j^* = Q^*$

Now, since firms are price takers, given p^*, firm j's equilibrium output level q_j^* must solve

$$\max_{q_j \geq 0} p^* q_j - c(q_j, Q)$$

where $Q = q_j + \sum_{k \neq j} q_k$. The first-order condition to this problem is

$$p^* \leq c_q\left(q_j^*, Q^*\right) + c_Q\left(q_j^*, Q^*\right) \text{ with equality if } q_j^* > 0$$

where $c_q + c_Q > 0$ by the assumption that $c_q + Jc_Q > 0$ and $c_Q < 0$, and the second-order condition is satisfied by the assumption

$$c_{qq} + 2c_{qQ} + c_{QQ} > 0$$

Thus, this problem has a unique solution q^*, independent of j, and the competitive equilibrium $Q^* = Jq^*$ will be determined by

$$p^* = c_q\left(\frac{Q^*}{J}, Q^*\right) + c_Q\left(\frac{Q^*}{J}, Q^*\right)$$

(b) (12.5 pts) Assume the partial equilibrium welfare approach applies. Derive the Pareto optimal output level. Is the competitive equilibrium Pareto optimal? Explain.

Assume that the social planner takes the number of firms, J, as given, so that total industry output is given by $Q = \sum_{j=1} q_j$. Then, the social planner's problem is to choose an output level q_j for each firm $j = 1, \ldots, J$ such that total surplus is maximized, where total surplus is the difference between gross consumer surplus and the total cost of producing output Q.

Denote the inverse of consumer demand $x(p)$ by $P(x)$. Then, gross consumer surplus is just the area "under" the inverse demand function, and we can write the social planner's problem as

$$\max_{q_1,\ldots,q_J \geq 0} \underbrace{\int_0^Q P(x)\,dx}_{\text{gross consumer surplus}} - \underbrace{\sum_{j=1} c(q_j, Q)}_{\text{total cost of production}}$$

The solution to this problem, $\left\{q_1^{PO}, \ldots, q_J^{PO}\right\}$, solves the following first-order conditions:

$$P(Q) = c_q(q_j, Q) + c_Q(q_j, Q) + \sum_{k \neq j} c_Q(q_k, Q)$$

Let q^{PO} denote the solution to this problem (again, independent of j), and let $Q^{PO} = Jq^{PO}$. The optimal Q^{PO} will then be determined by

$$P\left(Q^{PO}\right) = c_q\left(\frac{Q^{PO}}{J}, Q^{PO}\right) + Jc_Q\left(\frac{Q^{PO}}{J}, Q^{PO}\right)$$

and again, this problem has a solution for $P\left(Q^{PO}\right) > 0$ because of assumption $c_q + Jc_Q > 0$, and the second-order condition is satisfied by assumption $c_{qq} + 2Jc_{qQ} + J^2 c_{QQ} > 0$.

Now, let us compare the Pareto-optimal output level with the competitive level, characterized by

$$p^* = c_q\left(\frac{Q^*}{J}, Q^*\right) + c_Q\left(\frac{Q^*}{J}, Q^*\right)$$

We see immediately that they must be different, i.e. $q_j^* \neq q^{PO}$, because the right-hand side of the two first order conditions is different. In particular, the marginal cost of production perceived by a single firm in the competitive equilibrium is $c_q + c_Q$, while the marginal cost of production from the social planner's point of view is $c_q + Jc_Q$.

Now, since $c_Q < 0$, the social planner's first-order condition implies that

$$P\left(Q^{PO}\right) < c_q\left(\frac{Q^{PO}}{J}, Q^{PO}\right) + c_Q\left(\frac{Q^{PO}}{J}, Q^{PO}\right)$$

Also, since $P'(Q) < 0$, and

$$\frac{d}{dQ}\left(c_q\left(\frac{Q}{J}, Q\right) + c_Q\left(\frac{Q}{J}, Q\right)\right) = \frac{1}{J}\left(c_{qq} + (J+1)c_{qQ} + Jc_{QQ}\right) > 0$$

by assumption, then we must have that $Q^{PO} > Q^*$.

In this model, each firm that produces a strictly positive quantity has a positive externality on all other firms in the industry: the other firms' cost of production will decrease (i.e. they will become more efficient). When choosing their output in a competitive equilibrium, firms do not take this externality into account. Hence, competitive equilibrium output will fall short of Pareto-efficient output; firm j's private marginal cost of production is higher than the social marginal cost, and so firm j will undersupply.

