Politics & Economics: Theory and Applications
Solutions to Problem Set 1

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Exercise 2.4 Persson & Tabellini (PT) p.43
$y^i = y > 4$ exogenous income equal across individuals i
τ income tax
c^i private consumption level of individual i
q_1, q_2 per capita level of 2 types of public goods
α^i idiosyncratic preferences, endowments, risks, technical opportunities or other socio-economic variables of individual i
$\omega^i = U(c) + \alpha^i G(q_1) + (1-\alpha^i)F(q_2)$ utility function of agent i defined over c, q_1 and q_2
$U(.)$, $G(.)$ and $F(.)$ C^2, strictly increasing and concave

2.4.a
Government budget constraint $q_1 + q_2 \leq \tau y$
Individual budget constraint $c \leq (1-\tau)y$
Individual i's policy preferences

$$W(q_1, q_2, \alpha^i) = \underbrace{U(y - q_1 - q_2) + F(q_2)}_{J(q_1,q_2)\ +} + \underbrace{\alpha^i}_{K(\alpha^i)} \underbrace{[(G(q_1) - F(q_2)]}_{H(q_1,q_2)} \tag{1}$$

$K(\alpha^i) = \alpha^i$ is monotonic in α^i; then individuals have intermediate preferences.
This implies that a Condorcet winner exists and is given by the optimal policy quantities that the median voter i with the median value of α^i prefers by Proposition 3 PT p. 26.

The optimal quantities $q_1(q_2, \alpha^i)$ and $q_2(q_1, \alpha^i)$ for individual i solve

$$\max_{q_1,q_2} W(q_1, q_2, \alpha^i) = U(y - q_1 - q_2) + F(q_2) + \alpha^i((G(q_1) - F(q_2)) \tag{2}$$

Focs

$$\begin{aligned} \frac{\partial W(q_1, q_2, \alpha^i)}{\partial q_1} &= -U'_c(y - q_1(q_2, \alpha^i) - q_2) + \alpha^i G'_{q_1}(q_1(q_2, \alpha^i)) \\ \frac{\partial W(q_1, q_2, \alpha^i)}{\partial q_2} &= -U'_c(y - q_1 - q_2(q_1, \alpha^i)) + (1-\alpha^i) F'_{q_2}(q_2(q_1, \alpha^i)) \end{aligned} \tag{3}$$

Optimal quantities $q_1(q_2, \alpha^i)$ and $q_2(q_1, \alpha^i)$ are obtained from setting focs to zero and rearranging

$$\begin{aligned} q_1(q_2, \alpha^i) &: U'_c(y - q_1(q_2, \alpha^i) - q_2) = \alpha^i G'_{q_1}(q_1(q_2, \alpha^i)) \\ q_2(q_2, \alpha^i) &: U'_c(y - q_1 - q_2(q_1, \alpha^i)) = (1-\alpha^i) F'_{q_2}(q_2(q_1, \alpha^i)) \end{aligned}$$

Comparative statics to compute the effect of q_1 (q_2) on $q_1(q_2, \alpha^i)$ ($q_2(q_1, \alpha^i)$) by exploiting the assumption that $U(.)$, $G(.)$ and $F(.)$ C^2, strictly increasing and concave

$$\begin{aligned} -U''_{cc}(y - q_1(q_2, \alpha^i)\Big[\frac{\partial q_1(q_2, \alpha^i)}{\partial q_2} + 1\Big] &= \alpha^i G''_{q_1q_1}(q_1(q_2, \alpha^i))\frac{\partial q_1(q_2, \alpha^i)}{\partial q_2} \\ -U''_{cc}(y - q_1(q_2, \alpha^i)\Big[\frac{\partial q_2(q_1, \alpha^i)}{\partial q_1} + 1\Big] &= (1 - \alpha^i)F''_{q_2q_2}(q_2(q_1, \alpha^i))\frac{\partial q_2(q_1, \alpha^i)}{\partial q_1} \end{aligned} \tag{4}$$

By rearranging

$$\begin{aligned} \underbrace{\frac{\partial q_1(q_2, \alpha^i)}{\partial q_2}}_{\textit{then } <0} &= \underbrace{-U'_{cc}}_{<0 \textit{ by assumption}} \underbrace{[\alpha^i G'_{q_1q_1}(q_1(q_2, \alpha^i)) + U'_{cc}]^{-1}}_{>0 \textit{ by assumption}} \\ \underbrace{\frac{\partial q_2(q_1, \alpha^i)}{\partial q_1}}_{\textit{then } <0} &= \underbrace{-U''_{cc}}_{>0 \textit{ by assumption}} \underbrace{[(1 - \alpha^i)F''_{q_2q_2}(q_2(q_1, \alpha^i))]^{-1}}_{<0 \textit{ by assumption}} \end{aligned}$$

In words the publicly provided goods 1 and 2 are substitutes.

