Chesher—'Exogenous Impact and Conditional Quantile Functions'

The 'simple example', p.2:

$$\begin{aligned} Y_1 &= h_1(Y_2, X, \varepsilon, \nu) \\ Y_2 &= h_2(X, Z, \nu) \end{aligned}$$

Y_1, Y_2 scalar random variables.

X a list of covariates

Z a list of instrumental variables.

Some assumptions:

1. unobservable continuously distributed ε has conditional τ_ε –quantile 'given ν, X, Z' independent of υ, X, Z.
2. similarly ν has (conditional) (τ_ε –quantile $| X, Z$) independent of X and Z.
3. $h_1()$ differentiable w.r.t. scalar Y_2 and ν.
4. $h_2()$ differentiable w.r.t. scalar Z and ν.
5. $h_1()$ is monotonic increasing in ε.
6. $h_2()$ is monotonic increasing in ν.

The Exogenous Impact Function Defined

We want to know:
'The derivative of $h_1()$ with respect to Y_2, the variables X, ε, ν, and *therefore* Y_2, being held fixed at interesting values.' The system again:

$$\begin{aligned} Y_1 &= h_1(Y_2, X, \varepsilon, \nu) \\ Y_2 &= h_2(X, Z, \nu) \end{aligned}$$

X and Z 'lists'. Formally the exogenous impact function is denoted

$$\pi(\tau_\varepsilon, \tau_\nu, x, z) = \nabla_{y_2} h_1(Q_{Y_2|XZ}(\tau_\nu, x, z), x, Q_\varepsilon(\tau_\varepsilon), Q_\nu(\tau_\nu))$$

$\nabla_{y_2} h_1(\cdot, \cdot, \cdot) \equiv$partial derivative of $h_1()$ w.r.t. first argument.
$Q_\varepsilon(\tau_\varepsilon), Q_\nu(\tau_\nu)$ are τ_ε and τ_ν quantiles of the distributions of ε and ν (resp.)
$Q_{Y_2|XZ}(\tau_\nu, x, z)$ is the $\tau_\nu-$quantile of the conditional distribution of Y_2 given $X = x$ and $Z = z$.
$Q_{Y_2|XZ}(\tau_\nu, x, z) = h_2(x, z, Q_\nu(\tau_\nu))$ under the monotonicity assumption for $h_2(\cdot)$.

The Exogenous Impact Function ‘Interpreted’

‘The EIF $\pi(\tau_\varepsilon, \tau_\nu, x, z)$ is the rate at which Y_1 changes as the value of Y_2 is increased when $X = x$, $Z = z$, and when ε and ν have values equal to the specified quantiles.’

[So, e.g. it’s (possibly, perhaps usually) a different number at different ε,ν quantiles.]

The Exogenous Impact Function 'Computed': The Big Result

$$
\begin{aligned}
\pi(\tau_\varepsilon, \tau_\nu, x, z) &= \nabla_{y_2} h_1(Q_{Y_2|XZ}(\tau_\nu, x, z), x, Q_\varepsilon(\tau_\varepsilon), Q_\nu(\tau_\nu)) \\
&= \nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z) \\
&\quad + \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z)}{\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z)}
\end{aligned}
$$

Why is this a big deal?

It says: Do 2 quantile regressions

1. Y_1 on Y_2, X, Z
2. Y_2 on X, Z

Differentiate and manipulate as above. Then you have computed the 'exogenous impact function.'

