Politics & Economics: Theory and Applications

Solutions to Problem Set 3

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August 5, 2007

Exercise 6.1 Meltzer-Richard model, PT pp.154-155

c consumption level
x leisure
l per capita labour supply
f per capita lump sum transfer
θ proportional tax on labour income
$f = \tau l$ per capita government's budget constraint
$w^i = c^i - a(b - x^i)^2$ utility function describing preferences over consumption for individual i
$c^i = (1-\tau)l^i + f$ budget constraint for individual i
e^i individual i's productivity with density $f(e^i) = 2(1-e^i)$, $e^i \in [0,1]$
$1 + e^i = l^i + x^i$ time constraint for individual i

6.1.a. individual's and total labour supply

Individual's problem

$$\begin{aligned}
\max_{l^i} w^i &= c^i - a(b-x^i)^2 \text{ s.t.} \begin{cases} c^i = (1-\tau)l^i + f \\ 1 + e^i = l^i + x^i \end{cases} \\
\equiv \max_{l^i} w^i &= (1-\tau)l^i + f - a(b - 1 - e^i + l^i)^2 \\
(1-\tau) &= 2a(b-1-e^i+l^i) \text{ by setting foc to zero} \\
l^i(\tau) &= 1 + e^i - b + \frac{1-\tau}{2a} \text{ by substituting out for optimal individual's labour supply} \\
L(\tau) &= 1 + e - b + \frac{1-\tau}{2a} \text{ by substituting out for optimal total labour supply} \\
l^i(\tau) &= L(\tau) + e^i - e \text{ by substituting out the expression for } L(\tau) \text{ in that for } l^i(\tau);
\end{aligned}$$

By letting $E[e^i] = e$, then $f = \tau l = \tau L(\tau)$. By substituting the optimal individual's labour supply and the government budget constraint in the indirect utility function

$$\begin{aligned}
W^i &= (1-\tau)(l^i(\tau) + e^i - e) + \tau L(\tau) - a(b - 1 + l^i(\tau) - e^i)^2 \text{ by using } f = \tau L(\tau) \\
W^{i'}_{\tau^i} &= \frac{\partial W^i}{\partial \tau^i} = -\tau \frac{\partial l^i}{\partial \tau} - l^i + \tau L'(\tau) + L(\tau) - 2a(b - 1 + L(\tau) - e)\frac{\partial l^i}{\partial \tau} \\
W^{i'}_{\tau} &= -l^i + \tau L'(\tau) + L(\tau) \text{ as } \frac{\sum_i \partial l^i}{\partial \tau} = 0 \text{ by the envelope theorem} \\
&= e - e^i + \tau L'(\tau) = 0 \text{ using } l^i(\tau) = L(\tau) + e^i - e \\
\tau &= \frac{e^i - e}{L'(\tau)} \text{ by setting the foc } W^{i'}_{\tau} = 0 \\
\tau_i &= 2a(e - e^i) \text{ by taking the derivative } L'(\tau) = -\tfrac{1}{2a} \text{ of } L(\tau) \\
&= 2a(e - e^m) \text{ by monotonicity of } \tau_i \text{ in } e^i \text{ as } \tfrac{\partial \tau^i}{\partial e^i} > 0 \text{ and } \tau = \tau^m \text{ from median } (m) \text{ voter equilibrium}
\end{aligned}$$

in words, the greater is the difference between median and average income, a measure of tax inequality, the greater the taxes are.

compute the mean productivity e, the median one e^m and the equilibrium tax τ^m

$$\begin{aligned} e &= \int_0^1 2(1-x)xdx = \frac{1}{3} \\ & \int_0^{e^m} 2(1-x)xdx = \frac{1}{2} \;\Rightarrow\; e^m = 1 - \frac{1}{\sqrt{2}} \text{ by solving } -(e^m)^2 + 2e^m - \tfrac{1}{2} = 0 \\ \tau^m &= \left(\frac{1}{\sqrt{2}} - \frac{2}{3}\right) 2a \\ &\approx 0.081a \end{aligned}$$

6.1.b. equilibrium tax rate if only individuals with income above 0.05 vote
the electorate size shrinks to $\int_{0.05}^1 2(1-x)xdx = 0.9025$. then the median income is

$$\int_{0.05}^{e^{m1}} 2(1-x)xdx = \frac{0.9025}{2} \;\Rightarrow\; e^{m1} = 0.328$$

$$\begin{aligned} \text{the new equilibrium tax } \tau^{m1} &= \left(\frac{1}{3} - 0.328\right) 2a \\ &\approx 0.011a \\ &< \tau^m \end{aligned}$$

6.1.c. equilibrium tax rate with 10% citizens of each income level retiring
the median income is

$$0.1 + 0.9 \int_0^{e^m} 2(1-x)xdx = \frac{1}{2} \;\Rightarrow\; e^{m2} = 0.255$$

$$\begin{aligned} \text{the new equilibrium tax } \tau^{m2} &= \left(\frac{1}{3} - 0.255\right) 2a \\ &\approx 0.157a \\ &> \tau^m > \tau^{m1} \end{aligned}$$

