Basic Mathematics for Economics[1]
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Part I

Linear Algebra

Chapter 1

Systems of linear equations

1.1 Linear equations and solutions

Definition 1 *A*[1] *linear equation in the unknowns* $x_1, x_2, ..., x_n$ *is an equation of the form*

$$a_1x_1 + a_2x_2 + ...a_nx_n = b, \tag{1.1}$$

where $b \in \mathbb{R}$ *and* $\forall j \in \{1, ..., n\}$, $a_j \in \mathbb{R}$ *. The real number* a_j *is called the coefficient of* x_j *and* b *is called the constant of the equation.* a_j *for* $j \in \{1, ..., n\}$ *and* b *are also called parameters of system* (1.1).

Definition 2 *A solution to the linear equation* (1.1) *is an ordered n-tuple* $(\overline{x}_1, ..., \overline{x}_n) := (\overline{x}_j)_{j=1}^n$ *such*[2] *that the following statement (obtained by substituting* $\overline{x}_j$ *in the place of* x_j *for any* j *) is true:*

$$a_1\overline{x}_1 + a_2\overline{x}_2 + ...a_n\overline{x}_n = b,$$

The set of all such solutions is called the solution set or the general solution or, simply, the solution of equation (1.1).

The following fact is well known.

Proposition 3 *Let the linear equation*

$$ax = b \tag{1.2}$$

in the unknown (variable) $x \in \mathbb{R}$ *and parameters* $a, b \in \mathbb{R}$ *be given. Then,*

1. *if* $a \neq 0$, *then* $x = \frac{b}{a}$ *is the unique solution to* (1.2);
2. *if* $a = 0$ *and* $b \neq 0$, *then* (1.2) *has no solutions;*
3. *if* $a = 0$ *and* $b = 0$, *then any real number is a solution to* (1.2).

Definition 4 *A linear equation* (1.1) *is said to be degenerate if* $\forall j \in \{1, ..., n\}$, $a_j = 0$, *i.e., it has the form*

$$0x_1 + 0x_2 + ...0x_n = b, \tag{1.3}$$

Clearly,

1. if $b \neq 0$, then equation (1.3) has no solution,
2. if $b = 0$, any n-tuple $(\overline{x}_j)_{j=1}^n$ is a solution to (1.3).

Definition 5 *Let a nondegenerate equation of the form* (1.1) *be given. The leading unknown of the linear equation* (1.1) *is the first unknown with a nonzero coefficient, i.e.,* x_p *is the leading unknown if*

$$\forall j \in \{1, ..., p-1\}, a_j = 0 \quad and \quad a_p \neq 0.$$

For any $j \in \{1, ..., n\} \setminus \{p\}$, x_j *is called a free variable - consistently with the following obvious result.*

[1] In this part, I often follow Lipschutz (1991).
[2] ":=" means "equal by definition".

Proposition 6 *Consider a nondegenerate linear equation* $a_1x_1 + a_2x_2 + ...a_nx_n = b$ *with leading unknown* x_p. *Then the set of solutions to that equation is*

$$\left\{ (x_k)_{k=1}^n : \forall j \in \{1,...,n\} \setminus \{p\},\ x_j \in \mathbb{R} \text{ and } x_p = \frac{b - \sum_{j\in\{1,...,n\}\setminus\{,p\}} a_jx_j}{a_p} \right\}$$

1.2 Systems of linear equations, equivalent systems and elementary operations

Definition 7 *A system of* m *linear equations in the* n *unknowns* $x_1, x_2, ..., x_n$ *is a system of the form*

$$\begin{cases} a_{11}x_1 + ... + a_{1j}x_j + ... + a_{1n}x_n = b_1 \\ ... \\ a_{i1}x_1 + ... + a_{ij}x_j + ... + a_{in}x_n = b_i \\ ... \\ a_{m1}x_i + ... + a_{mj}x_j + ... + a_{mn}x_n = b_m \end{cases} \tag{1.4}$$

where $\forall i \in \{1,...,m\}$ *and* $\forall j \in \{1,...,n\}$, $a_{ij} \in \mathbb{R}$ *and* $\forall i \in \{1,...,m\}$, $b_i \in \mathbb{R}$. *We call* L_i *the* $i-th$ *linear equation of system* (1.4).

A solution to the above system is an ordered n-tuple $(\overline{x}_j)_{j=1}^n$ *which is a solution of each equation of the system. The set of all such solutions is called the solution set of the system.*

Definition 8 *Systems of linear equations are equivalent if their solutions set is the same.*

The following fact is obvious.

Proposition 9 *Assume that a system of linear equations contains the degenerate equation*

$$L: \qquad 0x_1 + 0x_2 + ...0x_n = b.$$

1. *If* $b = 0$, *then* L *may be deleted from the system without changing the solution set;*
2. *if* $b \neq 0$, *then the system has no solutions.*

A way to solve a system of linear equations is to transform it in an equivalent system whose solution set is "easy" to be found. In what follows we make precise the above sentence.

Definition 10 *An elementary operation on a system of linear equations* (1.4) *is one of the following operations:*

$[E_1]$ *Interchange* L_i *with* L_j, *an operation denoted by* $L_i \leftrightarrow L_j$ *(which we can read "put* L_i *in the place of* L_j *and* L_j *in the place of* L_i*");*

$[E_2]$ *Multiply* L_i *by* $k \in \mathbb{R}\setminus\{0\}$, *denoted by* $kL_i \to L_i$, $k \neq 0$ *(which we can read "put* kL_i *in the place of* L_i, *with* $k \neq 0$*");*

$[E_3]$ *Replace* L_i *by (k times* L_j *plus* L_i *), denoted by* $(L_i + kL_j) \to L_i$ *(which we can read "put* $L_i + kL_j$ *in the place of* L_i*").*

Sometimes we apply $[E_2]$ and $[E_3]$ in one step, i.e., we perform the following operation

$[E]$ Replace L_i by (k' times L_j and $k \in \mathbb{R}\setminus\{0\}$ times L_i), denoted by $(k'L_j + kL_i) \to L_i, k \neq 0$.

Elementary operations are important because of the following obvious result.

Proposition 11 *If* S_1 *is a system of linear equations obtained from a system* S_2 *of linear equations using a finite number of elementary operations, then system* S_1 *and* S_2 *are equivalent.*

In what follows, first we define two types of "simple" systems (triangular and echelon form systems), and we see why those systems are in fact "easy" to solve. Then, we show how to transform any system in one of those "simple" systems.

1.3 Systems in triangular and echelon form

Definition 12 *A linear system* (1.4) *is in triangular form if the number n of equations is equal to the number n of unknowns and $\forall i \in \{1,...,n\}$, x_i is the leading unknown of equation i, i.e., the system has the following form:*

$$\begin{cases} a_{11}x_1 & +a_{12}x_2 & +... & +a_{1,n-1}x_{n-1} & +a_{1n}x_n & = & b_1 \\ & a_{22}x_2 & +... & +a_{2,n-1}x_{n-1} & +a_{2n}x_n & = & b_2 \\ & & & & & ... & \\ & & & a_{n-1,n-1}x_{n-1} & +a_{n-1n}x_n & = & b_{n-1} \\ & & & & a_{nn}x_n & = & b_n \end{cases} \tag{1.5}$$

where $\forall i \in \{1,...,n\}$, $a_{ii} \neq 0$.

Proposition 13 *System* (1.5) *has a unique solution.*

Proof. We can compute the solution of system (1.5) using the following procedure, known as back-substitution.

First, since by assumption $a_{nn} \neq 0$, we solve the last equation with respect to the last unknown, i.e., we get

$$x_n = \frac{b_n}{a_{nn}}.$$

Second, we substitute that value of x_n in the next-to-the-last equation and solve it for the next-to-the-last unknown, i.e.,

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n} \cdot \frac{b_n}{a_{nn}}}{a_{n-1,n-1}}$$

and so on. The process ends when we have determined the first unknown, x_1.

Observe that the above procedure shows that the solution to a system in triangular form is unique since, at each step of the algorithm, the value of each x_i is uniquely determined, as a consequence of Proposition 3, conclusion 1. ■

Definition 14 *A linear system* (1.4) *is said to be in echelon form if*

1. *no equation is degenerate, and*

2. *the leading unknown in each equation is to the right of the leading unknown of the preceding equation.*

In other words, the system is of the form

$$\begin{cases} a_{11}x_1 & +... & +a_{1j_2}x_{j_2} & +... & & & & +a_{1s}x_s & =b_1 \\ & & a_{2j_2}x_{j_2} & +... & +a_{2,j_3}x_{j_3} & +... & & +a_{2s}x_s & =b_2 \\ & & & & a_{3,j_3}x_{j_3} & +... & & +a_{3s}x_s & =b_3 \\ & & & & & & & & ... \\ & & & & & a_{r,j_r}x_{j_r} & +a_{r,j_r+1}+... & +a_{rs}x_s & =b_r \end{cases} \tag{1.6}$$

with $j_1 := 1 < j_2 < ... < j_r$ and $a_{11}, a_{2j_2}, ..., a_{rj_r} \neq 0$. Observe that the above system has r equations and s variables and that $s \geq r$. The leading unknown in equation $i \in \{1,...,r\}$ is x_{j_i}.

Remark 15 *Systems with no degenerate equations are the "interesting" ones. If an equation is degenerate and the right hand side term is zero, then you can erase it; if the right hand side term is not zero, then the system has no solutions.*

Definition 16 *An unknown x_k in system* (1.6) *is called a free variable if x_k is not the leading unknown in any equation, i.e., $\forall i \in \{1,...,r\}$, $x_k \neq x_{j_i}$.*

In system (1.6), there are r leading unknowns, r equations and $s - r \geq 0$ free variables.

Proposition 17 *Let a system in echelon form with r equations and s variables be given. Then, the following results hold true.*

1. *If $s = r$, i.e., the number of unknowns is equal to the number of equations, then the system has a unique solution;*

2. *if $s > r$, i.e., the number of unknowns is greater than the number of equations, then we can arbitrarily assign values to the $n - r > 0$ free variables and obtain solutions of the system.*

Proof. We prove the theorem by induction on the number r of equations of the system.
Step 1. $r = 1$.
In this case, we have a single, nondegenerate linear equation, to which Proposition 6 applies if $s > r = 1$, and Proposition 3 applies if $s = r = 1$.
Step 2.
Assume that $r > 1$ and the desired conclusion is true for a system with $r-1$ equations. Consider the given system in the form (1.6) and erase the first equation, so obtaining the following system:

$$\begin{cases} a_{2j_2}x_{j_2} \quad +... \quad +a_{2,j_3}x_{j_3} \quad +... & = \quad b_2 \\ \qquad\qquad a_{3,j_3}x_{j_3} \quad +... & \\ \qquad\qquad\qquad ... & ... \\ \qquad\qquad\qquad a_{r,j_r}x_{j_r} \quad +a_{r,j_r+1} \qquad +a_{rs}x_s & = \quad b_r \end{cases} \tag{1.7}$$

in the unknowns $x_{j_2}, ..., x_s$. First of all observe that the above system is in echelon form and has $r-1$ equation; therefore we can apply the induction argument distinguishing the two case $s > r$ and $s = r$.

If $s > r$, then we can assign arbitrary values to the free variables, whose number is (the "old" number minus the erased ones)

$$s - r - (j_2 - j_1 - 1) = s - r - j_2 + 2$$

and obtain a solution of system (1.7). Consider the first equation of the original system

$$a_{11}x_1 \quad +a_{12}x_2 \quad +... \quad +a_{1,j_2-1}x_{j_2-1} \quad +a_{1j_2}x_{j_2} \quad +... \qquad = \quad b_1 \ . \tag{1.8}$$

We immediately see that the above found values together with arbitrary values for the additional

$$j_2 - 2$$

free variable of equation (1.8) yield a solution of that equation, as desired. Observe also that the values given to the variables $x_1, ..., x_{j_{2-1}}$ from the first equation do satisfy the other equations simply because their coefficients are zero there.

If $s = r$, the system in echelon form, in fact, becomes a system in triangular form and then the solution exists and it is unique. ■

Remark 18 *From the proof of the previous Proposition, if the echelon system* (1.6) *contains more unknowns than equations, i.e., $s > r$, then the system has an infinite number of solutions since each of the $s - r \geq 1$ free variables may be assigned an arbitrary real number.*

1.4 Reduction algorithm

The following algorithm (sometimes called row reduction) reduces system (1.4) of m equation and n unknowns to either echelon form, or triangular form, or shows that the system has no solution. The algorithm then gives a proof of the following result.

Proposition 19 *Any system of linear equations has either*

1. *infinite solutions, or*

2. *a unique solution, or*

3. *no solutions.*

Reduction algorithm.
Consider a system of the form (1.4) such that

$$\forall j \in \{1,...,n\}, \quad \exists i \in \{1,..,m\} \text{ such that } a_{ij} \neq 0, \tag{1.9}$$

i.e., a system in which each variable has a nonzero coefficient in at least one equation. If that is not the case, the remaining variables can renamed in order to have (1.9) satisfied.

Step 1. Interchange equations so that the first unknown, x_1, appears with a nonzero coefficient in the first equation; i.e., rearrange the equations in the system in order to have $a_{11} \neq 0$.

Step 2. Use a_{11} as a "pivot" to eliminate x_1 from all equations but the first equation. That is, for each $i > 1$, apply the elementary operation

$$[E_3] : -\left(\frac{a_{i1}}{a_{11}}\right) L_1 + L_i \to L_i$$

or

$$[E] : -a_{i1}L_1 + a_{11}L_i \to L_i.$$

Step 3. Examine each new equation L :

1. If L has the form

   $$0x_1 + 0x_2 + ... + 0x_n = 0,$$

   or if L is a multiple of another equation, then delete L from the system.[3]

2. If L has the form

   $$0x_1 + 0x_2 + ... + 0x_n = b,$$

   with $b \neq 0$, then exit the algorithm. The system has no solutions.

Step 4. Repeat Steps 1, 2 and 3 with the subsystem formed by all the equations, excluding the first equation.

Step 5. Continue the above process until the system is in echelon form or a degenerate equation is obtained in Step 3.2.

Summarizing, our method for solving system (1.4) consists of two steps:

Step A. Use the above reduction algorithm to reduce system (1.4) to an equivalent simpler system (in triangular form, system (1.5) or echelon form (1.6)).

Step B. If the system is in triangular form, use back-substitution to find the solution; if the system is in echelon form, bring the free variables on the right hand side of each equation, give them arbitrary values (say, the name of the free variable with an upper bar), and then use back-substitution.

Example 20

$$\begin{cases} x_1 & + & 2x_2 & + & (-3)x_3 & = & -1 \\ 3x_1 & + & (-1)\,x_2 & + & 2x_3 & = & 7 \\ 5x_1 & + & 3x_2 & + & (-4)\,x_3 & = & 2 \end{cases}$$

Step A.

Step 1. Nothing to do.

[3] The justification of Step 3 is Propositon 9 and the fact that if $L = kL'$ for some other equation L' in the system, then the operation $-kL' + L \to L$ replace L by $0x_1 + 0x_2 + ... + 0x_n = 0$, which again may be deleted by Propositon 9.

Step 2. Apply the operations

$$-3L_1 + L_2 \to L_2$$

and

$$-5L_1 + L_3 \to L_3,$$

to get

$$\begin{cases} x_1 & + & 2x_2 & + & (-3)x_3 & = & -1 \\ & & (-7)\,x_2 & + & 11x_3 & = & 10 \\ & & (-7)\,x_2 & + & 11x_3 & = & 7 \end{cases}$$

Step 3. Examine each new equations L_2 and L_3:

1. L_2 and L_3 do not have the form

$$0x_1 + 0x_2 + ... + 0x_n = 0;$$

   L_2 is not a multiple L_3;

2. L_2 and L_3 do not have the form

$$0x_1 + 0x_2 + ... + 0x_n = b,$$

Step 4.

Step 1.1 Nothing to do.

Step 2.1 Apply the operation

$$-L_2 + L_3 \to L_3$$

to get

$$\begin{cases} x_1 & + & 2x_2 & + & (-3)x_3 & = & -1 \\ & & (-7)\,x_2 & + & 11x_3 & = & 10 \\ 0x_1 & + & 0x_2 & + & 0x_3 & = & -3 \end{cases}$$

Step 3.1 L_3 has the form

$$0x_1 + 0x_2 + ... + 0x_n = b,$$

1. with $b = -3 \neq 0$, then exit the algorithm. The system has no solutions.

1.5 Matrices

Definition 21 *Given $m, n \in \mathbb{N}\setminus\{0\}$, a matrix (of real numbers) of order $m \times n$ is a table of real numbers with m rows and n columns as displayed below.*

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1j} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2j} & ... & a_{2n} \\ ... \\ a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{m1} & a_{m2} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}$$

For any $i \in \{1, ..., m\}$ and any $j \in \{1, ..., n\}$ the real numbers a_{ij} are called entries of the matrix; the first subscript i denotes the row the entries belongs to, the second subscript j denotes the column the entries belongs to. We will usually denote matrices with capital letters and we will write $A_{m\times n}$ to denote a matrix of order $m \times n$. Sometimes it is useful to denote a matrix by its "typical" element and we write$[a_{ij}]_{\substack{i\in\{1,...,m\} \\ j\in\{1,...,n\}}}$, or simply $[a_{ij}]$ if no ambiguity arises about the number of rows and columns. For $i \in \{1, ..., m\}$,

$$\begin{bmatrix} a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \end{bmatrix}$$

is called the $i-th$ *row of* A *and it denoted by* $R^i(A)$*. For* $j \in \{1,...,n\}$*,*

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ ... \\ a_{ij} \\ ... \\ a_{mj} \end{bmatrix}$$

is called the $j-th$ *column of* A *and it denoted by* $C^j(A)$*.*

We denote the set of $m \times n$ *matrices by* $\mathcal{M}_{m,n}$*, and we write, in an equivalent manner,* $A_{m\times n}$ *or* $A \in \mathcal{M}_{m,n}$*.*

Definition 22 *The matrix*

$$A_{m\times 1} = \begin{bmatrix} a_1 \\ ... \\ a_m \end{bmatrix}$$

is called column vector and the matrix

$$A_{1\times n} = \begin{bmatrix} a_1, & ... & a_n \end{bmatrix}$$

is called row vector. We usually denote row or column vectors by small Latin letters.

Definition 23 *The first nonzero entry in a row* R *of a matrix* $A_{m\times n}$ *is called the leading nonzero entry of* R*. If* R *has no leading nonzero entries, i.e., if every entry in* R *is zero, then* R *is called a zero row. If all the rows of* A *are zero, i.e., each entry of* A *is zero, then* A *is called a zero matrix, denoted by* $0_{m\times n}$ *or simply* 0*, if no confusion arises.*

In the previous sections, we defined triangular and echelon systems of linear equations. Below, we define triangular, echelon matrices and a special kind of echelon matrices. In Section (1.6), we will see that there is a simple relationship between systems and matrices.

Definition 24 *A matrix* $A_{m\times n}$ *is square if* $m = n$*. A square matrix* A *belonging to* $\mathcal{M}_{m,m}$ *is called square matrix of order* m*.*

Definition 25 *Given* $A = [a_{ij}] \in \mathcal{M}_{m,m}$*, the main diagonal of* A *is made up by the entries* a_{ii} *with* $i \in \{1,..,m\}$*.*

Definition 26 *A square matrix* $A = [a_{ij}] \in \mathcal{M}_{m,m}$ *is an upper triangular matrix or simply a triangular matrix if all entries below the main diagonal are equal to zero, i.e.,* $\forall i,j \in \{1,..,m\}$*, if* $i > j$*, then* $a_{ij} = 0$*.*

Definition 27 $A \in \mathcal{M}_{mm}$ *is called diagonal matrix of order* m *if any element outside the principal diagonal is equal to zero, i.e.,* $\forall i,j \in \{1,...,m\}$ *such that* $i \neq j$*,* $a_{ij} = 0$*.*

Definition 28 *A matrix* $A \in \mathcal{M}_{m,n}$ *is called an echelon (form) matrix, or it is said to be in echelon form, if the following two conditions hold:*

1. *All zero rows, if any, are on the bottom of the matrix.*
2. *The leading nonzero entry of each row is to the right of the leading nonzero entry in the preceding row.*

Definition 29 *If a matrix* A *is in echelon form, then its leading nonzero entries are called pivot entries, or simply, pivots*

Remark 30 *If a matrix* $A \in \mathcal{M}_{m,n}$ *is in echelon form and* r *is the number of its pivot entries, then* $r \leq \min\{m,n\}$*. In fact,* $r \leq m$*, because the matrix may have zero rows and* $r \leq n$*, because the leading nonzero entries of the first row maybe not in the first column, and the other leading nonzero entries may be "strictly to the right" of previous leading nonzero entry.*

Definition 31 *A matrix $A \in \mathcal{M}_{m,n}$ is called in row canonical form if*

1. *it is in echelon form,*
2. *each pivot is* 1*, and*
3. *each pivot is the only nonzero entry in its column.*

Example 32 *1. All the matrices below are echelon matrices; only the fourth one is in row canonical form.*

$$\begin{bmatrix} 0 & 7 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 2 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

2. Any zero matrix is in row canonical form.

Remark 33 *Let a matrix $A_{m\times n}$ in row canonical form be given. As a consequence of the definition, we have what follows.*
1. If some rows from A are erased, the resulting matrix is still in row canonical form.
2. If some columns of zeros are added, the resulting matrix is still in row canonical form.

Definition 34 *Denote by R^i the $i-th$ row of a matrix A. An elementary row operation is one of the following operations on the rows of A:*

$[E_1]$ *(Row interchange) Interchange R^i with R^j, an operation denoted by $R^i \leftrightarrow R^j$ (which we can read "put R^i in the place of R^j and R^j in the place of R^i");;*

$[E_2]$ *(Row scaling) Multiply R^i by $k \in \mathbb{R}\backslash\{0\}$, denoted by $kR^i \to R^i$, $k \neq 0$ (which we can read "put kR^i in the place of R^i, with $k \neq 0$");*

$[E_3]$ *(Row addition) Replace R^i by (k times R^j plus R^i), denoted by $\left(R^i + kR^j\right) \to R^i$ (which we can read "put $R^i + kR^j$ in the place of R^i").*

Sometimes we apply $[E_2]$ and $[E_3]$ in one step, i.e., we perform the following operation

$[E]$ Replace R^i by (k' times R^j and $k \in \mathbb{R}\backslash\{0\}$ times R^i), denoted by $\left(k'R^j + kR^i\right) \to R^i, k \neq 0$.

Definition 35 *A matrix $A \in \mathcal{M}_{m,n}$ is said to be row equivalent to a matrix $B \in \mathcal{M}_{m,n}$ if B can be obtained from A by a finite number of elementary row operations.*

It is hard not to recognize the similarity of the above operations and those used in solving systems of linear equations.

We use the expression "row reduce" as having the meaning of "transform a given matrix into another matrix using row operations". The following algorithm "row reduces" a matrix A into a matrix in echelon form.

Row reduction algorithm to echelon form.

Consider a matrix $A = [a_{ij}] \in \mathcal{M}_{m,n}$.

Step 1. Find the first column with a nonzero entry. Suppose it is column j_1.

Step 2. Interchange the rows so that a nonzero entry appears in the first row of column j_1, i.e., so that $a_{1j_1} \neq 0$.

Step 3. Use a_{1j_1} as a "pivot" to obtain zeros below a_{1j_1}, i.e., for each $i > 1$, apply the row operation

$$[E_3] : -\left(\frac{a_{ij_1}}{a_{1j_1}}\right) R^1 + R^i \to R^i$$

or

$$[E] : -a_{ij_1} R^1 + a_{11} R^i \to R^i.$$

Step 4. Repeat Steps 1, 2 and 3 with the submatrix formed by all the rows, excluding the first row.

Step 5. Continue the above process until the matrix is in echelon form.

Example 36 *Let's apply the above algorithm to the following matrix*

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{bmatrix}$$

Step 1. Find the first column with a nonzero entry: that is C^1, and therefore $j_1 = 1$.

Step 2. Interchange the rows so that a nonzero entry appears in the first row of column j_1, i.e., so that $a_{1j_1} \neq 0$: $a_{1j_1} = a_{11} = 1 \neq 0$.

Step 3. Use a_{11} as a "pivot" to obtain zeros below a_{11}. Apply the row operations

$$-3R^1 + R^2 \to R^2$$

and

$$-5R^1 + R^3 \to R^3,$$

to get

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{bmatrix}$$

Step 4. Apply the operation

$$-R^2 + R^3 \to R^3$$

to get

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

which is is in echelon form.

Row reduction algorithm from echelon form to row canonical form.
Consider a matrix $A = [a_{ij}] \in \mathcal{M}_{m,n}$ in echelon form, say with pivots

$$a_{1j_1}, a_{2j_2}, ..., a_{rj_r}.$$

Step 1. Multiply the last nonzero row R^r by $\frac{1}{a_{rj_r}}$,so that the leading nonzero entry of that row becomes 1.

Step 2. Use a_{rj_r} as a "pivot" to obtain zeros above the pivot, i.e., for each $i \in \{r-1, r-2, ..., 1\}$, apply the row operation

$$[E_3] : -a_{i,j_r} R^r + R^i \to R^i.$$

Step 3. Repeat Steps 1 and 2 for rows $R^{r-1}, R^{r-2}, ..., R^2$.

Step 4. Multiply R^1 by $\frac{1}{a_{1j_1}}$.

Example 37 *Consider the matrix*

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

in echelon form, with leading nonzero entries

$$a_{11} = 1, a_{22} = -7, a_{34} = -3.$$

Step 1. Multiply the last nonzero row R^3 by $\frac{1}{-3}$,so that the leading nonzero entry becomes 1:

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2. Use $a_{rj_r} = a_{34}$ as a "pivot" to obtain zeros above the pivot, i.e., for each $i \in \{r-1, r-2, ..., 1\} = \{2, 1\}$, apply the row operation

$$[E_3] : -a_{i,j_r} R^r + R^i \to R^i,$$

which in our case are

$$-a_{2,4}R^3 + R^2 \to R^2 \quad \text{i.e.,} \quad -10R^3 + R^2 \to R^2,$$
$$-a_{1,4}R^3 + R^1 \to R^1 \quad \text{i.e.,} \quad R^3 + R^1 \to R^1.$$

Then, we get

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -7 & 11 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3. Multiply R^2 by $\frac{1}{-7}$, and get

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -\frac{11}{7} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use a_{23} as a "pivot" to obtain zeros above the pivot, applying the operation:

$$-2R^2 + R^1 \to R^1,$$

to get

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} & 0 \\ 0 & 1 & -\frac{11}{7} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is in row reduced form.

Proposition 38 *Any matrix $A \in \mathcal{M}_{m,n}$ is row equivalent to a matrix in row canonical form.*

Proof. The two above algorithms show that any matrix is row equivalent to at least one matrix in row canonical form. ■

Remark 39 *In fact, in Proposition 153, we will show that: Any matrix $A \in \mathcal{M}_{m,n}$ is row equivalent to a* ***unique*** *matrix in row canonical form.*

1.6 Systems of linear equations and matrices

Definition 40 *Given system* (1.4)*, i.e., a system of m linear equation in the n unknowns $x_1, x_2, ..., x_n$*

$$\begin{cases} a_{11}x_1 + ... + a_{1j}x_j + ... + a_{1n}x_n = b_1 \\ ... \\ a_{i1}x_1 + ... + a_{ij}x_j + ... + a_{in}x_n = b_i \\ ... \\ a_{m1}x_i + ... + a_{mj}x_j + ... + a_{mn}x_n = b_m, \end{cases}$$

the matrix

$$\begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} & b_1 \\ ... \\ a_{i1} & ... & a_{ij} & ... & a_{in} & b_i \\ ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix M of system (1.4).

Each row of M corresponds to an equation of the system, and each column of M corresponds to the coefficients of an unknown, except the last column which corresponds to the constant of the system.

In an obvious way, given an arbitrary matrix M, we can find a unique system whose associated matrix is M; moreover, given a system of linear equations, there is only one matrix M associated with it. We can therefore identify system of linear equations with (augmented) matrices.

The coefficient matrix of the system is

$$A = \begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... & & & & \\ a_{i1} & ... & a_{ij} & ... & a_{in} \\ ... & & & & \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}$$

One way to solve a system of linear equations is as follows:

1. Reduce its augmented matrix M to echelon form, which tells if the system has solution; if M has a row of the form $(0, 0, ..., 0, b)$ with $b \neq 0$, then the system has no solution and you can stop. If the system admits solutions go to the step below.

2. Reduce the matrix in echelon form obtained in the above step to its row canonical form. Write the corresponding system. In each equation, bring the free variables on the right hand side, obtaining a triangular system. Solve by back-substitution.

The simple justification of this process comes from the following facts:

1. Any elementary row operation of the augmented matrix M of the system is equivalent to applying the corresponding operation on the system itself.

2. The system has a solution if and only if the echelon form of the augmented matrix M does not have a row of the form $(0, 0, ..., 0, b)$ with $b \neq 0$ - simply because that row corresponds to a degenerate equation.

3. In the row canonical form of the augmented matrix M (excluding zero rows) the coefficient of each nonfree variable is a leading nonzero entry which is equal to one and is the only nonzero entry in its respective column; hence the free variable form of the solution is obtained by simply transferring the free variable terms to the other side of each equation.

