Complex networks and local externalities: a strategic approach

Andrea Galeotti[*] Fernando Vega-Redondo[†]

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Abstract

In this paper, we illustrate a new approach to the study of how local externalities shape agents' strategic behavior when the underlying network is volatile and complex. We consider a large population who interacts as specified by a random network with a given degree distribution. Motivated by the complexity of the induced network, we assume that the only precise information agents have is local, i.e. it is restricted to their immediate neighborhood. Each agent chooses an investment level, which in turn imposes a payoff externality on her neighbors that is captured by a (local) Cobb-Douglas production/payoff function. We find that, in the unique interior equilibrium, the induced externality is positive or negative, depending on whether investment costs are (respectively) above or below a certain threshold. This also has implications on the nature of the equilibrium strategy, which is increasing in the degree, in the first case, and decreasing in the second. Finally, we also characterize how the equilibrium changes when the network topology varies and becomes more connected, or when its degree distribution becomes more polarized.

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[*]University of Essex, Department of Economics; email: agaleo@essex.ac.uk.

[†]Corresponding author, European University Institute, Department of Economics, and Instituto Valenciano de Investigaciones Económicas; email: fvega@eui.eu.
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1 Introduction

Local externalities are a phenomenon of great significance in a wide range of different contexts and the performance of an economic system hinges upon how agents respond to them. Local externalities are important, for example, in problems of human capital accumulation, learning and search, crime, productivity and growth, technological adoption, R&D collaboration.[1] By their very nature, local externalities greatly depend on the pattern of interaction, i.e. the social network. It is essential, therefore, to understand in detail the interplay between the topology of network and agents' incentives.

There are, however, few papers that explore this issue systematically and in some generality (see below for a summary). The main objective of this paper is to suggest an approach to the study of local effects in a context where the social network is complex, volatile, and agents have only local information about it. We say that the network is *complex* since its architecture displays substantial heterogeneities and no clear patterns. We posit that agents have only *local information* about the network since they are taken to know how many first neighbors they have but ignore the number of second neighbors (i.e. neighbors of neighbors). Finally, the network is best conceived as *volatile* (i.e. with links being short-lived), so players can only use probabilistic information on ex-ante regularities; in particular, they have no time to learn the type or behavior of their neighbors.

To accommodate these considerations in a strategic game, we use the framework provided by the theory of random networks.[2] In essence, a random network is to be conceived as a stochastic ensemble, i.e. a probability measure (typically uniform) defined on a given family of possible networks. This family is usually characterized in terms of certain overall properties such as a particular degree distribution, degree correlations, or clustering. The

[1] There is a vast empirical literature on peer effects on human capital accumulation (see e.g. Summers and Wolfe (1977), Henderson et al. (1978), Coleman et al. (1966), Glaeser *et al.* (1996)). There is also a large literature on the effects of local externalities on crime (Glaeser *et al.* (1996), Calvo *et al.* (2005)); productivity and growth (Glaeser *et al* (1992), Durlauf (1993), Ciccone and Hall (1996)); or technological adoption (Coleman (1988), Valente (1996), Conley and Udry (2000), Rogers (1962)).

[2] This theory has its precursor in the work of Erdös and Rényi (1959, 1960), who started their fruitful collaboration on this topic in the late 1950's. In recent times, this theory has been much extended to become a powerful tool in the study of large and complex networks (see Albert and Barabási (2002), Newman (2003), and Vega-Redondo (2007) for exhaustive surveys).

basic postulate is then that, while all eligible networks satisfy the properties required, the specific network realized is uncertain.

In our framework players are connected through a social network, whose statistical properties are solely characterized by a degree distribution. Each player knows this distribution and her own degree (which can be conceived as her "type"). With this information at hand, every player has to choose her costly effort. The equilibrium decision so taken by each player must depend on a number of factors. First, it has to reflect the intensity of interaction of the player in question, i.e. her degree. Second, it ought to hinge upon the overall distribution of types prevailing in the population. Finally, it must be shaped by the precise nature of local externalities. While our analysis is fully general in terms of the underlying degree distribution, concerning payoffs we focus on a paradigmatic case where an agent's gross payoffs are given by a Cobb-Douglas function of all efforts (or investments) displayed by herself and her neighbors and individual costs are quadratic. An interesting feature of this formulation is that the nature of the network externalities becomes an endogenous outcome of the model. Thus, whether the externalities induced by neighbors are positive or negative depends on equilibrium play – in particular, on whether their effort is high or low.

To understand better this feature of our model, let us fix ideas and conceive *interaction* as the mechanism through which agents accumulate human capital: an agent's level of skills and education is the result of combining her own effort with that of her partners. Our payoff formulation implies that, for a *given* set of those partners, the investment in human capital displays strategic complementarities. However, the effect of changes in the *number* of partners crucially depends on whether players exert high or low effort. In the former case, an increased number of partners yields positive externalities whereas it induces negative ones in the latter case. In this light, the following questions naturally suggest themselves. Under what conditions does players' interaction induce high effort and, therefore, positive spillover effects on one's own (investment in) education? Do more connected societies support a higher level of human capital, or quite the opposite? Are societies more homogeneous (in player connectivity) better-positioned to induce individual to invest more in human capital accumulation? Or, alternatively, do more polarized societies generate higher incentives to do so?