6.1.d. equilibrium tax rate with costly $(1-\theta)$ tax collection
one obtains the equilibrium tax rate as in 6.1.a. but now the government budget constraint is $f = \theta\tau L$.

$$\begin{aligned} W^{i\prime}_{\tau^i} &= \ldots \\ &= \theta e - e^i - +(\theta - 1)\left[1 - b + \frac{1}{2a}\right]\tau L'(\tau) + L(\tau) + \frac{(1-2\theta)\tau}{2a} = 0 \iff \\ \tau &= \frac{2a}{1-2\theta}\left[(1-\theta)\left(1 - b - \frac{1}{2a}\right) + e^i - \theta e\right] \end{aligned}$$

Exercise 6.3 Pensions and probabilistic voting, PT pp.156-157
3 generations: young (Y), middle-aged (M) and old (O);
n population growth rate;
τ proportional labour tax to finance pension scheme with lump sum transfers f to the old;
government intertemporal budget constraint $f = \sum_{iY} \tau l^{iY}(1+n)^2 + \sum_{iM} \tau l^{iM}(1+n)$;
subjective discount rate β equals the real interest rate ρ;

lifetime utility of a pension scheme to a young individual $w^{iY} = U(c^{iY}) + \frac{1}{1+\beta}U(c^{iM}) + \frac{1}{(1+\beta)^2}U(c^{iO}) + V(x^{iY}) + \frac{1}{(1+\beta)^2}V(x^{iM})$;
young individual intertemporal budget constraint $c^{iY} + \frac{c^{iM}}{1+\rho} + \frac{c^{iO}}{(1+\rho)^2} = l^{iY}(1-\tau) + \frac{l^{iM}(1-\tau)}{1+\rho} + \frac{f}{(1+\rho)^2}$;
2 political candidates A and B maximise votes by choosing τ to finance the pension scheme;
$\sigma^i \sim U[-\frac{1}{f^p}, \frac{1}{f^p}]$, $p \in \{Y, M, O\}$ utility of individual i if candidate B wins elections;
individuals choose labour l given the tax rate τ;
$W^{iY}(\tau)$ individual i's indirect utility function given by her utility at the optimum saving and labour quantities;

6.3.a. equilibrium level of pensions under no commitment (vote on pension system every period)

young individual's problem for consumption choices c^{iY} and c^{iM}

$$\begin{aligned}
\max_{c^{iY}} w^{iY} &= U(c^{iY}) + \frac{1}{1+\beta}U(c^{iM}) + \frac{1}{(1+\beta)^2}U(c^{iO}) + V(x^{iY}) + \frac{1}{(1+\beta)^2}V(x^{iM}) \\
\text{s.t. } c^{iY} + \frac{c^{iM}}{1+\rho} + \frac{c^{iO}}{(1+\rho)^2} &= l^{iY}(1-\tau) + \frac{l^{iM}(1-\tau)}{1+\rho} + \frac{f}{(1+\rho)^2}
\end{aligned}$$

$$\begin{aligned}
\equiv \max_{c^{iY}} w^{iY} &= U(c^{iY}) + \frac{1}{1+\beta}U\Bigg(\Bigg[-c^{iY} - \frac{c^{iO}}{(1+\rho)^2} + l^{iY}(1-\tau) + ...) + \\
&\quad \frac{1}{(1+\beta)^2}U(c^{iO})\Bigg](1+\rho)\Bigg) + V(x^{iY}) + \frac{1}{(1+\beta)^2}V(x^{iM})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w^{iY}}{\partial c^{iY}} &= U'(c^{iY} - \frac{1+\rho}{1+\beta}U'(c^{iY}) \\
&= 0 \iff U'(c^{iY}) = \frac{1+\rho}{1+\beta}U'(c^{iM})
\end{aligned}$$

$$\begin{aligned}
\max_{c^{iM}} w^{iY} &= U(c^{iY}) + \frac{1}{1+\beta}U(c^{iM}) + \frac{1}{(1+\beta)^2}U(c^{iO}) + \\
&\quad +V(x^{iY}) + \frac{1}{(1+\beta)^2}V(x^{iM}) \\
\text{s.t. } c^{iY} + \frac{c^{iM}}{1+\rho} + \frac{c^{iO}}{(1+\rho)^2} &= l^{iY}(1-\tau) + \frac{l^{iM}(1-\tau)}{1+\rho} + \frac{f}{(1+\rho)^2} \\
\equiv \max_{c^{iM}} w^{iY} &= U\Bigg(-\frac{c^{iM}}{(1+\rho)} - \frac{c^{iO}}{(1+\rho)^2} + l^{iY}(1-\tau) + ...) + \frac{1}{1+\beta}U(c^{iM}) + \\
&\quad \frac{1}{(1+\beta)^2}U(c^{iO})\Bigg) + V(x^{iY}) + \frac{1}{(1+\beta)^2}V(x^{iM}) \\
\frac{\partial w^{iY}}{\partial c^{iM}} &= \frac{1}{1+\beta}U'(c^{iM}) - \frac{1}{1+\rho}U'(c^{iY}) \\
&= 0 \iff \frac{U'(c^{iM})}{1+\rho} = \frac{U'(c^{iY})}{1+\beta} \\
U'(c^{iY}) &= \left[\frac{1+\rho}{1+\delta}\right]^2 \\
U'(c^{iM}) &= \frac{1+\rho}{1+\delta}
\end{aligned}$$