(c) (12.5 pts) What tax or subsidy restores efficiency?

Modify a competitive firm's profit maximization problem by adding a per-unit subsidy τ to their revenue:

$$\max_{q_j \geq 0} p^* q_j + \tau q_j - c(q_j, Q)$$

This problem has first-order condition

$$p^* + \tau = c_q + c_Q$$

For the q_j that solves this equation to coincide with the Pareto-optimal output level, defined by

$$P\left(Jq^{PO}\right) = c_q\left(q^{PO}, Jq^{PO}\right) + Jc_Q\left(q^{PO}, Jq^{PO}\right)$$

just set the subsidy

$$\tau = -(J-1)c_Q\left(q^{PO}, Jq^{PO}\right)$$

(d) (12.5 pts) Let J be an even number. Suppose that each firm j merges with one other firm in the industry, so that after the mergers a total of $J/2$ firms will be operating in the industry, each consisting of two separate plants under joint management. Will such mergers increase welfare?

Assume that the merged entities will continue to behave as price takers (i.e. market power is not an issue here). Let p^m be the competitive equilibrium price after the mergers, and denote by q_{ij} firm j's output produced in plant i (where $i = 1, 2$ as each of the $J/2$ firms has now two plants at its disposal), so that total industry output is now $Q = \sum_{j=1}^{J/2}(q_{1j} + q_{2j})$.

Then, the merged firm's output-choice problem is:

$$\max_{q_{1j},q_{2j}\geq 0} p^m(q_{1j}+q_{2j}) - \sum_{i=1} c(q_{ij},Q)$$

which has first-order conditions (for $i = 1, 2$):

$$p^m = c_q(q_{ij},Q) + [c_Q(q_{1j},Q) + c_Q(q_{2j},Q)]$$

Now, the equilibrium will again be symmetric, with $q_{1j} = q_{2j} = q_j$, and $q_j = q^m$ (independent of j), implying $Q^m = J2q^m$, and will be characterized by

$$p^m\left(\frac{Q^m}{J},Q^m\right) = c_q(\frac{Q^m}{J},Q^m) + 2c_Q(\frac{Q^m}{J},Q^m)$$

Comparing this expression to the corresponding ones obtained in (a) and (b), we see that the "merger equilibrium" will generate a higher industry output than the competitive equilibrium before the mergers, and will therefore be welfare improving. This is because part of the externality generated by production will now be internalized, namely the externality on the other plant belonging to the merged entity.

Note that if all firms in the industry were to merge to a single entity, then the merger equilibrium coincides with the social planner's problem, i.e. a merger to monopoly would restore the Pareto-optimal outcome.

Exercise 2 (40 pts)

Consider a 2-consumers 2-firms economy. The consumers have utility $u(x,l) = \ln(x) + \ln(l)$ where x is good's consumption and l is leisure consumption. Consumers are endowed with one unit of leisure, one unit of good, and also own half of each firm. The production technology is $y = f(z) = z^{1/2}$ where z is labor input and y is good output. Let w and p be the price of labor and good respectively.

(a) (10 pts) Assume consumers and firms are price takers. Write down the consumer's utility maximization problem and the firm's profit maximization problem.

The consumer's utility maximization problem is:

$$\max_{x,l\geq 0} u(x,l) = \ln(x) + \ln(l)$$

$$\text{subject to } px \leq p1 + w(1-l) + \sum_{j=1} \frac{1}{2}\pi_j(p,w)$$

The budget constraint deserves attention: it states that the consumer's total expenditure for the good must not exceed her total wealth, which is the sum of: the value of her good's endowment, $p1$, her total labor income, $w(1-l)$ (where labor supply is labor endowment minus leisure consumption, i.e. $1 - l$), and her share in the …rms' pro…ts, $\sum_{j=1} \frac{1}{2}\pi_j(p,w)$ (one half of each of the two …rms).

The …rm's pro…t maximization problem is:

$$\max_{z \geq 0} pz^{1/2} - wz$$

i.e. the …rm chooses the level of labor input z that will maximize it's pro…ts, given that one unit of labor costs w and z units of labor will produce $z^{1/2}$ units of output which can be sold at price p per unit.

(b) (10 pts) Characterize the conditions that a competitive equilibrium must satisfy and interpret these conditions. Show that one can set $p = 1$ without loss of generality. Compute the equilibrium.