An Agenda

1. An example with concepts/variables (returns to education.)

2. Proof of the big result.

3. An example with symbols (i.e. that the computation produces the expression that you would expect in a simple model.)

4. Commentary.

EIF: An example with ‘concepts’

Taken from ‘Identification in Nonseparable Models’

$$
\begin{aligned}
W &= h_W(S, Z, F, A) \\
S &= h_s(Z, A)
\end{aligned}
$$

$W =$ log wage

$S =$ schooling (‘investment in schooling’, ‘academic achievement’)

$F =$ ‘fortune’ (how one fares in the labour market)

$A =$ ‘ability’

$Z =$ a list of covariates (e.g. parental income, quality of school)

Returns to schooling: $\nabla_S\, h_W(\cdot, \cdot, \cdot)$

More specifically:

$$
\begin{aligned}
W &= \theta(A)S + \gamma' Z + F \\
S &= \beta' Z + A
\end{aligned}
$$

But (or ‘and’) there exists some covariate i such that $\gamma_i = 0$ but $\beta_i \neq 0$.

The 'Formal' Proof

The system:

$$\begin{aligned} Y_1 &= h_1(Y_2, X, \varepsilon, \nu) \\ Y_2 &= h_2(X, Z, \nu) \end{aligned}$$

so there is Z_i such that (1) it is excluded from $h_1(\cdot,\cdot,\cdot)$ and (2) $\nabla_{Z_i} = Q_{Y_2|XZ}(\tau_\nu, x, z) \neq 0$.
From monotonicity of h_1 w.r.t. ε and continuity:

$$Q_{Y_1|\nu XZ}(\tau_\varepsilon, \nu, x, z) = h_1(h_2(x, z, \nu), x, Q_{\varepsilon|\nu XZ}(\tau_\varepsilon, \nu, x, z), \nu) \qquad \text{(A1)}$$

Since the conditional τ_e-quantile of ε is independent of ν, x, z :

$$Q_{Y_1|\nu XZ}(\tau_\varepsilon, \nu, x, z) = h_1(h_2(x, z, \nu), x, Q_\varepsilon(\tau_\varepsilon), \nu)$$

From monotonicity of h_2 in ν:

$$\begin{aligned} \nu &= g_2(X_1, ..., X_k, Z_1, ...Z_m, Y_2) \qquad \text{inverse function of } h_2 \qquad \text{(A2)} \\ \nu &= g_2(X, Z, Y_2) \qquad \text{(abbreviated version)} \end{aligned}$$

The 'Formal' Proof (2)

Conditioning on ν, X, Z is the same as conditioning on Y_2, X, Z : putting (A2) into (A1):

$$Q_{Y_1|\nu XZ}(\tau_\varepsilon, \nu, x, z) = h_1(h_2(x, z, \nu), x, Q_{\varepsilon|\nu XZ}(\tau_\varepsilon, \nu, x, z), \nu) \qquad \text{(A1)}$$

$$\begin{aligned} \nu &= g_2(X_1, ..., X_k, Z_1, ...Z_m, Y_2) \qquad \text{inverse function of } h_2 \qquad \text{(A2)} \\ \nu &= g_2(X, Z, Y_2) \qquad \text{(abbreviated version)} \end{aligned}$$

$$Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2, x, z) = h_1(y_2, x, Q_\varepsilon(\tau_e), g_2(x, z, y_2)) \qquad \text{(A3)}$$

Let $y_2^* = Q_{Y_2|XZ}(\tau_\nu, x, z)$. Thus ∇_{y_2}:

$$\begin{aligned} \nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2^*, x, z) &= \nabla_{y_2}(h_1(y_2^*, x, Q_\varepsilon(\tau_e), g_2(x, z, y_2^*))) + \qquad \text{(A4)} \\ &\quad \nabla_{y_2} g_2(x, z, y_2^*) \; \nabla_\nu h_1(y_2^*, x, Q_\varepsilon(\tau_e), g_2(x, z, y_2^*)) \end{aligned}$$

∇z_i of (A3):

$$\nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2^*, x, z) = \nabla_{z_i} g_2(x, z, y_2^*) \; \nabla_\nu h_1(y_2^*, x, Q_\varepsilon(\tau_e), g_2(x, z, y_2^*)) \qquad \text{(A5)}$$

The ‘Formal’ Proof (3)