It is easy to see that the model always allows for a trivial noninterior equilibrium where *every* agent, independently of her degree, exerts zero effort. Naturally, our interest is on equilibria where agents play nontrivially as a function of their degree. In order to guarantee that any such equilibrium is interior (and thus can be characterized through marginal conditions), it is convenient to posit that the range of possible effort/investment is unbounded. Then we can show that there always exists a unique nontrivial equilibrium. In this equilibrium, there is a clear relation between effort profile, expected utilities, players' network degree, and investment costs. When the cost of effort is low (resp. high), equilibrium efforts and equilibrium utilities are declining (resp. increasing) in the degree. In this case, agents impose negative (resp. positive) externalities on their neighbors. The intuitive basis for this conclusion can be explained as follows. When investing in effort is relatively cheap, any effort profile in which neighbors generate positive externalities induce players to exert always additional effort. This positive "social-multiplier effect" is inconsistent with an interior equilibrium. So, in this case, neighbors must generate negative externalities at equilibrium and, consequently, the more connected a player is the lower her incentive to invest.

Our second set of results compare equilibria across different networks. We start by comparing degree distributions that are ranked according to the criterion of First Order Stochastic Dominance (FOSD). This amounts to comparing networks whose respective levels of connectivity can be unambiguously ordered. Then, as network connectivity is thus increased, we find that individual efforts uniformly adjust upward when costs are low, while they uniformly adjust downward otherwise. To understand the intuition, note that higher network connectivity in this sense simply implies that each player increases her probability of interacting with more connected players. This strengthens the negative (positive) externalities imposed on players by their neighbors when costs are low (high), thus leading players to uniformly increase (decrease) effort to offset such an effect.

Finally, to understand the effects of network polarization, we compare networks that have the same average connectivity but differ in the way the links (always the same number of them) are allocated across players. Specifically, we consider degree distributions that can be ordered according to the Mean Preserving Spread criterion. We show that efforts are systematically lower in networks displaying broader degree distributions. The intuition here relies on the curvature of the equilibrium strategies. Since equilibrium effort happens to be given by a convex function of the degree, whenever the degree distribution turns broader (keeping the same mean) each link becomes, on average, more valuable. At equilibrium, therefore, players are led to exerting lower effort in order to compensate for this effect.

Local knowledge, network complexity, and network volatility are the three aspects that distinguish our paper from the existing literature on local externalities. In this literature, there are two polar approaches which have been used to model local effects. One approach posits that agents interact with their neighbors in a *fixed* (or relatively stable) socio-economic network, whose architecture is common knowledge.[3] Hence, players know who are their neighbors, the neighbors of their neighbors, etc., and can fully analyze the situation as a game of complete information. In the second approach, the specific pattern of interaction is not known beforehand but each agent anticipates that she will be interacting with some fixed number of individuals randomly sampled from the whole population. Thus, even though players do not enjoy a precise knowledge of the realized network, social interaction has an extremely simple structure and players confront, *ex-ante*, a fully *symmetric* social environment.[4]

As compared to these two branches of literature, our approach introduces an incomplete-information scenario that seems better suited to understanding strategic behavior in large and typically complex social networks in the real world. Furthermore, it delivers predictions that are markedly different from those obtained in models where the network architecture is common knowledge.

The present approach was first explored in our original version of this paper, Galeotti and Vega-Redondo (2005), where we focused on scenarios given by three paradigmatic families of (parametrized) degree distributions: Poisson, Geometric, or Scale-Free. In that paper we characterized the equilibria for these three scenarios and also conducted some basic exercises of comparative statics. A development of this approach has been recently undertaken by Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2009). They carry out a general study of how the network topology impinges on strategic behavior in games where payoffs display what they call either degree complements or degree substitutes.[5] Their payoff formulation, however, is crucially different

[3]This is the framework considered in much of the theory of networks – c.f. Bramoulle and Kranton (2004), Calvó and Zenou (2004), Galeotti (2009), Goyal and Moraga-Gonzalez (2001). A good survey of this literature can be found in Jackson (2005).

[4]This scenario is present in much of the theory of evolution and learning (Weibull (1995), Vega-Redondo (1996), Young (1998), Fudenberg and Levine (1998)); the literature on bargaining in population environments (Rubinstein and Wolinsky (1985), Gale(1987)); or the study of how social norms arise in large populations (Kandori (1992), Okuno-Fujiwara and Postlewaite (1995)).

[5]Heuristically, these notions are appropriate extensions of the usual notions of strategic

from the present one. For example, in their model, whether there is degree complementarity or substitutability is an exogenous assumption while, in our case, it is (as advanced above) an endogenous outcome of the equilibrium. We borrow, however, from Galeotti *et al.* (2009) the tools used to compare different degree distributions – essentially, first- and second-order dominance criteria. More specifically, our comparative-static analysis applies that methodology to a setup that generalizes to *any* random network the analysis we originally undertook for specific contexts in Galeotti and Vega-Redondo (2005).