Example 41 *Consider the system presented in Example 20:*

$$\begin{cases} x_1 & + & 2x_2 & + & (-3)x_3 & = & -1 \\ 3x_1 & + & (-1)\,x_2 & + & 2x_3 & = & 7 \\ 5x_1 & + & 3x_2 & + & (-4)\,x_3 & = & 2 \end{cases}$$

The associated augmented matrix is:

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{bmatrix}$$

In example 36, we have see that the echelon form of the above matrix is

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

which has its last row of the form $(0, 0, ..., 0, b)$ *with* $b = -3 \neq 0$, *and therefore the system has no solution.*

Chapter 2

The Euclidean Space $\mathbb{R}^n$

2.1 Sum and scalar multiplication

It is well known that the real line is a representation of the set $\mathbb{R}$ of real numbers. Similarly, a ordered pair (x, y) of real numbers can be used to represent a point in the plane and a triple (x, y, z) or (x_1, x_2, x_3) a point in the space. In general, if $n \in \mathbb{N}_+ := \{1, 2, ..., \}$, we can define $(x_1, x_2, ..., x_n)$ or $(x_i)_{i=1}^n$ as a point in the $n - space$.

Definition 42 $\mathbb{R}^n := \mathbb{R} \times ... \times \mathbb{R}$.

In other words, $\mathbb{R}^n$ is the Cartesian product of $\mathbb{R}$ multiplied n times by itself.

Definition 43 *The elements of* $\mathbb{R}^n$ *are ordered n-tuple of real numbers and are denoted by*

$$x = (x_1, x_2, ..., x_n) \text{ or } x = (x_i)_{i=1}^n.$$

x_i is called $i - th$ component of $x \in \mathbb{R}^n$.

Definition 44 $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ *and* $y = (y_i)_{i=1}^n$ *are equal if*

$$\forall i \in \{1, ..., n\} , \ \ x_i = y_i.$$

In that case we write $x = y$.

Let us introduce two operations on $\mathbb{R}^n$ and analyze some properties they satisfy.

Definition 45 *Given* $x \in \mathbb{R}^n, y \in \mathbb{R}^n$, *we call addition or sum of x and y the element denoted by $x + y \in \mathbb{R}^n$ obtained as follows*

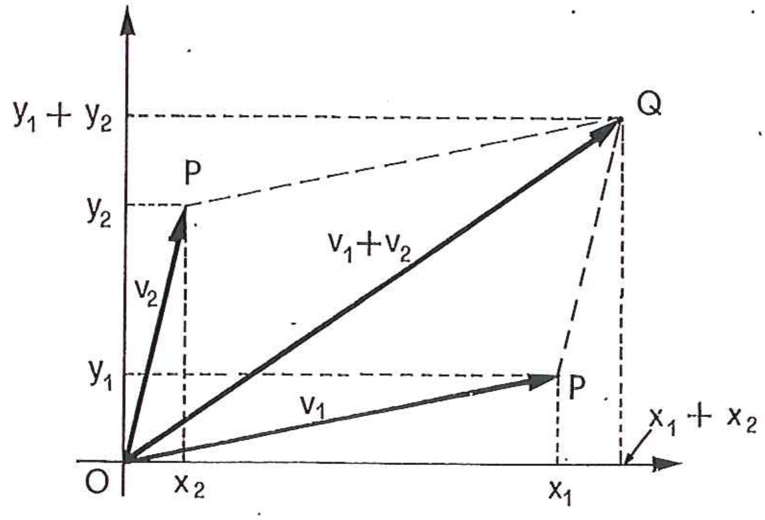
$$x + y := (x_i + y_i)_{i=1}^n.$$

Definition 46 *An element $\lambda \in \mathbb{R}$ is called scalar.*

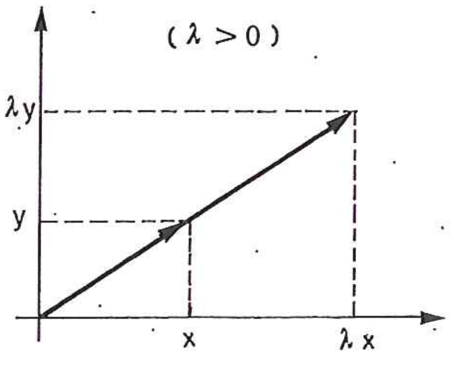
Definition 47 *Given $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we call scalar multiplication of x by λ the element $\lambda x \in \mathbb{R}^n$ obtained as follows*

$$\lambda x := (\lambda x_i)_{i=1}^n.$$

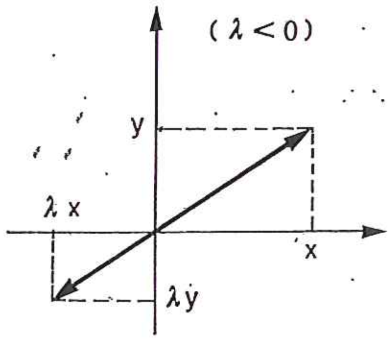
Geometrical interpretation of the two operations in the case $n = 2$.



Q
$y_1 + y_2$
P
y_2
v_1+v_2
v_2
y_1
P
v_1
$x_1 + x_2$
O
x_2
x_1



$(\lambda > 0)$
λy
y
x
λx



$(\lambda < 0)$
y
λx
x
λy

From the well known properties of the sum and product of real numbers it is possible to verify that the following properties of the above operations do hold true.

Properties of addition.

A1. (Associative) $\forall x, y \in \mathbb{R}^n, \quad (x+y)+z = x+(y+z)$;

A2. (existence of null element) there exists an element e in $\mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$, $x + e = x$; in fact such element is unique and it is denoted by 0;

A3. (existence of inverse element) $\forall x \in \mathbb{R}^n \ \exists y \in \mathbb{R}^n$ such that $x + y = 0$; in fact, that element is unique and denoted by $-x$;

A4. (Commutative) $\forall x, y \in \mathbb{R}^n$, $x + y = y + x$.

Properties of multiplication.

M1. (distributive) $\forall \alpha \in \mathbb{R}, \ x \in \mathbb{R}^n, \ y \in \mathbb{R}^n \quad \alpha(x+y) = \alpha x + \alpha y$;

M2. (distributive) $\forall \alpha \in \mathbb{R}, \ \beta \in \mathbb{R}, \ x \in \mathbb{R}^n, \ (\alpha+\beta)x = \alpha x + \beta x$

M3. $\forall \alpha \in \mathbb{R}, \ \beta \in \mathbb{R}, \ x \in \mathbb{R}^n, \quad (\alpha\beta)x = \alpha(\beta x)$;

M4. $\forall x \in \mathbb{R}^n$, $1x = x$.

2.2 Scalar product

Definition 48 *Given $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in \mathbb{R}^n$, we call dot, scalar or inner product of x and y, denoted by xy or $x \cdot y$, the scalar*

$$\sum_{i=1}^{n} x_i \cdot y_i \in \mathbb{R}.$$

Remark 49 *The scalar product of elements of $\mathbb{R}^n$ satisfies the following properties.*

1. $\forall x, y \in \mathbb{R}^n \quad x \cdot y = y \cdot x$;

2. $\forall \alpha, \beta \in \mathbb{R}, \forall x, y, z \in \mathbb{R}^n \quad (\alpha x + \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z)$;

3. $\forall x \in \mathbb{R}^n, \quad x \cdot x \geq 0$;

4. $\forall x \in \mathbb{R}^n \ , \quad x \cdot x = 0 \iff x = 0$.

Definition 50 *The set $\mathbb{R}^n$ with above described three operations (addition, scalar multiplication and dot product) is usually called Euclidean space of dimension n.*

Definition 51 *Given $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, we denote the (Euclidean) norm or length of x by*

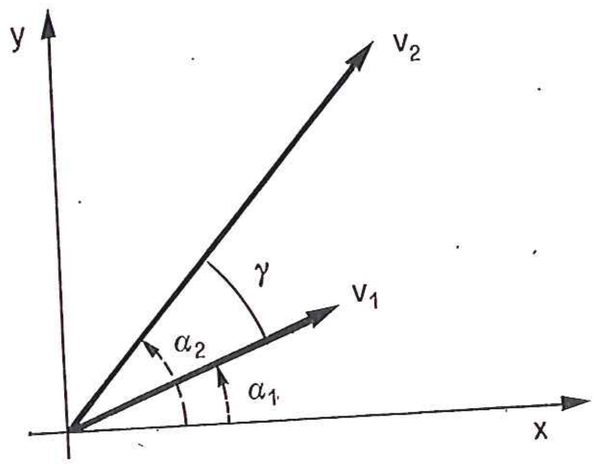
$$\|x\| := (x \cdot x)^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{n} (x_i)^2}.$$

Geometrical Interpretation of scalar products in $\mathbb{R}^2$.

Given $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$, from elementary trigonometry we know that

$$x = (\|x\| \cos\alpha, \|x\| \sin\alpha) \tag{2.1}$$

where α is the measure of the angle between the positive part of the horizontal axes and x itself.


y
v2
v1
γ
α2
α1
x

Using the above observation we can verify that given $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2 \setminus \{0\}$,

$$xy = \|x\| \cdot \|y\| \cdot \cos\gamma$$

where γ is an[1] angle between x and y.
`scan and insert picture (Marcellini-Sbordone page 179)`
From the picture and (2.1), we have

$$x = (\|x\| \cos\alpha_1, \|x\| \sin\alpha_1)$$

and

$$y = (\|y\| \cos\alpha_2, \|y\| \sin\alpha_2).$$

Then[2]

$$xy = \|x\| \|y\| (\cos\alpha_1 \cos\alpha_2 + \sin\alpha_1 \sin\alpha_2) = \|x\| \|y\| \cos(\alpha_2 - \alpha_1).$$

Taken x and y not belonging to the same line, define $\theta^* :=$ (angle between x and y with minimum measure). From the above equality, it follows that

$$\begin{array}{lcl} \theta^* = \frac{\pi}{2} & \Leftrightarrow & x \cdot y = 0 \\ \theta^* < \frac{\pi}{2} & \Leftrightarrow & x \cdot y > 0 \\ \theta^* > \frac{\pi}{2} & \Leftrightarrow & x \cdot y < 0. \end{array}$$

Definition 52 *$x, y \in \mathbb{R}^n \setminus \{0\}$ are orthogonal if $xy = 0$.*

2.3 Norms and Distances

Proposition 53 *(Properties of the norm). Let $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.*
1. *$\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$,*
2. *$\|\alpha x\| = |\alpha| \cdot \|x\|$,*
3. *$\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality),*
4. *$|xy| \leq \|x\| \cdot \|y\|$ (Cauchy-Schwarz inequality).*

Proof. 1. By definition $\|x\| = \sqrt{\sum_{i=1}^n (x_i)^2} \geq 0$. Moreover, $\|x\| = 0 \Leftrightarrow \|x\|^2 = 0 \Leftrightarrow \sum_{i=1}^n (x_i)^2 = 0 \Leftrightarrow x = 0$.

2. $\|\alpha x\| = \sqrt{\sum_{i=1}^n \alpha^2 (x_i)^2} = |\alpha| \sqrt{\sum_{i=1}^n (x_i)^2} = |\alpha| \cdot \|x\|$.

4. (3 is proved using 4)

We want to show that $|xy| \leq \|x\| \cdot \|y\|$ or $|xy|^2 \leq \|x\|^2 \cdot \|y\|^2$, i.e.,

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \left(\sum_{i=1}^n x_i^2 \right) \cdot \left(\sum_{i=1}^n y_i^2 \right)$$

Defined $X := \sum_{i=1}^n x_i^2, Y := \sum_{i=1}^n y_i^2$ and $Z := \sum_{i=1}^n x_i y_i$, we have to prove that

$$Z^2 \leq XY. \tag{2.2}$$

Observe that

$$\forall a \in \mathbb{R}, \quad \begin{array}{l} 1. \ \sum_{i=1}^n (a x_i + y_i)^2 \geq 0, \quad \text{and} \\ 2. \ \sum_{i=1}^n (a x_i + y_i)^2 = 0 \quad \Leftrightarrow \quad \forall i \in \{1, ..., n\}, a x_i + y_i = 0 \end{array}$$

[1] Recall that $\forall x \in \mathbb{R}$, $\cos x = \cos(-x) = \cos(2\pi - x)$.

[2] Recall that

$$\cos(x_1 \pm x_2) = \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2.$$

Moreover,

$$\sum_{i=1}^{n}(ax_i+y_i)^2 = a^2\sum_{i=1}^{n}x_i^2+2a\sum_{i=1}^{n}x_iy_i+\sum_{i=1}^{n}y_i^2 = a^2X+2aZ+Y \geq 0 \tag{2.3}$$

If $X>0$, we can take $a=-\frac{Z}{X}$, and from (2.3), we get

$$0 \leq \frac{Z^2}{X^2}X - 2\frac{Z^2}{X}+Y$$

or

$$Z^2 \leq XY,$$

as desired.

If $X=0$, then $x=0$ and $Z=0$, and (2.2) is true simply because $0 \leq 0$.

3. It suffices to show that $\|x+y\|^2 \leq (\|x\|+\|y\|)^2$.

$$\|x+y\|^2 = \sum_{i=1}^{n}(x_i+y_i)^2 = \sum_{i=1}^{n}\left((x_i)^2+2x_i\cdot y_i+(y_i)^2\right) =$$

$$= \|x\|^2+2xy+\|y\|^2 \leq \|x\|^2+2\,|xy|+\|y\|^2 \overset{(4\ \text{above})}{\leq} \|x\|^2+2\,\|x\|\cdot\|y\|+\|y\|^2 = (\|x\|+\|y\|)^2.$$

■

Remark 54 $|\|x\|-\|y\|| \leq \|x-y\|$.

Recall that $\forall a,b\in\mathbb{R}$

$$-b\leq a\leq b \Leftrightarrow |a|\leq b.$$

From Proposition 53.3,identifying x *with* $x-y$ *and* y *with* y, *we get* $\|x-y+y\|\leq\|x-y\|+\|y\|$, *i.e.,*

$$\|x\|-\|y\|\leq\|x-y\|$$

From Proposition 53.3, identifying x *with* $y-x$ *and* y *with* x, *we get* $\|y-x+x\|\leq\|y-x\|+\|x\|$, *i.e.,*

$$\|y\|-\|x\|\leq\|y-x\|=\|x-y\|$$

and

$$-\|x-y\|\leq\|x\|-\|y\|$$

Definition 55 *For any* $n\in\mathbb{N}\backslash\{0\}$ *and for any* $i\in\{1,...,n\}$, $e_n^i := \left(e_{j,n}^i\right)_{j=1}^n\in\mathbb{R}^n$ *with*

$$e_{n,j}^i = \begin{cases} 0 & if \quad i\neq j \\ 1 & if \quad i=j \end{cases}$$

In other words, e_n^i *is an element of* $\mathbb{R}^n$ *whose components are all zero, but the* $i-th$ *component which is equal to* 1. *The vector* e_n^i *is called the* $i-th$ *canonical vector in* $\mathbb{R}^n$.

Remark 56 $\forall x\in\mathbb{R}^n$,

$$\|x\|\leq\sum_{i=1}^{n}|x_i|,$$

as verified below.

$$\|x\| = \left\|\sum_{i=1}^{n}x_ie^i\right\| \overset{(1)}{\leq} \sum_{i=1}^{n}\left\|x_ie^i\right\| \overset{(2)}{=} \sum_{i=1}^{n}|x_i|\cdot\left\|e^i\right\| = \sum_{i=1}^{n}|x_i|,$$

where (1) *follows from the triangle inequality, i.e., Proposition 53.3, and* (2) *from Proposition 53.2.*

Definition 57 *Given* $x,y\in\mathbb{R}^n$,*we denote the (Euclidean) distance between* x *and* y *by*

$$d(x,y) := \|x-y\|$$

Proposition 58 *(Properties of the distance). Let* $x, y, z \in \mathbb{R}^n$.

1. $d(x,y) \geq 0$, *and* $d(x,y) = 0 \Leftrightarrow x = y$,
2. $d(x,y) = d(y,x)$,
3. $d(x,z) \leq d(x,y) + d(y,z)$ *(Triangle inequality).*

Proof. 1. It follows from property 1 of the norm.
2. It follows from the definition of the distance as a norm.
3. Identifying x with $x - y$ and y with $y - z$ in property 3 of the norm, we get $\|(x-y)+(y-z)\| \leq \|x-y\| + \|y-z\|$, i.e., the desired result. ■

Chapter 3

Matrices

We presented the concept of matrix in Definition 21. In this chapter, we study further properties of matrices.

Definition 59 *The transpose of a matrix $A \in \mathcal{M}_{m,n}$, denoted by A^T belongs to $M_{n,m}$ and it is the matrix obtained by writing the rows of A, in order, as columns:*

$$A^T = \begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... \\ a_{i1} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & ... & a_{i1} & ... & a_{m1} \\ ... \\ a_{1j} & ... & a_{ij} & ... & a_{mj} \\ ... \\ a_{1n} & ... & a_{in} & ... & a_{mn} \end{bmatrix}.$$

In other words, row 1 of the matrix A becomes column 1 of A^T, row 2 of A becomes column 2 of A^T, and so on, up to row m which becomes column m of A^T. Same results is obtained proceeding as follows: column 1of A becomes row 1 of A^T, column 2 of A becomes row 2 of A^T, and so on, up to column n which becomes row n of A^T. More formally, given $A = [a_{ij}]_{\substack{i\in\{1,...,m\} \\ j\in\{1,...,n\}}} \in \mathrm{M}_{m,n}$, then

$$A^T = [a_{ji}]_{\substack{j\in\{1,...,n\} \\ i\in\{1,...,m\}}} \in \mathcal{M}_{n,m}.$$

Definition 60 *A matrix $A \in M_{n,n}$ is said to be symmetric if $A = A^T$, i.e., $\forall i,j \in \{1,...,n\}$, $a_{ij} = a_{ji}$.*

Remark 61 *We can write a matrix $A_{m\times n} = [a_{ij}]$ as*

$$A = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix} = \left[C^1(A), ..., C^j(A), ..., C^n(A)\right]$$

where

$$R^i(A) = [a_{i1}, ..., a_{ij}, ...a_{in}] := \left[R^{i1}(A), ..., R^{ij}(A), ...R^{in}(A)\right] \in \mathbb{R}^n \quad \text{for } i \in \{1,...,m\} \quad \text{and}$$

$$C^j(A) = \begin{bmatrix} a_{1j} \\ \\ a_{ij} \\ \\ a_{mj} \end{bmatrix} := \begin{bmatrix} C^{j1}(A) \\ \\ C^{ji}(A) \\ \\ C^{jm}(A) \end{bmatrix} \in \mathbb{R}^n \qquad \text{for } j \in \{1,...,n\}.$$

In other words, $R^i(A)$ denotes row i of the matrix A and $C^j(A)$ denotes column j of matrix A.

3.1 Matrix operations

Definition 62 *Two matrices* $A_{m\times n} := [a_{ij}]$ *and* $B_{m\times n} := [b_{ij}]$ *are equal if for* $i \in \{1,...,m\}$, $j \in \{1,...,n\}$

$$\forall i \in \{1,...,m\}, j \in \{1,...,n\}, \quad a_{ij} = b_{ij}.$$

Definition 63 *Given the matrices* $A_{m\times n} := [a_{ij}]$ *and* $B_{m\times n} := [b_{ij}]$, *the sum of* A *and* B, *denoted by* $A+B$ *is the matrix* $C_{m\times n} = [c_{ij}]$ *such that*

$$\forall i \in \{1,...,m\}, j \in \{1,...,n\}, \qquad c_{ij} = a_{ij} + b_{ij}$$

Definition 64 *Given the matrices* $A_{m\times n} := [a_{ij}]$ *and the scalar* α, *the product of the matrix* A *by the scalar* α, *denoted by* $\alpha \cdot A$ *or* αA, *is the matrix obtained by multiplying each entry* A *by* α :

$$\alpha A := [\alpha a_{ij}]$$

Remark 65 *It is easy to verify that the set of matrices* $\mathcal{M}_{m,n}$ *with the above defined sum and scalar multiplication satisfies all the properties listed for elements of* $\mathbb{R}^n$ *in Section 2.1.*

Definition 66 *Given* $A = [a_{ij}] \in \mathcal{M}_{m,n}$, $B = [b_{jk}] \in \mathcal{M}_{n,p}$, *the product* $A \cdot B$ *is a matrix* $C = [c_{ik}] \in \mathcal{M}_{m,p}$ *such that*

$$\forall i \in \{1,...m\}, \forall k \in \{1,...,p\}, \qquad c_{ik} := \sum_{j=1}^{n} a_{ij} b_{jk} = R^i(A) \cdot C^k(B)$$

i.e., since

$$A = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix}, B = \left[C^1(B), ..., C^k(B), ..., C^p(B)\right] \tag{3.1}$$

$$AB = \begin{bmatrix} R^1(A)\cdot C^1(B) & ... & R^1(A)\cdot C^k(B) & ... & R^1(A)\cdot C^p(B) \\ ... \\ R^i(A)\cdot C^1(B) & ... & R^i(A)\cdot C^k(B) & ... & R^i(A)\cdot C^p(B) \\ ... \\ R^m(A)\cdot C^1(B) & ... & R^m(A)\cdot C^k(B) & ... & R^m(A)\cdot C^p(B) \end{bmatrix} \tag{3.2}$$

Remark 67 *If* $A \in \mathcal{M}_{1n}$, $B \in \mathcal{M}_{n,1}$, *the above definition coincides with the definition of scalar product between elements of* $\mathbb{R}^n$. *In what follows, we often identify an element of* $\mathbb{R}^n$ *with a row or a column vectors (- see Definition 22) consistently with we what write. In other words* $A_{m\times n}x = y$ *means that* x *and* y *are column vector with* n *entries, and* $wA_{m\times n} = z$ *means that* w *and* z *are row vectors with* m *entries.*

Definition 68 *If two matrices are such that a given operation between them is well defined, we say that they are conformable with respect to that operation.*

Remark 69 *If* $A, B \in M_{m,n}$, *they are conformable with respect to matrix addition. If* $A \in M_{m,n}$ *and* $B \in M_{n,p}$, *they are conformable with respect to multiplying* A *on the left of* B. *We often say the two matrices are conformable and let the context define precisely the sense in which conformability is to be understood.*

Remark 70 *(For future use)* $\forall k \in \{1,...,p\}$,

$$A \cdot C^k(B) = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix} \cdot C^k(B) = \begin{bmatrix} R^1(A)\cdot C^k(B) \\ ... \\ R^i(A)\cdot C^k(B) \\ ... \\ R^m(A)\cdot C^k(B) \end{bmatrix} \tag{3.3}$$

Then, just comparing (3.2) and (3.3), we get

$$AB = \begin{bmatrix} A \cdot C^1(B) & ... & A \cdot C^k(B) & ... & A \cdot C^p(B) \end{bmatrix} \tag{3.4}$$

Similarly, $\forall i \in \{1, ..., m\}$,

$$\begin{aligned} R^i(A) \cdot B = R^i(A) \cdot \begin{bmatrix} C^1(B) & ... & C^k(B) & ... & C^p(B) \end{bmatrix} = \\ = \begin{bmatrix} R^i(A) \cdot C^1(B) & ... & R^i(A) \cdot C^k(B) & ... & R^i(A) \cdot C^p(B) \end{bmatrix} \end{aligned} \tag{3.5}$$

Then, just comparing (3.2) and (3.5), we get

$$AB = \begin{bmatrix} R^1(A)B \\ ... \\ R^i(A)B \\ ... \\ R^m(A)B \end{bmatrix} \tag{3.6}$$

Definition 71 *A submatrix of a matrix* $A \in M_{m,n}$ *is a matrix obtained from* A *erasing some rows and columns.*

Definition 72 *A matrix* $A \in M_{m,n}$ *is partitioned in blocks if it is written as submatrices using a system of horizontal and vertical lines.*

Example 73 *The matrix*

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \end{bmatrix}$$

can be partitioned in block submatrices in several ways. For example as follows

$$\begin{bmatrix} 1 & 2 & | & 3 & 4 & 5 \\ 6 & 7 & | & 8 & 9 & 0 \\ - & - & | & - & - & - \\ 1 & 2 & | & 3 & 4 & 5 \\ 6 & 7 & | & 8 & 9 & 0 \end{bmatrix},$$

whose blocks are

$$\begin{bmatrix} 1 & 2 \\ 6 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 5 \\ 8 & 9 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 6 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 5 \\ 8 & 9 & 0 \end{bmatrix}.$$

The reason of the partition into blocks is that the result of operations on block matrices can obtained by carrying out the computation with blocks, just as if they were actual scalar entries of the matrices, as described below.

Remark 74 *We verify below that for matrix multiplication, we do not commit an error if, upon conformably partitioning two matrices, we proceed to regard the partitioned blocks as real numbers and apply the usual rules.*

1. Take $a := (a_i)_{i=1}^{n_1} \in \mathbb{R}^{n_1}, b := (b_j)_{j=1}^{n_2} \in \mathbb{R}^{n_2}, c := (c_i)_{i=1}^{n_1} \in \mathbb{R}^{n_1}, d := (d_j)_{j=1}^{n_2} \in \mathbb{R}^{n_2}$,

$$\begin{bmatrix} a & | & b \end{bmatrix}_{1\times(n_1+n_2)} \begin{bmatrix} c \\ - \\ d \end{bmatrix}_{(n_1+n_2)\times 1} = \sum_{i=1}^{n_1} a_i c_i + \sum_{j=1}^{n_2} b_j d_j = a \cdot c + b \cdot d. \tag{3.7}$$

2.

Take $A \in M_{m,n_1}, B \in M_{m,n_2}, C \in M_{n_1,p}, D \in M_{n_2,p}$, *with*

$$A = \begin{bmatrix} R^1(A) \\ ... \\ R^m(A) \end{bmatrix}, B = \begin{bmatrix} R^1(B) \\ ... \\ R^m(B) \end{bmatrix},$$

$$C = \left[C^1(C), ..., C^p(C)\right],$$

$$D = \left[C^1(D), ..., C^p(D)\right]$$

Then,

$$\begin{bmatrix} A & B \end{bmatrix}_{m\times(n_1+n_2)} \begin{bmatrix} C \\ D \end{bmatrix}_{(n_1+n_2)\times p} = \begin{bmatrix} R^1(A) & R^1(B) \\ ... & ... \\ R^m(A) & R^m(B) \end{bmatrix} \begin{bmatrix} C^1(C), ..., C^p(C) \\ C^1(D), ..., C^p(D) \end{bmatrix} =$$

$$= \begin{bmatrix} R^1(A)\cdot C^1(C) + R^1(B)\cdot C^1(D) & ... & R^1(A)\cdot + R^1(B)\cdot C^p(D) \\ ... & & \\ R^m(A)\cdot C^1(C) + R^m(B)\cdot C^1(D) & & R^m(A)\cdot C^p(C) + R^m(B)\cdot C^p(D) \end{bmatrix} =$$

$$= \begin{bmatrix} R^1(A)\cdot C^1(C) & ... & R^1(A)\cdot C^p(C) \\ ... & & \\ R^m(A)\cdot C^1(C) & & R^m(A)\cdot C^p(C) \end{bmatrix} + \begin{bmatrix} R^1(B)\cdot C^1(D) & ... & R^1(B)\cdot C^p(D) \\ ... & & \\ R^m(B)\cdot C^1(D) & & R^m(B)\cdot C^p(D) \end{bmatrix} =$$

$$= AC + BD.$$

Definition 75 *Let the matrices $A_i \in \mathcal{M}(n_i, n_i)$ for $i \in \{1, ..., K\}$, then the matrix*

$$A = \begin{bmatrix} A_1 & & & & & \\ & A_2 & & & & \\ & & \ddots & & & \\ & & & A_i & & \\ & & & & \ddots & \\ & & & & & A_K \end{bmatrix} \in \mathbb{M}\left(\sum_{i=1}^{K} n_i, \sum_{i=1}^{K} n_i\right)$$

is called block diagonal matrix.

Very often having information on the matrices A_i gives information on A.

Remark 76 *It is easy, but cumbersome, to verify the following properties.*

1. (associative) $\forall A \in \mathcal{M}_{m,n}$, $\forall B \in \mathcal{M}_{np}$, $\forall C \in \mathcal{M}_{pq}$, $A(BC) = (AB)C$;
2. (distributive) $\forall A \in \mathcal{M}_{m,n}$, $\forall B \in \mathcal{M}_{m,n}$, $\forall C \in \mathcal{M}_{np}$, $(A + B)C = AC + BC$.
3. $\forall x, y \in \mathbb{R}^n$ and $\forall \alpha, \beta \in \mathbb{R}$,

$$A(\alpha x + \beta y) = A(\alpha x) + B(\beta y) = \alpha Ax + \beta Ay$$

It is false that:
1. (commutative) $\forall A \in \mathcal{M}_{m,n}$, $\forall B \in \mathcal{M}_{np}$, $AB = BA$;
2. $\forall A \in \mathcal{M}_{m,n}$, $\forall B, C \in \mathcal{M}_{np}$, $\langle A \neq 0, AB = AC\rangle \Longrightarrow \langle B = C\rangle$;
3. $\forall A \in \mathcal{M}_{m,n}$, $\forall B \in \mathcal{M}_{np}$, $\langle A \neq 0, AB = 0\rangle \Longrightarrow \langle B = 0\rangle$.
Let's show why the above statements are false.
1.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 5 \\ -1 & 1 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$CD = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 2 & 2 \end{bmatrix}$$

$$DC = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 8 \end{bmatrix}$$

Observe that since the commutative property does not hold true, we have to distinguish between "left factor out" and "right factor out" and also between "left multiplication or pre-multiplication" and "right multiplication or post-multiplication":

$$\begin{array}{c} AB + AC = A(B + C) \\ EF + GF = (E + G)F \\ AB + CA \neq A(B + C) \\ AB + CA \neq (B + C)A \end{array}$$

2.

Given

$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 \\ -5 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 14 & 18 \end{bmatrix}$$

$$AC = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 14 & 18 \end{bmatrix}$$

3.

Observe that 3. $\Rightarrow$ 2. and therefore $\neg$2. $\Rightarrow$ $\neg$3. Otherwise, you can simply observe that 3. follows from 2., choosing A in 3. equal to A in 2., and B in 3. equal to $B - C$ in 2.:

$$A(B - C) = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 4 & 1 \\ -5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since the associative property of the product between matrices does hold true we can give the following definition.