Comparative statics to compute the effect of α^i on optimal quantities $q_1(q_2, \alpha^i)$ and $q_2(q_1, \alpha^i)$

$$\begin{aligned} -U''_{cc}(y - q_1(q_2, \alpha^i) - q_2)\frac{\partial q_1(q_2, \alpha^i)}{\partial \alpha^i} &= \alpha^i G''_{q_1q_1}(q_1(q_2, \alpha^i))\frac{\partial q_1(q_2, \alpha^i)}{\partial \alpha^i} + G'_{q_1}(q_1(q_2, \alpha^i)) \\ -U''_{cc}(y - -q_1 - q_2(q_1, \alpha^i))\frac{\partial q_2(q_1, \alpha^i)}{\partial \alpha^i} &= (1 - \alpha^i)F''_{q_2q_2}(q_2(q_1, \alpha^i))\frac{\partial q_2(q_1, \alpha^i)}{\partial \alpha^i} - F'_{q_2}(q_2(q_1, \alpha^i)) \end{aligned} \tag{5}$$

By rearranging

$$\begin{aligned} \underbrace{\frac{\partial q_1(q_2, \alpha^i)}{\partial \alpha^i}}_{\textit{then } >0} &= \underbrace{-G'_{q_1}(q_1(q_2, \alpha^i))}_{<0 \textit{ by assumption}} \underbrace{[U''_{cc} + \alpha^i G''_{q_1q_1}(q_1(q_2, \alpha^i))]^{-1}}_{<0 \textit{ by assumption}} \\ \underbrace{\frac{\partial q_2(q_1, \alpha^i)}{\partial \alpha^i}}_{\textit{then } <0} &= \underbrace{F'_{q_2}(q_2(q_1, \alpha^i))}_{>0 \textit{ by assumption}} \underbrace{[(1 - \alpha^i)F''_{q_2q_2}(q_2(q_1, \alpha^i)) + U''_{cc}]^{-1}}_{<0 \textit{ by assumption}} \end{aligned}$$

In words, the greater α^i the greater (smaller) is the quantity of public good 1 (2) individual i would like at the optimum

2.4.b

By replacing $U(x) = F(x) = G(x) = ln(x)$, one gets the optimal quantities $q_1(q_2, \alpha^i)$ and $q_2(q_1, \alpha^i)$

$$\begin{aligned} q_1(q_2, \alpha^i) &= \frac{\alpha^i}{1 + \alpha^i}(y - q_2) \\ q_2(q_1, \alpha^i) &= \frac{1 - \alpha^i}{2 - \alpha^i}(y - q_1) \end{aligned}$$

Then, one gets the equilibrium levels of public consumption $q_1(\alpha^i)$ and $q_2(\alpha^i)$ by substituting from the optimal quantities obtained above

$$\begin{aligned} q_1(\alpha^i) &= \frac{\alpha^i}{2}y \\ q_2(\alpha^i) &= \frac{1 - \alpha^i}{2}y \end{aligned}$$

Given individuals $i = \{1, 2, 3\}$ and their preferences $\alpha^1 = 0$, $\alpha^2 = \frac{1}{2}$ and $\alpha^3 = 1$, one obtains the optimal policy quantities for each agent

$$\begin{aligned} q_1(0) = 0 &\quad , q_2(0) = \frac{y}{2} \\ q_1(\frac{1}{2}) = \frac{y}{4} &\quad , q_2(\frac{1}{2}) = \frac{y}{4} \\ q_1(1) = \frac{y}{2} &\quad , q_2(1) = 0 \end{aligned}$$

Individual $i = 2$ (or a group of voters with the same preferences as individual 2) represents the median voter. Then her choice is the selected one under a majority rule by Proposition 3 in PT p.26 and recalling the separation argument used to prove it.

