Markov-Perfect Optimal Fiscal Policy: The Case of Unbalanced Budgets

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Abstract

We study time-consistent, optimal fiscal policy set by a benevolent government in a neoclassical economy with infinitely-lived households. The government makes sequential decisions on the provision of a valued public good, on income taxation and the issue of public debt. We characterize and compute Markov-perfect optimal fiscal policy in this economy with two payoff-relevant state variables: physical capital and public debt. We find two stable, steady-state equilibria: one with no income taxation and positive government asset holdings, and another with positive taxation and public debt issuances. We prove that the two steady states are associated with different policy rules, which implies a multiplicity of (expectation-driven) Markov-perfect equilibria. In the calibrated economy, optimal fiscal policy in the steady-state equilibrium with positive distortions yields a 20% income tax rate and a debt-GDP ratio of 60%.

Keywords: Optimal fiscal policy; Markov-perfect equilibrium; Time-consistent policy.

JEL Classification Numbers: E61; E62; H21; H63.

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1 Introduction

In most developed economies, income tax rates and government debt levels are positive and sizable. In the US, effective income tax rates have been in the order of 20%, and outstanding public debt represents about 60% of GDP.[1] The main question we pose in this paper is: can these numbers be accounted for as the outcome of a government's welfare maximization program in a neoclassical economy? To address this question we adopt the standard framework in the literature of optimal fiscal policy, and drop the assumption of government's full commitment to future policies. Instead, we assume that the government has no access to commitment devices nor to reputation mechanisms, and, therefore, we restrict our attention to Markovian optimal policies. Our answer to the above question is in the affirmative, provided this is the policy expected by households and all successive governments.

The observation of positive income taxes and, especially, of positive levels of public debt has been at odds with most neoclassical theories of optimal fiscal policy. Chamley (1986) and Judd (1985) established that a committed government will not tax capital income in the long run. Furthermore, they showed that the optimal policy set by a government with the ability to borrow and lend involves high capital taxation in the short run in order to build up enough assets to finance future government expenditure, so that all distortionary taxation can be disposed of in the long run. In economies with government's full commitment, this result has been proved to be robust to a number of non-trivial departures from the standard framework.

Theories as to why governments run deficits have been offered in terms of political disagreement. That is, policymakers have different preferences on the type of public good that should be provided. With alternation in office, public debt becomes a tool to influence the choices of future policymakers [see Alesina and Tabellini (1990) and Persson and Svensson (1989)]. In our paper, we abstract both from political disagreement and alternation in office, and show that public deficits can be understood as the optimal policy of a government that lacks commitment to future policies. We use a standard neoclassical economy where a benevolent government makes sequential decisions on the provision of a valued public good, on income taxation and the issue of public debt. We characterize and compute Markov-perfect optimal fiscal policy in this economy with two payoff-relevant state variables: physical capital and public debt.

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[1] According to OECD data (see OEDC Economic Outlook), the debt-GDP ratio in the U.S. economy averaged 64% during the period 1991-2007. In Germany this ratio averaged 58% and in France 63%. Belgium and Italy, on the other hand, showed ratios above 100%.

The main contribution of our analysis can be summarized as follows. In the class of economies outlined above, optimal fiscal policy in a Markov-perfect equilibrium with differentiable strategies is not uniquely determined. We find two stable, steady-state equilibria: in one of them taxes are zero and debt is negative, such that there are no distortions in the consumption/savings margin nor in the private/public consumption margin. In the second equilibrium taxes and debt are positive, implying distortions to these two margins. We prove that convergence to either of the two steady states is not pinned down by initial conditions but by expectations on government policy. That is, Markov-perfect optimal fiscal policy is driven by expectations. In a calibrated version of the model the steady state with distortions yields a 20% tax rate and a debt-GDP ratio of 60%.

The government's ability to run unbalanced budgets, along with the assumption of an infinitely-lived economy, are at the heart of the expectation-driven multiplicity of Markov-perfect equilibrium. The intuition for this multiplicity in our framework is as follows. Each period, the government sets its policy so that: (*i*) it is indifferent between using taxes or debt to finance the provision of the public good, and (*ii*) the marginal value of increasing taxes (or debt) equals the marginal value of investment in physical capital. Since the continuation value for a Markovian government depends on future policies, the occurrence of the first type of equilibrium (i.e., the one with negative debt levels and no wedges in the long-run) can be understood as a case where lack of commitment is non binding and the economy reaches the efficient solution. Expectations that all future governments will dispose of distortionary taxation if given enough assets to finance the public good, will render such a policy optimal for the current government. This equilibrium is the limit of the Markov-perfect equilibrium in truncated economies as the time horizon goes to infinity. On the other hand, along the equilibrium with positive distortions lack of commitment is binding. A Markovian government does not internalize the effects of its policy on past investment, and is thus prone to overtax. When its predecessor anticipates overtaxation, and thus a distortion to the consumption/savings margin, it will find it optimal to issue debt to affect next-period's consumption of the private good. It will do so up to the point in which debt is no longer net wealth. As we show below, the existence of this equilibrium with positive wedges hinges on the assumption of an infinite time horizon, that is, on that there is no last government unable to pass on the burden of debt.[2]

In economies without capital, the existence of two steady states has been recently shown by

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[2] Our multiplicity is thus of a different nature to that found by Calvo (1988) in a two-period economy with costly debt repudiation. This author finds two expectation-driven equilibria: a "good" Pareto-efficient equilibrium with no debt repudiation, and a "bad" Pareto-inefficient equilibrium with partial debt repudiation.

a number of authors. In these economies, however, equilibrium dynamics are drastically different from what we find in our economy with capital and debt. Martin (2009a) and Díaz-Giménez, Giovannetti, Marimon and Teles (2008), study optimal monetary policy in economies with debt and find two steady-state debt levels, only one of which is stable. These authors show that the two steady states are generated by the same policy function and, therefore, the Markov-perfect equilibrium is unique. Krusell, Martin and Ríos-Rull (2006) study optimal debt policy in a model with exogenous government expenditure, labor taxation and no capital. In their economy, the interior steady-state equilibrium with positive distortions is unstable. The authors then consider non-differentiable strategies and show that the equilibrium contains a large, countable set of long-run debt levels. Initial conditions pin down the element in this set to which the economy converges in a maximum of two periods. By contrast, our results in this paper show that when physical capital and government expenditure are endogenous there is a stable, interior steady-state equilibrium with distortions. This equilibrium is not the limit of finite-horizon equilibria and is new in the literature of Markov-perfect optimal fiscal policy.

The paper is organized as follows. Section 2 presents the model, characterizes Markov-perfect equilibria and shows the existence of the two steady states. In Section 3 we parameterize and calibrate our model economy and compute Markov-perfect equilibrium. We also compare Markovian policies with those arising in the efficient equilibrium. In Section 4 we assess the role of debt in the Markov-perfect equilibrium by comparing economies with and without a balanced budget constraint. Section 5 presents some concluding remarks and discusses related literature. Section 6 contains two Appendixes.

2 The Model

Our framework is the standard, non-stochastic neoclassical model of capital accumulation, extended to include a benevolent government that provides a valued public good. In order to finance the provision of such public good the government can levy a tax on household's income and issue public debt. Thus, fiscal policy in each period consists of the amount of the public good provided, G_t, the tax rate on income, τ_t, and the issue of public debt, B_{t+1}, which matures in period $t+1$.

We begin by describing the problem solved by each agent in this economy. We then characterize the fiscal policy set by the benevolent government lacking the ability to commit to future policies. In order to help compare our results with the case of full commitment, we also present a brief review of fiscal policy in the Ramsey equilibrium.