Definition 77 *Given* $A \in \mathcal{M}_{m,m}$*,*

$$A^k := \underset{1}{A} \cdot \underset{2}{A} \cdot \ldots \cdot \underset{k\ times}{A}.$$

Observe that if $A \in \mathcal{M}_{m,m}$ and $k, l \in \mathbb{N} \setminus \{0\}$, then

$$A^k \cdot A^l = A^{k+l}.$$

Remark 78 ***Properties of transpose matrices.***

1.	$\forall A \in \mathcal{M}_{m,n}$	$(A^T)^T = A$
2.	$\forall A, B \in \mathcal{M}_{m,n}$	$(A + B)^T = A^T + B^T$
3.	$\forall \alpha \in \mathbb{R}, \forall A \in \mathcal{M}_{m,n}$	$(\alpha A)^T = \alpha A^T$
4.	$\forall A \in \mathcal{M}_{m,n}, \forall B \in \mathcal{M}_{n,m}$	$(AB)^T = B^T A^T$

Matrices and linear systems.

In Section 1.6, we have seen that a system of m linear equation in the n unknowns $x_1, x_2, ..., x_n$ and parameters a_{ij}, for $i \in \{1, ..., m\}$, $j \in \{1, ..., n\}$, $(b_i)_{i=1}^n \in \mathbb{R}^n$ is displayed below:

$$\begin{cases} a_{11}x_1 + ... + a_{1j}x_j + ... + a_{1n}x_n = b_1 \\ ... \\ a_{i1}x_1 + ... + a_{ij}x_j + ... + a_{in}x_n = b_i \\ ... \\ a_{m1}x_i + ... + a_{mj}x_j + ... + a_{mn}x_n = b_m \end{cases} \tag{3.8}$$

Moreover, the matrix

$$\begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} & b_1 \\ ... \\ a_{i1} & ... & a_{ij} & ... & a_{in} & b_i \\ ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix M of system (1.4). The coefficient matrix A of the system is

$$A = \begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... \\ a_{i1} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}$$

Using the notations we described in the present section, we can rewrite linear equations and systems of linear equations in a convenient and short manner, as described below.

The linear equation in the unknowns $x_1, ..., x_n$ and parameters $a_1, ..., a_i, ..., a_n$, $b \in \mathbb{R}$

$$a_1x_1 + ... + a_ix_i + ... + a_nx_n = b$$

can be rewritten as

$$\sum_{i=1}^{n} a_ix_i = b$$

or

$$a \cdot x = b$$

where $a = [a_1, ..., a_n]$ and $x = \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix}$.

The linear system (3.8) can be rewritten as

$$\begin{cases} \sum_{j=1}^{n} a_{1j}x_j = b_1 \\ ... \\ \sum_{j=1}^{n} a_{mj}x_j = b_m \end{cases}$$

or

$$\begin{cases} R^1(A)\, x = b_1 \\ ... \\ R^m(A)\, x = b_m \end{cases}$$

or

$$Ax = b$$

where $A = [a_{ij}]$.

Definition 79 *The trace of $A \in \mathcal{M}_{mm}$, written* tr A, *is the sum of the diagonal entries, i.e.,*

$$\text{tr } A = \sum_{i=1}^{m} a_{ii}.$$

Definition 80 *The identity matrix I_m is a diagonal matrix of order m with each element on the principal diagonal equal to 1. If no confusion arises, we simply write I in the place of I_m.*

Remark 81 ***1.*** $\forall n \in \mathbb{N}\setminus\{0\}$, $(I_m)^n = I_m$;
2. $\forall A \in \mathcal{M}_{m,n}$, $I_mA = AI_n = A$.

Proposition 82 *Let $A, B \in \mathcal{M}(m,m)$ and $k \in \mathbb{R}$. Then*
1. tr $(A + B) =$ tr $A +$ tr B;
2. tr $kA = k \cdot$ tr A;
3. tr $AB =$ tr BA.

Proof. Exercise. ■

3.2 Inverse matrices

Definition 83 *Given a matrix* $A_{n\times n}$, *, a matrix* $B_{n\times n}$ *is called an inverse of* A *if*

$$AB = BA = I_n.$$

We then say that A *is invertible, or that* A *admits an inverse.*

Proposition 84 *If* A *admits an inverse, then the inverse is unique.*

Proof. Let the inverse matrices B and C of A be given. Then

$$AB = BA = I_n \tag{3.9}$$

and

$$AC = CA = I_n \tag{3.10}$$

Left multiplying the first two terms in the equality (3.9) by C, we get

$$(CA)\,B = C\,(BA)$$

and from (3.10) and (3.9) we get $B = C$, as desired. ■
Thanks to the above Proposition, we can present the following definition.

Definition 85 *If the inverse of* A *does exist, then it is denoted by* A^{-1}.

Example 86 *Assume that for* $i \in \{1, ..., n\}$, $\lambda_i \neq 0$. *The diagonal matrix*

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

is invertible and its inverse is

$$\begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{bmatrix}.$$

Remark 87 *If a row or a column of* A *is zero, then* A *is not invertible, as verified below.*
Without loss of generality, assume the first row of A *is equal to zero. Assume that* B *is the inverse of* A*. But then, since* $I = AB$*, we wold have* $1 = R^1(A) \cdot C^1(B) = 0$*, a contradiction.*

Proposition 88 *If* $A \in \mathcal{M}_{m,m}$ *and* $B \in \mathcal{M}_{m,m}$ *are invertible matrices, then* AB *is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof.

$$(AB)\,B^{-1}A^{-1} = A\left(BB^{-1}\right)A^{-1} = AIA^{-1} = AA^{-1} = I.$$

$$B^{-1}A^{-1}\,(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}IB = B^{-1}B = I.$$

■

Remark 89 *The existence of the inverse matrix gives an obvious way of solving systems of linear equations with the same number of equations and unknowns.*
Given the system

$$A_{n\times n}x = b,$$

if A^{-1} *exists, then*

$$x = A^{-1}b.$$

Proposition 90 *(Some other properties of the inverse matrix)*
Let the invertible matrix A be given.
1. A^{-1} in invertible and $\left(A^{-1}\right)^{-1} = A$;
2. A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$;

Proof. 1. We want to verify that the inverse of A^{-1} is A, i.e.,

$$A^{-1}A = I \text{ and } AA^{-1} = I,$$

which is obvious.
2. Observe that

$$A^T \cdot \left(A^{-1}\right)^T = \left(A^{-1} \cdot A\right)^T = I^T = I,$$

and

$$\left(A^{-1}\right)^T \cdot A^T = \left(A \cdot A^{-1}\right)^T = I.$$

■

3.3 Elementary matrices

Below, we recall the definition of elementary row operations on a matrix $A \in \mathrm{M}_{m\times n}$ presented in Definition 34.

Definition 91 *An elementary row operation on a matrix $A \in M_{m\times n}$ is one of the following operations on the rows of A:*

$[\mathcal{E}_1]$ *(Row interchange) Interchange R^i with R^j, denoted by $R^i \leftrightarrow R^j$;*

$[\mathcal{E}_2]$ *(Row scaling) Multiply R^i by $k \in \mathbb{R}\backslash\{0\}$, denoted by $kR^i \to R^i$, $k \neq 0$;*

$[\mathcal{E}_3]$ *(Row addition) Replace R^i by (k times R^j plus R^i), denoted by $\left(R^i + kR^j\right) \to R^i$.*

Sometimes we apply $[\mathcal{E}_2]$ and $[\mathcal{E}_3]$ in one step, i.e., we perform the following operation

$[\mathcal{E}']$ Replace R^i by (k' times R^j and $k \in \mathbb{R}\backslash\{0\}$ times R^i), denoted by $\left(k'R^j + kR^i\right) \to R^i, k \neq 0$.

Definition 92 *Let $\mathfrak{E}$ be the set of functions $\mathcal{E} : \mathcal{M}_{m,n} \to \mathcal{M}_{m,n}$ which associate with any matrix $A \in \mathcal{M}_{m,n}$ a matrix $\mathcal{E}(A)$ obtained from A via an elementary row operation presented in Definition 91. For $i \in \{1, 2, 3\}$, let $\mathfrak{E}_i \subseteq \mathfrak{E}$ be the set of elementary row operation functions of type i presented in Definition 91.*

Definition 93 *For any $\mathcal{E} \in \mathfrak{E}$, define*

$$E_{\mathcal{E}} = \mathcal{E}(I_m) \in \mathcal{M}_{m,m}.$$

$E_{\mathcal{E}}$ is called the elementary matrix corresponding to the elementary row operation function $\mathcal{E}$.

With some abuse of terminology, we call any $\mathcal{E} \in \mathfrak{E}$ an elementary row operation (omitting the word "function"), and we sometimes omit the subscript $\mathcal{E}$.

Proposition 94 *Each elementary row operations $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ has an inverse, and that inverse is of the same type, i.e., for $i \in \{1, 2, 3\}$, $\mathcal{E} \in \mathfrak{E}_i \Leftrightarrow \mathcal{E}^{-1} \in \mathfrak{E}_i$.*

Proof. 1. The inverse of $R^i \leftrightarrow R^j$ is $R^j \leftrightarrow R^i$.
2. The inverse of $kR^i \to R^i$, $k \neq 0$ is $k^{-1}R^i \to R^i$.
3. The inverse of $\left(R^i + kR^j\right) \to R^i$ is $\left(-kR^j + R^i\right) \to R^i$. ■

Remark 95 *Given the row, canonical*[1] *vectors* e_m^i, *for* $i \in \{1, ..., m\}$,

$$I_m = \begin{bmatrix} e_m^1 \\ ... \\ e_m^i \\ ... \\ e_m^j \\ ... \\ e_m^n \end{bmatrix}$$

The following Proposition shows that the result of applying an elementary row operation $\mathcal{E}$ to a matrix A can be obtained by premultiplying A by the corresponding elementary matrix $E_{\mathcal{E}}$.

Proposition 96 *For any* $A \in M_{m,n}$ *and for any* $\mathcal{E} \in \mathfrak{E}$,

$$\mathcal{E}(A) = \mathcal{E}(I_m) \cdot A := E_{\mathcal{E}} A. \tag{3.11}$$

Proof. *Recall that*

$$A = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^j(A) \\ ... \\ R^m(A) \end{bmatrix}$$

We have to prove that (3.11) *does hold true* $\forall \mathcal{E} \in \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$.

1. $\mathcal{E} \in \mathfrak{E}_1$.

First of all observe that

$$\mathcal{E}(I_m) = \begin{bmatrix} e_m^1 \\ ... \\ e_m^j \\ ... \\ e_m^i \\ ... \\ e_m^m \end{bmatrix} \quad \textit{and} \quad \mathcal{E}(A) = \begin{bmatrix} R^1(A) \\ ... \\ R^j(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix}.$$

From (3.6),

$$\mathcal{E}(I_m) \cdot A = \begin{bmatrix} e_m^1 \cdot A \\ ... \\ e_m^j \cdot A \\ ... \\ e_m^i \cdot A \\ ... \\ e_m^m \cdot A \end{bmatrix} = \begin{bmatrix} R^1(A) \\ ... \\ R^j(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix},$$

as desired.

2. $\mathcal{E} \in \mathfrak{E}_2$.

Observe that

$$\mathcal{E}(I_m) = \begin{bmatrix} e_m^1 \\ ... \\ k \cdot e_m^i \\ ... \\ e_m^j \\ ... \\ e_m^m \end{bmatrix} \quad \textit{and} \quad \mathcal{E}(A) = \begin{bmatrix} R^1(A) \\ ... \\ k \cdot R^j(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix}.$$

[1] See Definition 55.

$$\mathcal{E}(I_m)\cdot A=\begin{bmatrix} e_m^1\cdot A\\ ...\\ k\cdot e_m^i\cdot A\\ ...\\ e_m^j\cdot A\\ ...\\ e_m^m\cdot A\end{bmatrix}=\begin{bmatrix} R^1(A)\\ ...\\ k\cdot R^i(A)\\ ...\\ R^j(A)\\ ...\\ R^m(A)\end{bmatrix},$$

as desired.
3. $\mathcal{E}\in\mathfrak{E}_3$.
Observe that

$$\mathcal{E}(I_m)=\begin{bmatrix} e_m^1\\ ...\\ e_m^i+k\cdot e_m^j\\ ...\\ e_m^j\\ ...\\ e_m^m\end{bmatrix}\quad\text{and}\quad\mathcal{E}(A)=\begin{bmatrix} R^1(A)\\ ...\\ R^i(A)+k\cdot R^j(A)\\ ...\\ R^i(A)\\ ...\\ R^m(A)\end{bmatrix}.$$

$$\mathcal{E}(I_m)\cdot A=\begin{bmatrix} e_m^1\\ ...\\ e_m^i+k\cdot e_m^j\\ ...\\ e_m^j\\ ...\\ e_m^m\end{bmatrix}A=\begin{bmatrix} e_m^1\cdot A\\ ...\\ \left(e_m^i+k\cdot e_m^j\right)\cdot A\\ ...\\ e_m^j\cdot A\\ ...\\ e_m^m\cdot A\end{bmatrix}=\begin{bmatrix} R^1(A)\\ ...\\ R^i(A)+k\cdot R^j(A)\\ ...\\ R^j(A)\\ ...\\ R^m(A)\end{bmatrix},$$

as desired. ■

Corollary 97 *If A is row equivalent to B, then there exist $k\in\mathbb{N}$ and elementary matrices $E_1,...,E_k$ such that*

$$B=E_1\cdot E_2\cdot...\cdot E_k\cdot A$$

Proof. It follows from the definition of row equivalence and Proposition 96. ■

Proposition 98 *Every elementary matrix $E_{\mathcal{E}}$ is invertible and $(E_{\mathcal{E}})^{-1}$ is an elementary matrix. In fact, $(E_{\mathcal{E}})^{-1}=E_{\mathcal{E}^{-1}}$.*

Proof. Given an elementary matrix E, from Definition 93, $\exists\mathcal{E}\in\mathfrak{E}$ such that

$$E=\mathcal{E}(I)\tag{3.12}$$

Define

$$E'=\mathcal{E}^{-1}(I).$$

Then

$$I\overset{\text{def. inv. func.}}{=}\mathcal{E}^{-1}(\mathcal{E}(I))\overset{(3.12)}{=}\mathcal{E}^{-1}(E)\overset{\text{Prop. (96).}}{=}\mathcal{E}^{-1}(I)\cdot E\overset{\text{def. }E'}{=}E'E$$

and

$$I\overset{\text{def. inv.}}{=}\mathcal{E}\left(\mathcal{E}^{-1}(I)\right)\overset{\text{def. }E'}{=}\mathcal{E}(E')\overset{\text{Prop. (96).}}{=}\mathcal{E}(I)\cdot E'\overset{(3.12)}{=}EE'.$$

■

Corollary 99 *If $E_1,...,E_k$ are elementary matrices, then*

$$P:=E_1\cdot E_2\cdot...\cdot E_k$$

is an invertible matrix.

Proof. It follows from Proposition 88 and Proposition 98. In fact, $\left(E_k^{-1}\cdot...E_2^{-1}\cdot E_1^{-1}\right)$ is the inverse of P. ■

Proposition 100 *Let $A \in M_{m\times n}$ be given. Then, there exist a matrix $B \in M_{m\times n}$ in row canonical form, $k \in \mathbb{N}$ and elementary matrices $E_1, ..., E_k$ such that*

$$B = E_1 \cdot E_2 \cdot ... \cdot E_k \cdot A.$$

Proof. From Proposition 38, there exist $k \in \mathbb{N}$ elementary operations $\mathcal{E}^1, ..., \mathcal{E}^k$ such that

$$\left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^k\right)(A) = B.$$

From Proposition 96, $\forall j \in \{1, ..., k\}$,

$$\mathcal{E}^j(M) = \mathcal{E}^j(I) \cdot M := E_j \cdot M.$$

Then,

$$\left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^k\right)(A) = \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^{k-1}\right)\left(\mathcal{E}^k(A)\right) = \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^{k-1}\right)(E_k \cdot A) =$$

$$= \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^{k-2}\right) \circ \mathcal{E}^{k-1}(E_k \cdot A) = \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^{k-2}\right)(E_{k-1} \cdot E_k \cdot A) =$$

$$= \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ...\mathcal{E}^{k-3}\right) \circ \mathcal{E}^{k-2}(E_{k-1} \cdot E_k \cdot A) = \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ...\mathcal{E}^{k-3}\right)(E_{k-2} \cdot E_{k-1} \cdot E_k \cdot A) =$$

$$... = E_1 \cdot E_2 \cdot ... \cdot E_k \cdot A,$$

as desired. ■

Remark 101 *In fact, in Proposition 153, we will show that the matrix B of the above Corollary is unique.*

Proposition 102 *To be row equivalent is an equivalence relation.*

Proof. Obvious. ■

Proposition 103 $\forall n \in \mathbb{N}\setminus\{0\}$ *, $A_{n\times n}$ is in row canonical form and it is invertible $\Leftrightarrow A = I$.*

Proof. [$\Leftarrow$]Obvious.
[$\Rightarrow$]
We proceed by induction on n.
Case 1. $n = 1$.
The case $n = 1$ is obvious. To try to better understand the logic of the proof, take $n = 2$, i.e., suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is in row canonical form and invertible. Observe that $A \neq 0$.
1. $a_{11} = 1$. Suppose $a_{11} = 0$. Then, from 1. in the definition of matrix in echelon form - see Definition 28 - $a_{12} \neq 0$ (otherwise, you would have a zero row not on the bottom of the matrix). Then, from 2. in that definition, we must have $a_{21} = 0$. But then the first column is zero, contradicting the fact that A is invertible - see Remark 87. Since $a_{11} \neq 0$, then from 2. in the Definition of row canonical form matrix - see Definition 31 - we get $a_{11} = 1$.
2. $a_{21} = 0$. It follows from the fact that $a_{11} = 1$ and 3. in Definition 31.
3. $a_{22} = 1$. Suppose $a_{22} = 0$, but then the last row would be zero, contradicting the fact that A is invertible and a_{22} is the leading nonzero entry of the second row, i.e., $a_{22} \neq 0$. Then from 2. in the Definition of row canonical form matrix, we get $a_{22} = 1$.
4. $a_{12} = 0$. It follows from the fact that $a_{22} = 1$ and 3. in Definition 31.
Case 2. Assume that statement is true for $n - 1$.
Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1j} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2j} & ... & a_{2n} \\ ... \\ a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{n1} & a_{n2} & ... & a_{nj} & ... & a_{nn} \end{bmatrix}$$

is in row canonical form and invertible.

1. $a_{11} = 1$. Suppose $a_{11} = 0$. Then, from 1. in the definition of matrix in echelon form - see Definition 28 -

$$(a_{12}, ..., a_{1n}) \neq 0.$$

Then, from 2. in that definition, we must have

$$\begin{bmatrix} a_{21} \\ ... \\ a_{i1} \\ ... \\ a_{n1} \end{bmatrix} = 0.$$

But then the first column is zero, contradicting the fact that A is invertible - see Remark 87. Since $a_{11} \neq 0$, then from 2. in the Definition of row canonical form matrix - see Definition 31 - we get $a_{11} = 1$.

2. Therefore, we can rewrite the matrix as follows

$$A = \begin{bmatrix} 1 & a_{12} & ... & a_{1j} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2j} & ... & a_{2n} \\ ... \\ 0 & a_{i2} & ... & a_{ij} & ... & a_{in} \\ ... \\ 0 & a_{n2} & ... & a_{nj} & ... & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & A_{22} \end{bmatrix} \tag{3.13}$$

with obvious definitions of a and A_{22}. Since, by assumption, A is invertible, there exists B which we can partition in the same we partitioned A, i.e.,

$$B = \begin{bmatrix} b_{11} & b \\ c & B_{22} \end{bmatrix}.$$

and such that B is invertible. Then,

$$I_n = BA = \begin{bmatrix} b_{11} & b \\ c & B_{22} \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{11} + bB_{22} \\ c & ca + B_{22}A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \end{bmatrix};$$

then $c = 0$ and $A_{22}B_{22} = I_{n-1}$.

Moreover,

$$I_n = AB = \begin{bmatrix} 1 & a \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b \\ c & B_{22} \end{bmatrix} = \begin{bmatrix} b_{11} + ac & b + aB_{22} \\ c & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \end{bmatrix}. \tag{3.14}$$

Therefore, A_{22} is invertible. From 3.13, A_{22}can be obtained from A erasing the first row and then erasing a column of zero, from Remark 33, A_{22} is a row reduced form matrix. Then, we can apply the assumption of the induction argument to conclude that $A_{22} = I_{n-1}$. Then, from 3.13,

$$A = \begin{bmatrix} 1 & a \\ 0 & I \end{bmatrix}.$$

Since, by assumption, $A_{n\times n}$ is in row canonical form, from 3. in Definition 31, $a = 0$, and, as desired $A = I$. ■

Proposition 104 *Let A belong to $M_{m,m}$. Then the following statements are equivalent.*

1. *A is invertible;*
2. *A is row equivalent to I_m;*
3. *A is the product of elementary matrices.*

Proof. 1. $\Rightarrow$ 2.

From Proposition 100, there exist a matrix $B \in \mathrm{M}_{m\times m}$ in row canonical form, $k \in \mathbb{N}$ and elementary matrices $E_1, ..., E_k$ such that

$$B = E_1 \cdot E_2 \cdot ... \cdot E_k \cdot A.$$

Since A is invertible and, from Corollary 99, $E_1 \cdot E_2 \cdot ... \cdot E_k$ is invertible as well, from Proposition 88, B is invertible as well. Then, from Proposition 103, $B = I$.

2. $\Rightarrow$ 3.

By assumption and from Corollary 97, there exist $k \in \mathbb{N}$ and elementary matrices $E_1, ..., E_k$ such that

$$A = E_1 \cdot E_2 \cdot ... \cdot E_k \cdot I,$$

Since $\forall i \in \{1, ..., k\}$, E_i is an elementary matrix, the desired result follows.

3. $\Rightarrow$ 1.

By assumption, there exist $k \in \mathbb{N}$ and elementary matrices $E_1, ..., E_k$ such that

$$A = E_1 \cdot E_2 \cdot ... \cdot E_k.$$

Since, from Proposition 98, $\forall i \in \{1, ..., k\}$, E_i is invertible, A is invertible as well, from Proposition 88. ∎

Proposition 105 *Let $A_{m\times n}$ be given.*

1. $B_{m\times n}$ is row equivalent to $A_{m\times n}$ $\Leftrightarrow$ there exists an invertible $P_{m\times m}$ such that $B = PA$.

2. $P_{m\times m}$ is an invertible matrix $\Rightarrow$ PA is row equivalent to A.

Proof. 1.

[$\Rightarrow$] From Corollaries 99 and 97, $B = E_1 \cdot ... \cdot E_k \cdot A$ with $(E_1 \cdot ... \cdot E_k)$ invertible matrix. Then, it suffices to take $P = E_1 \cdot ... \cdot E_k$.

[$\Leftarrow$] From Proposition 104, P is row equivalent to I, i.e., there exist $E_1, ..., E_k$ such that $P = E_1 \cdot ... \cdot E_k \cdot I$. Then by assumption $B = E_1 \cdot ... \cdot E_k \cdot I \cdot A$, i.e., B is row equivalent to A.

2.

From Proposition 104, P is the product of elementary matrices. Then, the desired result follows from Proposition 96. ∎

Proposition 106 *If A is row equivalent to a matrix with a zero row, then A is not invertible.*

Proof. Suppose otherwise, i.e., A is row equivalent to a matrix C with a zero row and A is invertible. From Proposition 105, there exists an invertible P such that $A = PC$ and then $P^{-1}A = C$. Since A and $P^{-1}$are invertible, then, from Proposition 88, $P^{-1}A$ is invertible, while C, from Remark 87, C is not invertible, a contradiction. ∎

Remark 107 *From Proposition 104, we know that if $A_{m\times m}$ is invertible, then there exist $E_1, ..., E_k$ such that*

$$I = E_1 \cdot ... \cdot E_k \cdot A \tag{3.15}$$

or

$$A^{-1} = E_1 \cdot ... \cdot E_k \cdot I. \tag{3.16}$$

Then, from (3.15)*and* (3.16)*, if A is invertible then A^{-1} is equal to the finite product of those elementary matrices which "transform" A in I, or, equivalently, can be obtained applying a finite number of corresponding elementary operations to the identity matrix I. That observation leads to the following (Gaussian elimination) algorithm, which either show that an arbitrary matrix $A_{m\times m}$ is not invertible or finds the inverse of A.*

An algorithm to find the inverse of a matrix $A_{m\times m}$ or to show the matrix is not invertible.

Step 1. Construct the following matrix $M_{m\times(2m)}$:

$$\begin{bmatrix} A & I_m \end{bmatrix}$$

Step 2. Row reduce M to echelon form. If the process generates a zero row in the part of M corresponding to A, then stop: A is not invertible : A is row equivalent to a matrix with a zero row and therefore, from Proposition 106 is not invertible. Otherwise, the part of M corresponding to A is a triangular matrix.

Step 3. Row reduce M to the row canonical form

$$\begin{bmatrix} I_m & B \end{bmatrix}$$

Then, from Remark 107, $A^{-1} = B$.

Example 108 *We find the inverse of*

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix},$$

applying the above algorithm.
Step 1.

$$M = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{bmatrix}.$$

Step 2.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right], \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

The matrix is invertible.
Step 3.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right], \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right],$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right], \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

Then

$$A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}.$$

Example 109

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -10 & -4 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{4}{10} & -\frac{1}{10} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{2}{10} & \frac{0}{3} \\ 0 & 1 & \frac{4}{10} & -\frac{1}{10} \end{array}\right]$$

3.4 Elementary column operations

This section repeats some of the discussion of the previous section using column instead of rows of a matrix.

Definition 110 *An elementary column operation is one of the following operations on the columns of $A_{m\times n}$:*

$[\mathcal{F}_1]$ *(Column interchange) Interchange C_i with C_j, denoted by $C_i \leftrightarrow C_j$;*

$[\mathcal{F}_2]$ *(Column scaling) Multiply C_i by $k \in \mathbb{R}\backslash\{0\}$, denoted by $kC_i \to C_i$, $k \neq 0$;*

$[\mathcal{F}_3]$ *(Column addition) Replace C_i by (k times C_j plus C_i), denoted by $(C_i + kC_j) \to C_i$..*

Each of the above column operation has an inverse operation of the same type just like the corresponding row operations.

Definition 111 *Let $\mathcal{F}$ be an elementary column operation on a matrix $A_{m\times n}$. We denote the resulting matrix by $\mathcal{F}(A)$. We define also*

$$F_{\mathcal{F}} = \mathcal{F}(I_n) \in \mathcal{M}_{n,n}.$$

$F_{\mathcal{F}}$ is then called an elementary matrix corresponding to the elementary column operation $\mathcal{F}$. We sometimes omit the subscript $\mathcal{F}$.

Definition 112 *Given an elementary row operation $\mathcal{E}$, define $\mathcal{F}_{\mathcal{E}}$, if it exists*[2]*, as the column operation obtained by $\mathcal{E}$ substituting the word row with the word column. Similarly, given an elementary column operation $\mathcal{F}$ define $\mathcal{E}_{\mathcal{F}}$, if it exists, as the row operation obtained by $\mathcal{F}$ substituting the word column with the word row.*

In what follows, $\mathcal{F}$ and $\mathcal{E}$ are such that $\mathcal{F} = \mathcal{F}_{\mathcal{E}}$ and $\mathcal{E}_{\mathcal{F}} = \mathcal{E}$.

Proposition 113 *Let a matrix $A_{m\times n}$ be given. Then*

$$\mathcal{F}(A) = \left[\mathcal{E}\left(A^T\right)\right]^T.$$

Proof. The above fact is equivalent to $\mathcal{E}\left(A^T\right) = (\mathcal{F}(A))^T$ and it is a consequence of the fact that the columns of A are the rows of A^T and vice versa. As an exercise, carefully do the proof in the case of each of the three elementary operation types. ■

Remark 114 *The above Proposition says that applying the column operation $\mathcal{F}$ to a matrix A gives the same result as applying the corresponding row operation $\mathcal{E}_{\mathcal{F}}$ to A^T and then taking the transpose.*

Proposition 115 *Let a matrix $A_{m\times n}$ be given. Then*

1.

$$\mathcal{F}(A) = A \cdot (\mathcal{E}(I))^T = A \cdot \mathcal{F}(I),$$

or, since $E := \mathcal{E}(I)$ and $F := \mathcal{F}(I)$,

$$\mathcal{F}(A) = A \cdot E^T = A \cdot F. \tag{3.17}$$

2. $F = E^T$ and F is invertible.

Proof. 1.

$$\mathcal{F}(A) \overset{Lemma\ 113}{=} \left[\mathcal{E}\left(A^T\right)\right]^T \overset{Lemma\ 96}{=} \left(\mathcal{E}(I) \cdot A^T\right)^T = A \cdot (\mathcal{E}(I))^T \overset{Lemma\ 113}{=} A \cdot \mathcal{F}(I).$$

2. From (3.17), we then get

$$F := \mathcal{F}(I) = I \cdot E^T = E^T.$$

From Proposition 90 and Proposition 98, it follows that F is invertible. ■

[2] Of course, if you exchange the first and the third row, and the matrix has only two columns, you cannot exchange the first and the third column.