Part I: Characterize the conditions that a competitive equilibrium must satisfy and interpret these conditions.

A competitive equilibrium is an allocation $(x_1^*, x_2^*, l_1^*, l_2^*, z_1^*, z_2^*, y_1^*, y_2^*)$ and a price vector (p^*, w^*) such that

(i) Consumers maximize utility (see (a))

(ii) Firms maximize pro…ts (see (a))

(iii) the goods and labor markets clear:

$$\begin{aligned} x_1^* + x_2^* &= 2 + y_1^* + y_2^* \\ z_1^* + z_2^* &= (1 - l_1^*) + (1 - l_2^*) \end{aligned}$$

Denoting by λ_i the Lagrangian multiplier of consumer i's utility maximization problem, its solution is characterized by

$$\begin{aligned} \frac{1}{x_i} - p\lambda_i &= 0 \\ \frac{1}{l_i} - w\lambda_i &= 0 \\ px_i + wl_i &= p + w + \sum_{j=1} \frac{1}{2}\pi_j(p,w) \end{aligned}$$

i.e. the per-dollar marginal utility of goods and leisure consumption must be equal, and the budget constraint must be satis…ed.

The corresponding …rst-order condition to …rm j's pro…t-maximization problem is:

$$p\frac{1}{2}z_j^{-1/2} = w$$

i.e. the value of the marginal product of labor input must equal its cost.

Part II: Show that one can set $p = 1$ without loss of generality.

see MWG, p. 519. Normalizing p to 1 just means that we multiply the utility function and budget constraint of consumers as well as the pro…t function of firms by a constant, namely $1/p$. This leaves both consumers' budget sets and firms' production sets unaltered, and hence the equilibrium allocation under price vector (p^*, w^*) will be identical to the one under price vector $\left(1, \frac{w^*}{p^*}\right)$.

To see this point more clearly, let us solve for the competitive equilibrium without imposing the normalization (i.e. in terms of p and w), which will demonstrate that only relative price w^*/p^* is determined in equilibrium.

Part III: Compute the equilibrium.

Start with the firms: firm j's first-order condition can immediately be solved for z_j to obtain optimal labor demand as a function of p and w:

$$z_j(p,w) = \left(\frac{p}{2w}\right)^2$$

Inserting this expression into the production function $f(z_j)$ and the profit function, we obtain firm's goods supply and profits as a function of p and w:

$$\begin{aligned} y_j(p,w) &= f(z_j(p,w)) = (z_j(p,w))^{1/2} = \frac{p}{2w} \\ \pi_j(p,w) &= p \cdot y_j(p,w) - w \cdot z_j(p,w) = \frac{1}{4}\frac{p^2}{w} \end{aligned}$$

Now, we can insert the last term, which expresses firm profits as a function of p and w, into the consumers' budget constraints (recall that firms are owned by consumers, and so firm profits are part of consumers' wealth).

The consumer's first-order conditions imply that $px_i = lx_i$; Thus, substitute for lx_i in the budget constraint to solve for consumers' goods demand and labor supply as functions of p and w:

$$\begin{aligned} x_i(p,w) &= \frac{1}{2}\left[1 + \frac{w}{p} + \frac{1}{4}\frac{p}{w}\right] \\ l_i(p,w) &= \frac{1}{2}\left[\frac{p}{w} + 1 + \frac{1}{4}\left(\frac{p}{w}\right)^2\right] \end{aligned}$$

Next, market clearing on the goods market means that aggregate goods demand equals total endowments plus firm production:

$$x_1(p,w) + x_2(p,w) = 2 + y_1(p,w) + y_2(p,w)$$

Insert the expressions obtained above for $x_i(p,w)$ and $y_j(p,w)$ and simplify to obtain a quadratic form in terms of $\frac{w}{p}$, namely $\left(\frac{w}{p}\right)^2 - \frac{w}{p} - \frac{3}{4} = 0$, which has two solutions,

$$\frac{w^*}{p^*} = \frac{3}{2} \text{ and } \frac{w^*}{p^*} = -\frac{1}{2}$$

(disregard the second solution, which is irrelevant as it implies that one of the two prices is negative).