2.1 Households

There is a continuum of homogeneous households with measure one. Each household supplies one unit of labor and chooses consumption and savings in order to maximize lifetime utility, subject to a budget constraint and initial endowments of physical capital and public debt,

$$\max_{\{c_t,k_{t+1},b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t, G_t), \tag{2.1}$$

$$s.t.$$

$$c_t + k_{t+1} + b_{t+1} = k_t + b_t + (1-\tau_t)\left[w_t + (r_t - \delta)\, k_t + q_t b_t\right] \quad \forall t \tag{2.2}$$

$$k_0 > 0 \text{ and } b_0 \text{ given,}$$

where small letters are used to denote individual variables and capital letters to denote economy-wide values. Function $U(\cdot)$ in equation (2.1) is the instantaneous utility function, which depends on the consumption of a private good, c_t, and the consumption of a public good, G_t. $U(\cdot)$ is assumed to be continuously differentiable, increasing and concave; and $0 < \beta < 1$ is the discount factor. Labor is supplied inelastically at a real wage rate w_t. Household's asset holdings are made up of physical capital, k_t, which is rented to firms at the rate r_t, and government's bonds, b_t, which bear an interest denoted by q_t. Physical capital depreciates at a rate denoted by $0 < \delta < 1$. Household's total income, net of capital depreciation, is taxed at the rate τ_t. If the government is a net lender to the private sector —i.e., the household borrows from the government, $b_t < 0$— taxable income is net of interest payments.[3]

2.2 Firms

Firms are competitive and produce an aggregate good with a neoclassical production technology. Total production is given by,

$$Y_t = F(K_t, L_t) = F(K_t, 1) = f(K_t) \ \ \forall t, \tag{2.3}$$

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[3] Our assumption that the only tax instrument is a general income tax, which implies that the government cannot tax capital and labor income separately, has no implications for our results below. We will elaborate further on this point.

where K_t denotes the aggregate or economy-wide stock of capital. First-order conditions to profits maximization imply the typical demand and zero-profits equations,

$$\begin{aligned} r_t &= f_K(K_t) && (2.4) \\ w_t &= f(K_t) - r_t K_t. && (2.5) \end{aligned}$$

2.3 Government

Government's fiscal policy involves the setting of both the provision of the public good and its financing through taxes and debt. The government is benevolent in the sense that it seeks to maximize households' lifetime utility, (2.1), subject to its budget constraint, to a feasibility restriction, and to private sector's first-order conditions. In addition, government's policies may be conditioned by its lack of commitment. The budget constraint of the government is,

$$G_t + (1+q_t)B_t = B_{t+1} + \tau_t \left[w_t + (r_t - \delta) K_t + q_t B_t \right], \qquad (2.6)$$

where the right-hand side of equation (2.6) represents government's revenues, which are made up of the issue of debt, B_{t+1}, plus revenues from income taxation. The left-hand side is government's total expenditure, including the provision of the public good, the repaying of outstanding public debt and financial expenses.

2.4 Ramsey Optimal Fiscal Policy

This Section presents a brief review of the Ramsey fiscal policy in our model economy. In a Ramsey equilibrium, the benevolent government is assumed to have full commitment to future policies, and, thus, it can credibly announce the whole sequence of public expenditure, income taxes and issues of debt from the first period onwards. This allows the government to anticipate the response of the private sector to its fiscal policy. Hence, the problem of the government in the Ramsey equilibrium is to choose sequences for taxes and public debt so that the competitive equilibrium maximizes social welfare [equation (2.1)].

Proposition 1 below presents the optimal fiscal policy in the steady-state of the Ramsey equilibrium for our economy. Since the result in Proposition 1 is well known in the literature of optimal fiscal policy we only provide a sketch of the proof (see the Appendix).

Proposition 1: *In the steady-state of the Ramsey equilibrium the tax rate on income is zero and the government holds positive assets, i.e.* $B < 0$.

2.5 Markov-Perfect Optimal Fiscal Policy

In this Section we drop the assumption of government's full commitment to future policies and study time-consistent optimal policies. More specifically, we will focus on differentiable Markov-perfect equilibria of this economy populated by a continuum of households and a government that acts sequentially, foreseeing its future behavior when choosing current levels of the public good, income taxes and the issue of debt. We look for all differentiable equilibria of the infinite-horizon economy, including the limit of finite-horizon equilibria as well as those equilibria that only emerge with infinite horizons.

Following the literature on Markovian policies, we assume that the government —although unable to commit to future policies—, does commit to honoring the tax rate it announces for the current period, and to repaying outstanding debt obligations. The commitment to current taxes implies an intra-period timing of actions that grants the government leadership in the setting of the tax rate. That is, at the beginning of period t, the time-t government sets the tax rate for the period; next, once that choice is publicly known, consumers choose consumption/savings and the composition of their portfolios, and the government chooses the provision of the public good (or equivalently, the issue of debt). Governments are thus (intra-period) *Stackelberg players* and can therefore anticipate the effects of current taxation on household's decisions.

In sum, we assume that the time-t government has intra-period commitment to time-t taxes but not to debt issues. In our opinion, this fits well the timing of actions in real economies, where, typically, governments make decisions on taxes at discrete times but issue debt continuously. It is important to note, however, that this particular timing of actions carries no consequences for our multiplicity result. Our computations under the alternative timing in which the government sets both taxes and debt issues at the same time as households choose consumption and savings also show the existence of multiple Markov-perfect equilibria. (Section 3.1 below characterizes Markov-perfect equilibria under this alternative timing of events.)

The optimization problem of a typical household

The household chooses *(i)* how much to consume and save; and *(ii)* how to allocate savings between physical capital and public debt. At the time the household makes these decisions the tax rate for the period is known, but it must foresee both the current government's debt policy and future governments' fiscal policy.

Hence, the problem of a household that holds k and b of the physical and government assets, respectively, that has to pay taxes on current income at rate τ, that expects the current and future governments to issue new debt according to the policy $\psi_B : (K \times B \times \tau) \to B'$, and expects future governments to set taxes according to the policy $\psi_\tau : (K \times B) \to \tau$, can be written as,

$$\begin{aligned} v(k,b,K,B;\tau) &= \max_{c,k',b'} \left\{U\left(c,G\right) + \beta\tilde{v}(k',b',K',B')\right\} \\ s.t. & \\ c + k' + b' &= k + b + (1-\tau)\left[w(K) + \left[r(K)-\delta\right]k + q(K)b\right], \end{aligned} \tag{2.7}$$

where $\tilde{v}(k',b',K',B')$ is the continuation value as foreseen by the household. $\omega(K), r(K)$ and $q(K)$ are pricing functions. The economy-wide stock of physical capital is expected to evolve according to the law $K' = \mathcal{H}(K,B,\tau)$, say. By using the assumption of a representative household, i.e., $k = K$ and $b = B$, and the government's budget constraint, it follows from the above maximization problem that the consumption function in a competitive equilibrium —where today's tax rate is τ, future taxes are set according to policy ψ_τ and current and futures issues of debt are set according to policy ψ_B— can be expressed in terms of K, B and τ, say $C(K,B,\tau)$, and must satisfy the following Euler equation,

$$U_c\left(C(K,B,\tau),G\right) = \beta U_c\left(C(K',B',\tau'),G'\right)\left[1 + \left(1-\tau'\right)\left(f_K(K')-\delta\right)\right], \tag{2.8}$$

where $B' = \psi_B(K,B,\tau)$ and $\tau' = \psi_\tau(K',B')$. In equilibrium K' is given by,

$$K' = K + B + (1-\tau)\left[f(K) - \delta K + q(K)B\right] - C\left(K,B,\tau\right) - B', \tag{2.9}$$

where G and G' are given by the time-t and time-$(t+1)$ governments' budget constraints, respectively. Finally, pricing functions $\omega(K)$ and $r(K)$ are given by (2.4) and (2.5), and $q(K)$ must satisfy the non-arbitrage condition between the two assets,

$$q(K) = f_K(K) - \delta. \tag{2.10}$$

In equilibrium, capital and debt yield the same return, meaning that q is independent of B. The fact that the interest rate on public debt is independent of B implies an important departure from economies without physical capital. We will comment further on this issue below.

Equation (2.8) has the usual interpretation: the marginal utility of consumption equals the present value of the last unit of income devoted to savings. Since physical capital and debt yield the same return in equilibrium, the supply of public debt determines the composition of the household's portfolio. This implies a one-to-one crowding out of investment in capital by public debt. Taxation, on the other hand, affects disposable income and the level of consumption, and thus translates into a non-one-to-one crowding out of capital investment. The problem of the government is shown next.

The problem of the government

As explained above, the government's lack of commitment to future policies and our focus on Markov-perfect equilibria allows us to think of the government as a sequence of governments, one for each time period. Thus the time-t government sets the tax rate for the period and issues new debt foreseeing the fiscal policy to be set by successive governments. Following the timing of actions established above, the time-t government is an intra-period *Stackelberg player* in our economy: At the beginning of the period, it chooses the income tax rate for that period taking into account the effect of τ on the level of consumption, as given by the consumption function, $C(K, B, \tau)$, that solves (2.8) and (2.9). In a second stage, the government sets the issue of debt. The problem of the time-t government is thus solved backwards. Given the initial choice for taxes, the issue of debt is the solution to,

$$\begin{aligned} V(K,B,\tau) &= \max_{B'} \left\{ U(C(K,B,\tau),G) + \beta \tilde{V}(K',B') \right\} \\ \text{s.t.}& \\ K' &= (1-\delta)K + f(K) - C(K,B,\tau) - G \\ G &= \tau\left[f(K) - \delta K + q(K)B\right] + B' - \left[1+q(K)\right]B, \\ &\quad \text{and equation (2.10),} \end{aligned} \tag{2.11}$$

where $V(K, B, \tau)$ is the value to the time-t government that has set the tax rate at τ and foresees the fiscal policy to be set by future governments. $\tilde{V}(K', B')$ is next-period value as foreseen by the time-t government. The issue of debt that solves this problem can thus be written as $B'(K, B, \tau)$. Therefore, the tax rate set by the time-t government is the solution to,

$$\begin{aligned} W(K,B) &= \max_{\tau} \left\{ U(C(K,B,\tau),G) + \beta \tilde{V}(K',B'(K,B,\tau)) \right\} \\ \text{s.t.}& \\ K' &= (1-\delta)K + f(K) - C(K,B,\tau) - G \\ G &= \tau\left[f(K) - \delta K + q(K)B\right] + B'(K,B,\tau) - \left[1+q(K)\right]B, \\ &\quad \text{and equation (2.10).} \end{aligned} \tag{2.12}$$

The following proposition characterizes the fiscal policy set by the time-t government.