Remark 116 *The above Proposition says that says that the result of applying an elementary column operation $\mathcal{F}$ on a matrix A can be obtained by postmultiplying A by the corresponding elementary matrix F.*

Definition 117 *A matrix $B_{m\times n}$ is said column equivalent to a matrix $A_{m\times n}$ if B can be obtained from A using a finite number of elementary column operations.*

Remark 118 *By definition of row equivalent, column equivalent and transpose of a matrix, we have that*

$$A \text{ and } B \text{ are row equivalent} \Leftrightarrow \quad A^T \text{ and } B^T \text{ are column equivalent},$$

and

$$A \text{ and } B \text{ are column equivalent} \Leftrightarrow \quad A^T \text{ and } B^T \text{ are row equivalent.}$$

Proposition 119 *1. $B_{m\times n}$ is column equivalent to $A_{m\times n}$ $\Leftrightarrow$ there exists an invertible $Q_{n\times n}$ such that $B_{m\times n} = A_{m\times n}Q_{n\times n}$.*
2. $Q_{n\times n}$ in invertible matrix $\Rightarrow$ AQ is column equivalent to A.

Proof. It is very similar to the proof of Proposition 105. ■

Definition 120 *A matrix $B_{m\times n}$ is said equivalent to a matrix $A_{m\times n}$ if B can be obtained from A using a finite number of elementary row and column operations.*

Proposition 121 *A matrix $B_{m\times n}$ is equivalent to a matrix $A_{m\times n}$ $\Leftrightarrow$ there exist invertible matrices $P_{m\times m}$ and $Q_{n\times n}$ such that $B_{m\times n} = P_{m\times m}A_{m\times n}Q_{n\times n}$.*

Proof. [$\Rightarrow$]
By assumption $B = E_1 \cdot ... \cdot E_k \cdot A \cdot F_1 \cdot ... \cdot F_h$.
[$\Leftarrow$]
Similar to the proof of Proposition 105. ■

Proposition 122 *For any matrix $A_{m\times n}$ there exists a number $r \in \{0, 1, ..., \min\{m, n\}\}$ such that A is equivalent to the block matrix of the form*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.18}$$

Proof. The proof is constructive in the form of an algorithm.

Step 1. Row reduce A to row canonical form, with leading nonzero entries $a_{11}, a_{2j_2}, ..., a_{j j_r}$.

Step 2. Interchange C^2 and C_{j_2}, C^3 and C_{j_3} and so on up to C_r and C_{j_r}.You then get a matrix of the form

$$\begin{bmatrix} I_r & B \\ 0 & 0 \end{bmatrix}.$$

Step 3. Use column operations to replace entries in B with zeros.

■

Remark 123 *From Proposition 153 the matrix in Step 2 is unique and therefore the resulting matrix in Step 3, i.e., matrix* (3.18) *is unique.*

Proposition 124 *For any $A \in M_{m,n}$, there exists invertible matrices $P \in M_{m,m}$ and $Q \in M_{n,n}$ and $r \in \{0, 1, ..., \min\{m, n\}\}$ such that*

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Proof. It follows immediately from Propositions 122 and 121. ■

Remark 125 *From Proposition 153 the number r in the statement of the previous Proposition is unique.*

Proposition 126 *If $A_{m\times m}B_{m\times m} = I$, then $BA = I$ and therefore A is invertible and $A^{-1} = B$.*

Proof. Suppose A is not invertible, the from Proposition 104 is not row equivalent to I_m and from Proposition 122, A is equivalent to a block matrix of the form displayed in (3.18) with $r < m$. Then, from Proposition 121, there exist invertible matrices $P_{m\times m}$ and $Q_{m\times m}$ such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

and from $AB = I$, we get

$$P = PAQQ^{-1}B$$

and

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}(Q^{-1}B) = P$$

Therefore, P has some zero rows and columns, contradicting that P is invertible. ■

Remark 127 *The previous Proposition says that to verify that A in invertible it is enough to check that $AB = I$.*

Remark 128 *We will come back to the analysis of further properties of the inverse and on another way of computing it in Section 5.3*

Chapter 4

Vector spaces

4.1 Definition

Definition 129 *Let a nonempty set F with the operations of*
addition which assigns to any $x, y \in F$ an element denoted by $x \oplus y \in F$, and
multiplication which assigns to any $x, y \in F$ an element denoted by $x \odot y \in F$
be given. $(F, \oplus, \odot)$ is called a field, if the following properties hold true.

1. *(Commutative) $\forall x, y \in F$, $x \oplus y = y \oplus x$ and $x \odot y = y \odot x$;*
2. *(Associative) $\forall x, y, z \in F, (x \oplus y) \oplus z = x \oplus (y \oplus z)$ and $(x \odot y) \odot z = x \odot (y \odot z)$;*
3. *(Distributive) $\forall x, y, z \in F$, $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$;*
4. *(Existence of null elements) $\exists f_0, f_1 \in F$ such that $\forall x \in F$, $f_0 \oplus x = x$ and $f_1 \odot x = x$;*
5. *(Existence of a negative element) $\forall x \in F$ $\exists y \in F$ such that $x \oplus y = f_0$;*

   From the above properties, it follows that f_0 and f_1 are unique.[1] *We denote f_0 and f_1 by 0 and 1, respectively.*
6. *(Existence of an inverse element) $\forall x \in F \setminus \{0\}$, $\exists y \in F$ such that $x \odot y = 1$.*

   Elements of a field are called scalars.

Example 130 *The set $\mathbb{R}$ of real numbers with the standard addition and multiplication is a field. From the above properties all the rules of "elementary" algebra can be deduced.*[2]
The set $\mathbb{C}$ of complex numbers is a field - see Appendix 10.

Definition 131 *Let $(F, \oplus, \odot)$ be a field and V be a nonempty set with the operations of*
addition which assigns to any $u, v \in V$ an element denoted by $u + v \in V$, and
scalar multiplication which assigns to any $u \in V$ and any $\alpha \in F$ an element $\alpha \cdot u \in V$.
Then $(V, +, \cdot)$ is called a vector space on the field $(F, \oplus, \odot)$ and its elements are called vectors if the following properties are satisfied.
A1. (Associative) $\forall u, v, w \in V$, $(u + v) + w = u + (v + w)$;
A2. (existence of zero element) there exists an element 0 in V such that $\forall u \in V$, $u + 0 = u$;
A3. (existence of inverse element) $\forall u \in V$ $\exists v \in V$ such that $u + v = 0$;
A4. (Commutative) $\forall u, v \in V$, $u + v = v + u$;
M1. (distributive) $\forall \alpha \in F$ and $\forall u, v \in V$, $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$;
M2. (distributive) $\forall \alpha, \beta \in F$ and $\forall u \in V$, $(\alpha \oplus \beta) \cdot u = \alpha \cdot u + \beta \cdot u$;
M3. $\forall \alpha, \beta \in F$ and $\forall u \in V$, $(\alpha \odot \beta) \cdot u = \alpha \cdot (\beta \cdot u)$;
M4. $\forall u \in V$, $1 \cdot u = u$.
Elements of a vector space are called vectors.

[1] The proof of that result is very similar to the proof of Proposition 133.1 and 2.
[2] See, for example, Apostol (1967), Section 13.2, page 17.

Remark 132 *In the remainder of these notes, if no confusion arises, for ease of notation, we will denote a field simply by F and a vector space by V. Moreover, we will write $+$ in the place of $\oplus$, and we will omit $\odot$ and $\cdot$, i.e., we will write xy instead of $x \odot y$ and αv instead of $\alpha \cdot v$.*

Proposition 133 *If V is a vector space, then (as a consequence of the first four properties)*
1. *The zero vector is unique and it is denoted by 0.*
2. *$\forall u \in V$, the inverse element of u is unique and it is denoted by $-u$.*
3. *(cancellation law) $\forall u, v, w \in V$,*

$$u + w = v + w \Rightarrow u = v.$$

Proof. 1. Assume that there exist $0_1, 0_2 \in V$ which are zero vectors. Then from $(A2)$,

$$0_1 + 0_2 = 0_1 \quad \text{and} \quad 0_2 + 0_1 = 0_2.$$

From $(A.4)$,

$$0_1 + 0_2 = 0_2 + 0_1,$$

and therefore $0_1 = 0_2$.
2. Given $u \in V$, assume there exist $v^1, v^2 \in V$ such that

$$u + v^1 = 0 \quad \text{and} \quad u + v^2 = 0.$$

Then

$$v^2 = v^2 + 0 = v^2 + \left(u + v^1\right) = \left(v^2 + u\right) + v^1 = \left(u + v^2\right) + v^1 = 0 + v^1 = v^1..$$

3.

$$u + w = v + w \overset{(1)}{\Rightarrow} u + w + (-w) = v + w + (-w) \overset{(2)}{\Rightarrow} u + 0 = v + 0 \overset{(3)}{\Rightarrow} u = v,$$

where (1) follows from the definition of operation, (2) from the definition of $-w$ and (3) from the definition of 0. ■

Remark 134 *From A2. in Definition 131, we have that for any vector space V, $0 \in V$.*

Proposition 135 *If V is a vector space over a field F, then*

1. *For $0 \in F$ and $\forall u \in V$, $0u = 0$.*
2. *For $0 \in V$ and $\forall \alpha \in F$, $\alpha 0 = 0$.*
3. *If $\alpha \in F$, $u \in V$ and $\alpha u = 0$, then either $\alpha = 0$ or $u = 0$ or both.*
4. *$\forall \alpha \in F$ and $\forall u \in V$, $(-\alpha) u = \alpha (-u) = -(\alpha u) := -\alpha u$.*

Proof. 1. From $(M1)$,

$$0u + 0u = (0 + 0) u = 0u.$$

Then, adding $-(0u)$ to both sides,

$$0u + 0u + (-(0u)) = 0u + (-(0u))$$

and, using $(A3)$,

$$0u + 0 = 0$$

and, using $(A2)$, we get the desired result.
2. From $(A2)$,

$$0 + 0 = 0;$$

then multiplying both sides by α and using $(M1)$,

$$\alpha 0 = \alpha (0 + 0) = \alpha 0 + \alpha 0;$$

and, using $(A3)$,

$$\alpha 0 + (-(\alpha 0)) = \alpha 0 + \alpha 0 + (-(\alpha 0))$$

and, using $(A2)$, we get the desired result.

3. Assume that $\alpha u = 0$ and $\alpha \neq 0$. Then

$$u = 1u = \left(\alpha^{-1} \cdot \alpha\right) u = \alpha^{-1} (\alpha u) = \alpha^{-1} \cdot 0 = 0.$$

Taking the contropositive of the above result, we get $\langle u \neq 0 \rangle \Rightarrow \langle \alpha u \neq 0 \vee \alpha = 0 \rangle$. Therefore $\langle u \neq 0 \wedge \alpha u = 0 \rangle \Rightarrow \langle \alpha = 0 \rangle$.

4. From $u + (-u) = 0$, we get $\alpha (u + (-u)) = \alpha 0$, and then $\alpha u + \alpha (-u) = 0$, and therefore $-(\alpha u) = \alpha (-u)$.

From $\alpha + (-\alpha) = 0$, we get $(\alpha + (-\alpha)) u = 0u$, and then $\alpha u + (-\alpha) u = 0$, and therefore $-(\alpha u) = (-\alpha) u$. ■

Remark 136 *From Proposition 135.4, and $(M4)$in Definition 131, we have*

$$(-1) u = 1 (-u) = -(1u) = -u.$$

We also define subtraction as follows:

$$v - u := v + (-u)$$

4.2 Examples

Euclidean spaces.

The Euclidean space $\mathbb{R}^n$ with sum and scalar product defined in Chapter 2 is a vector space over the field $\mathbb{R}$ with the standard addition and multiplication

Matrices on $\mathbb{R}$.

For any $m, n \in \mathbb{N} \backslash \{0\}$, the set $\mathcal{M}_{m,n}$ of matrices with elements belonging to the field $\mathbb{R}$ with the operation of addition and scalar multiplication, as defined in Section 2.3 is a vector space on the field $\mathbb{R}$ and it is denoted by

$$\mathbb{M}(m, n).$$

Matrices on a field F.

For any $m, n \in \mathbb{N} \backslash \{0\}$, we can also consider the set of matrices whose entries are elements belonging to an arbitrary field F. It is easy to check that set is a vector space on the field F, with the operation of addition and scalar multiplication inherited by F, is a vector space and it is denoted by

$$\mathbb{M}_F(m, n).$$

We do set

$$\mathbb{M}_{\mathbb{R}}(m, n) = \mathbb{M}(m, n).$$

Polynomials

The set of *all* polynomials

$$a_0 + a_1 x + a_2 x^2 + ... + a_n t^n$$

with $n \in \mathbb{N}$ and $a_0, a_1, a_2, ..., a_n \in \mathbb{R}$ is a vector space on $\mathbb{R}$ with respect to the standard sum between polynomials and scalar multiplication.

Function space $\mathcal{F}(X)$.

Given a nonempty set X, the set of all functions $f : X \to \mathbb{R}$ with obvious sum and scalar multiplication is a a vector space on $\mathbb{R}$.

Sets which are not vector spaces.

$(0, +\infty)$ and $[0, +\infty)$ are **not** a vector spaces in $\mathbb{R}$.

For any $n \in \mathbb{N} \backslash \{0\}$, the set of all polynomials of degree n is not a vector space on $\mathbb{R}$.

4.3 Vector subspaces

In what follows, if no ambiguity may arise, we will say "vector space" instead of "vector space on a field".

Definition 137 *Let W be a subset of a vector space V. W is called a vector subspace of V if W is a vector space with respect to the operation of vector addition and scalar multiplication defined on V. In other words, given a vector space $(V,+,\cdot)$ on a field $(F,\oplus,\odot)$ and a subset W of V, we say that W is a vector subspace of V if $\left(W,+_{|W},\cdot_{|W}\right)$ is a vector space on the field $(F,\oplus,\odot)$.*

Proposition 138 *Let W be a subset of a vector space V. The following three statements are equivalent.*

1. *W is a vector subspace of V.*
2. *a. $W\neq\varnothing$, i.e.,[3] $0\in W$;*
   b. $\forall u,v\in W$, $u+v\in W$;
   c. $\forall u\in W,\alpha\in F$, $\alpha u\in W$.
3. *a. $W\neq\varnothing$, i.e., $0\in W$;*
   b. $\forall u,v\in W$, $\forall\alpha,\beta\in F$, $\alpha u+\beta v\in W$.

Proof. 2. and 3. are clearly equivalent.
If W is a vector subspace, clearly 2. holds. To show the opposite implication we have to check only $(A2)$ and $(A3)$, simply by definition of restriction function.
$(A2)$: Since $W\neq\varnothing$, we can take $u\in W$. Then, from *c.*, taken $0\in F$, we have $0u=0\in W$.
$(A3)$: Taken $u\in W$, from *c.*, $(-1)\,u\in W$, but then $(-1)\,u=-u\in W$. ■

Example 139 *1. Given an arbitrary vector space V, $\{0\}$ and V are vector subspaces of V.*
2. Given $\mathbb{R}^3$,

$$W:=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3:x_3=0\right\}$$

is a vector subspace of $\mathbb{R}^3$.
3. Given the space V of polynomials, the set W of all polynomials of degree $\leq n$ is a vector subspace of V.
4. The set of all bounded or continuous or differentiable or integrable functions $f:X\to\mathbb{R}$ is a vector subspace of $\mathcal{F}(X)$.
5. If V and W are vector spaces, then $V\cap W$ is a vector subspace of V and W.
6. $[0,+\infty)$ is not a vector subspace of $\mathbb{R}$.
7. Let $V=\{0\}\times\mathbb{R}\subseteq\mathbb{R}^2$ and $W=\mathbb{R}\times\{0\}\subseteq\mathbb{R}^2$. Then $V\cup W$ is not a vector subspace of $\mathbb{R}^2$.

4.4 Linear combinations

Notation convention. Unless otherwise stated, a greek (or Latin) letter with a subscript denotes a scalar; a Latin letter with a superscript denotes a vector.

Definition 140 *Let V be a vector space, $m\in\mathbb{N}\backslash\{0\}$ and $v^1,v^2,...,v^m\in V$. A linear combination of vectors $v^1,v^2,...,v^m$ via coefficients $\alpha_1,\alpha_2,...,\alpha_m\in F$ is the vector*

$$\sum_{i=1}^{m}\alpha_i v^i.$$

The set of all such combinations

$$\left\{v\in V:\exists\,(\alpha_i)_{i=1}^m\in F^m\text{ such that }v=\sum_{i=1}^{m}\alpha_i v^i\right\}$$

[3] It follows from Proposition 135.1.

is called span of $\{v^1, v^2, ..., v^m\}$ *and it is denoted by*

$$\text{span}\left(\{v^1, v^2, ..., v^m\}\right)$$

or simply

$$\text{span}\left(v^1, v^2, ..., v^m\right).$$

Definition 141 *Let* V *be a vector space and* $S \subseteq V$*.* $\text{span}(S)$ *is the set of all linear combinations of a finite number of vectors in* S*.*

Proposition 142 *Let* V *be a vector space and* $S \neq \varnothing, S \subseteq V$*.*

1. *a.* $S \subseteq \text{span}(S)$ *and b.* $\text{span}(S)$ *is a vector subspace of* V*.*

2. *If* W *is a subspace of* V *and* $S \subseteq W$*, then* $\text{span}\ (S) \subseteq W$*.*

Proof. 1a. Given $v \in S$, $1v = v \in \text{span}(S)$. 1b. Since $S \neq \varnothing$, then $\text{span}(S) \neq \varnothing$. Given $\alpha, \beta \in F$ and $v, w \in$*span* S. Then $\exists \alpha_1, ..., \alpha_n \in F$, $v^1, ..., v^n \in S$ and $\beta_1, ..., \beta_m \in F$, $w^1, ..., w^m \in S$, such that $v = \sum_{i=1}^n \alpha_i v^i$ and $w = \sum_{j=1}^m \beta_j w^j$. Then

$$\alpha v + \beta w = \sum_{i=1}^n (\alpha\alpha_i) v^i + \sum_{j=1}^m \left(\beta\beta_j\right) w^j \in \text{span} S.$$

2. Take $v \in \text{span}\ S$. Then $\exists \alpha_1, ..., \alpha_n \in F$, $v^1, ..., v^n \in S \subseteq W$ such that $v = \sum_{i=1}^n \alpha_i v^i \in W$, as desired. ■

Definition 143 *Let* V *be a vector space and* $v^1, v^2, ..., v^m \in V$*. If* $V = \text{span}\left(v^1, v^2, ..., v^m\right)$*, we say that* V *is the vector space generated or spanned by the vectors* $v^1, v^2, ..., v^m$*.*

Example 144 *1.* $\mathbb{R}^3 = \text{span}\left(\{(1,0,0), (0,1,0), (0,0,1)\}\right)$*.*
2. $\text{span}\ \left(\{t^n\}_{n \in \mathbb{N}}\right)$ *is equal to the vector space of all polynomials.*

4.5 Row and column space of a matrix

Definition 145 *Given* $A \in \mathbb{M}(m, n)$*,*

$$\text{row span} A := \text{span}\left(R^1(A), .., R^i(A), ..., R^m(A)\right)$$

is called the row space of A *or* row span *of* A*.*
The column space of A *or* $\text{col span} A$ *is*

$$\text{col span} A := \text{span}\left(C^1(A), .., C^j(A), ..., C^n(A)\right).$$

Remark 146 *Given* $A \in \mathcal{M}(m, n)$

$$\text{col span} A = \text{row span} A^T.$$

Remark 147 ***Linear combinations of columns and rows of a matrix.***
Let $A \in \mathbb{M}(m, n), x \in \mathbb{R}^n$ *and* $y \in \mathbb{R}^m$ *. Then,* $\forall j \in \{1, .., n\}$ *,* $C^j(A) \in \mathbb{R}^m$ *and* $\forall i \in \{1, ..., m\}$*,* $R^i(A) \in \mathbb{R}^n$*. Then,*

$$Ax = \left[C^1(A), ..., C^j(A), ..., C^n(A)\right] \cdot \begin{bmatrix} x_1 \\ ... \\ x_j \\ ... \\ x_n \end{bmatrix} = \sum_{j=1}^n x_j \cdot C^j(A)$$

and

Ax is a linear combination of the <u>columns</u> *of A via the components of the vector x.*

Moreover,

$$yA = [y_1, ..., y_i, ..., y_m] \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix} = \sum_{i=1}^{m} y_i \cdot R^i(A)$$

and

yA is a linear combination of the rows of A via the components of the vector y.

As a consequence of the above observation, we have what follow.

1.

$$\text{row span } A = \{w \in \mathbb{R}^n : \exists \;\; y \in \mathbb{R}^m \text{ such that } w = yA\}.$$

2.

$$\text{col span } A = \{z \in \mathbb{R}^m : \exists \;\; x \in \mathbb{R}^n \text{ such that } z = Ax\};$$

Proposition 148 *Given $A, B \in \mathbb{M}(m, n)$,*
1. *if A is row equivalent to B, then* $\text{row span}A = \text{row span}B$*;*
2. *if A is column equivalent to B, then* $\text{col span}A = \text{col span}B$.

Proof. 1. B is obtained by A via elementary row operations. Therefore, $\forall i \in \{1, ..., m\}$, either
i. $R^i(B) = R^i(A)$, or
ii. $R^i(B)$ is a linear combination of rows of A.
Therefore, $\text{row span}B \subseteq \text{row span}A$. Since A is obtained by B via elementary row operations, $\text{row span}B \supseteq \text{row span}A$.

2. if A is column equivalent to B, then A^T is row equivalent to B^T and therefore, from i. above, $\text{row span}A^T = \text{row span}B^T$. Then the result follows from Remark 146. ■

Remark 149 *Let $A \in \mathbb{M}(m, n)$ be given and assume that*

$$b := (b_j)_{j=1}^{n} = \sum_{i=1}^{m} c_i \cdot R^i(A),$$

i.e., b is a linear combination of the rows of A. Then,

$$\forall j \in \{1, ..., n\}, \quad b_j = \sum_{i=1}^{m} c_i \cdot R^{ij}(A),$$

where $\forall i \in \{1, ...m\}$ and $\forall j \in \{1, ...n\}$, $R^{ij}(A)$ is the $j-th$ component of the $i-th$ row $R^i(A)$ of A.

Lemma 150 *Assume that $A, B \in \mathbb{M}(m, n)$ are in echelon form with pivots*

$$a_{1j_1}, ..., a_{ij_i}, ..., a_{rj_r},$$

and

$$b_{1k_1}, ..., b_{ik_i}, ..., b_{sk_s},$$

respectively, and[4] *$r, s \leq \min\{m, n\}$. Then*

$$\langle \text{row span}A = \text{row span}B \rangle \Rightarrow \langle s = r \text{ and for } i \in \{1, ..., s\}, \quad j_i = k_i \rangle.$$

[4] See Remark 30.

Proof. Preliminary remark 1. If $A = 0$, then $A = B$ and $s = r = 0$.

Preliminary remark 2. Assume that $A, B \neq 0$ and then $s, r \geq 1$. We want to verify that $j_1 = k_1$. Suppose $j_1 < k_1$. Then, by definition of echelon matrix, $C_{j_1}(B) = 0$, otherwise you would contradict Property 2 of the Definition 28 of echelon matrix. Then, from the assumption that row span$A$ = row span$B$,we have that $R^1(A)$ is a linear combination of the rows of B, via some coefficients $c_1, ..., c_m$, and from Remark 149 and the fact that $C^{j_1}(B) = 0$, we have that $a_{1j_1} = c_1 \cdot 0 + ...c_m \cdot 0 = 0$, contradicting the fact that a_{1j_1} is a pivot for A. Therefore, $j_1 \geq k_1$. A perfectly symmetric argument shows that $j_1 \leq k_1$.

We can now prove the result by induction on the number m of rows.

Step 1. $m = 1$.

It is basically the proof of Preliminary Remark 2.

Step 2.

Given $A, B \in \mathbb{M}(m, n)$, define $A', B' \in \mathbb{M}(m-1, n)$ as the matrices obtained erasing the first row in matrix A and B respectively. From Remark 33, A' and B' are still in echelon form. If we show that row spanA' = row spanB', from the induction assumption, and using Preliminary Remark 2, we get the desired result.

Let $R = (a_1, ..., a_n)$ be any row of A'. Since $R \in$ row span$B$, $\exists (d_i)_{i=1}^m$ such that

$$R = \sum_{i=1}^{m} d_i R^i(B).$$

Since A is in echelon form and we erased its first row, we have that if $i \leq j_1 = k_1$, then $a_i = 0$, otherwise you would contradict the definition of j_1. Since B is in echelon form, each entry in its $k_1 - th$ column are zero, but b_{1k_1} which is different from zero. Then,

$$a_{1k_1} = 0 = \sum_{i=1}^{m} d_i \cdot b_{ik_1} = d_1 \cdot b_{ik_1},$$

and therefore $d_1 = 0$, i.e., $R = \sum_{i=2}^{m} d_i R^i(B)$, or $R \in$ row spanB', as desired. Symmetric argument shows the other inclusion. ■

Remark 151 *Given*

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix},$$

clearly $A \neq B$ *and* row span$A$ = row span$B$.

Proposition 152 *Assume that* $A, B \in \mathbb{M}(m, n)$ *are in row canonical form. Then,*

$$\langle \text{row span} A = \text{row span} B \rangle \Leftrightarrow \langle A = B \rangle.$$

Proof. [⇐] Obvious.

[⇒] From Lemma 150, the number of pivots in A and B is the same. Therefore, A and B have the same number s of nonzero rows, which in fact are the first s rows. Take $i \in \{1, ..., s\}$. Since row span$A$ = row span$B$, there exists $(c_h)_{h=1}^s$ such that

$$R^i(A) = \sum_{h=1}^{s} c_h \cdot R^h(B). \tag{4.1}$$

We want then to show that $c_i = 1$ and $\forall l \in \{1, ..., s\} \setminus \{i\}$, $c_l = 0$.

Let a_{ij_i}be the pivot of $R^i(A)$, i.e., a_{ij_i} is the nonzero $j_i - th$ component of $R^i(A)$. Then, from Remark 149,

$$a_{ij_i} = \sum_{h=1}^{s} c_h \cdot R^{hj_i}(B) = \sum_{h=1}^{s} c_h \cdot b_{hj_i}. \tag{4.2}$$

From Lemma 150, for $i \in \{1, ..., s\}$, $j_i = k_i$, and therefore b_{ij_i} is a pivot entry for B, and since B is in row reduced form, b_{ij_i} is the only nonzero element in the j_i column of B. Therefore, from (4.2),

$$a_{ij_i} = \sum_{h=1}^{s} c_h \cdot R^{hj_i}(B) = c_i \cdot b_{ij_i}.$$

Since A and B are in row canonical form $a_{ij_i} = b_{ij_i} = 1$ and therefore

$$c_i = 1.$$

Now take $l \in \{1, ..., s\} \setminus \{i\}$ and consider the pivot element b_{lj_l} in $R^l(B)$. From (4.1) and Remark 149,

$$a_{ij_l} = \sum_{h=1}^{s} c_h \cdot b_{hj_l} = c_l, \tag{4.3}$$

where the last equalities follow from the fact that B is in row reduced form and therefore b_{lj_l} is the only nonzero element in the $j_l - th$ column of B, in fact, $b_{lj_l} = 1$. From Lemma 150, since b_{lj_l} is a pivot element for B, a_{lj_l} is a pivot element for A. Since A is in row reduced form, a_{lj_l} is the only nonzero element in column j_l of A. Therefore, since $l \neq i$, $a_{ij_l} = 0$, and from (4.3), the desired result,

$$\forall l \in \{1, ..., s\} \setminus \{i\}, \quad c_l = 0.,$$

does follow. ■

Proposition 153 *For every $A \in \mathbb{M}(m,n)$, there exists a unique $B \in \mathbb{M}(m,n)$ which is in row canonical form and row equivalent to A.*

Proof. The existence of at least one matrix with the desired properties is the content of Proposition 38. Suppose that there exists B_1 and B_2 with those properties. Then from Proposition 148, we get

$$\text{row span} A = \text{row span} B_1 = \text{row span} B_2.$$

From Proposition 152,

$$B_1 = B_2.$$

■

Corollary 154 *1. For any matrix $A \in \mathbb{M}(m,n)$ there exists a unique number $r \in \{0, 1, ..., \min\{m,n\}\}$ such that A is equivalent to the block matrix of the form*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

2. For any $A \in \mathbb{M}(m,n)$, there exist invertible matrices $P \in \mathbb{M}(m,m)$ and $Q \in \mathbb{M}(n,n)$ and a unique number $r \in \{0, 1, ..., \min\{m,n\}\}$ such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Proof. *1.*

From Step 1 in the proof of Proposition 122 and from Proposition 153, there exists a unique matrix A^* which is row equivalent to A and it is in row canonical form.

From Step 2 and 3 in the proof of Proposition 122 and from Proposition 153, there exist a unique matrix

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}^T$$

which is row equivalent to A^{*T} and it is in row canonical form. Therefore the desired result follows.

2.

From Proposition 105.2, PA is row equivalent to A; from Proposition 119.2, PAQ is column equivalent to PA. Therefore, PAQ is equivalent to A.From Proposition 124 ,

$$\begin{bmatrix} I_{r'} & 0 \\ 0 & 0 \end{bmatrix}$$

is equivalent to A. From part 1 of the present Proposition, the desired result then follows. ■

4.6 Linear dependence and independence

Definition 155 *Let V be a vector space on a field F, $m \in \mathbb{N}\setminus\{0\}$ and $v^1, v^2, ..., v^m \in V$. The set $S = \{v^1, v^2, ..., v^m\}$ is a set of linearly dependent vectors if*

either $m = 1$ and $v^1 = 0$, i.e., $S = \{0\}$,

or $m > 1$ and $\exists\ k \in \{1, ...m\}$ and there exist $(m-1) \geq 1$ coefficients $\alpha_j \in F$ with $j \in \{1, ...m\}\setminus\{k\}$ such that

$$v^k = \sum_{j\in\{1,...m\}\setminus\{k\}} \alpha_j v^j$$

or, shortly,

$$v^k = \sum_{j\neq k} \alpha_j v^j$$

i.e., there exists a vector equal to a linear combination of the other vectors.

`Geometrical example in` $\mathbb{R}^2$.