Now for a constant value of ν :

$$0 = \nabla_{z_i} g_2(x, z, y_2^*) dz_i + \nabla_{y_2} g_2(x, z, y_2^*) dy_2 \tag{A6}$$

$$dy_2 = \nabla_{z_i} h_2(x, z, Q_v(\tau_v)) dz_i \tag{A7}$$

Monotonicity of h_2 w.r.t. ν implies

$$Q_{Y_2|XZ}(\tau_\nu, x, z) = h_2(x, z, Q_v(\tau_v))$$

A recap:

$$Q_{Y_1|Y_2XZ}(\tau_\varepsilon, y_2, x, z) = h_1(y_2, x, Q_\varepsilon(\tau_e), g_2(x, z, y_2)) \tag{A3}$$

The derivative of (A3) w.r.t. y_2 :

$$\nabla_{y_2} Q_{Y_1|Y_2XZ} = \nabla_{y_2}(h_1(y_2^*, \cdot, \cdot, \cdot)) + \nabla_{y_2} g_2(x, z, y_2^*) \ \nabla_\nu h_1(y_2^*, \cdot, \cdot, \cdot) \tag{A4}$$

‘something we can observe’ = ‘what we want’ + ‘something to work on’

The 'Formal' Proof (4)

$$dy_2 = \nabla_{z_i} h_2(x, z, Q_v(\tau_v))dz_i \tag{A7}$$

$$dz_i = \frac{dy_2}{\nabla_{z_i} h_2(x, z, Q_v(\tau_v))}$$

$$0 = \nabla_{z_i} g_2(\ )dz_i + \nabla_{y_2} g_2(\ )dy_2 \tag{A6}$$

So substituting for dz_i

$$0 = \frac{\nabla_{z_i} g_2(\ )dy_2}{\nabla_{z_i} h_2(\ )} + \nabla_{y_2} g_2(\ )dy_2$$

whence

$$\nabla_{y_2} g_2(\ ) = -\frac{\nabla_{z_i} g_2(\ )}{\nabla_{z_i} h_2(\ )} \tag{A8}$$

The ‘Formal’ Proof (5)

$$\nabla_{y_2} Q_{Y_1|Y_2XZ} = \nabla_{y_2}(h_1(y_2^*, \cdot, \cdot, \cdot)) + \nabla_{y_2} g_2(x, z, y_2^*)\ \nabla_\nu h_1(y_2^*, \cdot, \cdot, \cdot) \quad \text{(A4)}$$

‘something we can observe’ = ‘what we want’ + ‘something to work on’

So the first part of ‘something to work on’ is done; we still need $\nabla_\nu h_1(y_2^*, \cdot, \cdot, \cdot)$

$$\nabla_{y_2} g_2(\ ) = -\frac{\nabla_{z_i} g_2(\ )}{\nabla_{z_i} h_2(\ )} = \frac{-\nabla_{z_i} g_2(\ )}{\nabla_{z_i} Q_{Y_2|XZ}()} \quad \text{(A8)}$$

$$\nabla_{z_i} Q_{Y_1|Y_2XZ}() = \nabla_{z_i} g_2()\ \nabla_\nu h_1(), \qquad \text{so} \quad \text{(A5)}$$

$$\nabla_{y_2} g_2()\ \nabla_\nu h_1() = \frac{-\nabla_{z_i} g_2(\ )}{\nabla_{z_i} Q_{Y_2|XZ}()} \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}()}{\nabla_{z_i} g_2()}$$

$$\nabla_{y_2} Q_{Y_1|Y_2XZ} = \nabla_{y_2}(h_1(y_2^*, \cdot, \cdot, \cdot)) - \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}()}{\nabla_{z_i} Q_{Y_2|XZ}()}$$

The ‘Formal’ Proof Concluded

$$\nabla_{y_2} Q_{Y_1|Y_2XZ} = \nabla_{y_2}(h_1(y_2^*, \cdot, \cdot, \cdot)) - \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}()}{\nabla_{z_i} Q_{Y_2|XZ}()}$$

or

$$\nabla_{y_2}(h_1(y_2^*, \cdot, \cdot, \cdot)) = \nabla_{y_2} Q_{Y_1|Y_2XZ} + \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}()}{\nabla_{z_i} Q_{Y_2|XZ}()}$$

Reiteration: Catching Our Breath

We want to e.g. identify the 'eif' of schooling on log wages.