given $\delta = \rho$, then $U'(c^{iY}) = U'(c^{iM}) = 1$

focs from the individual's problem for labour choice l^{iY} and l^{iM}

$$U'(c^{iY})(1-\tau) - V'(1+e^i - l^{iY}) = 0$$
$$U'(c^{iM})(1-\tau) - V'(1+e^i - l^{iM}) = 0$$

then $l^{iY} = l^{iM} = 1 + e^i + V'(1-\tau) = L(\tau) - e + e^i$

let $W^{iY} = W^{iM} = U(1-\tau l^i)$ and $W^{iO} = U(f)$ be the indirect utility functions of the 3 generations. under no commitment the young voter i votes for A if $\sigma^i \leq W^{iY}(\tau^A) - W^{iY}(\tau^B)$ whose cdf is $F^Y(W^{iY}(\tau^A) - W^{iY}(\tau^B))$.
problem of candidate A standing for elections (symmetric to candidate B's)

$$\begin{aligned}
\max_{\tau_A} E[v] &= \sum_{iY} F^Y(W^{iY}(\tau^A) - W^{iY}(\tau^B)) + \sum_{iM} F^M(W^{iM}(\tau^A) - W^{iM}(\tau^B)) + \\
&\quad \sum_{iO} F^O(W^{iO}(\tau^A) - W^{iO}(\tau^B)) \ s.t. \ L(\tau)\tau(1+n)(2+n) = f \\
0 &= -f^{YM}L + f^O(\tau L_\tau + L) \text{ by setting the foc to zero} \\
\tau &= (f^O - f^{YM})\frac{L}{|L_\tau|} \text{ by substituting out for } \tau
\end{aligned}$$

where the population weighted average of the density of the young and middle aged is

$$\begin{aligned}
f^{YM}L &= \frac{\partial \sum_{iY} F^Y(W^{iY}(\tau^A) - W^{iY}(\tau^B))}{\partial \tau_A} + \frac{\partial \sum_{iM} F^M(W^{iY}(\tau^A) - W^{iY}(\tau^B))}{\partial \tau_A} \\
&= \frac{(1+n)^2}{(1+n)+(1+n)^2} f^Y L + \frac{(1+n)}{(1+n)+(1+n)^2} f^M L \text{ where } f^j = \tfrac{\partial F^j}{\partial W^{ij}},\ j \in \{Y, M\}
\end{aligned}$$

the budget constraint of the candidate is obtained as

$$\begin{aligned}
f &= \tau l^M(1+n) + \tau l^Y(1+n)^2 \\
&= \ldots \\
&= \tau(1+n)(l^M + (1+n)l^Y) \\
&= \tau(1+n)(2+n)L(\tau) \text{ using } l^i = L(\tau) + e^i - e \text{ and } e = \tfrac{e^Y + e^M}{2}
\end{aligned}$$

the equilibrium tax rates is $\tau = \max\left(f^O - f^{YM}\frac{L}{|L_\tau|}, 0\right)$
in words, positive pensions arise in equilibrium if the voting preferences of the old are more sensitive to pensions than the ones of the young and middle aged are. Note that in this model a group of less than 50% of the voters but politically effective can obtain substantial redistribution while in the median voter model no platform is approved with less than 50% support.

6.3.b. equilibrium tax rate under commitment to implement a pension scheme forever
The indirect utilities are computed for the whole life span of the individuals when solving the same candidates' problem as in 6.3.a. The new equilibrium conditions are

$$\begin{aligned}
\sum_{iY} f^Y\left[-l^{iY} - \frac{l^i M}{1+\delta} + \frac{f_\tau}{(1+\delta)^2}\right] + \sum_{iM} f^M\left[-l^{iM} + \frac{f_\tau}{1+\delta}\right] + \sum_{iO} f^O f_\tau &= 0 \text{ or equivalently} \\
-L\left[f^Y(1+n)^2\left(\frac{2+\delta}{1+\delta}\right) + f^M(1+n)\right] + f_\tau\left[f^O + f^M\frac{1+n}{1+\delta} + f^Y\frac{(1+n)^2}{(1+\delta)^2}\right] &= 0
\end{aligned}$$

$$\tau = \frac{f^O + f^Y(1+n)^2\frac{(2+n)(1+n)-(2+\delta)(1+\delta)}{(1+\delta)^2} + f^M(1+n)\left(\frac{(1+n)(2+n)}{1+\delta} - 1\right)}{|L|} \frac{L}{f^O + f^M\frac{1+n}{1+\delta} + f^Y\frac{(1+n)^2}{(1+\delta)^2}}$$

if the population growth rate is greater than the real interest rate ($n > \rho$), then all genereations gain from the pension scheme and taxes are positive.