Proposition 156 *Let V be a vector space and $v^1, v^2, ..., v^m \in V$ on a field F. The set $S = \{v^1, v^2, ..., v^m\} \subseteq V$ is a set of linearly dependent vectors if and only if*

$$\exists\ (\beta_1, ..., \beta_i, ..., \beta_m) \in F^m\setminus\{0\} \text{ such that } \sum_{i=1}^{m} \beta_i \cdot v^i = 0, \tag{4.4}$$

i.e., there exists a linear combination of the vectors equal to the null vector and with some nonzero coefficient.

Proof. [⇒]

If $\#S = 1$, i.e., $S = \{0\}$, any $\beta \in \mathbb{R}\setminus\{0\}$ is such that $\beta \cdot 0 = 0$. Assume then that $\#S > 1$. Take

$$\beta_i = \begin{cases} \alpha_i & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}$$

[⇐]

If $S = \{v\}$ and $\exists \beta \in \mathbb{R}\setminus\{0\}$ is such that $\beta \cdot v = 0$,then from Proposition 58.3 $v = 0$. Assume then that $\#S > 1$. Without loss of generality take $\beta_1 \neq 0$. Then,

$$\beta_1 v^1 + \sum_{i\neq 1} \beta_i v^i = 0$$

and

$$v^1 = \sum_{i\neq 1} \frac{\beta_i}{\beta_1} v^i.$$

■

Proposition 157 *Let $m \geq 2$ and $v^1, ..., v^m$ be nonzero linearly dependent vectors. Then, one of the vectors is a linear combination of the preceding vectors, i.e., $\exists\ k > 1$ and $(\alpha^i)_{i=1}^{k-1}$ such that $v^k = \sum_{i=1}^{k-1} \alpha v^i$.*

Proof. Since $\{v^1, ..., v^m\}$ are linearly dependent, $\exists\,(\beta_i)_{i=1}^m \in \mathbb{R}^m\setminus\{0\}$ such that $\sum_{i=1}^m \beta_i v^i = 0$. Let k be the largest i such that $\beta_i \neq 0$, i.e.,

$$\exists k \in \{1, ..., m\} \text{ such that } \beta_k \neq 0 \text{ and } \forall i \in \{k+1, ..., m\}\,, \quad \beta_i = 0. \tag{4.5}$$

Consider the case $k = 1$. Then we would have $\beta_1 \neq 0$ and $\forall i > 1$, $\beta_i = 0$ and therefore $0 = \sum_{i=1}^m \beta_i v^i = \beta_1 v^1$, contradicting the assumption that $v^1, ..., v^m$ are nonzero vectors. Then, we must have $k > 1$, and from (4.5) , we have

$$0 = \sum_{i=1}^{m} \beta_i v^i = \sum_{i=1}^{k} \beta_i v^i$$

and

$$\beta_k v^k = \sum_{i=1}^{k-1} \beta_i v^i,$$

or, as desired,

$$v^k = \sum_{i=1}^{k-1} \frac{-\beta^i}{\beta_k} v^i.$$

It is then enough to choose $\alpha_i = \frac{-\beta^i}{\beta_k}$ for any $i \in \{1, ..., k-1\}$. ■

Example 158 *Take the vectors* $x^1 = (1,2)$, $x^2 = (-1,-2)$ *and* $x^3 = (0,4)$. $S := \{x^1, x^2, x^3\}$ *is a set of linearly dependent vectors:* $x^1 = -1 \cdot x^2 + 0 \cdot x^3$. *Observe that there are no* $\alpha_1, \alpha_2 \in \mathbb{R}$ *such that* $x^3 = \alpha_1 \cdot x^1 + \alpha_2 \cdot x^2$.

Definition 159 *The set of vectors* $S = \{v^1, ..., v^m\} \subseteq V$ *is called a set of linearly independent vectors if it is not linearly dependent, i.e.,* $\langle \neg (4.4) \rangle$ *,i.e., if*

$$\forall (\beta_1, ..., \beta_i, ..., \beta_m) \in F^m \setminus \{0\} \text{ it is the case that } \sum_{i=1}^m \beta_i \cdot v^i \neq 0,$$

or

$$(\beta_1, ..., \beta_i, ..., \beta_m) \in F^m \setminus \{0\} \quad \Rightarrow \quad \sum_{i=1}^m \beta_i \cdot v^i \neq 0$$

or

$$\sum_{i=1}^m \beta_i \cdot v^i = 0 \quad \Rightarrow \quad (\beta_1, ..., \beta_i, ..., \beta_m) = 0$$

or the only linear combination of the vectors which is equal to the null vector has each coefficient equal to zero.

Example 160 *The vectors* $(1,2)$, $(1,5)$ *are linearly independent.*

Example 161 *Let* V *be the vector space of all functions* $f : \mathbb{R} \to \mathbb{R}$. f, g, h *defined below are linearly independent:*

$$f(x) = e^{2x}, \qquad g(x) = x^2, \qquad h(x) = x.$$

Suppose there exists $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ *such that*

$$\alpha_1 \cdot f + \alpha_2 \cdot g + \alpha_3 \cdot h = 0_V,$$

which means that

$$\forall x \in \mathbb{R}, \quad \alpha_1 \cdot f(x) + \alpha_2 \cdot g(x) + \alpha_3 \cdot h(x) = 0.$$

We want to show that $(\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$. *The trick is to find appropriate values of* x *to get the desired value of* $(\alpha_1, \alpha_2, \alpha_3)$. *Choose* x *to take values* $0, 1, -1$. *We obtain the following system of equations*

$$\begin{cases} \alpha_1 & & & & & = & 0 \\ e \cdot \alpha_1 & + & \alpha_2 & + & \alpha_3 & = & 0 \\ e^{-1}\alpha_1 & + & \alpha_2 & + & (-1)\,\alpha_3 & = & 0 \end{cases}$$

It then follows that $(\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$, *as desired.*

`Geometrical example in` $\mathbb{R}^2$.

Remark 162 *From Remark 147, we have what follows:*

$$\langle Ax = 0 \Rightarrow x = 0 \rangle \Leftrightarrow \langle \text{the column vectors of } A \text{ are linearly independent} \rangle \tag{4.6}$$

$$\langle yA = 0 \Rightarrow y = 0 \rangle \Leftrightarrow \langle \text{the row vectors of } A \text{ are linearly independent} \rangle \tag{4.7}$$

Remark 163 $\varnothing$ *is a set of linearly independent vectors: see Definition 159.*

Example 164 *Consider the vectors* $x^1 = (1,0,0,0) \in \mathbb{R}^4$ *and* $x^2 = (0,1,0,0)$. *Observe that* $\alpha_1 x^1 + \alpha_2 x^2 = 0$ *means* $(\alpha_1, \alpha_2, 0, 0) = (0,0,0,0)$.

Exercise 165 *Say if the following set is a set of linearly dependent or independent vectors:*
$S = \{x^1, x^2, x^3\}$ *with* $x^1 = (3,2,1)$, $x^2 = (4,1,3)$, $x^3 = (3,-3,6)$.

Proposition 166 *Let* V *be a vector space and* $v^1, v^2, ..., v^m \in V$. *If* $S = \{v^1, ..., v^m\}$ *is a set of linearly dependent vectors and* $v^{m+1}, ..., v^{m+k} \in V$ *are arbitrary vectors, then*

$$S' := S \cup \{v^{m+1}, ..., v^{m+k}\} = \{v^1, ..., v^m, v^{m+1}, ..., v^{m+k}\}$$

is a set of linearly dependent vectors.

Proof. From the assumptions $\exists i^* \in \{1, ..., m\}$ and $(a_j)_{j \neq i^*}$ such that

$$v^j = \sum_{j \neq i^*} \alpha_j v^j$$

But then

$$v^j = \sum_{j \neq i^*} \alpha_j v^j + 0 \cdot v^{m+1} + 0 \cdot v^{m+k}$$

■

Proposition 167 *If* $S = \{v^1, ..., v^m\} \subseteq V$ *is a set of linearly independent vectors, then* $S' \subseteq S$ *is a set of linearly independent vectors*

Proof. Suppose otherwise, but then you contradict the previous proposition. ■

The above two Propositions allow to generalize the definition of linearly dependent and linearly independent sets of vectors for sets of arbitrary cardinality as follows.

Definition 168 *Let* V *be a vector space and* $S \subseteq V$. *The set* S *is linearly dependent if either* $S = \{0\}$ *or there exists a subset of* S *with finite cardinality which is linearly dependent.*

Definition 169 *Let* V *be a vector space and* $S \subseteq V$. *The set* S *is linearly independent if every subset of* S *with finite cardinality is linearly independent.*

Remark 170 *Consider vectors in* $\mathbb{R}^n$.

1. *Adding components to linearly dependent vectors gives raise to linearly dependent or independent vectors;*
2. *Eliminating components from linearly independent vectors gives raise to linearly dependent or independent vectors;*
3. *Adding components to linearly independent vectors gives raise to linearly independent vectors;*
4. *Eliminating components from linearly dependent vectors gives raise to linearly dependent vectors.*

To verify 1. 1nd 2. above consider the following two vectors:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

To verify 4., if you have a set of linearly dependent vectors, it is possible to express one vector as a linear combination of the others and then eliminate components leaving the equality still true. The proof of 3 is contained in the following Proposition.

Proposition 171 *If* $S_x = \{x^1, ..., x^m\} \subseteq \mathbb{R}^n$ *is a set of linearly independent vectors and* $S_y = \{y^1, ..., y^m\} \subseteq \mathbb{R}^k$ *is a set of vectors, then* $S = \{(x^1, y^1), ... (, x^m, y^m)\} \subseteq \mathbb{R}^{n+k}$ *is a set of linearly independent vectors.*

Proof. By assumption

$$\sum_{i=1}^{m} \beta_i \cdot v^i = 0 \quad \Rightarrow \quad (\beta_1, ..., \beta_i, ..., \beta_m) = 0.$$

S is a set of linearly independent vectors if

$$\sum_{i=1}^{m} \beta_i \cdot \left(v^i, y^i\right) = 0 \quad \Rightarrow \quad (\beta_1, ..., \beta_i, ..., \beta_m) = 0.$$

Since

$$\sum_{i=1}^{m} \beta_i \cdot \left(v^i, y^i\right) = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{m} \beta_i \cdot v^i = 0 \text{ and } \sum_{i=1}^{m} \beta_i \cdot y^i = 0$$

the desired result follows. ■

Corollary 172 *If* $S_x = \left\{x^1, ..., x^m\right\} \subseteq \mathbb{R}^n$ *is a set of linearly independent vectors and* $S_y = \left\{y^1, ..., y^m\right\} \subseteq \mathbb{R}^k$ *and* $S_z = \left\{z^1, ..., z^m\right\} \subseteq \mathbb{R}^l$ *are sets of vectors, then*

$$S = \left\{\left(z_1, x^1, y^1\right), ... \left(z_m, x^m, y^m\right)\right\} \subseteq \mathbb{R}^{l+n+k}$$

is a set of linearly independent vectors.

Example 173 *1. The set of vectors* $\left\{v^1, ..., v^m\right\}$ *is a linearly dependent set if* $\exists k \in \{1, ..., m\}$ *such that* $v^k = 0$*:*

$$v^k + \sum_{i \neq k} 0 \cdot v^i = 0$$

2. The set of vectors $\left\{v^1, ..., v^m\right\}$ *is a linearly dependent set if* $\exists k, k' \in \{1, ..., m\}$ *and* $\alpha \in \mathbb{R}$ *such that* $v^{k'} = \alpha v^k$*:*

$$v^{k'} - \alpha v^k + \sum_{i \neq k, k'} 0 \cdot v^i = 0$$

3. Two vectors are linearly dependent if and only if one is a multiple of the other.

Proposition 174 *The nonzero rows of a matrix* A *in echelon form are linearly independent.*

Proof. We will show that each row of A starting from the first one is not a linear combination of the subsequent rows. Then, as a consequence of Proposition 157, the desired result will follow.

Since A is in echelon form, the first row has a pivot below which all the elements are zero. Then that row cannot be a linear combination of the following rows. Similar argument applies to the other rows. ■

4.7 Basis and dimension

Definition 175 *An ordered set* S *(of arbitrary cardinality) in a vector space* V *on a field* F *is a basis of* V *if*

1. *S is a linearly independent set;*
2. span $(S) = V$.

Proposition 176 *A set* $S = \left\{u^1, u^2, ..., u^n\right\} \subseteq V$ *is a basis of* V *on a field* F *if and only if* $\forall v \in V$ *there exists a unique* $(\alpha_i)_{i=1}^n \in F^n$ *such that* $v = \sum_{i=1}^n \alpha_i u^i$*.*

Proof. [⇒] Suppose there exist $(\alpha_i)_{i=1}^n, (\beta_i)_{i=1}^n \in \mathbb{R}^n$ such that $v = \sum_{i=1}^n \alpha_i u^i = \sum_{i=1}^n \beta_i u^i$. Then

$$0 = \sum_{i=1}^{n} \alpha_i u^i - \sum_{i=1}^{n} \beta_i u^i = \sum_{i=1}^{n} (\alpha_i - \beta_i) u^i.$$

Since $\{u^1, u^2, ..., u^n\}$ are linearly independent,

$$\forall i \in \{1, ..., n\}, \ \alpha_i - \beta_i = 0,$$

as desired.
[$\Leftarrow$]
Clearly $V = \text{span}\ (S)$; we are left with showing that $\{u^1, u^2, ..., u^n\}$ are linearly independent. Consider $\sum_{i=1}^n \alpha_i u^i = 0$. Moreover, $\sum_{i=1}^n 0 \cdot u^i = 0$. But since there exists a unique $(\alpha_i)_{i=1}^n \in \mathbb{R}^n$ such that $v = \sum_{i=1}^n \alpha_i u^i$, it must be the case that $\forall i \in \{1, ..., n\}, \ \alpha_i = 0$. ■

Lemma 177 *Suppose that given a vector space V,* $\text{span}\left(v^1, ..., v^m\right) = V$*.*

1. *If $w \in V$, then $\{w, v^1, ..., v^m\}$ is linearly dependent and* $\text{span}\left(w, v^1, ..., v^m\right) = V$*;*
2. *If v^i is a linear combination of $\{v^j\}_{j=1}^{i-1}$, then* $\text{span}\left(v^1, ..., v^{i-1}, v^{i+1}, ..., v^m\right) = V$*.*

Proof. Obvious. ■

Lemma 178 *(Replacement Lemma) Given a vector space V, if*

1. $\text{span}\left(\{v^1, ..., v^n\}\right) = V$*,*
2. *$\{w^1, ..., w^m\} \subseteq V$ is linearly independent,*

then

1. *$n \geq m$,*
2. a. *If $n = m$, then* $\text{span}\left(\{w^1, ..., w^m\}\right) = V$*.*
   b. *if $n > m$, there exists $\{v^{i_1}, ..., v^{i_{n-m}}\} \subseteq \{v^1, ..., v^n\}$ such that*

$$\text{span}\left(\{w^1, ..., w^m, v^{i_1}, ..., v^{i_{n-m}}\}\right) = V.$$

Proof. Observe preliminary that since $\{w^1, ..., w^m\}$ is linearly independent, $\forall j \in \{1, ..., m\}$, $w^j \neq 0$.

Define I_1 as the subset of $\{1, ..., n\}$ such that $\forall i \in I_1$, $v^i \neq 0$. Then, clearly, $\text{span}\left(\{v^i\}_{i \in I_1}\right) = \text{span}\left(\{v^1, ..., v^n\}\right) = V$. We are going to show that $m \leq \#I_1$, and since $\#I_1 \leq n$, our result will imply conclusion 1. Moreover, we are going to show that there exists $\{v^{i_1}, ..., v^{i_{\#I_1 - m}}\} \subseteq \{v^i\}_{i \in I_1} \subseteq \{v^1, ..., v^n\}$ such that $\text{span}\left(\{w^1, ..., w^m, v^{i_1}, ..., v^{i_{\#I_1 - m}}\}\right) = V$, and that result will imply conclusion 2.

Using the above observation and to make notation easier we will assume that $I_1 = \{1, ..., n\}$, i.e., $\forall i \in \{1, ..., n\}, v^i \neq 0$.

Now consider the case $n = 1$. $\{w^1, ..., w^m\} \subseteq V$ implies that there exists $(\alpha_j)_{j=1}^m \in \mathbb{R}^m$ such that $\forall j$, $\alpha_j \neq 0$ and $w^j = \alpha^j v^1$, then it has to be $m = 1$ (and conclusion 1 holds) and since $w^1 = \alpha_1 v_1$, $\text{span}\left(w^1\right) = V$ (and conclusion 2 holds).

Consider now the case $n \geq 2$.

First of all, observe that from Lemma 177.1, $\{w^1, v^1, ..., v^n\}$ is linearly dependent and

$$\text{span}\left(\{w^1, v^1, ..., v^n\}\right) = V.$$

By Lemma 157, there exists $k_1 \in \{1, ..., n\}$ such that v^k is a linear combination of the preceding vectors. Then from Lemma 177.2, we have

$$\text{span}\left(w^1, \left(v^i\right)_{i \neq k_1}\right) = V.$$

Then again from Lemma 177.1, $\left\{w^1, w^2, \left(v^i\right)_{i \neq k_1}\right\}$ is linearly dependent and $\text{span}\left(\left\{w^1, w^2, \left(v^i\right)_{i \neq k_1}\right\}\right) = V$. By Lemma 157, there exists $k_2 \in \{2, ..., n\} \setminus \{k_1\}$ such that v^{k_2} or w^2 is a linear combination of the preceding vectors. That vector cannot be w^2 because of assumption 2. Therefore,

$$\text{span}\left(\left\{w^1, w^2, \left(v^i\right)_{i \neq k_1, k_2}\right\}\right) = V.$$

We can now distinguish three cases: $m < n,\ m = n$ and $m > n$.

Now if $m < n$, after m steps of the above procedure we get

$$\operatorname{span}\left(\left\{w^1, ..., w^m, \left(v^i\right)_{i \neq k_1, k_2, .., k_n}\right\}\right) = V,$$

which shows 2.a. If $m = n$,we have

$$\operatorname{span}\left(\left\{w^1, ..., w^m\right\}\right) = V,$$

which shows 2.b.

Let's now show that it cannot be $m > n$. Suppose that is the case. Then, after n of the above steps, we get $\operatorname{span}\left(w^1, ..., w^n\right) = V$ and therefore w^{n+1} is a linear combination of $\left(w^1, ..., w^n\right)$, contradicting assumption 2. ■

Proposition 179 *Assume that* $S = \left\{u^1, u^2, ..., u^n\right\}$ *and* $T = \left\{v^1, v^2, ..., v^m\right\}$ *are bases of* V*. Then* $n = m$*.*

Proof. By definition of basis we have that

$$\operatorname{span}\left(\left\{u^1, u^2, ..., u^n\right\}\right) = V \quad \text{and} \quad \left\{v^1, v^2, ..., v^m\right\} \text{ are linearly independent.}$$

Then from Lemma 178, $m \leq n$. Similarly,

$$\operatorname{span}\left(\left\{v^1, v^2, ..., v^m\right\}\right) = V \quad \text{and} \quad \left\{u^1, u^2, ..., u^n\right\} \text{ are linearly independent,}$$

and from Lemma 178, $n \leq m$. ■

The above Proposition allows to give the following Definition.

Definition 180 *A vector space* V *has dimension* n *if there exists a basis of* V *whose cardinality is* n*. In that case, we say that* V *has finite dimension (equal to* n*) and we write* $\dim V = n$*. If a vector space does not have finite dimension, it is said to be of infinite dimension.*

Definition 181 *The vector space* $\{0\}$ *has dimension* 0*.*

Example 182 *1. A basis of* $\mathbb{R}^n$ *is* $\left\{e^1, ..., e^i, ..., e^n\right\}$*, where* e^i *is defined in Definition 55. That basis is called canonical basis. Then* $\dim \mathbb{R}^n = n$*.*

2. Consider the vector space $\mathbb{P}_n(t)$ *of polynomials of degree* $\leq n$*. The set* $\left\{t^0, t^1, ...t^n\right\}$ *of polynomials is a basis of* $\mathbb{P}_n(t)$ *and therefore* $\dim \mathbb{P}_n(t) = n + 1$*.*

Proposition 183 *Let* V *be a vector space of dimension* n*.*

1. *$m > n$ vectors in V are linearly dependent;*
2. *If $S = \left\{u^1, ..., u^n\right\} \subseteq V$ is a linearly independent set, then it is a basis of V;*
3. *If $\operatorname{span}\left(u^1, ..., u^n\right) = V$, then $\left\{u^1, ..., u^n\right\}$ is a basis of V.*

Proof. Let $\left\{w^1, ..., w^n\right\}$ be a basis of V.

1. We want to show that $\left\{v^1, ..., v^m\right\}$ arbitrary vectors in V are linearly dependent. Suppose otherwise, then by Lemma 178, we would have $m \leq n$, a contradiction.
2. It is the content of Lemma 178.2.a.
3. We have to show that $\left\{u^1, ..., u^n\right\}$ are linearly independent. Suppose otherwise. Then there exists $k \in \{1, ..., n\}$ such that $\operatorname{span}\left(\left\{\left(u^i\right)_{i \neq k}\right\}\right) = V$, but since $\left\{w^1, ..., w^n\right\}$ is linearly independent, from Proposition 178 (the Replacement Lemma), we have $n \leq n - 1$, a contradiction.

■

Remark 184 *The above Proposition 847 shows that in the case of finite dimensional vector spaces, one of the two conditions defining a basis is sufficient to obtain a basis.*

Proposition 185 *(Completion Lemma) Let V be a vector space of dimension n and $\{w^1, ..., w^m\} \subseteq V$ be a linearly independent set, with*[5] *$m \leq n$. If $m < n$, then, there exists a set $\{u^1, ..., u^{n-m}\}$ such that*

$$\{w^1, ..., w^m, u^1, ..., u^{n-m}\}$$

is a basis of V.

Proof. Take a basis $\{v^1, ..., v^n\}$ of V. Then from Conclusion 2.b in Lemma 178,

$$\text{span}\left(w^1, ..., w^m, v^{i_1}, ..., v^{i_{n-m}}\right) = V.$$

Then from Proposition 847.3, we get the desired result. ■

Proposition 186 *Let W be a subspace of an $n-$dimensional vector space V. Then*

1. $\dim W \leq n$*;*

2. If $\dim W = n$*, then* $W = V$*.*

Proof. 1. From Proposition 847.1, $m > n$ vectors in V are linearly dependent. Since a basis of W is a set of linearly independent vectors then $\dim W \leq n$.

2. If $\{w^1, ..., w^n\}$ is a basis of W, then span $\left(w^1, ..., w^n\right) = W$. Moreover, those vectors are n linearly independent vectors in V. Therefore from Proposition 847.2., span $\left(w^1, ..., w^n\right) = V$. ■

Remark 187 *As a trivial consequence of Proposition 185, $V = \text{span}\{u^1, ..., u^r\} \Rightarrow \dim V \leq r$.*

Example 188 *Let W be a subspace of $\mathbb{R}^3$, whose dimension is 3. Then from the previous Proposition, $\dim W \in \{0, 1, 2, 3\}$. In fact,*
1. If $\dim W = 0$, then $W = \{0\}$, i.e., a point,
2. if $\dim W = 1$, then W is a straight line trough the origin,
3. if $\dim W = 2$, then W is a plane trough the origin,
4. if $\dim W = 3$, then $W = \mathbb{R}^3$.

Definition 189 *A maximal linearly independent subset S' of a set of vectors $S \subseteq V$ is a subset of S such that*
1. S' is linearly independent, and
2. $S' \subseteq S'' \subseteq S \Rightarrow S''$ is linearly dependent, i.e., if S'' is another subset of S whose cardinality is bigger than the cardinality of S', then S'' is a linearly dependent set.

Lemma 190 *Given a vector space V, if*
1. $S = \{v^1, ..., v^k\} \subseteq V$ are linearly independent, and
2. $S' = S \cup \{v^{k+1}\}$ are linearly dependent,
then
v^{k+1} is a linear combination of the vectors in S.

Proof. *First Proof.*
Since S' is a linearly dependent set,

$$\exists i \in \{1, ..., k+1\}, \left(\beta_j\right)_{j \in \{1,...,k+1\} \setminus \{i\}} \text{ such that } v^i = \sum_{j \in \{1,...,k+1\} \setminus \{i\}} \beta_j v^j.$$

If $i = k+1$,we are done. If $i \neq k+1$, without loss of generality, take $i = 1$. Then

$$\exists \left(\gamma_j\right)_{j \in \{1,...,k+1\} \setminus \{1\}} \text{ such that } v^1 = \sum_{j=2}^{k+1} \gamma_j v^j.$$

If $\gamma_{k+1} = 0$, we would have $v^1 - \sum_{j=2}^{k} \gamma_j v^j = 0$, contradicting Assumption 1. Then

$$v^{k+1} = \frac{1}{\gamma_{k+1}} \left(v^1 - \sum_{j=2}^{k} \gamma_j v \right).$$

[5] The inequality $m \leq n$ follows from Proposition 847.1.

Second Proof.

It follows from Proposition 157, as shown below. Observe that for any $i \in \{1, ..., k\}$, $v^k \neq 0$, otherwise S would be linearly dependent. If $v^{k+1} = 0$, we are done. If $v^{k+1} \neq 0$, assume the conclusion is false; then, from Proposition 157, $\exists k' \in \{1, ..., k\}$ such that $v^{k'}$ is a linear combination of the preceding vectors, contradicting assumption 1. ■

Remark 191 *Let V be a vector space and $S \subseteq T \subseteq V$. Then,*

1. span $S \subseteq$ span T*;*

2. span (span S) $=$ span S*.*

Proposition 192 *If S is a finite set of vectors in V and S' is a maximal linearly independent subset of S, then S' is a basis of* span S*.*

Proof. Let S' be equal to $\{v^1, ..., v^n\}$. If $w \in S'$, then $w \in$ span S'; if $w \in S \backslash S'$, then $\{v^1, ..., v^n, w\}$ is linearly dependent and $w \in$ span S'. Therefore, $w \in S$ implies that $w \in$ span S',i.e.,

$$S \subseteq \text{span } S'.$$

Then

$$\text{span } S \overset{(1)}{\subseteq} \text{span}\,(\text{span } S') \overset{(2)}{=} \text{span } S' \overset{(3)}{\subseteq} \text{span } S.$$

where (1) follows from Remark 191.1, (2) Remark 191.2, and (3) from Remark 191.1. Then span $S =$ span S' and therefore S' is a basis of span S. ■

Proposition 193 *Let S', S''be two maximal linearly independent subsets of a finite $S \subseteq V$. Then $\#S = \#S'$.*

Proof. From Proposition 192 , S' and $S''$are a basis of span S. Therefore, they have the same cardinality. ■

Proposition 193 allows to give the following definition.

Definition 194 *The row rank of $A \in \mathbb{M}(m,n)$ is the maximum number of linearly independent rows of A (i.e., the cardinality of each maximal linearly independent subset of the set of the row vectors of A).*

Definition 195 *The column rank of $A \in \mathbb{M}(m,n)$ is the maximum number of linearly independent columns of A.*

Proposition 196 *For any $A \in \mathbb{M}(m,n)$,*

1. *row rank of A is equal to* $\dim \text{row span}$ *of A;*

2. *column rank of A is equal to* $\dim \text{col span}$ *of A.*

Proof. 1.

Take a maximal linearly independent subset S of the rows of A. Then, by definition,

$$\#S = \text{row rank } A. \tag{4.8}$$

From Proposition 192, S is a basis of row span A, and therefore

$$\#S = \dim \text{row rank } A. \tag{4.9}$$

(4.8) and (4.9) give the desired result.

2.

$\dim \text{col span } A \overset{\text{Rmk.146}}{=} \dim \text{row span } A^T = \max \#\ \text{ l.i. rows in } A^T = \max \#\ \text{ l.i. columns in } A.$

■

Proposition 197 *For any $A \in \mathbb{M}(m,n)$, row rank of A is equal to column rank of A.*

Proof. Let A be an arbitrary $m \times n$ matrix

$$\begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... \\ a_{i1} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}$$

Suppose the row rank is $r \leq m$ and the following r vectors form a basis of the row space:

$$\begin{bmatrix} S_1 & = & [b_{11} & ... & b_{1j} & ... & b_{1n}] \\ & & ... \\ S_k & = & [b_{k1} & ... & b_{kj} & ... & b_{kn}] \\ & & ... \\ S_r & = & [b_{r1} & ... & b_{rj} & ... & b_{rn}] \end{bmatrix}$$

Then, each row vector of A is a linear combination of the above vectors, i.e., we have

$$\forall i \in \{1, ..., m\}, \quad R^i = \sum_{k=1}^{r} \alpha_{ik} S_k,$$

or

$$\forall i \in \{1, ..., m\}, \quad \begin{bmatrix} a_{i1} & ... & a_{ij} & ... & a_{in} \end{bmatrix} = \sum_{k=1}^{r} \alpha_{ik} \begin{bmatrix} b_{k1} & ... & b_{kj} & ... & b_{kn} \end{bmatrix},$$

and setting the j component of each of the above vector equations equal to each other, we have

$$\forall j \in \{1, .., n\} \quad \text{and} \quad \forall i \in \{1, ..., m\}, \quad a_{ij} = \sum_{k=1}^{r} \alpha_{ik} b_{kj},$$

and

$$\forall j \in \{1, .., n\}, \quad \begin{cases} a_{1j} & = & \sum_{k=1}^{r} \alpha_{1k} b_{kj}, \\ .... \\ a_{ij} & = & \sum_{k=1}^{r} \alpha_{ik} b_{kj}, \\ ... \\ a_{mj} & = & \sum_{k=1}^{r} \alpha_{mk} b_{kj}, \end{cases}$$

or

$$\forall j \in \{1, .., n\}, \quad \begin{bmatrix} a_{1j} \\ ... \\ a_{ij} \\ ... \\ a_{mj} \end{bmatrix} = \sum_{k=1}^{r} b_{kj} \begin{bmatrix} \alpha_{1k} \\ ... \\ \alpha_{ik} \\ ... \\ \alpha_{mk} \end{bmatrix},$$

i.e., each column of A is a linear combination of the r vectors

$$\left\{ \begin{pmatrix} \alpha_{11} \\ ... \\ \alpha_{i1} \\ ... \\ \alpha_{m1} \end{pmatrix}, ..., \begin{pmatrix} \alpha_{1k} \\ ... \\ \alpha_{ik} \\ ... \\ \alpha_{mk} \end{pmatrix}, ..., \begin{pmatrix} \alpha_{1r} \\ ... \\ \alpha_{ir} \\ ... \\ \alpha_{mr} \end{pmatrix} \right\}.$$

Then, from Remark 187,

$$\dim \operatorname{col} \operatorname{span} A \leq r = \operatorname{row} \operatorname{rank} A, \tag{4.10}$$

i.e.,

$$\operatorname{col} \operatorname{rank} A \leq \operatorname{row} \operatorname{rank} A.$$

From (4.10) ,which holds for arbitrary matrix A, we also get

$$\dim \operatorname{col} \operatorname{span} A^T \leq \operatorname{row} \operatorname{rank} A^T. \tag{4.11}$$

Moreover,

$$\dim \operatorname{col} \ \operatorname{span} A^T \stackrel{Rmk. \ 146}{=} \dim \operatorname{row} \operatorname{span} A := \operatorname{row} \operatorname{rank} A$$

and

$$\text{row rank } A^T := \dim \text{row span } A^T \overset{Rmk.\ 146}{=} \dim \text{col span } A.$$

Then, from the two above equalities and (4.11), we get

$$\text{row rank } A \leq \dim \text{col span } A, \tag{4.12}$$

and (4.10) and (4.12) gives the desired result. ■

We can summarize Propositions 196 and 197 as follows.