Proposition 2: *A tax and debt policy that solves the government's problem is the solution to the following Generalized Euler Equations:*

$$\frac{U_c C_\tau + U_G G_\tau}{G_\tau + C_\tau} = \beta \left[U'_{c'} C'_{K'} + U'_{G'} G'_{K'} + \frac{U'_{c'} C'_{\tau'} + U'_G G'_{\tau'}}{G'_{\tau'} + C'_{\tau'}} \left(f'_{K'} + 1 - \delta - C'_{K'} - G'_{K'} \right) \right] \tag{2.13}$$

and

$$\frac{U_c C_\tau + U_G G_\tau}{G_\tau + C_\tau} = U_G G_{B'} + \beta \left[U'_{c'} C'_{B'} + U'_{G'} G'_{B'} - \frac{U'_{c'} C'_{\tau'} + U'_G G'_{\tau'}}{G'_{\tau'} + C'_{\tau'}} \left(C'_{B'} + G'_{B'} \right) \right]. \tag{2.14}$$

Proof: See the Appendix.

Some comments on notation are in order. Function arguments in equations (2.13) and (2.14) have been omitted for expositional clarity. Subscripts denote the variable with respect to which the derivative is taken. A prime in a variable indicates next-period values, and a prime in a function indicates it is evaluated at next-period variables. Finally, G_τ and G_B denote the derivatives of G with respect to τ and B, respectively, holding B' constant.

Before providing an interpretation of the two Generalized Euler Equations presented in Proposition 2, we offer the following definition of a Markov-perfect equilibrium in our economy:

Definition: *A Markov-perfect equilibrium is a quadruplet of functions $C(K, B, \tau)$, $\psi_B(K, B, \tau)$, $\psi_\tau(K, B)$ and $W(K, B)$, such that:*

(i) Given ψ_B and ψ_τ, $C(K, B, \tau)$ solves the household's maximization problem.

(ii) Given $C(K, B, \tau)$, ψ_B and ψ_τ solve the government's maximization problem. That is, $B' = \psi_B(K, B, \tau)$ and $\tau = \psi_\tau(K, B)$.

(iii) $W(K, B)$ is the value function of the government.

An alternative definition of Markov-perfect equilibrium in an economy without debt has been suggested by Harald Uhlig [see Klein, Krusell and Ríos-Rull (2008) for such definition]. In that framework, the problem of the government is set as choosing the level expenditure and the stock of capital for the next period directly. In the present framework, we find our equilibrium definition above more transparent for two reasons: First, our timing of actions involves the government choosing debt issues simultaneously with the consumption/savings decision. This amounts to the government not being able to anticipate the response of current consumption to debt issues. With our timing of events it is thus more straightforward to set the problem of the government as choosing current policy rather than next-period's capital. Second, by defining the equilibrium in terms of a consumption function depending on capital, debt and the tax rate, it will allow us to show some results on the partial derivatives of this consumption function.

The two Generalized Euler Equations, (2.13) and (2.14), which characterize Markov-perfect taxation and debt policies, respectively, have the following interpretation. Equation (2.13) establishes that the tax rate has to equate the marginal value of taxation to the marginal value of investing in physical capital. Equation (2.14) establishes that the issue of debt has to equate the marginal value of issuing debt to the marginal value of investing in physical capital (and consequently to the marginal value of taxation). In a Markov-perfect equilibrium, the government is indifferent between using taxes or debt to finance the provision of the public good. Both equations involve only wedges between today and tomorrow, as subsequent wedges are implicitly handled optimally by an envelope argument. Consecutive governments, however, disagree on how much to tax tomorrow [the time-$(t+1)$ government does not internalize the distortionary effects of its policy on time-t investment]. The current government thus takes into account the effect of its policy on tomorrow's initial conditions, K' and B', in order to help compensate for that disagreement. Following this reasoning, one may interpret the different terms in (2.13) and (2.14) as follows.

The left-hand side of equation (2.13) is today's marginal utility of taxation *per* unit of savings crowded out. The numerator of this expression is the change in utility from a marginal increase in the tax rate, which is made up of the change in utility from the private good, $U_c C_\tau$, plus the change in utility from the public good, $U_G G_\tau$. The denominator is the amount of savings crowded out, or, equivalently, the change in consumption of the public and private good brought about by the increase in the tax rate.

The right-hand side of equation (2.13) is the marginal utility of investing in physical capital. An extra unit of investment today yields an increase in resources tomorrow by $f'_{K'} + 1 - \delta$. The breakdown of the value of these resources is: *(i)* $C'_{K'}$ of them are consumed as private good, yielding a value of $U'_{c'} C'_{K'}$; *(ii)* $G'_{K'}$ corresponds to the increase in the provision of the public good obtained from the increase in the tax base, which yields a value of $U'_{G'} G'_{K'}$; *(iii)* the remaining $f'_{K'} + 1 - \delta - C'_{K'} - G'_{K'}$ are taxed away, and the marginal value is the left-hand side of equation (2.13), updated one period ahead. Hence, the right-hand side of (2.13) results from adding up all these values and discounting.

Equation (2.14) is a non-arbitrage condition between taxation and public debt, and its interpretation is equally straightforward. The right-hand side is the value of issuing an extra unit of government debt today. The first term on the right-hand side is the value of today's extra public good financed with the increase in government debt. The second term is the present value of the implied changes in tomorrow's consumption of the private and public good, $C'_{B'}$ and $G'_{B'}$, respectively. Besides the direct effects on tomorrow's utility, these changes have an effect on tomorrow's taxation, which must be valued using the marginal utility of taxation. Equation (2.14) establishes that the value of issuing debt must equal the value of taxation (the left-hand side of the equation).

A re-arrangement of equation (2.14) offers an alternative interpretation of the non-arbitrage condition between taxes and bonds in terms of two wedges, $U_c - U_G$ and $U'_{c'} - U'_{G'}$. Such a re-arrangement yields,

$$(U_c - U_G)\,\frac{C_\tau}{G_\tau + C_\tau} + \beta \left\{ \left(U'_{c'} - U'_{G'}\right) \left(G'_{B'} + \frac{G'_{\tau'}}{G'_{\tau'} + C'_{\tau'}} K''_{B'} \right) \right\} = 0. \tag{2.15}$$

Equation (2.15) says that the value of using debt instead of taxes to finance the last unit of public expenditure equals zero in a Markov-perfect equilibrium. The first term is the net change in utility today of using debt instead of taxes per unit of forgone savings. The second term captures the change in future distortions induced by the extra unit of public debt.[4] The way the current government trades off these two wedges when choosing B' depends on expectations on future government policy. As will become clearer below, there is an equilibrium policy which renders a non-zero wedge $U_c - U_G$ in the long run.

2.5.1 Markov-Perfect Equilibrium: Steady States

The *steady state of a Markov-perfect equilibrium* is defined as a list of infinite sequences for quantities $\{C_t\}$, $\{K_t\}$, fiscal variables, $\{G_t\}$, $\{\tau_t\}$, $\{B_t\}$ and prices $\{\omega_t\}$, $\{r_t\}$, and $\{q_t\}$ such that they are generated by a Markov-perfect equilibrium, and its values do not change over time, i.e. $K_{t+1} = K_t,\ B_{t+1} = B_t,\ \tau_{t+1} = \tau_t$ for all t, and the same is true for consumption and prices.