Our theorem sez:

$$\pi(\tau_\varepsilon, \tau_\nu, x, z) = \nabla_{y_2} h_1(Q_{Y_2|XZ}(\tau_\nu, x, z), x, Q_\varepsilon(\tau_\varepsilon), Q_\nu(\tau_\nu)) \quad (*)$$

$$= \nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z)$$

$$+ \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z)}{\nabla_{z_i} Q_{Y_2|XZ}(\tau_\nu, x, z)}$$

1. Pick out y_2^*, x, z.
2. Run $Q_{Y_2|XZ}$; y_2^* determines τ_ν.
3. Run $Q_{Y_1|Y_2XZ}$; at y_2^*, x, z pick $y_1 \Leftrightarrow \tau_\varepsilon$
4. Differentiate as indicated in (*).

Now do this globally: for all $x, z, y_2^*, \tau_\varepsilon$. Now you know 'everything'.

Examples with Symbols: (1)

The linear model : (example from eicqf pp. 7-8):

$$\begin{aligned} Y_1 &= \theta Y_2 + X'\beta_1 + \varepsilon + \lambda\nu \\ Y_2 &= X'\beta_2 + Z'\delta + \nu \end{aligned}$$

$$\begin{aligned} Q_{Y_1|Y_2XZ} &= \theta y_2 + x'\beta_1 + Q_\varepsilon(\tau_e) + \lambda(y_2 - x'\beta_2 + z'\delta) \\ &= (\theta + \lambda)y_2 + x'(\beta_1 - \lambda\beta_2) - z'\lambda\delta + Q_\varepsilon(\tau_e) \\ Q_{Y_2|XZ} &= x'\beta_2 + z'\delta + Q_\nu(\tau_\nu) \end{aligned}$$

We need 3 derivatives to apply the magic formula:

$$\begin{aligned} \nabla_{y_2} Q_{Y_1|Y_2XZ}(\tau_\varepsilon, Q_{Y_2|XZ}(\tau_\nu, x, z), x, z) &= \theta + \lambda \\ \nabla_{z_i} Q_{Y_1|Y_2XZ}(\cdot, \cdot \cdot, \cdot) &= -\lambda\delta_i \\ \nabla_{z_i} Q_{Y_2|XZ}(\cdot, \cdot \cdot, \cdot) &= \delta_i \end{aligned}$$

Do the magic calculation:

$$\nabla_{y_2} Q_{Y_1|Y_2XZ} + \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}}{\nabla_{z_i} Q_{Y_2|XZ}} = (\theta + \lambda) + \frac{-\lambda\delta_i}{\delta_i} = \theta$$

Examples with Symbols: (2)

Returns to schooling with a random coefficient structure (taken from pp. 6–7, 'Identification in nonseparable models')

$$\begin{aligned} W &= \theta(A)S + \gamma' Z + F \\ S &= \beta' Z + A \end{aligned}$$

$$\begin{aligned} Q_{W|SZ} &= \theta(s - \beta' z)s + \gamma' z + Q_{F|AZ}(\tau_F, s - \beta' z, z) \\ Q_{S|Z} &= \beta' Z + Q_{A|Z}(\tau_A, z), \quad \text{with } \gamma_i = 0 \text{ and } \beta_i \neq 0 \end{aligned}$$