The rest of the paper is organized as follows. Section 2 presents the general theoretical framework. Section 3 specializes this general framework to a context with strategic complementarities. Section 4 concludes. The detailed proof of the results is relegated to the Appendix.

2 General framework

There is a countable infinity of players, N, who meet randomly. Each agent $i \in N$ meets a number of other agents, as determined by her degree k_i. We denote the degree distribution by $\mathbf{P}$. We assume that $\mathbf{P}$ is fixed, as given by a probability density

$$\mathbf{P} = \{p_k\}_{k=0}^{\bar{k}} \tag{1}$$

where each p_k denotes the fraction of individuals who have k neighbors. For simplicity, we suppose that $\bar{k}$ is finite, but nothing important depends on that – see Example 1 where we consider an example where the degree distribution has an infinite support.

Related to $\mathbf{P}$, we need to consider the degree distribution of a so-called "neighboring node," i.e. a node that is selected as the neighbor of some randomly selected node. Let this distribution be denoted by $\tilde{\mathbf{P}} = \{\tilde{p}_k\}_{k=1}^{\bar{k}}$. Since the probability of "finding" any such a node is proportional to its degree, it follows (cf. Newman (2003)) that

$$\tilde{p}_k = \frac{p_k k}{\sum_{l'=1}^{\bar{k}} p_{l'} l'}, \tag{2}$$

i.e. the frequency of finding any (neighboring) node with degree k is proportional to the product $p_k k$ of the frequency of those nodes in the population

complementarity and substitutability, coupled with an assumption on how payoffs change as the interaction involves a varying number of players.

and the degree (which determines the number of alternative routes that lead to each of them).

Players interact with each other as determined by the prevailing social network. This network is chosen equiprobably from all networks that display the given degree distribution $\mathbf{P}$. Ex ante, therefore, we are in the presence of a random network, which is simply defined through a uniform probability measure on the family of networks characterized by the given degree distribution.

Each player i knows her own degree k_i (i.e. the number of players she will meet) but ignores the degree of these players. The overall degree distribution, however, is common knowledge. Prior to interaction, each individual i has to choose an effort (or investment) level $x_i \in \mathbb{R}_+$. This choice can be tailored to her degree k_i (which she knows) but cannot depend on the identity, degree, or behavior of each of her future k_i partners (all of which she ignores). Given the profile of effort levels $[x_i, (x_j)_{j \in N_i}]$ chosen by player i and each of the k_i agents in her neighborhood N_i, the payoff earned by player i is given by:

$$\pi_i[x_i, (x_j)_{j \in N_i}] = f[x_i, (x_j)_{j \in N_i}] - c(x_i), \tag{3}$$

where, assuming ex-ante symmetry across players,

$$f : \mathbb{R}_+ \times \left[\bigcup_{m=0}^{\bar{k}} \mathbb{R}_+^m \right] \to \mathbb{R}_+$$

stands for the (symmetric)[6] gross payoff function of each player, and

$$c : \mathbb{R}_+ \to \mathbb{R}_+$$

is the cost function for individual effort. For presentational convenience, we posit that the effort levels of agents are *a priori* unbounded. It will be clear, however, that our results (e.g. Theorem 1 below) only require that they can be assumed to lie in some compact interval $[0, M]$ where, given the remaining parameters, M is large enough. This guarantees that an interior equilibrium always exists.

Consider any given agent with degree k who has to choose her effort level before knowing her future partners' characteristics. We posit that every such

[6]The function f is symmetric in the sense of being independent to any permutation in its arguments.

agent chooses an effort level x so as to maximize the expected value of (3) induced by the probability density $\tilde{\mathbf{P}}$ and some *predicted* degree-contingent (symmetric) strategy

$$\hat{\mathbf{x}} = \{\hat{x}(k')\}_{k'=1}^{\bar{k}}$$

that specifies how every other individual, depending on her degree k', is anticipated to choose her effort level. We denote by $\psi_k(x, \hat{\mathbf{x}})$ the expected payoff function embodying the aforementioned considerations for an agent of degree k choosing effort x.

To provide a precise specification of $\psi_k(x, \hat{\mathbf{x}})$, we need to introduce some additional notation. First, for any given player with degree $\kappa = 1, 2, ..., \bar{k}$, let

$$S_k \equiv \left\{ r = (r_1, r_2, ..., r_{\bar{k}}) \in (N \cup \{0\})^{\bar{k}} : \sum_{l=1}^{\bar{k}} r_l = k \right\}$$

with the following interpretation: each vector $r = (r_1, r_2, ..., r_{\bar{k}}) \in S_k$ specifies, for each $l = 1, 2, ..., \bar{k}$, a corresponding number of the player's neighbors that have degree l. Naturally, only those sequences for which $\sum_{l=1}^{\bar{k}} r_l = k$ are valid, since the agent in question is taken to have k neighbors. For any one of such neighbors, who is randomly chosen from the overall population, her degree is k' with a probability $\tilde{p}_{k'}$ given by (2). (This follows from the fact that, as explained above, the suitable degree distribution in this case is that of a *neighboring* node.) Therefore, the distribution induced on each $r \in S_k$ follows a multinomial distribution given by:

$$P_k(r) = \frac{\kappa!}{r_1! r_2! ... r_{\bar{k}}!} \prod_{l=1}^{\bar{k}} \tilde{p}_l^{r_l} \tag{4}$$