In this subsection we offer some insights on the steady state of our model economy, and prove three propositions. A first insight is related to the existence of two different steady-states, each

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[4] By way of clarity, the expression $K''_{B'}$ in this second term of the equation denotes the change in tomorrow's investment with respect to today's issue of debt. More specifically, if we combine the two restrictions in maximization problem (2.12), we can write K' as a function, say H, of K, B, B' and τ. Thus, $K''_{B'}$ is the derivative of function H evaluated at K', B', B'', τ', where $B'' = \psi_B(K', B', \tau')$ and $\tau' = \psi_\tau(K', B', \tau')$.

associated with a different Markov-perfect equilibrium. Evaluating equation (2.15) at the steady state of a Markov-perfect equilibrium yields,

$$(U_c - U_G)\left\{\frac{C_\tau}{G_\tau + C_\tau} + \beta\left(G_B + \frac{G_\tau}{G_\tau + C_\tau}K'_B\right)\right\} = 0. \tag{2.16}$$

This equation suggests that there may be two different taxation and debt policies consistent with the existence of a steady-state. The first one corresponds to the policy prescribed by the long-run Ramsey outcome. As shown in Proposition 1, the Ramsey equilibrium prescribes zero income taxes and positive government asset holdings in the steady state. The provision of the public good is financed entirely from the returns on government's assets, and therefore, $U_c = U_G$. The next proposition proves that this policy is a Markov-perfect equilibrium.

Proposition 3: *The steady-state Ramsey outcome is a Markov-perfect equilibrium.*

Proof: See Appendix.

In a related paper, Azzimonti-Renzo, Sarte and Soares (2006) study a model with differentiated taxes on capital and labor, and exogenous government expenditure. Within their framework, the authors find a Markov-perfect equilibrium which yields zero labor taxes from all initial conditions, K and B, and zero capital taxes from next-period onwards. As confirmed by our numerical computations, this result also holds in our model economy: when there are no exogenous bounds on income taxation, there exists a Markov-perfect equilibrium in which income taxes are zero after one period, and government assets converge to the long-run Ramsey value. Furthermore, for some initial conditions the initial income tax is negative, which amounts to a subsidy to households.

The second taxation and debt policy consistent with a steady-state Markov-perfect equilibrium involves positive income taxes and the issuing of government's bonds. Under this policy $U_c \neq U_G$, and the second term on the left-hand side of equation (2.16) is zero. The next proposition presents an important feature of the Markov-perfect equilibrium with positive taxation.

Proposition 4: *Along the steady state of a Markov-perfect equilibrium with positive distortions, government bonds are not net wealth, i.e.,* $C_B(K^*, B^*, \tau^*) = 0$.

Proof: See the Appendix.

Even though a proof of existence and uniqueness of this latter type of Markov-perfect equilibrium can not provided in our model economy, our numerical analysis, where we explored different subsets of the state space, produced only one steady state with positive taxation and public debt.

The existence of two stable, steady-state equilibria raises a question concerning equilibrium dynamics from initial values K_0 and B_0. Proposition 5 below proves that the government's policy rules generating the two steady states are different, which implies that steady-state multiplicity is expectational. Therefore, given initial conditions K_0 and B_0, expectations on government policy determine equilibrium dynamics and convergence to one of the two long-run equilibria. The basic idea of the proof relies on the fact that the steady-state equilibrium with positive distortions is not the limit of the finite-horizon economy's Markov-perfect equilibrium as the time horizon goes to infinity. Actually, we show that the steady-state equilibrium with no distortions is the only limit of the finite-horizon equilibrium.

Proposition 5: *The two steady states arising under Markov-perfect equilibrium are not associated with the same pair of decision rules* ψ_τ *and* ψ_B. *Hence, given* K_0 *and* B_0, *the Markov-perfect equilibrium is (globally) indeterminate.*

Proof: See the Appendix.

The next section presents a numerical analysis of the global dynamic properties of the steady state of the Markov-perfect equilibrium with positive distortions.

3 The Markov-Perfect Equilibrium in a Calibrated Economy

In this section we parameterize our model economy, set values to its parameters and compute Markov-perfect equilibria. A detailed explanation of our computational approach can be found in Appendix II. We find it convenient to emphasize that the two Markov-perfect equilibria are found as solutions to the three functional equations defined by the household's Euler equation and the two Generalized Euler equations, evaluated at equilibrium conditions. Our algorithm finds the two equilibria by searching for solutions in different subsets of the state space. We then check numerically that there are no profitable deviations. That is, no government wants to deviate from the policy prescribed in either of the two solutions. Special attention will be devoted first to the presentation of the Markov-perfect equilibrium rendering distortionary taxation and positive debt in the long run. Next, we compare Markov-perfect equilibria to the efficient solution (lump-sum taxation).

The instantaneous utility function is assumed to be of the CES form in the composite good $c_t G_t^\theta$, that is,

$$U(c, G) = \frac{(c\, G^\theta)^{1-\sigma} - 1}{1-\sigma}, \tag{3.1}$$

where $0 < \theta < 1$, and $1/\sigma$ denotes the elasticity of intertemporal substitution of the composite good. The functional form for the production technology is the standard Cobb-Douglas function, with α denoting the capital's share of income, i.e.,

$$f(K) = AK^{\alpha}, \quad A > 0. \tag{3.2}$$

Parameter values are set as follows. The constant in the production function, A, and the inverse of the elasticity of intertemporal substitution, σ, are both set equal to one. The value of α is set at 0.36, which is the capital's share of income in the US economy; the depreciation rate of capital is set at 0.09, which is a standard value in macroeconomic models; β is set at 0.96, and θ is 0.2 so that the public-to-private consumption ratio falls within the range $15 - 30\%$ for all equilibrium concepts mentioned above. These parameter values are in line with those in Klein, Ríos-Rull and Krusell (2008), Ortigueira (2006) and many others.

We start by presenting steady-state values for macroeconomic aggregates and fiscal policy under the equilibrium with lump-sum taxes and the Markov-perfect equilibrium. Table 1 below presents these steady-state values.

Table 1

Steady-State Equilibria

		Markov-perfect	
	Efficient	No wedges	Positive wedges
Y	1.7608	1.7608	1.6934
K	4.8144	4.8144	4.3201
C	1.1063	1.1063	1.1017
G	0.2213	0.2213	0.2032
G/C	0.2	0.2	0.1844
τ	indet.	0	0.1905
B/Y	indet.	-3.015	0.5639
W	-5.0157	-5.0157	-5.5525

Notes: Steady-state values for the efficient and Markov-perfect equilibria.

The first column in Table 1 shows the efficient equilibrium when lump-sum taxation is available. The second and third columns show the two steady states associated with Markov-perfect equilibria. The one in the second column is the equilibrium with zero wedges, which coincides with the steady state of the Ramsey outcome. In this equilibrium, the government does not distort long-run investment and sets income taxes equal to zero. Public expenditure is financed entirely from the income generated by the assets owned by the government. That is, negative public debt (positive asset holdings) is the only source of income for the government in this steady-state. In the calibrated economy the value of the assets held by the government is larger than the assets held by the private sector, and more than three times the value of output.

The steady state of the Markov-perfect equilibrium with positive wedges is shown in the third column of Table 1. In this equilibrium, income is taxed at a rate of 19.05% and the debt-GDP ratio is 56.39%. These numbers fall well within the range of observed values in the U.S. and in most developed economies.

Figures 1 to 7 below display equilibrium dynamics converging to the steady state of the Markov-perfect equilibrium with long-run distortions.[5] (Details on our method to compute Markov-perfect equilibria can be seen below and in Appendix II.) Figures 1 to 3 show government's optimal fiscal policy along the Markov-perfect equilibrium converging to the steady state in the last column of Table 1. The optimal income tax, as a function of K and B, is shown in Figure 1. The tax rate increases both with capital and debt. Figure 2 shows government's debt policy. The issue of debt decreases sharply with capital, indicating that capital-rich economies rely relatively less on public debt to finance government expenditure. Figure 3 shows public expenditure as a function of K and B. The private-good consumption function is displayed in Figure 4.

The stability of the steady-state is shown in Figures 5 to 7. Net investment in physical capital, $K' - K$, is presented in Figure 5. In Figure 6 we plot the change in the level of outstanding debt, $B' - B$. Finally, Figure 7 presents the two loci, $K' = K$ and $B' = B$. The point in which these two loci intersect corresponds to the steady-state values for K and B. The arrows indicate the direction of the trajectories starting in the different regions of the state space.

It should be noted that the Markov-perfect equilibrium shown in Figures 1 to 7 has been computed using a global method, where the equilibrium must be computed without prior knowledge of steady-state values. Thus, the subset of the state space must be changed in a trial-and-error process until it contains the steady-state equilibrium. Figures 8, 9 and 10 plot relative residuals in the Euler equation and the two Generalized Euler equations, respectively. Figure 11 shows relative residuals in the Bellman equation. It should be noted that the errors are very small, less than 0.001 of 1 per cent. In addition to this, errors satisfy relatively well the equioscillation property: The sign of the errors alternates between positive and negative. Overestimation

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[5] Equilibrium dynamics converging to the steady state without distortions are not presented here as they are well known from the literature on Ramsey optimal policy. (See the paragraph following Proposition 3.)

and underestimations alternate between collocation points and each error function achieves its extreme points about ten times. This property of the errors indicates that our approximations are close to being optimal, in the sense that there are no better polynomials to approximate the unknown functions.