Corollary 198 *For every $A \in \mathbb{M}(m,n)$,*

$$\text{row rank } A = \dim \text{row span} A = \text{col rank } A = \dim \text{col span } \ A.$$

Exercise 199 *Check the above result on the following matrix*

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 1 & 6 \\ 3 & 7 & 10 \end{bmatrix}.$$

4.8 Change of basis

Definition 200 *Let V be a vector space over a field F with a basis $S = \{v^1, ..., v^n\}$. Then, from Proposition 176 $\forall v \in V$ $\exists! \, (a_i)_{i=1}^n \in F^n$ such that $v = \sum_{i=1}^n \alpha_i v^i$. The scalars $(a_i)_{i=1}^n$ are called the coordinates of v relative to the basis S and are denoted by $[v]_S$, or simply by $[v]$, when no ambiguity arises. We also denote the $i-th$ component of the vector $[v]_S$ by $[v]_S^i$ and therefore we have $[v]_S = \left([v]_S^i\right)_{i=1}^n$.*

Remark 201 *If $V = \mathbb{R}^n$ and $S = \{e_n^i\}_{i=1}^n := \mathbf{e}_n$, i.e., the canonical basis, then*

$$\forall x \in \mathbb{R}^n, \quad [x]_{\mathbf{e}_n} = \left[\sum_{i=1}^n x_i e_n^i\right] = x.$$

Exercise 202 *Consider the vector space V of polynomials of degree ≤ 2*

$$\{ax^2 + bx + c : a, b, c \in \mathbb{R}\}.$$

Find the coordinates of an arbitrary element of V with respect to the basis

$$\left\{v^1 = 1, v^2 = t - 1, v^3 = (t-1)^2\right\}$$

Definition 203 *Consider a vector space V of dimension n and two bases $\mathbf{v} = \{v^i\}_{i=1}^n$ and $\mathbf{u} = \{u^k\}_{k=1}^n$ of V. Then,*

$$\forall k \in \{1, ..., n\}, \exists! \, (a_{ik})_{i=1}^n = \begin{bmatrix} a_{1k} \\ ... \\ a_{ik} \\ \\ a_{nk} \end{bmatrix} \text{ such that } u^k = \sum_{i=1}^n a_{ik} v^i. \tag{4.13}$$

The matrix

$$P = \begin{bmatrix} a_{11} & ... & a_{1k} & ... & a_{1n} \\ & & ... & & \\ a_{i1} & & a_{ik} & & a_{in} \\ & & & & \\ a_{n1} & & a_{nk} & & a_{nn} \end{bmatrix} \in \mathbb{M}(n,n), \tag{4.14}$$

i.e.,

$$P = \left[\ [u^1]_{\mathbf{v}} \ \ ... \ \ [u^k]_{\mathbf{v}} \ \ ... \ \ [u^n]_{\mathbf{v}} \ \right] \in \mathbb{M}(n,n),$$

is called the change-of-basis matrix from the basis $\mathbf{v}$ to the basis $\mathbf{u}$.

Remark 204 *The name of the matrix P in the above Definition follows from the fact that the entries of P are used to "transform" (in the way described in (4.13)) vectors of the basis* $\mathbf{v}$ *in vectors of the basis* $\mathbf{u}$.

Proposition 205 *If P is the change-of-basis matrix from the basis* $\mathbf{v}$ *to the basis* $\mathbf{u}$ *and Q the change-of-basis matrix from the basis* $\mathbf{u}$ *to the basis* $\mathbf{v}$, *then P and Q are invertible matrices and $P = Q^{-1}$.*

Proof. By assumption,

$$\forall k \in \{1, ..., n\}, \exists (a_{ik})_{i=1}^n \text{ such that } u^k = \sum_{i=1}^n a_{ik} v^i, \tag{4.15}$$

$$P = [a_{ik}] \in \mathbb{M}(n, n),$$

$$\forall i \in \{1, ..., n\}, \exists (b_j)_{j=1}^n = \begin{bmatrix} b_{1i} \\ ... \\ b_{ji} \\ \\ b_{ni} \end{bmatrix} \text{ such that } v^i = \sum_{j=1}^n b_{ji} u^j, \tag{4.16}$$

$$Q = \begin{bmatrix} b_{11} & ... & b_{1i} & ... & b_{1n} \\ & & ... & & \\ b_{j1} & & b_{ji} & & b_{jn} \\ & & & & \\ b_{n1} & & b_{ni} & & b_{nn} \end{bmatrix} \in \mathbb{M}(n, n),$$

Substituting (4.16) in (4.15), we get

$$u^k = \sum_{i=1}^n a_{ik} \left(\sum_{j=1}^n b_{ji} u^j \right) = \sum_{j=1}^n \left(\sum_{i=1}^n b_{ji} a_{ik} \right) u^j$$

Now, observe that $\forall k \in \{1, ..., n\}$,

$$[u^k]_{\mathbf{u}} = \left(\sum_{i=1}^n b_{ji} a_{ik} \right)_{j=1}^n = e_n^k$$

and

$$\left(\sum_{i=1}^n b_{ji} a_{ik} \right)_{j=1}^n = \left(\begin{bmatrix} b_{j1} & b_{ji} & b_{jn} \end{bmatrix} \begin{bmatrix} a_{1k} \\ ... \\ a_{ik} \\ \\ a_{nk} \end{bmatrix} \right)_{j=1}^n = \left(b_{j\cdot}^T a_{\cdot k} \right)_{j=1}^n \text{ is the } k\text{-}th \text{ column of } BA.$$

Therefore

$$BA = I$$

and we got the desired result. ■

The next Proposition explains how the coordinate vectors are affected by a change of basis.

Proposition 206 *Let P be the change-of-basis matrix from* $\mathbf{v}$ *to* $\mathbf{u}$. *Then,*

$$\forall w \in V, \quad [w]_{\mathbf{u}} = P^{-1} \cdot [w]_{\mathbf{v}} \quad \textit{and}$$

$$[w]_{\mathbf{v}} = P \cdot [w]_{\mathbf{u}}$$

Proof. By definition of $[w]_{\mathbf{u}}$, there exists $(\alpha_k)_{k=1}^n \in \mathbb{R}^n$ such that $[w]_{\mathbf{u}} = (\alpha_k)_{k=1}^n$ and

$$w = \sum_{k=1}^{n} \alpha_k u^k.$$

Moreover, as said at the beginning of the previous proof,

$$\forall k \in \{1, ..., n\}, \exists (a_{ik})_{i=1}^n \text{ such that } u^k = \sum_{i=1}^n a_{ik} v^i.$$

Then,

$$w = \sum_{k=1}^{n} \alpha_k \left(\sum_{i=1}^{n} a_{ik} v^i \right) = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \alpha_k a_{ik} \right) v^i,$$

i.e.,

$$[w]_{\mathbf{v}} = \left(\sum_{k=1}^{n} a_{ik} \cdot \alpha_k \right)_{i=1}^{n}.$$

Moreover, using the definition of P - see (4.14) - we get

$$P \cdot [w]_{\mathbf{u}} = \begin{bmatrix} a_{11} & ... & a_{1k} & ... & a_{1n} \\ & & ... & & \\ a_{i1} & & a_{ik} & & a_{in} \\ & & & & \\ a_{n1} & & a_{nk} & & a_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ ... \\ a_k \\ \\ a_n \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n a_{1k} \cdot \alpha_k \\ ... \\ \sum_{k=1}^n a_{ik} \cdot \alpha_k \\ ... \\ \sum_{k=1}^n a_{nk} \cdot \alpha_k \end{bmatrix},$$

as desired. ■

Remark 207 *Although P is called the change-of-basis matrix from $\mathbf{v}$ to $\mathbf{u}$, for the reason explained in Remark 204, it is P^{-1} which transforms the coordinates of $w \in V$ relative to the original basis $\mathbf{v}$ into the coordinates of w relative to the new basis $\mathbf{u}$.*

Proposition 208 *Let $S = \{v^1, ..., v^n\}$ be a subset of a vector space V. Let T be a subset of V obtained from S using one of the following "elementary operations":*

1. *interchange two vectors,*
2. *multiply a vector by a nonzero scalar,*
3. *add a multiple of one vector to another one.*

Then,

1. $\text{span} S = \text{span} T$*, and*
2. *S is independent $\Leftrightarrow$ T is independent.*

Proof. 1. Any element in T is a linear combination of vectors of S. Since any operation has an inverse, i.e., an operation which brings the set back to its original nature - similarly to what said in Proposition 94 - any element in S is a linear combination of vectors in T.

2. S is independent $\overset{(1)}{\Leftrightarrow}$ S is a basis of $\text{span} S$ $\overset{(2)}{\Leftrightarrow}$ $\dim \text{span} S = n$ $\overset{(3)}{\Leftrightarrow}$ $\dim \text{span} T = n \Leftrightarrow T$ is a basis of $\text{span} T \Leftrightarrow T$ is independent

where (1) follows from the definitions of basis and span, the [$\Leftarrow$] part in (2) from Proposition 847.3, (3) from above conclusion 1. . ■

Proposition 209 *Let $A = [a_{ij}], B = [b_{ij}] \in \mathbb{M}(m, n)$ be row equivalent matrices over a field F and $v^1, ...v^n$ vectors in a vector space V. Define*

$$\begin{array}{rcl} u^1 & = & a_{11}v^1 + \quad ... \quad a_{1j}v^j + \quad ... \quad a_{1n}v^n \\ & & \qquad\qquad ... \\ u^i & = & a_{i1}v^1 \qquad a_{ij}v^j + \qquad a_{in}v^n \\ u^m & = & a_{m1}v^1 \qquad a_{mj}v^j + \qquad a_{mn}v^n \end{array}$$

and

$$\begin{array}{ccccccc} w^1 & = & b_{11}v^1+ & ... & b_{1j}v^j+ & ... & b_{1n}v^n \\ & & & & ... & & \\ w^i & = & b_{i1}v^1 & & b_{ij}v^j+ & & b_{in}v^n \\ & & & & & & \\ w^m & = & b_{m1}v^1 & & b_{mj}v^j+ & & b_{mn}v^n \end{array} .$$

Then

$$\text{span}\left(u^1, ..., u^m\right) = \text{span}\left(w^1, ..., w^m\right).$$

Proof. Observe that applying "elementary operations" of the type defined in the previous Proposition to the set of vectors $\left\{u^1, ..., u^m\right\}$ is equivalent to applying elementary row operations to the matrix A.

Since B can be obtained via row operations from A, then the set of vectors $\left\{w^1, ..., w^m\right\}$ can be obtained applying "elementary operations" of the type defined in the previous Proposition to the set of vectors $\left\{u^1, ..., u^m\right\}$, and from that Proposition the desired result follows. ■

Proposition 210 *Let $v^1, ..., v^n$ belong to a vector space V. Assume that $\forall k \in \{1, ..., n\}$, $\exists \left(a_{ik}\right)_{i=1}^n$ such that $u^k = \sum_{k=1}^n a_{ik}v^k$, and define $P = [a_{ik}] \in \mathbb{M}(n, n)$. Then,*

1. If P is invertible, then

$$\left\langle \left\{v^i\right\}_{i=1}^n \text{ is linearly independent} \right\rangle \Leftrightarrow \left\langle \left\{u^k\right\}_{k=1}^n \text{ is linearly independent} \right\rangle;$$

2. If P is not invertible, then $\left\{u^k\right\}_{k=1}^n$ is linearly dependent.

Proof. 1.

From Proposition 104, P is row equivalent to the identity matrix I_n. Therefore, from Proposition 209,

$$\text{span}\left\{v^i\right\}_{i=1}^n = \text{span}\left\{u^k\right\}_{k=1}^n,$$

and from Proposition 208, the desired result follows.

2. Since P is not invertible, then P is not row equivalent to the identity matrix and from Proposition 122, it is equivalent to a matrix with a zero row, say the last one. Then $\forall i \in \{1, ..., n\}$, $a_{ni} = 0$ and $u^n = \sum_{k=1}^n a_{nk}v^k = 0$. ■

Corollary 211 *Let $\mathbf{v} = \left\{v^1, ..., v^n\right\}$ be a basis of a vector space V. Let $P = [a_{ik}] \in \mathbb{M}(n, n)$ be invertible. Then $\mathbf{u} = \left\{u^1, ..., u^n\right\}$ such that $\forall k \in \{1, ..., n\}$, $u^k = \sum_{k=1}^n a_{ik}v^k$ is a basis of V and P is the change-of-basis matrix from $\mathbf{v}$ to $\mathbf{u}$.*

Proposition 212 *If $\mathbf{v} = \left\{v^i\right\}_{i=1}^n$ is a basis of $\mathbb{R}^n$,*

$$B := \begin{bmatrix} v^1 & ... & v^n \end{bmatrix} \in \mathbb{M}(n, n),$$

i.e., B is the matrix whose columns are the vectors of S, and $P \in \mathbb{M}(n, n)$ is an invertible matrix, then

1.

$$BP := \begin{bmatrix} u^1 & ... & u^n \end{bmatrix} \in \mathbb{M}(n, n),$$

is a matrix whose columns are another basis of $\mathbb{R}^n$, and P is the change-of-basis matrix from $\mathbf{v}$ to $\mathbf{u} := \left\{u^1, ..., u^n\right\}$, and

2. $\forall w \in \mathbb{R}^n$, $[w]_{\mathbf{u}} = P^{-1} \cdot [w]_{\mathbf{v}}$.

Proof. 1.

$$\begin{array}{l} BP \overset{(3.4)\text{in Rmk. } 70}{=} \begin{bmatrix} B \cdot C^1(P) & ... & B \cdot C^n(P) \end{bmatrix} \overset{\text{Rmk. } 147}{=} \\ = \begin{bmatrix} \sum_{k=1}^n C^1(P) \cdot v^k & ... & \sum_{k=1}^n C^{nk}(P) \cdot v^k \end{bmatrix} \end{array} .$$

Therefore,

$$\forall i \in \{1, ..., n\}, \quad u^i = \sum_{k=1}^n C^{ki}(P) \cdot v^k,$$

i.e., each element in the basis $\mathbf{u}$ is a linear combinations of elements in $\mathbf{v}$. Then from Corollary 211, the desired result follows.

2. It follows from Proposition 206. ■

Remark 213 *In the above Proposition 212, if*

$$B := \begin{bmatrix} e^1 & ... & e^n \end{bmatrix},$$

i.e., $\mathbf{e} = \{e^1, ..., e^n\}$ *is the canonical basis (and* P *is invertible), then*
1. the column vectors of P *are a new basis of* $\mathbb{R}^n$ *, call it* $\mathbf{u}$*, and*
2. $\forall x \in \mathbb{R}^n$,

$$y := [x]_{\mathbf{u}} = P^{-1} \cdot [x]_{\mathbf{e}} = P^{-1} \cdot x.$$

Chapter 5

Determinant and rank of a matrix

In this chapter we are going to introduce the definition of determinant, an useful tool to study linear dependence, invertibility of matrices and solutions to systems of linear equations.

5.1 Definition and properties of the determinant of a matrix

To motivate the definition of determinant, we present an informal discussion of a way to find solutions to the linear system with two equations and two unknowns, shown below.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \tag{5.1}$$

The system can be rewritten as follows

$$Ax = b$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Let's informally discuss how to find solutions to system (5.1). If $a_{22} \neq 0$ and $a_{12} \neq 0$, multiplying both sides of the first equation by a_{22}, of the second equation by $-a_{12}$ and adding up, we get

$$a_{11}a_{22}x_1 + a_{12}a_{22}x_2 - a_{12}a_{21}x_1 - a_{12}a_{22}x_2 = a_{22}b_1 - a_{12}b_2$$

Therefore, if

$$a_{11}a_{22} - a_{12}a_{21} \neq 0$$

we have

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \tag{5.2}$$

In a similar manner[1] we have

$$x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \tag{5.3}$$

We can then the following preliminary definition: given $A \in \mathcal{M}_{22}$, the determinant of A is

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} := a_{11}a_{22} - a_{12}a_{21}$$

Using the definition of determinant, we can rewrite (5.2) and (5.3)as follows.

$$x_1 = \frac{\det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det A} \quad \text{and} \quad x_2 = \frac{\det \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}}{\det A}$$

[1] Assuming $a_{21} \neq 0$ and $a_{11} \neq 0$, multiply both sides of the first equation by a_{21}, of the second equation by $-a_{11}$ and then add up.

We can now present the definition of the determinant of a **square** matrix $A_{n\times n}$ for arbitrary $n\in\mathbb{N}\setminus\{0\}$.

Definition 214 *Given $n>1$ and $A\in\mathbb{M}(n,n)$, $\forall i,j\in\{1,...,n\}$, we call $A_{ij}\in\mathbb{M}(n-1,n-1)$ the matrix obtained from A erasing the $i-th$ row and the $j-th$ column.*

Definition 215 *Given $A\in\mathbb{M}(1,1)$, i.e., $A=[a]$ with $a\in\mathbb{R}$. The determinant of A is denoted by $\det A$ and we let $\det A:=a$. For $n\in\mathbb{N}\setminus\{0,1\}$, given $A\in\mathbb{M}(n,n)$, we define the determinant of A as*

$$\det A:=\sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j}$$

Observe that $[a_{1j}]_{j=1}^{n}$ is the first row of A, i.e.,

$$\det A:=\sum_{j=1}^{n}(-1)^{1+j}R^{1j}(A)\det A_{1j}$$

Example 216 *For $n=2$, we have*

$$\det\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}=\sum_{j=1}^{2}(-1)^{1+j}a_{1j}\det A_{1j}=(-1)^{1+1}a_{11}\det A_{11}+(-1)^{1+2}a_{12}\det A_{12}=$$

$$=a_{11}a_{22}-a_{12}a_{21}$$

and we get the informal definition given above.

Example 217 *For $n=3$, we have*

$$\det A=\det\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}=(-1)^{1+1}a_{11}\det A_{11}\ +\ (-1)^{1+2}a_{12}\det A_{12}\ +\ (-1)^{1+3}a_{13}\det A_{13}$$

Definition 218 *Given $A=[a_{ij}]\in\mathbb{M}(n,n)$, , $\forall i,j$, $\det A_{ij}$ is called minor of a_{ij} in A;*

$$(-1)^{i+j}\det A_{ij}$$

is called cofactor of a_{ij} in A.

Theorem 219 *Given $A\in\mathbb{M}(n,n)$, $\det A$ is equal to the sum of the products of the elements of any rows or column for the corresponding cofactors, i.e.,*

$$\forall i\in\{1,...n\},\ \det A=\sum_{j=1}^{n}(-1)^{i+j}R^{ij}(A)\det A_{ij}\tag{5.4}$$

and

$$\forall j\in\{1,...n\},\ \det A=\sum_{i=1}^{n}(-1)^{i+j}C^{ji}(A)\det A_{ij}\tag{5.5}$$

Proof. Omitted. We are going to omit several proofs about determinants. There are different ways of introducing the concept of determinant of a square matrix. One of them uses the concept of permutations - see, for example, Lipschutz (1991), Chapter 7. Another one is an axiomatic approach - see, for example, Lang (1971) - he introduces (three) properties that a function $f:\mathbb{M}(n,n)\to\mathbb{R}$ has to satisfy and then shows that there exists a unique such function, called determinant. Following the first approach the proof of the present theorem can be found on page 252, in Lipschutz (1991) Theorem 7.8, or following the second approach, in Lang (1971), page 128, Theorem 4. ■

Definition 220 *The expression used above for the computation of $\det A$ in (5.4) is called "(Laplace) expansion" of the determinant by row i.*

The expression used above for the computation of $\det A$ in (5.5) is called "(Laplace) expansion" of the determinant by column j.

Definition 221 *Consider a matrix $A_{n\times n}$. Let $1 \le k \le n$. A $k-th$ order principal submatrix (minor) of A is the (determinant of the) square submatrix of A obtained deleting $(n-k)$ rows and $(n-k)$ columns in the same position.*

Theorem 222 *(Properties of determinants)*

Let the matrix $A = [a_{ij}] \in \mathbb{M}(n, n)$ be given. Properties presented below hold true even if words "column, columns" are substituted by the words "row, rows".

1. $\det A = \det A^T$.
2. *if two columns are interchanged, the determinant changes its sign,.*
3. *if there exists $j \in \{1, .., n\}$ such that $C^j(A) = \sum_{k=1}^{p} \beta_k b^k$, then*

$$\det\left[C^1(A), ..., \sum_{k=1}^{p} \beta_k b^k, ..., C^n(A)\right] = \sum_{k=1}^{p} \beta_k \det\left[C^1(A), ..., b^k, ..., C^n(A)\right],$$

   i.e., the determinant of a matrix which has a column equal to the linear combination of some vectors is equal to the linear combination of the determinants of the matrices in which the column under analysis is each of the vector of the initial linear combination, and, therefore, $\forall \beta \in \mathbb{R}$ and $\forall j \in \{1, ..., n\}$,

$$\det\left[C^1(A), ..., \beta C^j(A), ..., C^n(A)\right] = \beta \det\left[C^1(A), ..., C^j(A), ..., C^n(A)\right] = \beta \det A.$$

4. *if $\exists j \in \{1, ..., n\}$ such that $C^j(A) = 0$, then $\det A = 0$, i.e., if a matrix has column equal to zero, then the determinant is zero.*
5. *if $\exists\ j, k \in \{1, ..., n\}$ and $\beta \in \mathbb{R}$such that $C^j(A) = \beta C^k(A)$, then $\det A = 0$, i.e., the determinant of a matrix with two columns proportional one to the other is zero.*
6. *If $\exists k \in \{1, ..., n\}$ and $\beta_1, ..., \beta_{k-1}, \beta_{k+1}, ..., \beta_n \in \mathbb{R}$ such that $C^k(A) = \sum_{j\ne k} \beta_j C^j(A)$, then $\det A = 0$, i.e., if a column is equal to a linear combination of the other columns, then $\det A = 0$.*
7.

$$\det\left[C^1(A), ..., C^k(A) + \sum_{j\ne k} \beta_j \cdot C^j(A), ..., C^n(A)\right] = \det A$$

8. $\forall j, j^* \in \{1, ..., n\}$, $\sum_{i=1}^{n} a_{ij} \cdot (-1)^{i+j^*} \det A_{ij^*} = 0$, *i.e., the sum of the products of the elements of a column times the cofactor of the analogous elements of another column is equal to zero.*
9. *If A is triangular, $\det A = a_{11} \cdot ... \cdot a_{22} \cdot ... a_{nn}$, i.e., if A is triangular (for example, diagonal), the determinant is the product of the elements on the diagonal.*

Proof. 1.

Consider the expansion of the determinant by the first row for the matrix A and the expansion of the determinant by the first column for the matrix A^T.

2.

We proceed by induction. Let A be the starting matrix and $A'$the matrix with the interchanged columns.

$\mathcal{P}(2)$ is obviously true.

$\mathcal{P}(n-1) \Rightarrow \mathcal{P}(n)$

Expand $\det A$ and $\det A'$ by a column which is not any of the interchanged ones:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} C_j^i(A) \det A_{ij}$$

$$\det A' = \sum_{i=1}^{n} (-1)^{i+k} C_j^i(A) \det A'_{ij}$$

Since $\forall k \in \{1, ..., n\}$, A_{kj}, $A'_{kj} \in \mathbb{M}(n-1, n-1)$, and they have interchanged column, by the induction argument, $\det A_{kj} = -\det A'_{kj}$, and the desired result follows.

3.

Observe that

$$\sum_{k=1}^{p} \beta_k b^k = \left(\sum_{k=1}^{p} \beta_k b_i^k \right)_{i=1}^{n}.$$

Then,

$$\det \left[C^1(A), ..., \sum_{k=1}^{p} \beta_k b^k, ..., C^n(A) \right] =$$

$$= \det \left[C^1(A), ..., \sum_{k=1}^{p} \beta_k \begin{bmatrix} b_1^k \\ ... \\ b_i^k \\ ... \\ b_n^k \end{bmatrix}, ..., C^n(A) \right] =$$

$$= \det \left[C^1(A), ..., \begin{bmatrix} \sum_{k=1}^{p} \beta_k b_1^k \\ ... \\ \sum_{k=1}^{p} \beta_k b_i^k \\ ... \\ \sum_{k=1}^{p} \beta_k b_n^k \end{bmatrix}, ..., C^n(A) \right] =$$

$$= \sum_{i=1}^{n} (-1)^{i+j} \left(\sum_{k=1}^{p} \beta_k b_i^k \right) \det A_{ij} =$$

$$= \sum_{k=1}^{p} \beta_k \sum_{i=1}^{n} (-1)^{i+j} b_i^k \det A_{ij} = \sum_{k=1}^{p} \beta_k \det \left[C^1(A), ..., b^k, ..., C^n(A) \right].$$

4.

It is sufficient to expand the determinant by the column equal to zero.

5.

Let $A := \left[C^1(A), \beta C^1(A), C^3(A), ..., C^n(A) \right]$ and $\widetilde{A} := \left[C^1(A), C^1(A), C^3(A), ..., C^n(A) \right]$ be given. Then $\det A = \beta \det \left[C^1(A), C^1(A), C^3(A), ..., C^n(A) \right] = \beta \det \widetilde{A}$. Interchanging the first column with the second column of the matrix $\widetilde{A}$, from property 2, we have that $\det \widetilde{A} = -\det \widetilde{A}$ and therefore $\det \widetilde{A} = 0$, and $\det A = \beta \det \widetilde{A} = 0$.

6.

It follows from 3 and 5.

7.

It follows from 3 and 6.

8.

It follows from the fact that the obtained expression is the determinant of a matrix with two equal columns.

9.

It can be shown by induction and expanding the determinant by the firs row or column, choosing one which has all the elements equal to zero excluding at most the first element. In other words, in the case of an upper triangular matrix, we can say what follows.

$$\det \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = a_{11} \cdot a_{22}.$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1n} \\ 0 & a_{22} & a_{23} & ... & a_{2n} \\ 0 & 0 & a_{33} & & a_{3n} \\ ... & & & \ddots & \\ 0 & ... & 0 & ... & a_{nn} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & a_{23} & ... & a_{2n} \\ 0 & a_{33} & & a_{3n} \\ & & \ddots & \\ ... & 0 & ... & a_{nn} \end{bmatrix} = a_{11} \cdot a_{22} \cdot a_{33} \cdot ... a_{nn}.$$

■

Theorem 223 *$\forall A, B \in \mathbb{M}(n, n)$, $\det(AB) = \det A \cdot \det B$.*

Proof. Exercise. ■

Definition 224 *$A \in \mathbb{M}(n, n)$ is called nonsingular if $\det A \neq 0$.*

5.2 Rank of a matrix

Definition 225 *Given $A \in \mathbb{M}(m,n)$, a square submatrix of A of order $k \leq \min\{m,n\}$ is a matrix obtained considering the elements belonging to k rows and k columns of A.*

Definition 226 *Given $A \in \mathbb{M}(m,n)$, the* rank *of A is the greatest order of square nonsingular submatrices of A.*

Remark 227 $\text{rank}A \leq \min\{m,n\}$.

To compute rank A, with $A \in \mathbb{M}(m,n)$, we can proceed as follows.

1. Consider $k = \min\{m,n\}$, and the set of square submatrices of A of order k. If there exists a nonsingular matrix among them, then rank $A = k$. If all the square submatrices of A of order k are singular, go to step 2 below.

2. Consider $k-1$, and then the set of the square submatrices of A of order $k-1$. If there exists a nonsingular matrix among them, then rank $A = k-1$. If all square submatrices of order $k-1$are singular, go to step 3.

3. Consider $k-2$ ...

and so on.

Remark 228 *1.* rank $I_n = n$.

2. The rank of a matrix with a zero row or column is equal to the rank of that matrix without that row or column, i.e.,

$$\text{rank} \begin{bmatrix} A \\ 0 \end{bmatrix} = \text{rank} \begin{bmatrix} A & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \text{rank } A$$

That result follows from the fact that the determinant of any square submatrix of A involving that zero row or columns is zero.