In terms of these probabilities, the expected-payoff function $\psi_k(x, \hat{\mathbf{x}})$ can be formally defined as follows:[7]

$$\psi_k(x, \hat{\mathbf{x}}) = \sum_{r \in S_k} P_k(r) \; f[x; \underbrace{\hat{x}(1), ..., \hat{x}(1)}_{r_1 \text{ times}}, ..., \underbrace{\hat{x}(\bar{k}), ..., \hat{x}(\bar{k})}_{r_{\bar{k}} \text{ times}}] - c(x) \tag{5}$$

Then, as customary, we say that a profile $\mathbf{x}^* = \{x^*(k)\}_{k=1}^{\bar{k}}$ defines a

[7] The expected-payoff function of a player with $k = 0$ is $\psi_{k=0}(x, \bar{x}) = f(x, \varnothing) - c(x)$.

(symmetric) *Nash equilibrium* strategy if it satisfies:[8]

$$x^*(k) \in \arg\max_{x\in\mathbb{R}_+} \psi_k(x, \mathbf{x}^*) \qquad (k = 0, 1, ..., \bar{k}). \tag{6}$$

Note that this equilibrium can also be regarded as a Bayes-Nash equilibrium of a (Bayesian) incomplete-information game where the type space of every agent coincides with the set of possible degrees and their beliefs on the types of others are induced by $\tilde{\mathbf{P}}$. In this sense, the type and beliefs of an agent define her perception of the local topology of interaction she faces.

3 Strategic Complementarities

We now restrict attention to the case in which individuals' efforts are strategic complements. More specifically, we posit that the *gross* payoff of a player is the product of her own efforts and the efforts exerted by each of her neighbors. On the other hand, we suppose that the agent's investment cost is quadratic, the magnitude of these costs being parametrized by some $\alpha > 0$. Combining both payoff components (gross payoffs and costs), and relying on (4), the expected *net* payoffs for an agent with k neighbors can be written as follows:

$$\psi_k(x, \hat{\mathbf{x}}) = \sum_{r\in S_\kappa} \left\{ \frac{\kappa!}{r_1!r_2!...r_{\bar{k}}!}\; x \; \left[\prod_{k'=1}^{\bar{k}} \left[\tilde{p}_{k'} \cdot x\left(k'\right)\right]^{r_{k'}} \right] \right\} - \frac{\alpha}{2}x^2. \tag{7}$$

To compute the equilibria in this case, note that the functions $\psi_k(x, \hat{\mathbf{x}})$ are obviously differentiable with respect to x. Thus, the conditions (6) that define a symmetric Nash equilibrium $\mathbf{x}^* = \{x^*(k)\}_{k=1}^{\bar{k}}$ imply:

$$\left.\frac{\partial \psi_k}{\partial x}(x, \mathbf{x}^*)\right|_{x=x^*(k)} \leq 0 \qquad (k = 0, 1, ..., \bar{k}) \tag{8}$$

which, assuming interiority and using (7), yields:

$$\alpha x^*(k) = \left(\sum_{k'=1}^{\bar{k}} \tilde{p}_{k'} x^*\left(k'\right) \right)^k \qquad (k = 1, 2, ..., \bar{k}). \tag{9}$$

[8] Often, for simplicity, we shall speak of $\mathbf{x}^*$ as an "equilibrium" although, strictly speaking, it is only the common strategy played by every player in a symmetric equilibrium.

Particularizing (9) for $k = 1$ we have:

$$\alpha x^*(1) = \sum_{k'=1}^{\bar{k}} \tilde{p}_{k'} x^*(k')$$

which can then be introduced again in (9) for $k \geq 2$ to obtain:

$$x^*(k) = \frac{1}{\alpha} [\alpha x^*(1)]^k, \qquad (k = 2, 3, ..., \bar{k}), \tag{10}$$

while the value of $x^*(1)$ can then be solved from the equation

$$\frac{1}{\alpha} \sum_{k'=1}^{\bar{k}} \tilde{p}_{k'} [\alpha x^*(1)]^{k'-1} = 1. \tag{11}$$

Since second-order conditions are obviously satisfied in this context, (10)-(11) can be used to characterize the interior Nash equilibria. This we do in the next subsection, where we also explore how they depend on the parameters of the environment.

3.1 Equilibria

In general, of course, Nash equilibria depend on the degree distribution of the network. It is clear, however, that, regardless of this distribution, there always exists a symmetric Nash equilibrium where *no* player exerts *any* effort at all. Note as well that there are not any asymmetric equilibria, i.e. equilibria where players with the same degree choose a different effort. This follows from the fact that the expected utility of every player is strictly concave in her own choice. This observation also rules out symmetric mixed strategy equilibria. Our main concern, therefore, is to identify conditions under which symmetric pure strategy interior equilibria exist, i.e. equilibria in which connected players exert positive effort.