Now, note that all our demand functions (labor demand $z_j(p,w)$, goods demand $x_i(p,w)$, leisure demand $l_i(p,w)$) are functions of the relative price of labor, w/p (or rather its inverse, p/w), and so we can immediately solve for the equilibrium allocation:

$$\begin{aligned} z_1^* &= z_2^* = \frac{1}{9} \\ y_1^* &= y_2^* = \frac{1}{3} \\ x_1^* &= x_2^* = \frac{4}{3} \\ l_1^* &= l_2^* = \frac{8}{9} \end{aligned}$$

Note that our calculations confirm what we said above about the insensitivity of equilibrium allocations with respect to normalizing one price to 1: since all equilibrium quantities turned out to be functions of relative price w^*/p^* only (rather than the absolute levels of w and p), we find that setting $p = 1$ is indeed without loss of generality.

(c) (10 pts) Assume one consumer owns both firms and can make a take-it-or-leave-it offer to the other consumer of the kind "exchange l units of labor against y units of good". What is the optimal offer?

Let us call the consumer who owns the two firms "firm owner" (index all variables relating to the firm owner by o), and the other consumer "worker" (index all variables relating to the worker by w).

First, reason in the following way: what is the worker's situation of she decides to reject the firm owner's offer? Well, all she can do is to consume her endowments (recall: $w_x = w_l = 1$), which will leave her with utility $u(w_x, w_l) = \ln(1) + \ln(1) = 0$. Thus, any offer by the firm owner which leaves the worker with a utility level of at least zero will be accepted, while all offers implying a utility level strictly less than zero will be rejected.

More formally, the firm owner's offer is a pair (x^w, z^w), where x^w denotes the units of goods that the firm owner promises the worker in return for z^w units of labor supplied by the latter. Now, any offer (x^w, z^w) such that

$$u(w_x + x^w, w_l - z^w) = \ln(1 + x^w) + \ln(1 - z^w) \geq 0$$

will be accepted by the worker.

Second, note that the production technology is convex, i.e. the higher a firm's output level, the more labor input is necessary to produce an additional unit of output. Thus, no matter what the aggregate level of labor supply (sum of worker's and firm owner's labor supply $z^w + z^o$) will be, the firm owner will always want to split this labor supply equally among the two firms, as this allows for higher output production than having only one firm producing (i.e. $2\sqrt{\frac{1}{2}(z^w + z^o)} > \sqrt{(z^w + z^o)}$).

Now, the firm owner's problem is to choose an "acceptable" offer (x^w, z^w) as well as his own labor supply, z^o, such that his utility from goods and leisure consumption is maximized. More formally, his problem is

$$
\begin{aligned}
\max_{\{z^o, x^w, z^w\} \geq 0} u(x^o, l^o) &= \ln(x^o) + \ln(w_l - z^o) \\
\text{subject to } x^o &\leq w_x + 2\left[\frac{1}{2}(z^w + z^o)\right]^{1/2} - x^w \\
\text{and } 0 &\leq \ln(1 + x^w) + \ln(1 - z^w)
\end{aligned}
$$

Note that the firm owner's goods consumption, x^o, is determined by his goods endowment plus the total output produced by his two firms, minus the units he has to give the worker in return for her labor.

The worker's "acceptance" condition, $\ln(1 + x^w) + \ln(1 - z^w) = 0$, can be solved for x^w in terms of z^w, to have $x^w = z^w/(1 - z^w)$. Thus, the firm owner's problem can be rewritten as one of choosing own and worker's labor supply:

$$
\max_{\{z^o, z^w\} \geq 0} u(x^o, l^o) = \ln(1 + 2\left[\frac{1}{2}(z^w + z^o)\right]^{1/2} - \frac{z^w}{1 - z^w}) + \ln(w_l - z^o)
$$

which has first-order conditions

$$
\begin{aligned}
\text{with respect to } z^o &: \quad \frac{2\frac{1}{2}\left[\frac{1}{2}(z^w + z^o)\right]^{-1/2}\frac{1}{2}}{1 + 2\left[\frac{1}{2}(z^w + z^o)\right]^{1/2} - \frac{z^w}{1 - z^w}} = \frac{1}{1 - z^o} \\
\text{with respect to } z^w &: \quad \frac{2\frac{1}{2}\left[\frac{1}{2}(z^w + z^o)\right]^{-1/2}\frac{1}{2} - \frac{1}{(1 - z^w)^2}}{1 + 2\left[\frac{1}{2}(z^w + z^o)\right]^{1/2} - \frac{z^w}{1 - z^w}} = 0
\end{aligned}
$$