Derivatives for the magic formula (at $\bar{\tau}_F, s^*, \bar{z}$):

$$\begin{aligned} \nabla_S Q_{W|SZ} &= \theta(s^* - \beta' \bar{z}) + \theta'(s^* - \beta' \bar{z})s^* + \nabla_A Q_{F|AZ}(\bar{\tau}_F, s^* - \beta' \bar{z}, \bar{z}) \\ \nabla_{z_i} Q_{W|SZ} &= -\beta_i \theta'(s^* - \beta' \bar{z})s^* + [\nabla_A Q_{F|AZ}(\bar{\tau}_F, s^* - \beta' \bar{z}, \bar{z})](-\beta_i) \\ \nabla_{z_i} Q_{S|Z} &= \beta_i \end{aligned}$$

Examples with Symbols: (2) (Concluded)

$$\begin{aligned}
\nabla_S Q_{W|SZ} &= \theta(s^* - \beta'\bar{z}) + \theta'(s^* - \beta'\bar{z})s^* + \nabla_A Q_{F|AZ}(\bar{\tau}_F, s^* - \beta'\bar{z}, \bar{z}) \\
\nabla_{z_i} Q_{W|SZ} &= -\beta_i\theta'(s^* - \beta'\bar{z})s^* + [\nabla_A Q_{F|AZ}(\bar{\tau}_F, s^* - \beta'\bar{z}, \bar{z})](-\beta_i) \\
\nabla_{z_i} Q_{S|Z} &= \beta_i
\end{aligned}$$

The magic formula:

$$\begin{aligned}
& \theta(s^* - \beta'\bar{z}) + \theta'(s^* - \beta'\bar{z})s^* + \nabla_A Q_{F|AZ}(\bar{\tau}_F, s^* - \beta'\bar{z}, \bar{z}) \\
& + \frac{-\beta_i\theta'(s^* - \beta'\bar{z})s^* + [\nabla_A Q_{F|AZ}(\bar{\tau}_F, s^* - \beta'\bar{z}, \bar{z})](-\beta_i)}{\beta_i} \\
= \; & \theta(s^* - \beta'\bar{z}) + \theta'(s^* - \beta'\bar{z})s^* + \nabla_A Q_{F|AZ}(\bar{\tau}_F, s^* - \beta'\bar{z}, \bar{z}) \\
& -\{\theta'(s^* - \beta'\bar{z})s^* + [\nabla_A Q_{F|AZ}(\bar{\tau}_F, s^* - \beta'\bar{z}, \bar{z})]\} \\
= \; & \theta(s^* - \beta'\bar{z})
\end{aligned}$$

Discussion

1. The method can be extended to the 'triangular' or 'recursive' case:

$$
\begin{aligned}
Y_1 &= h_1(Y_2, Y_3, ..., Y_m, Z, \varepsilon_1, \varepsilon_2, ..., \varepsilon_m) \\
Y_2 &= h_2(Y_3, ..., Y_m, Z, \varepsilon_2, ..., \varepsilon_m) \\
&\vdots \\
Y_m &= h_m(Z, \varepsilon_m)
\end{aligned}
$$

   N.B. That an equivalent representation can be constructed by recursively substituting for $Y_m, Y_{m-1}, ..., Y_3$ so that (apparently) if one is interested in a single 'structural effect' the two equation system will do.

2. There is an ambiguity in our answer:

$$\nabla_{y_2} Q_{Y_1|Y_2XZ}() + \frac{\nabla_{z_i} Q_{Y_1|Y_2XZ}()}{\nabla_{z_i} Q_{Y_2|XZ}()}$$

   because there maybe (usually is!) more than one candidate for z_i. Then the EIF is 'overidentified'. (Discussion).

3. Our ability to compute the magic formula clearly depends on $\nabla_{z_i} Q_{Y_2|XZ}() \neq 0$, a condition which may obtain for some X, Z values and not others. Indeed, all the steps of the derivation may hold only 'locally.' Thus e.g. effects at some ε, ν quantiles may be identifiable while others are not. (This is discussed in 'Identification in nonseparable models' (2003) and the 2005 paper (originally 'Instrumental values').)

4. Notice that if you have 'the effect' at all quantiles you can get the 'average effect.'

5. The meaning of it all (changes in Y_2 induced by changes in instrument.)