Let z_P be the average degree associated to the degree distribution $\mathbf{P}$, i.e. $z_P = \sum_{k=1}^{\bar{k}} k p_k$. The next result provides a complete qualitative characterization of symmetric interior equilibria.

Theorem 1 *Consider any degree distribution* $\mathbf{P}$*. A symmetric interior equilibrium exists if and only if* $\alpha > \frac{p_1}{z_P}$*. In this case, there exists a unique symmetric interior equilibrium that displays the following properties:*

(a) $x_k^* < x_{k+1}^*$ *and* $\psi_k^* < \psi_{k+1}^*$ *for all* $k = 0, 1, ...\bar{k}$ *if and only if* $\alpha > 1$.

(b) $x_k^* = x_{k+1}^*$ *and* $\psi_k^* = \psi_{k+1}^*$ *for all* $k = 0, 1, ...\bar{k}$ *if and only if* $\alpha = 1$.

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Theorem 1 establishes a clear relation between equilibrium efforts, equilibrium expected utilities, players' network degree and costs of investment. Interior equilibria exists (and are unique) if, and only if, effort costs are sufficiently high. In general the higher is the average connectivity of a network the wider the range of costs that allows for an interior equilibrium. And when such an equilibrium exists, a somewhat paradoxical conclusion obtains: if costs are high, equilibrium efforts are increasing with the player's degree, whereas the opposite happens if costs are low.

The intuition for this equilibrium pattern can be explained as follows. When effort is not very costly, an increasing equilibrium strategy would generate a snow-ball effect that, in the end, would be incompatible with the self-consistency requirement of an interior Nash equilibrium. The alternative for the population must then be to settle on a situation where externalities are negative and thus more connected players are worst hit by them. The situation, therefore, becomes one where those that are in the best position to generate positive externalities are not provided with the necessary incentives to do so. This mechanism reinforces itself, thus leading to an obviously inefficient allocation of efforts.

It is easy to check that the same monotonicity properties that characterize the equilibrium strategy also hold for equilibrium expected utilities. Combining this observation with the monotonicity displayed by the equilibrium strategy, it follows that, regardless of costs, the players who obtain higher expected utilities are always the ones who exert higher effort – i.e. the less connected players when costs are low and the more connected ones in the opposite case.

Before turning to the comparison of equilibria across networks with different degree distributions we provide two examples that may help illustrate the relevant features of the analysis. The first example involves the canonical degree distribution in the theory of random networks: the Poisson distribution, which arises from the original model of random connectivity proposed by Erdös and Rényi (1960). This example is intended to illustrate that the explicit computation of equilibrium strategies is often simple in our framework and that the finite-support assumption adopted throughout can be dispensed with. The second example illustrates that the incomplete-information assumption characterizing our approach is key in delivering some of the conclusions. In particular, it yields the equilibrium monotonicity established in Theorem 1, which does not generally arise if the prevailing network architecture is common knowledge.

Example 1 *Poisson networks*

Let the network degree be Poisson distributed. Then, the degree distribution $\mathbf{P} = \{p_k\}_{k=1}^{\infty}$ is given by:

$$p\,(k) = \exp(-z)\frac{z^k}{k!} \qquad (k = 0, 1, 2, ...) \tag{12}$$

where z is the average network degree. Correspondingly, the degree distribution of a neighboring agent $\tilde{\mathbf{P}} = \{\tilde{p}_k\}_{k=1}^{\infty}$ is given by:

$$\tilde{p}\,(k) = \frac{p\,(k)\,k}{\sum_{k'=0}^{\infty} p'_{k'} k'} = \exp(-z)\frac{z^{k-1}}{(k-1)!} \qquad (k = 1, 2, 3, ...) \tag{13}$$

We can rewrite (11) as follows:

$$\frac{1}{\alpha^2 x^*\,(1)} \sum_{k'=1}^{\infty} \tilde{p}_{k'} (\alpha x^*\,(1))^{k'} = \frac{1}{\alpha^2 x^*\,(1)} G_1(\alpha x^*\,(1)) = 1, \tag{14}$$

where $G_1(\cdot)$ is the generating function of $\tilde{\mathbf{P}}$, the degree distribution of a neighboring node. Denote by $G_0(x) = \sum_{k'=0}^{\infty} p_{k'} x^{k'}$ the generating function of the original degree distribution $\mathbf{P}$. It is easy to verify that

$$G_1(x) = \frac{x G'_0(x)}{G'_0(1)}$$

and therefore

$$G_1(\alpha x^*\,(1)) = \frac{\alpha x^*\,(1)\, G'_0(\alpha x^*\,(1))}{z}.$$

Since in the case of a Poisson distribution we have:

$$G'_0(x) = z \exp(z(x-1))$$

we can write expression (14) as follows:

$$\frac{1}{\alpha}\exp(z(\alpha x^*(1)-1)) = \frac{\exp(z\alpha x^*(1))}{\alpha\exp(z)} = 1,$$

and solving for $x^*(1)$ we obtain:

$$x^*(1) = \frac{\ln\alpha + z}{\alpha z}$$

Therefore:

$$x^*(k) = \frac{1}{\alpha}\left(\frac{\ln\alpha + z}{z}\right)^k, \; k = 0, 1, ... \tag{15}$$

We now must require that for any possible degree k, $x^*(k) > 0$. This holds if, and only if, $\ln\alpha + z > 0$, which is satisfied if, and only if,

$$\alpha > \frac{1}{\exp(z)} = \frac{z\exp(-z)}{z} = \frac{p_1}{z}.$$

Finally, to verify that the equilibrium strategy defined in (15) satisfies the general properties established in (a)-(c) of Theorem 1 can be verified directly in a straightforward fashion.