In our economy with physical capital accumulation and public debt, the configuration of differentiable Markov-perfect equilibria differs drastically from that of the economy without capital, studied by Krusell, Martin and Ríos-Rull (2006). Contrary to their results, we find an interior, stable steady state with positive distortions. I.e. there is a Markov-perfect equilibrium whose time paths converge to this steady state, both for economies starting with debt levels below and above the steady-state value. The key ingredients for our result are physical capital and endogenous government expenditure. These two variables are exogenous in the work of Krusell, Martin and Ríos-Rull (2006). On the one hand, endogenous physical capital gives rise to a Markov-perfect equilibrium which is not the limit of finite-horizon equilibria. On the other hand, physical capital determines the equilibrium interest rate on public debt. The interest rate is now independent of the level of outstanding debt and the current government can affect next period's interest rate only through the stock of capital. This is in contrast with the economy without capital where the government nails down next period's interest rate when setting today's debt issues. Finally, while endogenous government expenditure is not key for the emergence of the steady-state equilibrium with distortions, it is key for its stability. Without the ability to adjust the expenditure margin, successive governments will set non-converging paths of debt when trying to influence next-period's consumption of the private good.

We close this Section by emphasizing that the multiplicity of Markov-perfect equilibria stems neither from our assumption on the timing of events within the period nor from our assumption of a general income tax. Allowing the government to tax capital and labor income separately, or to tax only capital income, will not remove the multiplicity result. For instance, we fixed a tax rate on labor income and let the government choose capital income taxes and public debt and found the same multiplicity. Our results will also hold in economies with endogenous labor. Martin (2009b) shows, in an economy with a balanced budget constraint and endogenous labor, that if the government is able to tax labor and capital income separately it will set a negative tax on labor. If subsidies are not allowed, then the optimal choice is to set the labor tax to zero. Thus, introducing public debt in this framework will give rise to the same type of multiplicity we found in our model, although labor income would be zero in this case.

As mentioned above, our timing of events within the period is not key for the multiplicity of Markov-perfect equilibrium. With the aim of shedding further light on our results above, the next subsection characterizes the Markov-perfect equilibrium under an alternative timing of events. We will use this new timing to showing robustness of our numerical computations of the value for a given government of unilateral deviations from the Markov-perfect equilibrium. More specifically, this timing of events will allow us to consider large deviations from Markov-perfect policies, and thus show that there are no profitable deviations from the Markov-perfect equilibrium with positive distortions, even when negative debt levels can be chosen.

3.1 Markov-perfect Equilibrium with an Alternative Timing of Events

In this subsection we briefly discuss Markov-perfect optimal policy in an economy where the government sets its policy —both taxes and debt issues— at the same time households choose consumption and savings. In contrast to our previous timing of events, the tax rate for the current period is not announced before households make their decision, implying that they must now forecast this rate as well as debt issues. The consumption function that solves the household's Euler equation can now be expressed in terms of K and B, say $C(K,B)$,

$$U_c\left(C(K,B),G\right) = \beta U_c\left(C(K',B'),G'\right)\left[1+\left(1-\tau'\right)\left(f_K(K')-\delta\right)\right], \tag{3.3}$$

where $B' = \psi_B(K,B)$ and $\tau' = \psi_\tau(K',B')$. And K' is given by,

$$K' = K + B + (1-\tau)\left[f(K) - \delta K + q(K)B\right] - C\left(K,B\right) - B'. \tag{3.4}$$

Since households do not know the tax rate for the current period, they anticipate that the government will set τ as a function of K and B, $\psi_\tau(K,B)$.

The government is no longer an intra-period *Stackelberg player*, and it sets τ and B' taking as given the consumption function and the policy of future governments. As will become apparent below, the functional equations characterizing a Markov-perfect equilibrium are now substantially simpler.

The problem of the current government can then be written as,

$$\begin{aligned} V(K,B) &= \max_{\tau,B'}\left\{U(C(K,B),G) + \beta\tilde{V}(K',B')\right\} \qquad (3.5)\\ &\text{s.t.}\\ K' &= (1-\delta)K + f(K) - C(K,B) - G\\ G &= \tau\left[f(K) - \delta K + q(K)B\right] + B' - \left[1+q(K)\right]B,\\ &\quad \text{and equation (2.10).} \end{aligned}$$

By assuming that savings is the residual variable, the government is unable to affect current private-good consumption, thus leaving consumption of the public good and the continuation value as its only trade-off. First-order conditions with respect to τ and B' are, respectively, $U_G = \beta \tilde{V}'_{K'}$ and $G_{B'}(\beta \tilde{V}'_{K'} - U_G) = \beta \tilde{V}'_{B'}$. From these two conditions it is straightforward to see that $\tilde{V}'_{B'} = 0$. Using the envelope conditions we obtain the two generalized Euler equations,

$$(U'_{C'} - U'_{G'})C'_{B'} = 0 \tag{3.6}$$

$$U_G = \beta \left[U'_{C'} C'_{K'} + U'_{G'}(1 + f'_{K'} - \delta - C'_{K'}) \right]. \tag{3.7}$$

Under this alternative timing, the possibility of multiple Markov-perfect equilibria becomes readily apparent from equation (3.6). Both a Markov-perfect equilibrium with no distortions, $U'_{C'} = U'_{G'}$, and a Markov-perfect equilibrium with distortions and with $C'_{B'} = 0$ satisfy this condition. The first of these equilibria yields a steady state that coincides with the Ramsey outcome. The second equilibrium yields a steady state with positive taxes and debt.

An important insight from the exercise in this subsection is that $V_B = 0$ along a Markov-perfect equilibrium, independently of whether or not the equilibrium involves distortions. This is a feature that turns out to be very useful when showing that the solution to the system of functional equations involving distortions is indeed a Markov-perfect equilibrium. To show this, we must check that no government wants to deviate from the policy prescribed by this solution when households and all successive governments are expected to follow it. Thus, since the continuation value for a given government does not depend on debt (nor does current private-good consumption depend on current taxes), we can consider large deviations both in taxes and debt issues, despite the fact that we confined the computation of this Markov-perfect equilibrium to a relatively narrow subset, $[K_{min}, K_{max}] \times [B_{min}, B_{max}]$, around its steady state. Actually, we can consider deviations to negative debt levels and arbitrarily large tax rates, provided the implied level of capital for the next period remains within our interval $[K_{min}, K_{max}]$. Our computations show that no government would gain by unilaterally deviating from the policy prescribed by either of the two Markov-perfect equilibria. That is, the problem of the time-t government is concave under either of the two equilibria.

4 The Role of Debt in the Markov-Perfect Equilibrium

We now assess the role of public debt in economies without commitment. The fact that governments, when setting fiscal policy for the current period, do not internalize the effects on past decisions raises a question on the role of debt in curving the resulting overtaxation. In this section, we solve for the Markov-perfect equilibrium under a balanced-budget constraint and compare the results with the ones we obtained when the government was allowed to run deficits or surpluses.

As discussed above, the multiplicity of Markov-perfect equilibrium does not hold in the economy with balanced budgets. The unique steady state in the economy without debt is shown in Table 2.

Table 2

Steady State with balanced budgets

	Markov-perfect
Y	1.6710
K	4.1632
C	0.9911
G	0.3052
G/C	0.3079
τ	0.2354
W	-6.1565

Notes: Steady state of the Markov-perfect equilibrium under balanced budgets.

This steady state yields higher income taxes than either of the steady states under unbalanced budgets. Consumption of the private good is slightly lower, and consumption of the public good is higher. The proneness of Markovian governments to overtax and overspend in the economy without debt is especially acute due to: *(i)* their lack of ability to internalize the distortionary effects of current taxation on past investment, and *(ii)* their leadership to set taxes before households choose consumption, which allows them to anticipate the response of current consumption to taxes, and then diminishing the perceived crowding out of physical investment. On the contrary, when governments are unable to weigh the reaction of private consumption to an increase in taxes, and perceive that such an increase will fall entirely on investment, they will be less prone to tax. As shown by Martin (2009c), this latter effect is not a generic result. This author presents different versions of the model without debt and finds parameterizations where a Stackelberg government sets lower taxes than under simultaneous moves. The assumption on the tax deductibility of capital depreciation seems to play a key role.

Public debt seems thus to play an important role in curbing the tendency of Markovian governments to overtax. More importantly, this role of debt is not limited to the equilibrium where the government can implement the efficient solution by accumulating assets and setting taxes to zero in the long run. Indeed, in the steady state of the Markov-perfect equilibrium with distortions, public debt brings consumption, public expenditure and capital closer to the efficient allocation, as compared to the economy with balanced budgets. The ability of Markovian governments to issue debt is thus bound to have positive effects on welfare.