3. From the above results, we also have that

$$\text{rank} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = r$$

We now describe an easier way to the compute the rank of A, which in fact involves elementary row and column operations we studied in Chapter 1.

Proposition 229 *Given $A, A' \in \mathbb{M}(m,n)$,*

$$\langle\ A \text{ is equivalent to } A' \rangle \Leftrightarrow \langle \text{rank } A = \text{rank } A' \rangle$$

Proof. $[\Rightarrow]$ Since A is equivalent to A', it is possible to go from A to A' through a finite number of elementary row or column operations. In each step, in any square submatrix A^* of A which has been changed accordingly to those operations, the elementary row or column operations 1, 2 and 3 (i.e., 1. row or column interchange, 2. row or column scaling and 3. row or column addition) are such that the determinant of A^* remains unchanged or changes its sign (Property 2, Theorem 222), it is multiplied by a nonzero constant (Property 3), remains unchanged (Property 7), respectively.

Therefore, each submatrix A^* whose determinant is different from zero remains with determinant different from zero and any submatrix A^* whose determinant is zero remains with zero determinant.

$[\Leftarrow]$

From Corollary 154.2 [2], we have that there exist unique $\widehat{r}$ and $\widehat{r}'$ such that

$$A \text{ is equivalent to } \begin{bmatrix} I_{\widehat{r}} & 0 \\ 0 & 0 \end{bmatrix}$$

[2] That results says what follows:
For any $A \in \mathbb{M}(m,n)$, there exist invertible matrices $P \in \mathbb{M}(m,m)$ and $Q \in \mathbb{M}(n,n)$ and a unique number $r \in \{0,1,...,\min\{m,n\}\}$ such that A is equivalent to

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$A' \text{ is equivalent to } \begin{bmatrix} I_{\widehat{r}'} & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, from [⇒] part of the present proposition, and Remark 228

$$rank A = \operatorname{rank} \begin{bmatrix} I_{\widehat{r}} & 0 \\ 0 & 0 \end{bmatrix} = \widehat{r}$$

and

$$rank A' = \operatorname{rank} \begin{bmatrix} I_{\widehat{r}'} & 0 \\ 0 & 0 \end{bmatrix} = \widehat{r}'.$$

Then, by assumption, $\widehat{r} = \widehat{r}' := r$, and A and A' are equivalent to

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

and therefore A is equivalent to A'. ■

Example 230 *Given*

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

we can perform the following elementary rows and column operations, and cancellation of zero row and columns on the matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the rank of the matrix is 2.

5.3 Inverse matrices (continued)

Using the notion of determinant, we can find another way of analyzing the problems of i. existence and ii. computation of the inverse matrix. To do that, we introduce the concept of adjoint matrix.

Definition 231 *Given a matrix $A_{n\times n}$, we call adjoint matrix of A, and we denote it by Adj A, the matrix whose elements are the cofactors*[3] *of the corresponding elements of A^T.*

Remark 232 *In other words to construct Adj A,*

1. construct A^T,

2. consider the cofactors of each element of A^T.

Example 233

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \; A^T = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, \; Adj\ A = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$$

Observe that

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = (\det A) \cdot I.$$

[3] From Definition 218, recall that given $A = [a_{ij}] \in \mathbb{M}(m, m)$, $\forall i, j$,

$$(-1)^{i+j} \det A_{ij}$$

is called cofacor of a_{ij} in A.

Example 234

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \; A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}, \; Adj\ A = \begin{bmatrix} -4 & 6 & 1 \\ 2 & -3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

Proposition 235 *Given $A_{n\times n}$, we have*

$$A \cdot Adj\ A = Adj\ A \cdot A = \det A \cdot I \tag{5.6}$$

Proof. Making the product $A \cdot Adj\ \ A := B$, we have

1. $\forall i \in \{1, ..., n\}$, the $i - th$ element on the diagonal of B is the expansion of the determinant by the $i - th$ row and therefore is equal to $\det A$.

2. any element not on the diagonal of B is the product of the elements of a row times the corresponding cofactor a parallel row and it is therefore equal to zero due to Property 8 of the determinants stated in Theorem 222). ■

Example 236

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = -3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -4 & 6 & 1 \\ 2 & -3 & -2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = -3 \cdot I$$

Proposition 237 *Given an $n \times n$ matrix A, the following statements are equivalent:*

1. $\det A \neq 0$, *i.e., A is nonsingular;*

2. A^{-1} *exists, i.e., A is invertible;*

3. *rank* $A = n$;

4. $\operatorname{col\,rank}\ A = n$, *i.e., the column vectors of the matrix A are linearly independent;*

5. $\operatorname{row\,rank}\ A = n$, *i.e., the row vectors of the matrix A are linearly independent;*

6. $\dim \operatorname{col\,span}\ A = n$;

7. $\dim \operatorname{row\,span}\ A = n$.

Proof. $1 \Rightarrow 2$

From (5.6) and from the fact that $\det A \neq 0$, we have

$$A \cdot \frac{Adj\ A}{\det A} = \frac{Adj\ A}{\det A} \cdot A = I$$

and therefore

$$A^{-1} = \frac{Adj\ A}{\det A} \tag{5.7}$$

$1 \Leftarrow 2$

$AA^{-1} = I \Rightarrow \det\left(AA^{-1}\right) = \det I \Rightarrow \det A \cdot \det A^{-1} = 1 \Rightarrow \det A \neq 0$ (and $\det A^{-1} \neq 0$).

$1 \Leftrightarrow 3$

It follows from the definition of rank and the fact that A is $n \times n$ matrix.

$2 \Rightarrow 4$

From (4.6), it suffices to show that $\langle Ax = 0 \Rightarrow x = 0\rangle$. Since A^{-1} exists, $Ax = 0 \Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow x = 0$.

$2 \Leftarrow 4$

From Proposition 847.2, the n linearly independent column vectors $\left[C^1(A), ..., C_i(A), ..., C^n(A)\right]$ are a basis of $\mathbb{R}^n$. Therefore, each vector in $\mathbb{R}^n$ is equal to a linear combination of those vectors.

Then $\forall k \in \{1, ..., n\}$ $\exists b^k \in \mathbb{R}^n$ such that the $k-th$ vector e^k in the canonical basis is equal to $\left[C^1(A), ..., C_i(A), ..., C^n(A)\right] \cdot b^k = Ab^k$, i.e.,

$$\begin{bmatrix} e^1 & ... & e^k & ... & e^n \end{bmatrix} = \begin{bmatrix} Ab^1 & ... & Ab^k & ... & Ab^n \end{bmatrix}$$

or, from (3.4) in Remark 70, defined

$$B := \begin{bmatrix} b^1 & ... & b^k & ... & b^n \end{bmatrix}$$

$$I = AB$$

i.e., A^{-1} exists (and it is equal to B).

The remaining equivalences follow from Corollary 198. ■

Remark 238 *From the proof of the previous Proposition, we also have that, if* $\det A \neq 0$, *then* $\det A^{-1} = (\det A)^{-1}$.

Remark 239 *The previous theorem gives a way to compute the inverse matrix as explained in* (5.7).

Example 240 *1.*

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}^{-1} = -\frac{1}{3} \begin{bmatrix} -4 & 6 & 1 \\ 2 & -3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

2.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

3.

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}^{-1} \textit{ does not exist because } \det \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} = 0$$

4.

$$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{3}{4} & -\frac{1}{2} \\ 1 & 0 & 0 \\ -1 & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

5.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$$

if $a, b, c \neq 0$.

5.4 Span of a matrix, linearly independent rows and columns, rank

Proposition 241 *Given* $A \in \mathbb{M}(m, n)$, *then*

$$\dim \text{row span } A = \text{rank } A.$$

Proof. First proof.

The following result which is the content of Corollary 154.5 plays a crucial role in the proof:

For any $A \in \mathbb{M}(m, n)$, there exist invertible matrices $P \in \mathbb{M}(m, m)$ and $Q \in \mathbb{M}(n, n)$ and a unique number $r \in \{0, 1, ..., \min\{m, n\}\}$ such that

$$A \text{ is equivalent to } PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

From the above result, Proposition 229 and Remark 228 , we have that

$$\operatorname{rank} A = \operatorname{rank} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = r. \tag{5.8}$$

From Propositions 105 and 148, we have

$$\text{row span } A = \text{row span } PA;$$

from Propositions 119 and 148, we have

$$\text{col span } PA = \text{col span } PAQ = \text{col span } \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

From Corollary 198,

$$\dim \text{row span } PA = \dim \text{col span } PA.$$

Therefore

$$\dim \text{row span } A = \dim \text{col span } \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = r, \tag{5.9}$$

where the last equality follows simply because

$$\text{col span } \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \text{col span } \begin{bmatrix} I_r \\ 0 \end{bmatrix},$$

and the r column vectors of the matrix $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ are linearly independent and therefore, from Proposition 192 , they are a basis of span $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$.

From (5.8) and (5.9), the desired result follows.

Second proof.

We are going to show that row rank $A = \operatorname{rank} A$.

Recall that

$$\text{row rank } A := \left\langle \begin{array}{l} r \in \mathbb{N} \text{ such that} \\ \text{i . } r \text{ row vectors of } A \text{ are linearly independent,} \\ \text{ii. if } m > r \text{, any set of rows of } A \text{ of cardinality } > r \text{ is linearly dependent.} \end{array} \right\rangle$$

We ant to show that

1. if row rank $A = r$, then rank $A = r$, and
2. if rank $A = r$, then row rank $A = r$.

1.

Consider the r l.i. row vectors of A. Since r is the maximal number of l.i. row vectors, from Lemma 190, each of the remaining $(m-r)$ row vectors is a linear combination of the r l.i. ones. Then, up to reordering of the rows of A, which do not change either row rank A or rank A, there exist matrices $A_1 \in \mathbb{M}(r,n)$ and $A_2 \in \mathbb{M}(m-r,n)$ such that

$$\operatorname{rank} A = \operatorname{rank} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \operatorname{rank} A_1$$

where the last equality follows from Proposition 229. Then r is the maximum number of l.i. row vectors of A_1 and therefore, from Proposition 197, the maximum number of l.i. column vectors of A_1. Then, again from Lemma 190, we have that there exist $A_{11} \in \mathbb{M}(r,r)$ and $A_{12} \in \mathbb{M}(r,n-r)$ such that

$$\operatorname{rank} A_1 = \operatorname{rank} \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} = \operatorname{rank} A_{11}$$

Then the square $r \times r$ matrix A_{11} contains r linearly independent vectors. Then from Proposition 237, the result follows.

2.

By Assumption, up to reordering of rows, which do not change either row rank A or rank A,

$$A = \begin{bmatrix} & s & r & n-r-s \\ & & & \\ r & A_{11} & A_{12} & A_{13} \\ m-r & A_{21} & A_{22} & A_{23} \end{bmatrix}$$

with

$$\text{rank } A_{11} = r.$$

Then from Proposition 237, row, and column, vectors of A_{12} are linearly independent. Then from Corollary 172, the r row vectors of

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \end{bmatrix}$$

are l.i.

Now suppose that the maximum number of l.i. row vectors of A are $r' > r$ (and the other $m - r'$ row vectors of A are linear combinations of them). Then from part 1 of the present proof, *rank* $A = r' > r$, contradicting the assumption. ■

Remark 242 *From Corollary 198 and the above Proposition, we have for any matrix $A_{m \times n}$, the following numbers are equal:*

1. rank A :=*greatest order of square nonsingular submatrices of A,*
2. dim row span A*,*
3. dim col span A
4. row rank $A :=$ max *number of linear independent rows of A,*
5. col rank $A :=$ max *number of linear independent columns of A.*

Chapter 6

Eigenvalues and eigenvectors

In the present chapter, we try to answer the following basic question: given a real matrix $A \in \mathbb{M}(n,n)$, can we find a nonsingular matrix P (which can be thought to represent a change-of-basis matrix (see Corollary 212 and Remark after it) such that

$$P^{-1}AP \text{ is a diagonal matrix?}$$

6.1 Characteristic polynomial

We first introduce the notion of similar matrices.

Definition 243 *A matrix $B \in \mathbb{M}(n,n)$ is similar to a matrix $A \in \mathbb{M}(n,n)$ if there exists an invertible matrix $P \in \mathbb{M}(n,n)$ such that*

$$B = P^{-1}AP$$

Remark 244 *Similarity is an equivalence relation. Therefore, we say that A and B are similar matrices if $B = P^{-1}AP$.*

Definition 245 *A matrix $A \in \mathbb{M}(n,n)$ is said to be diagonalizable if there exists an invertible matrix P such that $B = P^{-1}AP$ is a diagonal matrix, i.e., if there exists a similar diagonal matrix.*

Definition 246 *Given a matrix $A \in \mathbb{M}_F(n,n)$, the matrix*

$$tI - A,$$

with $t \in F$, is called the characteristic matrix of A;

$$\det[tI - A]$$

is called the characteristic polynomial (in t) and it is denoted by $\Delta_A(t)$;

$$\Delta_A(t) = 0$$

is the characteristic equation of A .

Proposition 247 *Similar matrix have the same determinant and the same characteristic polynomial.*

Proof. Assume that A and B are similar matrices; then, by definition, there exists a nonsingular matrix P such that $B = P^{-1}AP$. Then

$$\begin{aligned}&\det B = \det\left[P^{-1}AP\right] = \det P^{-1} \det A \det P = \\ &= \det P^{-1} \det P \det A = \det\left(P^{-1}P\right) \det A = \\ &= \det A.\end{aligned}$$

and very similarly,

$$\begin{aligned}
&\det [tI - B] = \det \left[tI - P^{-1}AP\right] = \det \left[P^{-1}tIP - P^{-1}AP\right] = \\
&= \det \left[P^{-1}(tI - A) P\right] = \det P^{-1} \det [tI - A] \det P = \\
&= \det P^{-1} \det P \det [(tI - A)] = \det \left(P^{-1}P\right) \det [(tI - A)] = \\
&= \det [(tI - A)] .
\end{aligned}$$

■

The proof of the following result goes beyond the scope of these notes and it is therefore omitted.

Proposition 248 *Let $A \in \mathbb{M}_F(n,n)$ be given. Then its characteristic polynomial is*

$$t^n - S_1 t^{n-1} + S_2 t^{n-2} + ... + (-1)^n S_n,$$

where $\forall i \in \{1, ..., n\}$, S_i is the sum of the principal minors of order i in A - see Definition 221.

Exercise 249 *Verify the statement of the above Proposition for $n \in \{2, 3\}$.*

6.2 Eigenvalues and eigenvectors

Definition 250 *Let $A \in \mathbb{M}(n,n)$ on a field F be given. A scalar $\lambda \in F$ is called an eigenvalue of A if there exists a* **nonzero** *vector $v \in F^n$ such that*

$$Av = \lambda v.$$

Every vector v satisfying the above relationship is called an eigenvector of A associated with (or belonging to) the eigenvalue λ.

Remark 251 *The two main reasons to require an eigenvector to be different from zero are explained in Proposition 253, part 1, and the first line in both steps of the induction proof of Proposition 260 below.*

Definition 252 *Let E_λ be the set of eigenvectors belonging to λ.*

Proposition 253 *1. There is only one eigenvalue associated with an eigenvector.*
2. The set $E_\lambda \cup \{0\}$ is a subspace of F^n.

Proof. 1. If $Av = \lambda v = \mu v$, then $(\lambda - \mu) v = 0$. Since $v \neq 0$, the result follows.
2. Step 1. If λ is an eigenvalue of A and v is an associated eigenvector, then $\forall k \in F \setminus \{0\}$, kv is an associated eigenvector as well: $Av = \lambda v \Rightarrow A(kv) = kAv = k(\lambda v)$.
Step 2. If λ is an eigenvalue of A and v, u are associated eigenvectors, then $v + u$ is an associated eigenvector as well: $A(v + u) = Av + Au = \lambda v + \lambda u = \lambda (v + u)$. ■

Proposition 254 *Let A be a matrix in $\mathbb{M}(n,n)$ over a field F. Then, the following statements are equivalent.*

1. *$\lambda \in F$ is an eigenvalue of A.*
2. *$\lambda I - A$ is a singular matrix.*
3. *$\lambda \in F$ is a solution to the characteristic equation $\Delta_A(t) = 0$.*

Proof. $\lambda \in F$ is an eigenvalue of $A \Leftrightarrow \exists v \in F^n \setminus \{0\}$ such that $Av = \lambda v \Leftrightarrow \exists v \in F^n \setminus \{0\}$ such that $(\lambda I - A) v = 0 \overset{\text{Exercise}}{\Leftrightarrow} \lambda I - A$ is a singular matrix $\Leftrightarrow \det(\lambda I - A) = 0 \Leftrightarrow \lambda \in F$ is a solution to the characteristic equation $\Delta_A(t) = 0$. ■

Theorem 255 ***(The fundamental theorem of algebra)*** [1] *Let*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 \qquad \textit{with } a_n \neq 0$$

[1] For a brief description of the field of complex numbers, see Chapter 10.

be a polynomial of degree $n \geq 1$ with complex coefficients $a_0, ..., a_n$. Then

$$p(z) = 0$$

for at least one $z \in \mathbb{C}$.

In fact, there exist $r \in \mathbb{N} \setminus \{0\}$, pairwise distinct $(z_i)_{i=1}^r \in \mathbb{C}^r$ and $(n_i)_{i=1}^r \in \mathbb{N}^r$, $c \in \mathbb{C} \setminus \{0\}$ such that

$$p(z) = c \cdot \Pi_{i=1}^r (z - z_i)^{n_i}$$

and

$$\sum_{i=1}^r n_i = n.$$

Definition 256 *For any $i \in \{1, ..., r\}$, n_i is called the algebraic multiplicity of the solution z_i.*

As a consequence of the Fundamental Theorem of Algebra, we have the following result.

Proposition 257 *Let A be a matrix in $\mathbb{M}(n, n)$ over the complex field $\mathbb{C}$. Then, A has at least one eigenvalue.*

Definition 258 *Let λ be an eigenvalue of A. The algebraic multiplicity of λ is the multiplicity of λ as a solution to the characteristic equation $\Delta_A(t) = 0$. The geometric multiplicity of λ is the dimension of $E_\lambda \cup \{0\}$.*

In Section 8.8, we will prove the following result: Let λ be an eigenvalue of A. The geometric multiplicity of λ is smaller or equal than the algebraic multiplicity of λ.

6.3 Similar matrices and diagonalizable matrices

Proposition 259 *$A \in \mathbb{M}(n, n)$ is diagonalizable, i.e., A is similar to a diagonal matrix $D \Leftrightarrow$ A has n linearly independent eigenvectors. In that case, the elements on the diagonal of D are the associated eigenvalues and $D = P^{-1}AP$, where P is the matrix whose columns are the n eigenvectors.*

Proof. [$\Leftarrow$] Assume that $\{v^k\}_{k=1}^n$ is the set of linearly independent eigenvectors associated with A and for every $k \in \{1, ..., n\}$, λ_k is the unique eigenvalue associated with v^k. Then,

$$\forall k \in \{1, ..., n\}, \quad Av^k = \lambda_k v^k$$

i.e.,

$$AP = P \cdot \text{diag} \quad [(\lambda_k)_{k=1}^n],$$

where P is the matrix whose columns are the n eigenvectors. Since they are linearly independent, P is invertible and

$$P^{-1}AP = \text{diag} \quad [(\lambda_k)_{k=1}^n].$$

[$\Rightarrow$] Assume that there exists an invertible matrix $P \in \mathbb{M}(n, n)$ and a diagonal matrix $D =$diag $[(\lambda_k)_{k=1}^n]$ such that $P^{-1}AP = D$. Then

$$AP = PD.$$

Let $\{v^k\}_{k=1}^n$ be the column vectors of P. Then the above expression can be rewritten as

$$\forall k \in \{1, ..., n\}, \quad Av^k = \lambda_k v^k.$$

First of all, $\forall k \in \{1, ..., n\}, \quad v^k \neq 0$, otherwise P would not be invertible. Then v^k, λ_k are eigenvectors-eigenvalues associated with A. Finally $\{v^k\}_{k=1}^n$ are linearly independent because P is invertible. ■

Proposition 260 *Let $v^1, ..., v^n$ be eigenvectors of $A \in \mathbb{M}(m, m)$ belonging to pairwise distinct eigenvalues $\lambda_1, ..., \lambda_n$, respectively. Then $v^1, ..., v^n$ are linearly independent.*

Proof. We prove the result by induction on n.
Step 1. $n = 1$.
Since, by definition, eigenvectors are nonzero, the result follows.
Step 2.
Assume that

$$\sum_{i=1}^{n} a_i v^i = 0. \tag{6.1}$$

Multiplying (6.1) by A, and using the assumption that $\forall i \in \{1, ..., n\}$, $Av^i = \lambda_i v^i$, we get

$$0 = \sum_{i=1}^{n} a_i A v^i = \sum_{i=1}^{n} a_i \lambda_i v^i. \tag{6.2}$$

Multiplying (6.1) by λ_n, we get

$$0 = \sum_{i=1}^{n} a_i \lambda_n v^i. \tag{6.3}$$

Subtracting (6.3) from (6.2), we get

$$0 = \sum_{i=1}^{n-1} a_i (\lambda_i - \lambda_n) v^i. \tag{6.4}$$

From the assumption of Step 2 of the proof by induction and from (6.4), we have that $\left(v^1, ..., v^{n-1}\right)$ are linearly independent, and therefore, $\forall i \in \{1, ..., n-1\}$, $a_i (\lambda_i - \lambda_n) = 0$, and since eigenvalue are pairwise distinct,

$$\forall i \in \{1, ..., n-1\}, a_i = 0.$$

Substituting the above conditions in (6.1), we get $a_n v^n = 0$, and since v^n is an eigenvector and therefore different from zero,

$$a_n = 0.$$

■

Remark 261 *$A \in \mathbb{M}(m, m)$ admits at most m distinct eigenvalues. Otherwise, from Proposition 260, you would have $m + 1$ linearly independent vectors in $\mathbb{R}^m$, contradicting Proposition 847 .1.*

Proposition 262 *Let $A \in \mathbb{M}(n, n)$ be given. Assume that $\Delta_A(t) = \Pi_{i=1}^{n} (t - a_i)$, with $a_1, ..., a_n$ pairwise distinct. Then, A is similar to diag $\left[(a_i)_{i=1}^n\right]$, and, in fact, diag $\left[(a_i)_{i=1}^n\right] = P^{-1}AP$, where P is the matrix whose columns are the n eigenvectors.*

Proof. From Proposition 254, $\forall i \in \{1, ..., n\}$, a_i is an eigenvalue of A. For every $i \in \{1, ..., n\}$, let v^i be the eigenvector associated with a_i. Since $a_1, ..., a_n$ are pairwise distinct, from Proposition 260, the eigenvectors $v^1, ..., v^n$ are linearly independent. From Proposition 259, the desired result follows. ■

In Section 9.2, we will describe an algorithm to find eigenvalues and eigenvectors of A and to show if A is diagonalizable.

Chapter 7

Linear functions

7.1 Definition

Definition 263 *Given the vector spaces V and U over the same field F, a function $l : V \to U$ is linear if*

1. $\forall v, w \in V$, $l(v + w) = l(v) + l(w)$, *and*

2. $\forall \alpha \in F$, $\forall v \in V$, $l(\alpha v) = \alpha l(v)$.

Call $L(V, U)$ the set of all such functions. Any time we write $L(V, U)$, we implicitly assume that V and U are vector spaces on the same field F.

In other words, l is linear if it "preserves" the two basic operations of vector spaces.

Remark 264 *1. $l \in L(V, U)$ iff $\forall v, w \in V$ and $\forall \alpha, \beta \in F, l(\alpha v + \beta w) = \alpha l(v) + \beta l(w)$;*
2. If $l \in L(V, U)$, then $l(0) = 0$: for arbitrary $x \in V$, $l(0) = l(0x) = 0l(x) = 0$.

Example 265 *Let V and U be vector spaces. The following functions are linear.*
1. (identity function)

$$l_1 : V \to V, \qquad l_1(v) = v.$$

2. (null function)

$$l_2 : V \to U, \qquad l_2(v) = 0.$$

3.

$$\forall a \in F, \qquad l_a : V \to V, \qquad l_a(v) = av.$$

4. (projection function)
$\forall n \in \mathbb{N}, \forall k \in \mathbb{N} \setminus \{0\}$,

$$proj_{n+k,n} : \mathbb{R}^{n+k} \to \mathbb{R}^n, \qquad proj_{n+k,n} : (x_i)_{i=1}^{n+k} \mapsto (x_i)_{i=1}^{n};$$

5. (immersion function)
$\forall n \in \mathbb{N}, \forall k \in \mathbb{N} \setminus \{0\}$,

$$i_{n,n+k} : \mathbb{R}^n \to \mathbb{R}^{n+k}, \qquad i_{n,n+k} : (x_i)_{i=1}^{n} \mapsto \left((x_i)_{i=1}^{n}, 0\right) \qquad \text{with } 0 \in \mathbb{R}^k.$$

Example 266 *Taken $A \in \mathbb{M}(m, n)$, then*

$$l : \mathbb{R}^n \to \mathbb{R}^m, \qquad l(x) = Ax$$

is a linear function, as shown in part 3 in Remark 76.

Example 267 *Let V be the vector space of polynomials in the variable t. The following functions are linear*

1. The derivative defines a function $\mathbb{D} : V \to V$ as

$$\mathbb{D} : p \mapsto p'$$

where p' is the derivative function of p.

2. The definite integral from 0 to 1 defines a function $i : V \to \mathbb{R}$ as

$$i : p \mapsto \int_0^1 p(t)\, dt.$$

Proposition 268 *If $l \in L(V, U)$ is invertible, then its inverse l^{-1} is linear.*

Proof. Take arbitrary $u, u' \in U$ and $\alpha, \beta \in F$. Then, since l is onto, there exist $v, v' \in V$ such that

$$l(v) = u \quad \text{and} \quad l(v') = u'$$

and by definition of inverse function

$$l^{-1}(u) = v \quad \text{and} \quad l^{-1}(u') = v' \ .$$

Then

$$\alpha u + \beta u' = \alpha l(v) + \beta l(v') = l(\alpha v + \beta v')$$

where last equality comes from the linearity of l. Then again by definition of inverse,

$$l^{-1}(\alpha u + \beta u') = \alpha v + \beta v' = \alpha l^{-1}(u) + \beta l^{-1}(u').$$

■

7.2 Kernel and Image of a linear function

Definition 269 *Assume that $l \in L(V, U)$. The kernel of l, denoted by $\ker l$ is the set*

$$\{v \in V : l(v) = 0\} = l^{-1}(0).$$

The Image of l, denoted by $\operatorname{Im} l$ is the set

$$\{u \in U : \exists v \in V \text{ such that } f(v) = u\} = l(V).$$

Proposition 270 *Given $l \in L(V, U)$, $\ker l$ is a vector subspace of V and $\operatorname{Im} l$ is a vector subspace of U.*

Proof. Take $v^1, v^2 \in \ker l$ and $\alpha, \beta \in F$. Then

$$l\left(\alpha v^1 + \beta v^2\right) = \alpha l\left(v^1\right) + \beta l\left(v^2\right) = 0$$

i.e., $\alpha v^1 + \beta v^2 \in \ker l$.

Take $w^1, w^2 \in \operatorname{Im} l$ and $\alpha, \beta \in F$. Then for $i \in \{1, 2\}$, $\exists v^i \in V$ such that $l\left(v^i\right) = w_i$. Moreover,

$$\alpha w^1 + \beta w^2 = \alpha l(v_i) + \beta l\left(v^2\right) = l\left(\alpha v^1 + \beta v^2\right)$$

i.e., $\alpha w^1 + \beta w^2 \in \operatorname{Im} l$. ■

Proposition 271 *If* $\operatorname{span}\left(\left\{v^1, ..., v^n\right\}\right) = V$ *and $l \in L(V, U)$, then* $\operatorname{span}\left(\left\{l\left(v^i\right)\right\}_{i=1}^n\right) = \operatorname{Im} l$.

Proof. Taken $u \in \operatorname{Im} l$, there exists $v \in V$ such that $l(v) = u$. Moreover, $\exists\ (\alpha_i)_{i=1}^n \in \mathbb{R}^n$ such that $v = \sum_{i=1}^n \alpha_i v^i$. Then

$$u = l(v) = l\left(\sum_{i=1}^n \alpha_i v^i\right) = \sum_{i=1}^n \alpha_i l\left(v^i\right),$$

as desired. ■

Remark 272 *From the previous proposition, we have that if $\{v^1, ..., v^n\}$ is a basis of V, then*

$$n \geq \dim \operatorname{span}\left(\left\{l\left(v^i\right)\right\}_{i=1}^n\right) = \dim \operatorname{Im} l.$$

Example 273 *Let V the vector space of polynomials and $\mathbb{D}^3 : V \to V$, $p \mapsto p'''$, i.e., the third derivative of p. Then*

$$\ker \mathbb{D}^3 = \text{ set of polynomials of degree } \leq 2,$$

since $\mathbb{D}^3\left(at^2 + bt + c\right) = 0$ and $\mathbb{D}^3(t^n) \neq 0$ for $n > 2$. Moreover,

$$\operatorname{Im} \mathbb{D}^3 = V,$$

since every polynomial is the third derivative of some polynomial.

Proposition 274 ***(Dimension Theorem)****If V is a finite dimensional vector space and $l \in L(V, U)$, then*

$$\dim V = \dim \ker l + \dim \operatorname{Im} l$$

Proof. (Idea of the proof.

1. Using a basis of $\ker l$ (with n_1 elements) and a basis of $\operatorname{Im} l$ (with n_2 elements) construct a set with $n_1 + n_2$ elements which generates V.

2. Show that set is linearly independent (by contradiction), and therefore a basis of V, and therefore $\dim V = n_1 + n_2$.)