Example 2 *Common knowledge of the network*

Suppose that there are seven players. Player 1 is linked to player 2. On the other hand, the "odd players" $3, 5$, and 7 only have one link to player 1, while the "even players" 4 and 6 have only one link to player 2. Thus, the degree of player 1 is $k_1 = 4$, the degree of player 2 is $k_1 = 3$, and $k_j = 1$ for all $j = 3, ..., 7$. It is immediate to compute that, under complete information and payoff functions as specified by our model, the unique symmetric and interior Nash equilibrium $\hat{x} = (\hat{x}(i))_{i=1}^7$ has $\hat{x}(1) = \alpha$, $\hat{x}(2) = \alpha^2$, $\hat{x}(i) = 1$ for $i = 1, 3, 5$, and $\hat{x}(j) = \alpha$ for $j = 2, 4$. Thus, *both* degree-symmetry and degree-monotonicity are violated for *any* value $\alpha \neq 1$ of the cost parameter.

3.2 Comparative Statics

We now investigate whether there are systematic relationships between equilibria across different topologies of social interaction, here identified with different degree distributions of the underlying random network. We first analyze the impact of an increase in the connectivity of a network on the equilibrium effort profile and expected utilities. In our setup, a natural way of doing this is to compare equilibria across degree distributions that can be ranked by the First Order Stochastic Dominance (FOSD) criterion.[9] Let us denote by $x^*_{\mathbf{P}}(k)$ the effort of a player with degree k in the interior equilibrium corresponding to the degree distribution $\mathbf{P}$. Similarly, denote by $\psi^*_{\mathbf{P}}(k)$ the expected equilibrium utility to a player k corresponding to $\mathbf{P}$.

Proposition 1 *Consider two degree distributions* $\mathbf{P}$ *and* $\mathbf{P}'$*. Assume that* $\alpha > \max[\frac{p_1}{z_{\mathbf{P}}}, \frac{p'_1}{z_{\mathbf{P}'}}]$*. If* $\tilde{\mathbf{P}}'$ *FOSD* $\tilde{\mathbf{P}}$ *(where* $\tilde{\mathbf{P}}$ *and* $\tilde{\mathbf{P}}'$ *are the corresponding neighboring-node distributions) then:*

(i) if $\alpha > 1$ *then* $x^*_{\mathbf{P}'}(k) < x^*_{\mathbf{P}}(k)$ *and* $\psi^*_{\mathbf{P}'}(k) < \psi^*_{\mathbf{P}}(k)$ *for all* $k = 1, ..., \bar{k}$.

(ii) if $\alpha = 1$ *then* $x^*_{\mathbf{P}'}(k) = x^*_{\mathbf{P}}(k)$ *and* $\psi^*_{\mathbf{P}'}(k) = \psi^*_{\mathbf{P}}(k)$ *for all* $k = 1, ..., \bar{k}$.

(iii) if $\alpha < 1$ *then* $x^*_{\mathbf{P}'}(k) > x^*_{\mathbf{P}}(k)$ *and* $\psi^*_{\mathbf{P}'}(k) > \psi^*_{\mathbf{P}}(k)$*, for all* $k = 1, ..., \bar{k}$.

Proposition 1 establishes that an increase in connectivity has a qualitatively different effect on equilibrium effort levels depending on the cost of effort. In essence, this result reflects the idea that individual efforts must adjust upward or downward, uniformly, so as to *offset the change in the externality* that is induced by a shift in the connectivity distribution of neighbors. To understand the intuition underlying such an "equilibrating adjustment," let us focus on the case in which effort costs are low. Under these conditions, highly connected players are those who display a lower effort. Then, if the degree distribution shifts in the FOSD sense with an increasing probability for higher-degree neighbors, the negative effect of the externality channeled through every link is reinforced. If equilibrium is to be restored, this effect must be mitigated through a uniform increase in the effort levels.

We now turn to analyzing the impact of allocating (the same number of) links in a more or less disperse manner, while keeping the average connectivity constant. Formally, we shall formalize this by comparing degree

[9] $\mathbf{P}$ is said to FOSD $\mathbf{P}'$ if, for all $\ell = 0, 1, 2, ..., \bar{k}$, we have $\sum_{k=0}^{\ell} p_k \leq \sum_{k=0}^{\ell} p'_k$.

distributions that can be ordered according to the Mean Preserving Spread (MPS) criterion.[10]

Proposition 2 *Consider two degree distributions* $\mathbf{P}$ *and* $\mathbf{P}'$*. Assume that* $\alpha > \frac{p_1}{z_{\mathbf{P}}}$*. If* $\tilde{\mathbf{P}}'$ *is a MPS of* $\mathbf{P}$ *(where* $\tilde{\mathbf{P}}$ *and* $\tilde{\mathbf{P}}'$ *are the corresponding neighboring-node distributions) then:*

(i) if $\alpha \neq 1$ *then* $x^*_{\mathbf{P}'}(k) < x^*_{\mathbf{P}}(k)$ *and* $\psi^*_{\mathbf{P}'}(k) < \psi^*_{\mathbf{P}}(k)$ *for all* $k = 1, ..., \bar{k}$.