5 Conclusions

This paper studies Markov-perfect optimal fiscal policy in a neoclassical economy with physical capital and public debt. We extend a recent literature on time-consistent policies to economies where the government can issue debt to finance government expenditures and households hold physical capital and public debt in their portfolios. Previous studies on Markov-perfect policy abstract from either public debt, by assuming a government's period-by-period balanced budget constraint, or from physical capital, assuming that labor is the only factor of production.

We characterize and compute Markov-perfect optimal fiscal policy in our model economy and find two equilibria. We prove that the long-run Ramsey outcome is the steady state of a Markov-perfect equilibrium. In addition, our numerical computations find a stable, steady state with positive income taxes and positive public debt associated with a different Markov-perfect equilibrium. In a calibrated version of the model, this latter steady state yields an income tax rate close to 20% and a debt-GDP ratio in the order of 60%. These numbers are in line with those observed in most developed economies. Although the framework presented in this paper displays an expectations-driven multiplicity of equilibria —and thus fails to provide predictions on optimal policy— we argue that it can however be useful as a positive theory of fiscal policy. That is, to help understand how actual policies are determined. The equilibrium with no distortions involves initial tax rates and levels of government asset holdings that may be unfeasible in most fiscal constitutions. This could leave the equilibrium with positive long-run distortions as the only Markov-perfect equilibrium of our economy.

We build on a recent body of literature dealing with optimal fiscal policy in environments with no commitment. In economies without debt, Markov-perfect optimal taxation has been studied by Klein and Ríos-Rull (2003), Klein, Krusell and Ríos-Rull (2008) and Ortigueira (2006). Our paper extends the framework presented in Klein, Krusell and Ríos-Rull (2008) to include public debt. Thus, our focus is on time-consistent tax and debt policy in economies without reputation mechanisms.[6]

Our paper is also related to the work of Song, Storesletten and Zilibotti (2007) who study optimal fiscal policy in economies where subsequent generations of agents (young and old) vote on policy. These authors focus on the Markov-perfect political equilibrium in an economy without physical capital and find that the long-run level of debt depends crucially on the distortions brought about by labor taxation. When these distortions are large enough debt converges to an interior value, otherwise debt accumulation depletes the economy. Debortoli and Nunes (2007) study the evolution of public debt under different degrees of commitment in an economy with political disagreement. Abstracting from physical capital, the authors show that it is political disagreement what explains positive values of long-run debt and not lack of commitment.

A different body of literature has developed after the paper by Lucas and Stokey (1983), who study the role of public debt as a substitute for commitment in Ramsey economies without capital. They show that the Ramsey policy is consistent if governments can commit to inherited debt contracts. Specifically, they show that future governments will comply with the fiscal plans chosen today if the current government delegates rich enough state-contingent multiple-period debt contracts. Later, Persson, Persson and Svenson (1988) extend this line of research to monetary policy inconsistency. Finally, Aiyagari, Marcet, Sargent and Seppälä (2002) modify the Stokey and Lucas (1993) model by dropping the complete markets assumption, which introduces a history dependence on the debt path, as opposed to a contingency to future states. They show that when there are no exogenous bounds on debt the Ramsey planner in their economy lets public debt converge to a negative level.

6 Appendix I: Proofs

Proof of Proposition 1:

The problem solved by a government with full commitment is to set infinite sequences $\{G_t, \tau_t, B_t\}$

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[6] For an analysis of time-consistent fiscal policy with history-dependent strategies, see Dominguez (2007) and Reis (2006). These authors study sustainable equilibria and find that the best sustainable equilibrium prescribes zero long-run capital taxation. A non-balanced budget constraint is key in obtaining this result, as it allows the government to increase its assets until the lack of commitment is no longer binding.

so that the implied competitive equilibrium maximizes welfare. That is,

$$\max_{\{G_t,\tau_t,B_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(C_t, G_t) \tag{6.1}$$

$s.t.$

$$C_t + K_{t+1} + G_t = f(K_t) + (1-\delta)K_t \tag{6.2}$$

$$G_t + [1 + r_t - \delta]B_t = B_{t+1} + \tau_t[(r_t - \delta)(K_t + B_t) + \omega_t] \tag{6.3}$$

$$U_c(C_t, G_t) = \beta U_c(C_{t+1}, G_{t+1})[1 + (1-\tau_{t+1})(r_{t+1} - \delta)], \quad t = 0...\infty, \tag{6.4}$$

K_0 and B_0 are given.

After defining new variables $\tilde{r}_t \equiv (1-\tau_t)(r_t - \delta)$ and $\tilde{\omega}_t \equiv (1-\tau_t)\omega_t$, and formulating the problem of the government as choosing after-tax rental prices, the first-order condition with respect to K_{t+1} (by using the primal approach) can be written as,

$$\Gamma_t = \beta \left[\Lambda_{t+1}(r_{t+1} - \tilde{r}_{t+1})) + \Gamma_{t+1}(1 + r_{t+1} - \delta)\right], \tag{6.5}$$

where Γ_t and Λ_t are Lagrange multipliers. Using the Euler equation, equation (6.5) in a steady-state equilibrium is,

$$(\Gamma + \Lambda)(r - \tilde{r}) = 0, \tag{6.6}$$

from which it follows that $\tau = 0$ in the steady-state equilibrium, and, consequently, $B < 0$.

Proof of Proposition 2:

The first-order condition to B' in government's maximization problem (3.5) is given by,

$$U_G G_{B'} - \beta \tilde{V}'_{K'} G_{B'} + \beta \tilde{V}'_{B'} = 0. \tag{6.7}$$

The first-order condition to τ in government's maximization (2.12) is,

$$U_c C_\tau + U_G \left(G_\tau + G_{B'} B'_\tau\right) - \beta \tilde{V}'_{K'} \left(C_\tau + G_\tau + G_{B'} B'_\tau\right) + \beta \tilde{V}'_{B'} B'_\tau = 0, \tag{6.8}$$

which, after making use of (6.7), simplifies to,

$$U_c C_\tau + U_G G_\tau - \beta \tilde{V}'_{K'} \left(C_\tau + G_\tau\right) = 0. \tag{6.9}$$

Envelope conditions, along with $W(K, B) = \tilde{V}(K, B)$, yield,

$$W_K = U_c C_K + U_G G_K + \beta W'_{K'} \left[1 + f_K - \delta - C_K - G_K\right] \tag{6.10}$$

$$W_B = U_c C_B + U_G G_B + \beta W'_{K'} \left[-C_B - G_B \right]. \tag{6.11}$$

Forwarding these envelope conditions one period and using the above first-order conditions, we obtain the two Generalized Euler Equations, (2.13) and (2.14) , presented in Proposition 2.

Proof of Proposition 3:

As shown in Proposition 1, in the steady state of the Ramsey equilibrium income taxes are zero and the government holds negative debt (assets) to finance the provision of the public good. The government does not rely on distortionary taxation, and the efficiency condition, $U_c = U_G$, is attained. In this proof we show that the system of equations characterizing the steady states of Markov-perfect equilibria has a solution with these properties.

Let us start by assuming that $U_c = U_G$. Then, from (6.9) it follows that $U_c = \beta W_K$. From (6.11) it is then easy to see that $W_B = 0$. Finally, equation (6.10) becomes,

$$\frac{1}{\beta} = 1 + f_K - \delta, \tag{6.12}$$

which, along with the consumer's Euler equation, implies that $\tau = 0$.

Proof of Proposition 4:

The proof follows directly from (2.16) along with the non-arbitrage, Euler and feasibility conditions. In a Markov-perfect equilibrium with positive distortions $U_c \neq U_G$. Thus, from (2.16),

$$\frac{C_\tau}{G_\tau + C_\tau} + \beta \left(G_B + \frac{G_\tau}{G_\tau + C_\tau} K'_B \right) = 0. \tag{6.13}$$

By feasibility, this is equivalent to

$$\frac{C_\tau}{G_\tau + C_\tau} \left(\frac{1}{\beta} + C_B + G_B \right) = C_B. \tag{6.14}$$

Then, plugging $G_B = -(1-\tau)q - 1$ and the non-arbitrage condition, $q = f_K - \delta$, into equation (6.14) and using the household's Euler equation, it follows that $C_B = 0$ at the steady state.

Proof of Proposition 5:

Here we prove that the two steady states —one with positive distortions and one without— are not associated with the same pair of Markov-perfect decision rules ψ_τ and ψ_B. To do this, we show that the policy rules generating the steady state with no distortions are the only limit of policy rules in the finite-horizon economy as the planning horizon goes to infinity. The proof, although algebraically tedious, is straightforward.