2.4.c

Individual i's policy preferences

$$W(q_1, q_2, \alpha^i) = ln(y - q_1 - q_2) + \alpha^i ln(q_1) + (1 - \alpha^i) ln(q_2) + h(\alpha^i) ln(q_1 q_2) \tag{6}$$

One cannot rewrite such policy preferences in the intermediate preferences form $W(q_1, q_2, \alpha^i) = J(q_1, q_2) + K(\alpha^i)H(q_1, q_2)$ as there is no such $K(\alpha^i)$ that is monotonic in α^i in this specification. Then a Condorcet winner may not exist.

Focs

$$\begin{aligned} \frac{\partial W(q_1, q_2, \alpha^i)}{\partial q_1} &= -\frac{1}{y - q_1 - q_2} + \frac{\alpha^i + h(\alpha^i)}{q_1} \\ \frac{\partial W(q_1, q_2, \alpha^i)}{\partial q_2} &= -\frac{1}{y - q_1 - q_2} + \frac{1 - \alpha^i + h(\alpha^i)}{q_2} \end{aligned}$$

Optimal quantities $q_1(q_2, \alpha^i)$ and $q_2(q_1, \alpha^i)$ are obtained from setting focs to zero and rearranging

$$q_1(q_2, \alpha^i) = a(\alpha^i)[y - q_2]$$

$$q_2(q_1, \alpha^i) = b(\alpha^i)[y - q_1]$$

where

$$\begin{aligned} a(\alpha^i) &= \frac{\alpha^i + h(\alpha^i)}{1 + \alpha^i + h(\alpha^i)} \\ b(\alpha^i) &= \frac{1 - \alpha^i + h(\alpha^i)}{2 - \alpha^i + h(\alpha^i)} \end{aligned}$$

One clearly sees that public goods 1 and 2 are substitutes as $\frac{\partial q_1(q_2, \alpha^i)}{\partial q_2} < 0$ and $\frac{\partial q_2(q_1, \alpha^i)}{\partial q_1} < 0$

The equilibrium public consumption levels of public goods 1 and 2 are

$$\begin{aligned} q_1(\alpha^i) &= \frac{a(\alpha^i)[1 - b(\alpha^i)]}{1 - a(\alpha^i)b(\alpha^i)} y \\ q_2(\alpha^i) &= \frac{b(\alpha^i)[1 - a(\alpha^i)]}{1 - a(\alpha^i)b(\alpha^i)} y \end{aligned}$$

Comparative statics to compute the effect of a change in α^i on the optimal quantities $q_1(\alpha^i)$ and $q_2(\alpha^i)$

$$\begin{aligned}
a'(\alpha^i) &= \frac{1+h'(\alpha^i)(1+\alpha^i+h(\alpha^i))-(\alpha^i+h(\alpha^i))(1+h'(\alpha^i))}{(1+\alpha^i+h(\alpha^i))^2} \\
&= ... \\
&= \frac{1+h'(\alpha^i)}{(1+\alpha^i+h(\alpha^i))^2} \\
b'(\alpha^i) &= \frac{(-1+h'(\alpha^i))(2-\alpha^i+h(\alpha^i))-(1+h'(\alpha^i))(-1+h'(\alpha^i))}{(2-\alpha^i+h(\alpha^i))^2} \\
&= ... \\
&= \frac{h(\alpha^i)-1}{(2-\alpha^i+h(\alpha^i))^2}
\end{aligned}$$

Let $h(\alpha^i) = (\alpha^i)^2$. Then, compute the optimal quantities of public goods 1 and 2

$$\begin{aligned}
q_1(q_2,\alpha^i) &= \frac{\alpha^i(1+\alpha^i)}{1+\alpha^i+h(\alpha^i)}[y-q_2] \\
q_2(q_1,\alpha^i) &= \frac{\alpha^i(1-\alpha^i+(\alpha^i)^2}{2-\alpha^i+(\alpha^i)^2}[y-q_1]
\end{aligned}$$