(ii) if $\alpha = 1$ *then* $x^*_{\mathbf{P}'}(k) = x^*_{\mathbf{P}}(k)$ *and* $\psi^*_{\mathbf{P}'}(k) = \psi^*_{\mathbf{P}}(k)$ *for all* $k = 1, ..., \bar{k}$.

Proposition 2 indicates that when the neighboring-node degree distribution spreads out (while the average connectivity is kept constant) players uniformly choose lower efforts at equilibrium, irrespectively of the level of costs. This effect results from the curvature of the equilibrium strategy, which is convex in the degree both for low- and high-effort costs. Hence an increase in the dispersion of the distribution leads to an increase in the effort that any given player expects from each of her neighbors. Thus, when costs are low the entailed *negative* externalities are less acute, while in the case when costs are high the induced *positive* externalities are stronger. In both cases, therefore, the effort levels must fall uniformly at equilibrium in order to offset that effect.

Proposition 1 and 2 elucidate an interesting contrast between the changes in the degree distribution conducted according to the FOSD and MPS criteria. The FOSD-based changes shift the connectivity distribution of neighbors and thus their effect on the externality (positive or negative) depends on the slope (positive or negative) of the equilibrium strategy. Instead, changes which are MPS-based keep the average connectivity of neighbors fixed and have the effect on the externality depend on the curvature of the equilibrium strategy. The fact that this strategy is always convex (independently of α) explains why the effect of any such change always affects equilibrium behavior in the same direction, independently of costs.

[10] There are different equivalent ways of formulating the MPS criterion. For example, we may say that $\mathbf{P}'$ is a MPS of $\mathbf{P}$ if they both have the same mean ($\sum_{k=0}^{\bar{k}} kp_k = \sum_{k=0}^{\bar{k}} kp'_k$) and, moreover, $\mathbf{P}'$ is dominated by $\mathbf{P}$ in the Second-Order Stochastic sense, i.e. for all $m = 0, 1, 2, ..., \bar{k}$, $\sum_{\ell=0}^{m} \sum_{k=0}^{\ell} p_k \leq \sum_{\ell=0}^{m} \sum_{k=0}^{\ell} p'_k$.

4 Summary and conclusions

In this paper, we propose a new framework geared towards the study of local effects when players have partial information about the pattern of interaction and the topology of social interaction is both complex and volatile. We provide an application of this framework for a case of strategic complements. The equilibrium predictions of the model are sharp and illustrate a rich and subtle interplay between the network topology and strategic individuals' behavior.

Our approach to the study of strategic interaction in network setups attempts to make some progress over much of the existing network literature in the following two important, and related, respects.

First, it does not shun contexts where the underlying social network displays significant interagent heterogeneity and substantial topological complexity. These two features – a mark of many interesting social networks in the real world – are accommodated by modelling the system as a large stochastic system that, despite its intrinsic complexity, displays *given* overall *statistical* regularities. The analysis may then rely on the versatile tools afforded by the modern theory of complex systems.

Second, in view of such network complexity, players are postulated to hold only imprecise information on their individual circumstances (i.e. the type of their neighbors), although they all share the same global information (accurate but "anonymous") on the whole network. Interestingly, these natural informational constraints reduce the vast multiplicity of equilibria that are typically found in many network models under complete information, and allow for definite theoretical predictions as well as clear-cut comparative analysis.

The present paper represents a first incursion into a new terrain – one that could be labelled "strategic complex-network analysis." Thus, obviously, it should be extended and enriched in a number of important directions. For example, a key objective should be to understand how the interplay of payoffs and the network topology shapes equilibrium behavior. This paper has illustrated such an interplay by focusing on a stylized context where strategic complementarities are captured by a Cobb-Douglas formulation. As indicated in the Introduction, alternative payoff scenarios are studied in a paper by Galeotti *et al.* (2009) that considers abstract contexts displaying what they call degree complementarity and substitutability. A further important issue (briefly discussed in that paper) is how changes in players' knowledge about the underlying random network affects equilibrium behavior. This, in particular, must be a key component of any dynamic model of learning in this context, and therefore a central feature as well in understanding the implications of complexity and limited information in network-mediated strategic interaction.