In the finite-horizon economy with last period denoted by T, we have $K_{T+1} = B_{T+1} = 0$. Therefore, in period T households simply consume all their resources. The problem of the time-T government is then,

$$\max_{\tau_T} \quad U(C_T, G_T)$$

$$s.t.$$

$$C_T = K_T + B_T + (1-\tau_T)\left[f(K_T) - \delta K_T + q_T B_T\right] \tag{6.15}$$

$$G_T = \tau_T\left[f(K_T) - \delta K_T + q_T B_T\right] - (1+q_T)\,B_T. \tag{6.16}$$

The first-order condition to this problem is,

$$U_c(T) = U_G(T), \tag{6.17}$$

where $U_c(T)$ denotes $U_c(C_T, G_T)$.

In period $T-1$, the households' Euler equation is,

$$U_c(C_{T-1}, G_{T-1}) = \beta U_c(C_T, G_T)\left[1 + (1-\tau_T)\left(f_K\left(K_T\right) - \delta\right)\right], \tag{6.18}$$

and the non-arbitrage condition between the two assets is $q_T = f_K\left(K_T\right) - \delta$. The fiscal policy chosen by the time-$(T-1)$ government is obtained in a two-step maximization problem. First, given τ_{T-1}, the issue of debt solves,

$$\max_{B_T} \quad \left\{U(C_{T-1}, G_{T-1}) + \beta U(C_T, g_T)\right\}$$

$$s.t.$$

$$G_{T-1} = B_T + \tau_{T-1}\left[f(K_{T-1}) - \delta + q_{T-1}B_{T-1}\right] - (1+q_{T-1})\,B_{T-1} \tag{6.19}$$

$$K_T = f(K_{T-1}) + (1-\delta)\,K_{T-1} - G_{T-1} - C_{T-1} \tag{6.20}$$

and equations (6.15), (6.16), (6.17) and (6.18).

The first-order condition to this problem is,

$$U_G(T-1)\frac{\partial G_{T-1}}{\partial B_T} + \beta\left[\left(U_c(T)\frac{dC_T}{dK_T} + U_G(T)\frac{dG_T}{dK_T}\right)\frac{dK_T}{dB_T} + \left(U_c(T)\frac{dC_T}{dB_T} + U_G(T)\frac{dG_T}{dB_T}\right)\right] = 0 \tag{6.21}$$

Then, τ_{T-1} is the solution to,

$$\max_{\tau_{T-1}} \quad \left\{U(C_{T-1}, G_{T-1}) + \beta U(C_T, g_T)\right\}$$

$$s.t.$$

equations (6.15), (6.16), (6.17), (6.18), (6.19), (6.20) and (6.21).

The first-order condition is,

$$U_c(T-1)\frac{\partial C_{T-1}}{\partial \tau_{T-1}} + U_G(T-1)\frac{\partial G_{T-1}}{\partial \tau_{T-1}} + \beta\left[U_c(T)\frac{dC_T}{dK_T} + U_G(T)\frac{dG_T}{dK_T}\right]\frac{\partial K_T}{\partial \tau_{T-1}} = 0 \tag{6.22}$$

Now, from feasibility conditions at T and $T-1$ we obtain that $\frac{dC_T}{dB_T} = -\frac{dG_T}{dB_T}$, $\frac{\partial K_T}{\partial \tau_{T-1}} = -\frac{\partial C_{T-1}}{\partial \tau_{T-1}} - \frac{\partial G_{T-1}}{\partial \tau_{T-1}}$ and $\frac{dK_T}{dB_T} = -\frac{dG_{T-1}}{dB_T} = -1$. Using these values in equations (6.21) and (6.22), we have

$$\left[U_c(T-1) - U_G(T-1)\right]\frac{\partial C_{T-1}}{\partial \tau_{T-1}} = 0. \tag{6.23}$$

Since $\frac{\partial C_{T-1}}{\partial \tau_{T-1}} \neq 0$, it thus follows that $U_c(T-1) = U_G(T-1)$. Then, using the fact that $\frac{dC_T}{dK_T} + \frac{dG_T}{dK_T} = 1 + f_K(K_T) - \delta$, equation (6.22) yields,

$$U_c(T-1) = \beta U_c(T)\left[1 + f_K(K_T) - \delta\right]. \tag{6.24}$$

From this equation and the household's Euler equation it follows that $\tau_T = 0$.

Solving the problem for period $T-2$, yields $\tau_{T-1} = 0$. By proceeding in this way up to the initial period, it can be shown that all taxes are zero but the initial one. That is, $\tau_0 \neq 0$ and $\tau_t = 0$ for all t from 1 to T.

Appendix II: Numerical Approach

In this appendix we outline our strategy for the computation of the Markov-perfect equilibrium. The first challenge in the computation of the three unknown functions $C(K, B, \tau)$, $\psi_\tau(K, B)$, and $\psi_B(K, B, \tau)$ stems from the presence of the derivatives of the consumption function in the two generalized Euler equations, (2.13) and (2.14). In a steady state, these derivatives must be solved for, thus making the number of unknowns exceed the number of equations.

Our computational method is an application of a projection method which approximates the three unknown functions with a combination of Chebyshev polynomials. Within the class of orthogonal polynomials, Chebyshev polynomials stand ut for its efficiency to approximate smooth functions.[7] The unknown coefficients in the approximate functions are then obtained so that they satisfy the three Euler equations at some collocation points within a subset of the state space, $[K_{min}, K_{max}] \times [B_{min}, B_{max}]$.

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[7] For a complete characterization of their properties and a rigorous exposition of projection techniques see Judd (1992, 1998). For a previous application of these ideas to the computation of Markovian optimal taxes see Ortigueira (2006).

Thus, we approximate functions for consumption, taxes and the issue of debt by[8]:

$$\hat{C}(K,B,\tau;\vec{a}) = \sum_{i=0}^{n_k^c}\sum_{j=0}^{n_b^c}\sum_{\ell=0}^{n_\tau^c} a_{ij\ell}\ \phi_{ij\ell}(K,B,\tau) \tag{6.25}$$

$$\hat{\psi}_\tau(K,B,\vec{d}) = \sum_{i=0}^{n_k^b}\sum_{j=0}^{n_b^b} d_{ij}\ \phi_{ij}(K,B) \tag{6.26}$$

$$\hat{\psi}_B(K,B,\vec{h}) = \sum_{i=0}^{n_k^\tau}\sum_{j=0}^{n_b^\tau} h_{ij}\phi_{ij}(K,B), \tag{6.27}$$

where $\phi_{ij\ell}(K,B,\tau)$ and $\phi_{ij}(K,B)$ are tensor products of univariate Chebyshev polynomials, which form the multidimensional basis for approximation. For instance, $\phi_{ij}(K,B) = \phi_i(K)\phi_j(B)$, with $\phi_i(K)$ denoting the Chebyshev polynomial of order i in K and $\phi_j(B)$ the Chebyshev polynomial of order j in B. Since Chebyshev polynomials are only defined in the interval $[-1,1]$, K and B must be re-scaled accordingly, using the chosen $K_{\min}, K_{\max}, B_{\min}, B_{\max}$. That is,

$$\phi_{ij}(K,B) = \phi_i\left(\frac{2(K-K_{\min})}{K_{\max}-K_{\min}} - 1\right) \times \phi_j\left(\frac{2(B-B_{\min})}{B_{\max}-B_{\min}} - 1\right). \tag{6.28}$$

Vectors $\vec{a}$, $\vec{d}$, $\vec{h}$ in (6.25) – (6.27) are the unknown coefficients, which are pinned down by imposing that $\hat{C}(K,B,\tau;\vec{a})$, $\hat{\psi}_\tau(K,B,\vec{d})$ and $\hat{\psi}_B(K,B,\vec{h})$ satisfy the three Euler equations and the laws of motion at a number of collocation points. The number of collocation points is set so that the number of equations equals the number of unknown coefficients. In our exercise we choose Chebyshev collocation. It should be noted that the approximation of the debt policy, equation (6.27), embeds already the approximation of the tax policy in terms of K and B. On the other hand, the approximation of the consumption function, (6.25), must be done in terms of K, B and τ, in order to obtain the derivatives of the consumption function which show up in the Generalized Euler equations. Note, however, that we compute the consumption function, and its derivatives, only along the equilibrium path, i.e., for $\tau = \hat{\psi}_\tau(K,B)$. That is, both the consumption function and its derivatives are computed as functions of K and B.

The value function, $W(K,B)$, can then be easily computed as follows. Using the solutions for consumption, taxation and the issue of debt, the value function is approximated by,

$$\hat{W}(K,B,\vec{e}) = \sum_{i=0}^{n_k^v}\sum_{j=0}^{n_b^v} e_{ij}\phi_{ij}(K,B), \tag{6.29}$$

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[8] Since the derivative of $\psi_B(K,B,\tau)$ with respect to τ is not needed to solve our system of functional equations, we approximate the debt policy rule at equilibrium by a function of K and B alone, replacing τ by $\psi_\tau(K,B)$.

where the vector $\vec{e}$ contains the unknown coefficients in the value function, which are pinned down so that (6.29) solves the government's Bellman equation at a number of collocation points.