Since $\alpha^i \in [0,1]$,

$$\begin{aligned}
a'(\alpha^i) &= \frac{1+2\alpha^i}{[1+\alpha^i+(\alpha^i)^2]^2} > 0 \\
b'(\alpha^i) &= \frac{(2\alpha^i-1)((\alpha^i)^2-1)}{[2-\alpha^i+(\alpha^i)^2]^2} < 0
\end{aligned}$$

Then $\frac{\partial q_1(q_2,\alpha^i)}{\partial \alpha^i} > 0$ and $\frac{\partial q_2(q_1,\alpha^i)}{\partial \alpha^i} < 0$

Exercise 3.2 Persson & Tabellini (PT) p.65
y level of publicly provided good
c^i level of private consumption
α^i random variable with cdf $F(\alpha^i)$ and $E[\alpha^i] = \alpha$
$\omega^i = c^i + \alpha^i V(y)$ individual i's utility function defined over c^i and y and with $V(y)$ C^2, $V'(y) > 0$ and $V''(y) < 0$
$e^i = 1$ $\forall i$ individual i's initial endowment in the private good
1 unit of private good yields 1 unit of the public good (cost of transforming the private good into the public one)
q lump sum government tax
$y \leq q$ government budget constraint
$c^i \leq 1 - q$ individual i's budget constraint

3.2.a
By replacing individual i's and the government budget constraints in individual i's policy preferences, one rewrites the policy preferences as a function of the public good level q only

$$W(q,\alpha^i) = 1 - q + \alpha^i V(q) \tag{7}$$

Individual i's problem over the public good provision

$$\max_q \int_{\alpha^i} W(q,\alpha^i)dF(\alpha^i) \quad s.t. \int_{\alpha^i}(1-c-y)dF(\alpha^i) \geq 0 \tag{8}$$

Focs

$$\frac{\partial \int_{\alpha^i}[1-q+\alpha^i V(q)]dF(\alpha^i)}{\partial q} = -1 + V_q'(q)\underbrace{\int_{\alpha^i} dF(\alpha^i)}_{E[\alpha^i]} \tag{9}$$

The optimum provision q^* of the public good is obtained by setting focs to zero

$$q^* = V^{-1}(\frac{1}{2}) \tag{10}$$

From the government budget constraint, it follows that $y^* = q^* = V^{-1}(\frac{1}{2})$ is the optimal provision of the public good.

3.2.b

2 politicians $P = A, B$:
i) propose electoral platforms q_a and q_b to voters
ii) maximise the expected value of the "ego-rent" R where π_P is the vote share of politician P, $p_P = Pr(\pi_P \geq \frac{1}{2})$ is the probability of winning elections and $p_P R$ is the expected utility

The timing of the game between voters and politicians is:
1) politicians announce the platform non cooperatively and simultaneously
2) voters express their preferences in the elections
3) the elected politician implements the policy announced in 1)

Voters are utility maximisers. Then the probability p_A of winning elections for politician A is

$$p_A = \begin{cases} 0 & \text{if } W(q^A,\alpha) < W(q^B,\alpha) \\ \frac{1}{2} & \text{if } W(q^A,\alpha) = W(q^B,\alpha) \\ 1 & \text{if } W(q^A,\alpha) > W(q^B,\alpha) \end{cases} \tag{11}$$

The probability that politician B wins is $p_B = 1 - p_A$.
By additional assumption voters prefer y^* as $\alpha^i = \alpha \ \forall i$. Then each politician can increase the proability p_P of winning elections by converging to a policy q closer and closer to y^*. Then the unique and socially optimum equilibrium is given by

$$q_A = q_B = q^* \tag{12}$$

3.2.c

Differently from 3.2.b the introduction of heterogeneity in individuals' types α^i fails to deliver a social optimum as only those with preferences $\alpha^i = \alpha$ vote for it.
Since individuals' preferences are single peaked, the chosen policy is the one preferred by the median voter with preferences $\alpha^i = \alpha^m$ by Proposition 3 PT p. 26. Then politicians maximise the probability of winning elections by choosing a platform that is preferred by the median voter

$$q_A = q_B = q^m \tag{13}$$

3.2.d

The model predicts that:
i) the social optimum is achieved if the agents have identical characteristics $\alpha^i = \alpha \ \forall i$
ii) introducing heterogeneity in agents' preferences/types yields a suboptimal provision of the public good