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Appendix

Proof of Theorem 1. We first prove existence. Define the function $\phi(x)$ by

$$\phi(x) \equiv \frac{1}{\alpha} \sum_{k=1}^{\bar{k}} \tilde{p}_k (\alpha x)^{k-1}. \tag{16}$$

Then, the first-order equilibrium condition for a player with degree $k = 1$, as given in (11), can be written as follows:

$$\phi(x(1)) = 1.$$

Using (2), rewrite $\phi(x)$ as follows:

$$\phi(x) = \frac{1}{\alpha z_P} \sum_{k=1}^{\bar{k}} p_k k \alpha^{k-1} x^{k-1}$$

and note that

$$\phi(0) = \frac{1}{\alpha z_P} p_1.$$

On the other hand, it is clear that for all $x > 0$

$$\frac{\partial \phi(x)}{\partial x} = \frac{1}{\alpha z_P} \sum_{k=1}^{\bar{k}} p_k k(k-1) \alpha^{k-1} x^{k-2} > 0$$

and

$$\frac{\partial^2 \phi(x)}{\partial x^2} = \frac{1}{\alpha z_P} \sum_{k=1}^{\bar{k}} p_k k(k-1)(k-2) \alpha^{k-1} x^{k-3} > 0. \tag{17}$$

Hence, $\phi(x)$ is strictly increasing and strictly convex for all $x > 0$. These observations imply that an interior solution of condition (11) exists if, and only if,

$$\phi(0) = \frac{1}{\alpha z_P} p_1 < 1.$$

Thus suppose that $\frac{1}{\alpha z_P} p_1 < 1$. Since $x^*(1) > 0$, condition (10) implies that $x^*(k) > 0$, for all $k = 2, ..., \bar{k}$. In order to establish existence, therefore, we only need to check that the equilibrium expected utility to a player with degree $k = 1, ..., \bar{k}$ is non-negative. To verify it, compute the expected utility to each player with degree k:

$$\psi_k(x^*(k), \mathbf{x}^*) = x^*(k) \left(\sum_{k'=1}^{\bar{k}} \tilde{p}_{k'} x^*(k') \right)^k - \alpha \frac{x^*(k)^2}{2}$$

so that, using the equilibrium conditions, it follows that:

$$\psi_k(x^*(k), \mathbf{x}^*) = \frac{\alpha}{2} x^*(k)^2 > 0. \tag{18}$$

Finally, we prove that the agents' payoffs satisfy the stated monotonicity conditions. The key observation is that, by virtue of the equilibrium conditions (10),

$$x^*(k+1) = \frac{1}{\alpha} [\alpha x^*(1)]^{k+1} > x^*(k) = \frac{1}{\alpha} [\alpha x^*(1)]^k$$

if and only if

$$x^*(1) > \frac{1}{\alpha}.$$

On the other hand, from (16), we have that $x^*(1) > \frac{1}{\alpha}$ if, and only if,

$$\phi\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha z_P} \sum_{k=1}^{\bar{k}} p_k k = \frac{1}{\alpha} < 1$$

Combining (18) with the above considerations, the desired result on equilibrium utilities follows. This completes the proof of the Theorem. ■

Proof of Proposition 1. Start with the interior equilibrium under P and recall that, in equilibrium, the following condition holds:

$$\frac{1}{\alpha} \sum_{k=1}^{\bar{k}} \tilde{p}_k \left[\alpha x_{\tilde{P}}(1)\right]^{k-1} = 1$$

First, suppose that $\alpha < 1$. This implies (as explained in the proof of Theorem 1) that $\alpha x_{\tilde{P}}(1) < 1$. Then, $\frac{1}{\alpha} \left[\alpha x_{\tilde{P}}(1)\right]^{k-1}$ is strictly decreasing in k and, since $\tilde{P}'$ FOSD $\tilde{P}$, it follows that

$$\frac{1}{\alpha}\sum_{k=1}^{\bar{k}}\tilde{p}_k'\left[\alpha x_{\tilde{P}}(1)\right]^{k-1} < \frac{1}{\alpha}\sum_{k=1}^{\bar{k}}\tilde{p}_k\left[\alpha x_{\tilde{P}}(1)\right]^{k-1} = 1$$

Hence, $x_{\tilde{P}'}(k) > x_{\tilde{P}}(k)$, as claimed. On the other hand, the monotonicity on expected utilities also stated is straightforward to verify. This proves matters for $\alpha < 1$, while the proof for the case $\alpha \geq 1$ is analogous and therefore omitted. ■

Proof of Proposition 2. Recall, from the proof of Theorem 1, that the function $\phi(x)$ defined in (16) is strictly convex (cf. (17)). Suppose $\alpha \neq 1$. Note that $\frac{1}{\alpha}\left[\alpha x_{\tilde{P}}(1)\right]^{k-1}$ is strictly convex in k. Then, since $\tilde{\mathbf{P}}'$ is a MPS of $\mathbf{P}$, it follows that

$$\frac{1}{\alpha}\sum_{k=1}^{\bar{k}}\tilde{p}_k'\left[\alpha x_{\tilde{P}}(1)\right]^{k-1} > \frac{1}{\alpha}\sum_{k=1}^{\bar{k}}\tilde{p}_k\left[\alpha x_{\tilde{P}}(1)\right]^{k-1} = 1,$$

where the equality is imposed by equilibrium. This implies that $x_{\tilde{P}}(1) > x_{\tilde{P}'}(1)$. Correspondingly, in view of (10), the same holds for players with degree $k > 1$. On the other hand, the claim on expected utilities directly follows from (18). Finally, the case where $\alpha = 1$ is trivial and therefore omitted. ■