The second condition can be simplified to have $z^o = \frac{1}{2}(1 - z^w)^4 - z^w$. Substituting for z^o in the first condition, we can reduce this condition to a quadratic form in $(1 - z^w)^2$, namely $\frac{3}{2}(1 - z^w)^4 + 2(1 - z^w)^2 - 2 = 0$, which has two solutions:

$$
(1 - z^w)^2 = \frac{2}{3} \text{ and } (1 - z^w)^2 = -2
$$

Ignoring the second solution (which is negative), we can solve for the firm owner's optimal offer as

$$
\begin{aligned}
z^w &= 1 - \sqrt{2/3} \\
x^w &= \frac{z^w}{1 - z^w} = \frac{1 - \sqrt{2/3}}{\sqrt{2/3}}
\end{aligned}
$$

as well as for the remaining variables of interest, namely total and firm owner's labor supply, total goods output, and firm owner's consumption:

$$
\begin{aligned}
z^w + z^o &= \frac{1}{2}(1 - z^w)^4 = \frac{2}{9} \\
z^o &= \frac{2}{9} - z^w = \frac{2}{9} - \left(1 - \sqrt{\frac{2}{3}}\right) \\
y_1 + y_2 &= 2\left(\frac{1}{2}(z^w + z^o)\right)^{1/2} = \frac{2}{3} \\
x^o &= 1 + (y_1 + y_2) - x^w = \frac{5}{3} - \frac{1 - \sqrt{2/3}}{\sqrt{2/3}}
\end{aligned}
$$

(d) (10 pts) Derive the Pareto optimal allocation. Are the outcomes in (b) and/or (c) efficient? Explain

The social planner's problem is to maximize the sum of consumers' utilities subject to the economy's resource constraint:

$$
\begin{aligned}
\max_{\{x_1, x_2, z_1, z_2\} \geq 0} \sum_{i=1} u(x_i, l_i) &= \sum_{i=1} \{\ln(x_i) + \ln(w_{li} - z_i)\} \\
x_1 + x_2 &\leq 2 + 2\left(\frac{1}{2}(z_1 + z_2)\right)^{1/2}
\end{aligned}
$$

Denoting the Lagrangian multiplier by λ, this problem has first-order conditions

$$
\begin{aligned}
\frac{1}{x_i} + \lambda &= 0 \\
-\frac{1}{1 - z_i} &= \lambda 2 \frac{1}{2}\left(\frac{1}{2}(z_1 + z_2)\right)^{-1/2} \frac{1}{2} \\
x_1 + x_2 &= 2 + 2\left(\frac{1}{2}(z_1 + z_2)\right)^{1/2}
\end{aligned}
$$

We see immediately that the first-order conditions regarding consumption levels and labor supply are the same for the two consumers $i = 1, 2$, which implies that the social planner would want them to work and consume the same amounts (i.e. $x_1 + x_2 = 2x$ and $z_1 + z_2 = 2z$).

Now, insert $\lambda = -1/x$ into the second first-order condition to have

$$
x = \frac{1 - z}{2\sqrt{z}}
$$

which can be inserted into the resource constraint, and reducing the latter to a quadratic form in $\sqrt{z}$, we can solve $3z + 2\sqrt{z} - 1 = 0$ to have

$$
\sqrt{z} = \frac{1}{3} \text{ and } \sqrt{z} = -1
$$

Again, ignoring the second solution (which is negative), we can immediately solve for the Pareto-optimal allocation to have

$$z^{PO} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$x^{PO} = \frac{1 - z^{PO}}{2\sqrt{z^{PO}}} = \frac{4}{3}$$

We see that the Pareto-optimal allocation coincides with the labor supply and consumption levels in (b) (i.e. the competitive equilibrium with equal ownership of firms), while it is different from the one obtained in (c).

Thus, (b) generates the efficient outcome (which is what we would expect considering the First Welfare Theorem), while (c) does not. In the latter, it is bargaining rather than competitive markets which determine the allocation.

We saw that efficiency requires both agents to work and consume equally. Now, under (c), while total labor supply (and thus total goods output) are still at efficient levels ($z^w + z^o = \frac{2}{9} = 2z^{PO}$, and $(w_x^w + x^w) + x^o = 2 + \frac{2}{3} = 2x^{PO}$), the distribution of labor supply and goods consumption is unequal, and thus inefficient (the worker works "too much", $z^w > z^{PO} > z^o$, and consumes "too little", $1 + x^w < x^{PO} < x^o$, and vice versa for the firm owner).