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Policy Functions

Figure 1. Tax Policy Function

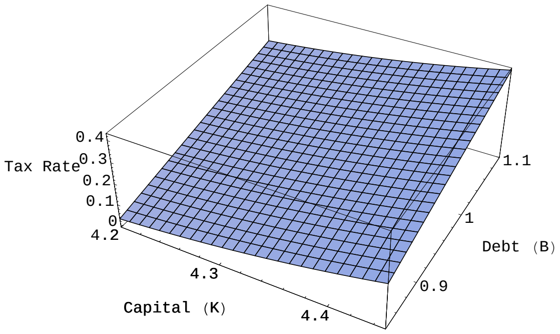


Figure 2. Debt Policy Function

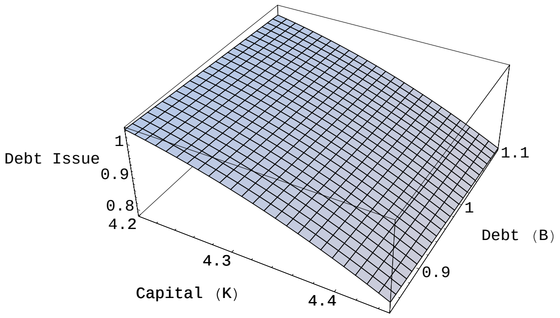


Figure 3. Gov. Expenditure Function

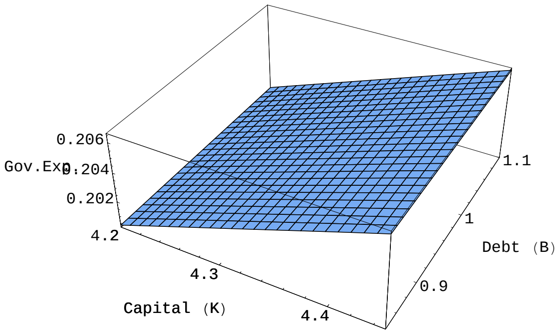
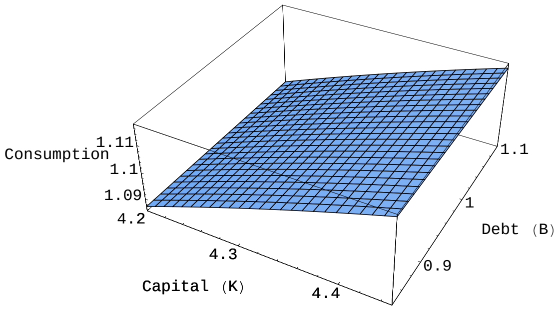


Figure 4. Consumption Function



Notes: Figures 1 to 4 show the policy functions in a Markov-perfect equilibrium. The government's tax policy is shown in Figure 1. The government's debt policy is shown in Figure 2. The government's spending policy is displayed in Figure 3. Finally, the private consumption function is shown in Figure 4.

Equilibrium Dynamics

Figure 5. Net Investment, K′–K

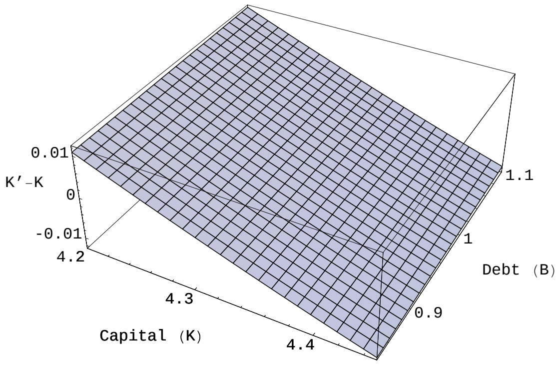


Figure 6. Change in Public Debt, B′–B

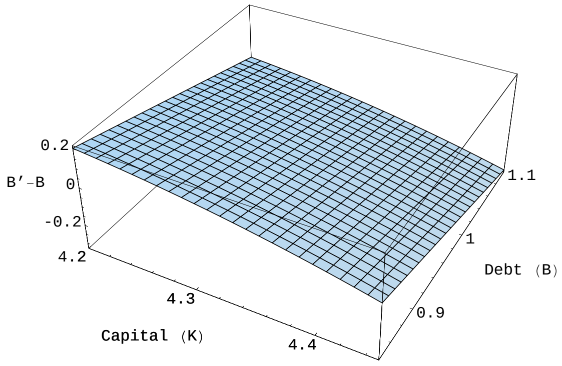
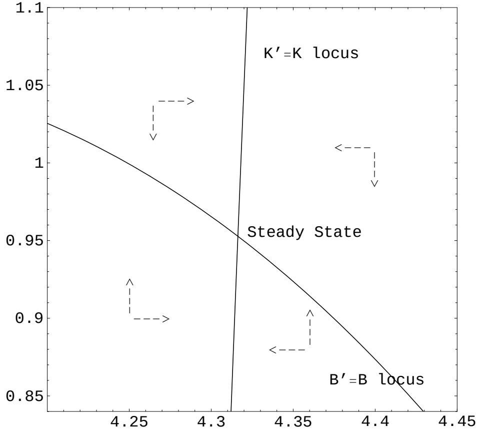


Figure 7. The K′=K and B′=B Loci



Notes: Figures 5 to 7 show the dynamics around the steady-state equilibrium with positive income taxation. Figure 5 shows net investment; Figure 6 shows the change in government debt; and Figure 7 shows the $K' = K$ and $B' = B$ loci.

Errors

Figure 8. Euler Equation Errors

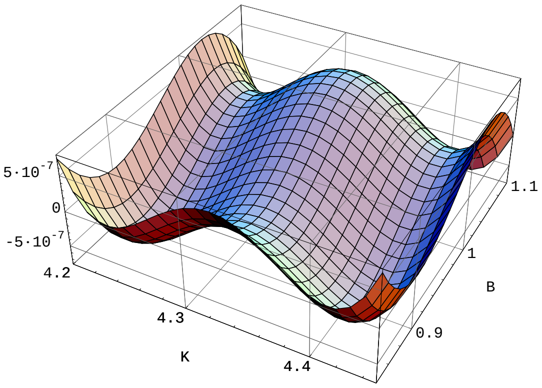


Figure 9. Generalized Euler Equation 1 Errors

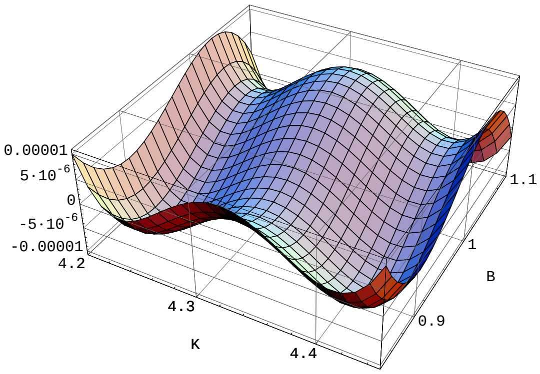


Figure10. Generalized Euler Equation 2 Errors

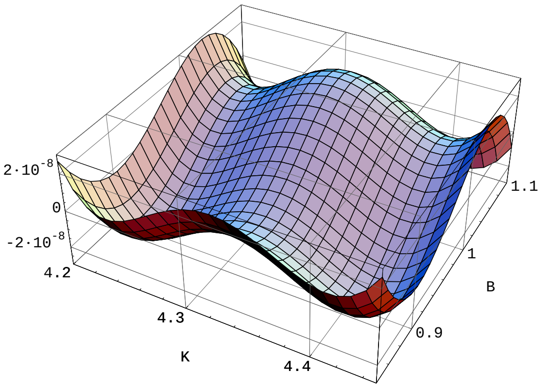
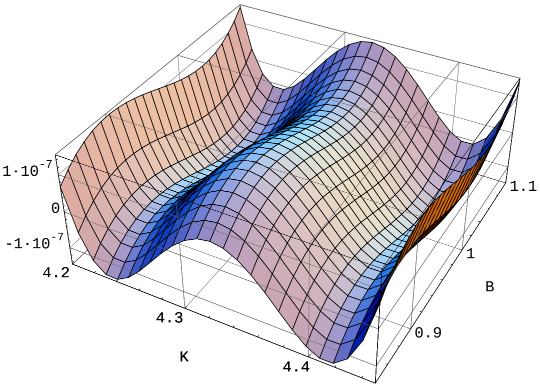


Figure 11. Bellman Equation Errors



Notes: Figures 8 to 11 show relative errors of Chebyshev collocation for the Euler equation (Figure 8), Generalized Euler equations (Figures 9 and 10) and the Bellman equation (Figure 11).