Since $\ker l \subseteq V$ and from Remark 272, $\ker l$ and $\operatorname{Im} l$ have finite dimension. Therefore, we can define $n_1 = \dim \ker l$ and $n_2 = \dim \operatorname{Im} l$.

Take an arbitrary $v \in V$. Let

$$\left\{w^1, .., w^{n_1}\right\} \text{ be a basis of } \ker l \tag{7.1}$$

and

$$\left\{u^1, .., u^{n_2}\right\} \text{ be a basis of } \operatorname{Im} l \tag{7.2}$$

Then,

$$\forall i \in \{1, .., n_2\}, \ \exists v^i \in V \text{ such that } u^i = l\left(v^i\right) \tag{7.3}$$

From (7.2),

$$\exists c = (c_i)_{i=1}^{n_2} \text{ such that } l(v) = \sum_{i=1}^{n_2} c_i u^i \tag{7.4}$$

Then, from (7.4) and (7.3) ,we get

$$l(v) = \sum_{i=1}^{n_2} c_i u^i = \sum_{i=1}^{n_2} c_i l\left(v^i\right)$$

and from linearity of l

$$0 = l(v) - \sum_{i=1}^{n_2} c_i l\left(v^i\right) = l(v) - l\left(\sum_{i=1}^{n_2} c_i v^i\right) = l\left(v - \sum_{i=1}^{n_2} c_i v^i\right)$$

i.e.,

$$v - \sum_{i=1}^{n_2} c_i v^i \in \ker l \tag{7.5}$$

From (7.1),

$$\exists (d_j)_{j=1}^n \text{ such that } v - \sum_{i=1}^{n_2} c_i v^i = \sum_{j=1}^{n_1} d_j w^j$$

Summarizing, we have

$$\forall v \in V, \exists (c_i)_{i=1}^{n_2} \text{ and } (d_j)_{j=1}^{n_1} \text{ such that } v = \sum_{i=1}^{n_2} c_i v^i + \sum_{j=1}^{n_1} d_j w^j$$

Therefore, we found $n_1 + n_2$ vectors which generate V; if we show that they are l .i., then, by definition, they are a basis and therefore $n = n_1 + n_2$ as desired.

We want to show that

$$\sum_{i=1}^{n_2} \alpha_i v^i + \sum_{j=1}^{n_1} \beta_j w^j = 0 \Rightarrow \left((\alpha_i)_{i=1}^{n_2}, (\beta_j)_{j=1}^{n_1} \right) = 0 \tag{7.6}$$

Then

$$0 = l \left(\sum_{i=1}^{n_2} \alpha_i v^i + \sum_{j=1}^{n_1} \beta_j w^j \right) = \sum_{i=1}^{n_2} \alpha_i l\left(v^i\right) + \sum_{j=1}^{n_1} \beta_j l\left(w^j\right)$$

From (7.1), i.e., $\left\{w^1, .., w^{n_1}\right\}$ is a basis of $\ker l$, and from (7.3), we get

$$\sum_{i=1}^{n_2} \alpha_i u_i = 0$$

From (7.2), i.e., $\left\{u^1, .., u^{n_2}\right\}$ be a basis of $\operatorname{Im} l$,

$$(\alpha_i)_{i=1}^{n_2} = 0 \tag{7.7}$$

But from the assumption in (7.6) and (7.7) we have that

$$\sum_{j=1}^{n_1} \beta_j w^j = 0$$

and since $\left\{w^1, .., w^{n_1}\right\}$ is a basis of $\ker l$, we get also that

$$(\beta_j)_{j=1}^{n_1} = 0,$$

as desired. ■

Example 275 *Let V and U be vector spaces, with $\dim V = n$.*

In 1. and 2. below, we verify the statement of the Dimension Theorem: in 3. and 4., we use that statement.

1. Identity function id_V.

$$\dim \operatorname{Im} id_V = n$$
$$\dim \ker l = 0.$$

2. Null function $0 \in L(V, U)$

$$\dim \operatorname{Im} 0 = 0$$
$$\dim \ker 0 = n.$$

3. $l \in L\left(\mathbb{R}^2, \mathbb{R}\right)$,

$$l\left((x_1, x_2)\right) = x_1.$$

Since $\ker l = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\right\}$, $\{(0,1)\}$ *is a basis*[1] *of* $\ker l$ *and*

$$\dim \ker l = 1,$$
$$\dim \operatorname{Im} l = 2 - 1 = 1.$$

4. $l \in L\left(\mathbb{R}^3, \mathbb{R}^2\right)$,

$$l\left((x_1, x_2, x_3)\right) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Defined

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix},$$

since

$$\operatorname{Im} l = \left\{y \in \mathbb{R}^2 : \exists x \in \mathbb{R}^3 \textit{ such that } Ax = y\right\} = \operatorname{span} \operatorname{col} A = rank\ A,$$

and since the first two column vectors of A are linearly independent, we have that

$$\dim \operatorname{Im} l = 2$$
$$\dim \ker l = 3 - 2 = 1.$$

[1] In Remark 365 we will present an algorithm to compute a basis of $\ker l$.

7.3 Nonsingular functions and isomorphisms

Definition 276 *$l \in L(V,U)$ is singular if $\exists v \in V \backslash \{0\}$ such that $l(v) = 0$.*

Remark 277 *Thus $l \in L(V,U)$ is nonsingular if $\forall v \in V \backslash \{0\}$, $l(v) \neq 0$ i.e., $\ker l = \{0\}$. Briefly,*

$$l \in L(V,U) \text{ is nonsingular } \Leftrightarrow \quad \ker l = \{0\}.$$

Remark 278 *In Remark 312, we will discuss the relationship between singular matrices and singular linear functions.*

Proposition 279 *If $l \in L(V,U)$ is nonsingular, then the image of any linearly independent set is linearly independent.*

Proof. Suppose $\{v^1, ..., v^n\}$ are linearly independent. We want to show that $\{l(v^1), ..., l(v^n)\}$ are linearly independent as well. Suppose

$$\sum_{i=1}^{n} \alpha_i \cdot l(v^i) = 0.$$

Then

$$l\left(\sum_{i=1}^{n} \alpha_i \cdot v^i\right) = 0.$$

and therefore $\left(\sum_{i=1}^{n} \alpha_i \cdot v^i\right) \in \ker l = \{0\}$, where the last equality comes from the fact that l is nonsingular. Then $\sum_{i=1}^{n} \alpha_i \cdot v^i = 0$ and, since $\{v^1, ..., v^n\}$ are linearly independent, $(\alpha_i)_{i=1}^{n} = 0$, as desired. ■

Definition 280 *Let two vector spaces V and U be given. U is isomorphic to V if there exists a function $l \in L(V,U)$ which is one-to-one and onto. l is called an isomorphism from V to U.*

Remark 281 *By definition of isomorphism , if l is an isomorphism, the l is invertible and therefore, from Proposition 268, $l^{-1}$is linear.*

Remark 282 *"To be isomorphic" is an equivalence relation.*

Proposition 283 *Any n-dimensional vector space Von a field F is isomorphic to F^n.*

Proof. Since V and F^n are vector spaces, we are left with showing that there exists an isomorphism between them. Let $\mathbf{v} = \{v^1, ..., v^n\}$ be a basis of V. Define

$$cr : V \to F^n, \quad v \mapsto [v]_{\mathbf{v}},$$

where cr stands for "coordinates".

1. cr is linear. Given $v, w \in V$, suppose

$$v = \sum_{i=1}^{n} a_i v^i \quad \text{and} \quad w = \sum_{i=1}^{n} b_i v^i$$

i.e.,

$$[v]_{\mathbf{v}} = [a_i]_{i=1}^{n} \quad \text{and} \quad [w]_{\mathbf{v}} = [b_i]_{i=1}^{n} .$$

$\forall \alpha, \beta \in F$ and $\forall v^1, v^2 \in V$,

$$\alpha v + \beta w = \alpha \sum_{i=1}^{n} a_i v^i + \beta \sum_{i=1}^{n} b_i v^i = \sum_{i=1}^{n} (\alpha a_i + \beta b_i) v^i$$

i.e.,

$$[\alpha v + \beta w]_{\mathbf{v}} = \alpha [a_i]_{i=1}^{n} + \beta [b_i]_{i=1i}^{n} = \alpha \ [v]_{\mathbf{v}} + \beta [w]_{\mathbf{v}}.$$

2. cr is onto. $\forall (a)_{i=1}^{n} \in \mathbb{R}^n$, $cr\left(\sum_{i=1}^{n} a_i v^i\right) = (a)_{i=1}^{n}$.

3. cr is one-to-one. $cr(v) = cr(w)$ implies that $v = w$, simply because $v = \sum_{i=1}^{n} cr_i(v) u^i$ and $w = \sum_{i=1}^{n} cr_i(w) u^i$. ■

Proposition 284 *Let V and U be finite dimensional vectors spaces on the same field F such that $S = \{v^1, ..., v^n\}$ is a basis of V and $\{u^1, ..., u^n\}$ is a set of arbitrary vectors in U. Then there exists a unique linear function $l : V \to U$ such that $\forall i \in \{1, ..., n\}$, $l(v^i) = u^i$.*

Proof. The proof goes the following three steps.
1. Define l;
2. Show that l is linear;
3. Show that l is unique.
1. Using Definition 200, $\forall v \in V$, define

$$l : V \to U, \quad v \mapsto \sum_{i=1}^{n} [v]_s^i \cdot u^i.$$

Recall that $e_n^j \in \mathbb{R}^n$ is the $j - th$ element in the canonical basis of $\mathbb{R}^n$ and defined $e_n^j := \left(e_{n,i}^j\right)_{i=1}^n$, we have

$$e_{n,i}^j = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j. \end{cases}$$

Then $\forall j \in \{1, ..., n\}$, $[v^j]_S = e_n^j$ and

$$l(v^j) = \sum_{i=1}^{n} [v^j]_s^i \cdot u^i = \sum_{i=1}^{n} e_{n,i}^j \cdot u^i = u^j.$$

2. Let $v, w \in V$ and $\alpha, \beta \in F$. Then

$$l(\alpha v + \beta w) = \sum_{i=1}^{n} [\alpha v + \beta w]_S^i \cdot u^i = \sum_{i=1}^{n} \left(\alpha [v]_S^i + \beta [w]_S^i\right) \cdot u^i = \alpha \sum_{i=1}^{n} [v]_S^i \cdot u^i + \beta \sum_{i=1}^{n} [w]_S^i \cdot u^i,$$

where the before the last equality follows from the linearity of $[.]$ - see the proof of Proposition 283
3. Suppose $g \in L(V, U)$ and $\forall i \in \{1, ..., n\}$, $g(v^i) = u^i$. Then $\forall v \in V$,

$$g(v) = g\left(\sum_{i=1}^{n} [v]_S^i \cdot v^i\right) = \sum_{i=1}^{n} [v]_S^i \cdot g(v^i) = \sum_{i=1}^{n} [v]_S^i \cdot u^i = l(v)$$

where the last equality follows from the definition of l. ■

Remark 285 *Observe that if V and U are finite(nonzero) dimensional vector spaces, there is a multitude of functions from V to U. The above Proposition says that linear functions are completely determined by what "they do to the elements of a basis"of V.*

Proposition 286 *Assume that $l \in L(V, U)$. Then,*

$$l \text{ is one-to-one} \Leftrightarrow l \text{ is nonsingular}$$

Proof. $[\Rightarrow]$
Take $v \in \ker l$. Then

$$l(v) = 0 = l(0)$$

where last equality follows from Remark 264. Since l is one-to-one, $v = 0$.
$[\Leftarrow]$ If $l(v) = l(w)$, then $l(v - w) = 0$ and, since l is nonsingular, $v - w = 0$. ■

Proposition 287 *Assume that V and U are finite dimensional vector spaces and $l \in L(V, U)$. Then,*
1. *l is one-to-one* $\Rightarrow \dim V \leq \dim U$;
2. *l is onto* $\Rightarrow \dim V \geq \dim U$;
3. l is invertible $\Rightarrow \dim V = \dim U$.

Proof. The main ingredient in the proof is Proposition 274 , i.e., the Dimension Theorem.

1. Since l is one-to-one, from the previous Proposition, $\dim \ker l = 0$. Then, from Proposition 274 (the Dimension Theorem), $\dim V = \dim \operatorname{Im} l$. Since $\operatorname{Im} l$ is a subspace of U, then $\dim \operatorname{Im} l \leq \dim U$.

2. Since l is onto iff $\operatorname{Im} l = U$, from Proposition 274 (the Dimension Theorem), we get

$$\dim V = \dim \ker l + \dim U \geq \dim U.$$

3. l is invertible iff l is one-to-one and onto. ■

Proposition 288 *Let V and U be finite dimensional vector space on the same field F. Then,*

$$U \text{ and } V \text{ are isomorphic } \Leftrightarrow \dim U = \dim V.$$

Proof. $[\Rightarrow]$

It follows from the definition of isomorphism and part 3 in the previous Proposition.

$[\Leftarrow]$

Assume that V and U are vector spaces such that $\dim V = \dim U = n$. Then, from Proposition 283, V and U are isomorphic to F^n and from Remark 282, the result follows. ■

Proposition 289 *Suppose V and U are vector spaces such that $\dim V = \dim U = n$ and $l \in L(V, U)$. Then the following statements are equivalent.*

1. l is nonsingular, i.e., $\ker l = \{0\}$,
2. l is one-to-one,
3. l is onto,
4. l is an isomorphism.

Proof. $[1 \Leftrightarrow 2]$.

It is the content of Proposition 286.

$[1 \Rightarrow 3]$

Since l is nonsingular, then $\ker l = \{0\}$ and $\dim \ker l = 0$. Then, from Proposition 274 (the Dimension Theorem), i.e., $\dim V = \dim \ker l + \dim \operatorname{Im} l$, and the fact $\dim V = \dim U$, we get $\dim U = \dim \operatorname{Im} l$. Since $\operatorname{Im} l \subseteq U$ and U is finite dimensional, from Proposition 186, $\operatorname{Im} l = U$, i.e., l is onto, as desired.

$[3 \Rightarrow 1]$

Since l is onto, $\dim \operatorname{Im} l = \dim V$ and from Proposition 274 (the Dimension Theorem), $\dim V = \dim \ker t + \dim V$, and therefore $\dim \ker l = 0$, i.e., l is nonsingular.

$[1 \Rightarrow 4]$

It follows from the definition of isomorphism and the facts that $[1 \Leftrightarrow 2]$ and $[1 \Rightarrow 3]$.

$[4 \Rightarrow 1]$

It follows from the definition of isomorphism and the facts that $[2 \Leftrightarrow 1]$. ■

Chapter 8

Linear functions and matrices

In Remark 65 we have seen that the set of $m \times n$ matrices with the standard sum and scalar multiplication is a vector space, called $\mathbb{M}(m,n)$. We are going to show that:

1. the set $L(V,U)$ with naturally defined sum and scalar multiplication is a vector space, called $\mathcal{L}(V,U)$;
2. If $\dim V = n$ and $\dim U = m$, then $\mathcal{L}(V,U)$ and $\mathbb{M}(m,n)$ are isomorphic.

8.1 From a linear function to the associated matrix

Definition 290 *Suppose V and U are vector spaces over a field F and $l_1, l_2 \in L(V,U)$ and $\alpha \in F$.*

$$l_1 + l_2 : V \to U, \qquad v \mapsto l_1(v) + l_2(v)$$

$$\alpha l_1 : V \to U, \qquad v \mapsto \alpha l_1(v)$$

Proposition 291 *$L(V,U)$ with the above defined operations is a vector space on F, denoted by $\mathcal{L}(V,U)$.*

Proof. Exercise. ■

Proposition 292 *Compositions of linear functions are linear.*

Proof. Suppose V, U, W are vector spaces over a field F, $l_1 \in \mathcal{L}(V,U)$ and $l_2 \in \mathcal{L}(U,W)$. We want to show that $l := l_2 \circ l_1 \in \mathcal{L}(V,W)$. Indeed, for any$\alpha_1, \alpha_2 \in F$ and for any $v^1, v^2 \in V$, we have that

$$l\left(\alpha_1 v^1 + \alpha_2 v^2\right) := (l_2 \circ l_1)\left(\alpha_1 v^1 + \alpha_2 v^2\right) = l_2\left(l_1\left(\alpha_1 v^1 + \alpha_2 v^2\right)\right) =$$

$$= l_2\left(\alpha_1 l_1\left(v^1\right) + \alpha_2 l_1\left(v^2\right)\right) = \alpha_1 l_2\left(l_1\left(v^1\right)\right) + \alpha_2 l_2\left(l_1\left(v^2\right)\right) = \alpha_1 l\left(v^1\right) + \alpha_2 l\left(v^2\right),$$

as desired. ■

Definition 293 *Suppose $l \in \mathcal{L}(V,U)$, $\mathbf{v} = \{v^1, ..., v^n\}$ is a basis of V, $\mathbf{u} = \{u^1, ..., u^m\}$ is a basis of U. Then,*

$$[l]_{\mathbf{v}}^{\mathbf{u}} := \left[\ \left[l\left(v^1\right)\right]_{\mathbf{u}} \ \ ... \ \ \left[l\left(v^j\right)\right]_{\mathbf{u}} \ \ ... \ \ \left[l\left(v^n\right)\right]_{\mathbf{u}}\ \right] \in \mathbb{M}(m,n), \tag{8.1}$$

where for any $j \in \{1, ..., n\}$, $\left[l\left(v^j\right)\right]_{\mathbf{u}}$ is a column vector, is called the matrix representation of l relative to the basis $\mathbf{v}$ and $\mathbf{u}$. In words, $[l]_{\mathbf{v}}^{\mathbf{u}}$ is the matrix whose columns are the coordinates relative to the basis of the codomain of l of the images of each vector in the basis of the domain of l.

Remark 294 *Observe that since a basis is an order set there is a unique matrix representation of a linear function.*

Definition 295 *Suppose $l \in \mathcal{L}(V,U)$, $\mathbf{v} = \{v^1, ..., v^n\}$ is a basis of V, $\mathbf{u} = \{u^1, ..., u^m\}$ is a basis of U.*

$$\varphi_{\mathbf{v}}^{\mathbf{u}} : \mathcal{L}(V,U) \to \mathbb{M}(m,n), \quad l \mapsto [l]_{\mathbf{v}}^{\mathbf{u}} \ \text{defined in (8.1)}.$$

If no confusion may arise, we will denote $\varphi_{\mathbf{v}}^{\mathbf{u}}$ simply by φ.

The proposition below shows that multiplying the coordinate vector of v relative to the basis $\mathbf{v}$ by the matrix $[l]_{\mathbf{v}}^{\mathbf{u}}$, we get the coordinate vector of $l(v)$ relative to the basis $\mathbf{u}$.

Proposition 296 *$\forall v \in V$,*

$$[l]_{\mathbf{v}}^{\mathbf{u}} \cdot [v]_{\mathbf{v}} = [l(v)]_{\mathbf{u}} \tag{8.2}$$

Proof. Assume $v \in V$. First of all observe that

$$[l]_{\mathbf{v}}^{\mathbf{u}} \cdot [v]_{\mathbf{v}} = \left[\ [l(v^1)]_{\mathbf{u}} \ \ \dots \ \ [l(v^j)]_{\mathbf{u}} \ \ \dots \ \ [l(v^n)]_{\mathbf{u}}\ \right] \begin{bmatrix} [v]_{\mathbf{v}}^1 \\ \dots \\ [v]_{\mathbf{v}}^j \\ \dots \\ [v]_{\mathbf{v}}^n \end{bmatrix} = \sum_{j=1}^{n} [v]_{\mathbf{v}}^j \cdot [l(v^j)]_{\mathbf{u}}.$$

Moreover, from the linearity of the function $cr_{\mathbf{u}} := [.]_{\mathbf{u}}$, and using the fact that the composition of linear functions is a continuous function, we get:

$$[l(v)]_{\mathbf{u}} = cr_{\mathbf{u}}(l(v)) = (cr_{\mathbf{u}} \circ l)\left(\sum_{j=1}^{n} [v]_{\mathbf{v}}^j \cdot v^j\right)_{\mathbf{u}} = \sum_{j=1}^{n} [v]_{\mathbf{v}}^j \cdot (cr_{\mathbf{u}} \circ l)(v^j) = \sum_{j=1}^{n} [v]_{\mathbf{v}}^j \cdot [l(v^j)]_{\mathbf{u}}.$$

■

Example 297 *Let's verify equality* (8.2) *in the case in which*

a.

$$l : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

b. the basis $\mathbf{v}$ of the domain of l is

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

c. the basis $\mathbf{u}$ of the codomain of l is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\},$$

d.

$$v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

The main needed computations are presented below.

$$[l]_{\mathbf{v}}^{\mathbf{u}} := \left[\left[l \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_{\mathbf{u}}, \left[l \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_{\mathbf{u}} \right] = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathbf{u}}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathbf{u}} \right] = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix},$$

$$[v]_{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

$$[l(v)]_{\mathbf{u}} = \left[\begin{pmatrix} 7 \\ -1 \end{pmatrix} \right]_{\mathbf{u}} = \begin{bmatrix} -9 \\ 8 \end{bmatrix},$$

$$[l]_{\mathbf{v}}^{\mathbf{u}} \cdot [v]_{\mathbf{v}} = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \end{bmatrix}.$$

8.2 From a matrix to the associated linear function

Given $A \in \mathbb{M}(m,n)$, recall that $\forall i \in \{1,..,m\}$, $R^i(A)$ denotes the $i-th$ row vector of A, i.e.,

$$A = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix}_{m\times n}$$

Definition 298 *Consider vector spaces V and U with basis $\mathbf{v} = \{v^1, ..., v^n\}$ and $\mathbf{u} = \{u^1, ..., u^m\}$, respectively. Given $A \in \mathbb{M}(m,n)$, define*

$$l^{\mathbf{u}}_{A,\mathbf{v}} : V \to U, \quad v \mapsto \sum_{i=1}^{m} \left(R^i(A) \cdot [v]_{\mathbf{v}}\right) \cdot u^i.$$

Example 299 *Take $U = V = \mathbb{R}^2$, $\mathbf{v}= \mathcal{E}_2$, $\mathbf{u} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ and $A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$. Then,*

$$l^{\mathbf{u}}_{A,\mathbf{v}}(x_1,x_2) := \sum_{i=1}^{2} \left(R^i(A) \cdot [v]_{\mathcal{E}_2}\right) \cdot u^i = \left(\begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

$$= (x_1 - 3x_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 + 4x_2 \\ x_1 - 3x_2 + 2x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

Proposition 300 *$l^{\mathbf{u}}_{A,\mathbf{v}}$ defined above is linear, i.e., $l^{\mathbf{u}}_{A,\mathbf{v}} \in \mathcal{L}(V,U)$.*

Proof. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall v^1, v^2 \in V$,

$$l^{\mathbf{u}}_{A,\mathbf{v}}\left(\alpha v^1 + \beta v^2\right) = \sum_{i=1}^{m} R^i(A) \cdot \left[\alpha v^1 + \beta v^2\right]_{\mathbf{v}} \cdot u^i = \sum_{i=1}^{m} R^i(A) \cdot \left(\alpha \left[v^1\right]_{\mathbf{v}} + \beta \left[v^2\right]_{\mathbf{v}}\right) \cdot u^i =$$

$$= \alpha \sum_{i=1}^{m} R^i(A) \cdot \left[v^1\right]_{\mathbf{v}} \cdot u^i + \beta \sum_{i=1}^{m} R^i(A) \cdot \left[v^2\right]_{\mathbf{v}} \cdot u^i = \alpha l^{\mathbf{u}}_{A,\mathbf{v}}\left(v^1\right) + \beta l^{\mathbf{u}}_{A,\mathbf{v}}\left(v^2\right).$$

where the second equality follows from the proof of Proposition 283. ■

Definition 301 *Given the vector spaces V and U with basis $\mathbf{v} = \{v^1, ..., v^n\}$ and $\mathbf{u} = \{u^1, ..., u^m\}$, respectively, define*

$$\psi^{\mathbf{u}}_{\mathbf{v}} : \mathbb{M}(m,n) \to \mathcal{L}(V,U) \qquad : A \mapsto l^{\mathbf{u}}_{A,\mathbf{v}} .$$

If no confusion may arise, we will denote $\psi^{\mathbf{u}}_{\mathbf{v}}$ simply by ψ.

Proposition 302 *ψ defined above is linear.*

Proof. We want to show that $\forall \alpha, \beta \in \mathbb{R}$ and $\forall A, B \in \mathbb{M}(m,n)$,

$$\psi(\alpha A + \beta B) = \alpha \psi(A) + \beta \psi(B)$$

i.e.,

$$l^{\mathbf{v}}_{\alpha A + \beta B, \mathbf{u}} = \alpha l^{\mathbf{u}}_{A,\mathbf{v}} + \beta l^{\mathbf{v}}_{B,\mathbf{u}}$$

i.e., $\forall v \in V$,

$$l^{\mathbf{v}}_{\alpha A + \beta B, \mathbf{u}}(v) = \alpha l^{\mathbf{u}}_{A,\mathbf{v}}(v) + \beta l^{\mathbf{v}}_{B,\mathbf{u}}(v).$$

Now,

$$l^{\mathbf{v}}_{\alpha A + \beta B, \mathbf{u}}(v) = \sum_{i=1}^{m} \left(\alpha \cdot R^i(A) + \beta \cdot R^i(B)\right) \cdot [v]_{\mathbf{v}} \cdot u^i =$$

$$= \alpha \sum_{i=1}^{m} R^i(A) \cdot [v]_{\mathbf{v}} \cdot u^i + \beta \sum_{i=1}^{m} R^i(B) \cdot [v]_{\mathbf{v}} \cdot u^i = \alpha l^{\mathbf{u}}_{A,\mathbf{v}}(v) + \beta l^{\mathbf{v}}_{B,\mathbf{u}}(v),$$

where the first equality come from Definition 298. ■

8.3 $\mathbb{M}(m,n)$ and $\mathcal{L}(V,U)$ are isomorphic

Proposition 303 *Given the vector space V and U with dimension n and m, respectively,*

$$\mathbb{M}(m,n) \text{ and } \mathcal{L}(V,U) \text{ are isomorphic,}$$

and

$$\dim \mathcal{L}(V,U) = mn.$$

Proof. Linearity of the two spaces was proved above. We want now to check that ψ presented in Definition 301 is an isomorphism, i.e., ψ is linear, one-to-one and onto. In fact, thanks to Proposition 302, it is enough to show that ψ is invertible.

First proof.

1. ψ is one-to-one: see Theorem 2, page 105 in Lang (1971);
2. ψ is onto: see bottom of page 107 in Lang (1970).

Second proof.

1. $\psi \circ \varphi = id_{\mathcal{L}(V,U)}$.

Given $l \in \mathcal{L}(V,U)$, we want to show that $\forall v \in V$,

$$l(v) = ((\psi \circ \varphi)(l))(v)$$

i.e., from Proposition 283,

$$[l(v)]_{\mathbf{u}} = [((\psi \circ \varphi)(l))(v)]_{\mathbf{u}}.$$

First of all, observe that from (8.2), we have

$$[l(v)]_{\mathbf{u}} = [l]_{\mathbf{v}}^{\mathbf{u}} [v]_{\mathbf{v}}.$$

Moreover,

$$[((\psi \circ \varphi)(l))(v)]_{\mathbf{u}} \overset{(1)}{=} \left[\psi\left([l]_{\mathbf{v}}^{\mathbf{u}}\right)(v)\right]_{\mathbf{u}} \overset{(2)}{=} \left[\sum_{i=1}^{n} \left(i-th \text{ row of } [l]_{\mathbf{v}}^{\mathbf{u}}\right) \cdot [v]_{\mathbf{v}} \cdot u^i\right]_{\mathbf{u}} \overset{(3)}{=}$$

$$= \left[\left(i-th \text{ row of } [l]_{\mathbf{v}}^{\mathbf{u}}\right) \cdot [v]_{\mathbf{v}}\right]_{i=1}^{n} \overset{(4)}{=} [l]_{\mathbf{v}}^{\mathbf{u}} \cdot [v]_{\mathbf{v}}$$

where (1) comes from the definition of φ, (2) from the definition of ψ, (3) from the definition of $[.]_{\mathbf{u}}$, (4) from the definition of product between matrices.

2. $\varphi \circ \psi = id_{\mathbb{M}(m,n)}$.

Given $A \in \mathbb{M}(m,n)$, we want to show that $(\varphi \circ \psi)(A) = A$. By definition of ψ,

$$\psi(A) = l_{A,\mathbf{v}}^{\mathbf{u}} \text{ such that } \forall v \in V,\ l_{A,\mathbf{v}}^{\mathbf{u}}(v) = \sum_{i=1}^{m} R^i(A) \cdot [v]_{\mathbf{v}} \cdot u^i. \tag{8.3}$$

By definition of φ,

$$\varphi(\psi(A)) = \left[l_{A,\mathbf{v}}^{\mathbf{u}}\right]_{\mathbf{v}}^{\mathbf{u}}.$$

Therefore, we want to show that $\left[l_{A,\mathbf{v}}^{\mathbf{u}}\right]_{\mathbf{v}}^{\mathbf{u}} = A$. Observe that from 8.3,

$$\begin{array}{lllllll}
l_{A,\mathbf{v}}^{\mathbf{u}}(v^1) & = & \sum_{i=1}^{m} R^i(A)\cdot[v^1]_{\mathbf{v}}\cdot u^i & = & \sum_{i=1}^{m}[a_{i1},...,a_{ij},...,a_{in}]\cdot[1,...,0,...,0]\cdot u^i & = & a_{11}u^1+...+a_{i1}u^i+....+a_{m1}u^m \\
.... & & & & & & \\
l_{A,\mathbf{v}}^{\mathbf{u}}(v^j) & = & \sum_{i=1}^{m} R^i(A)\cdot[v^j]_{\mathbf{v}}\cdot u^i & = & \sum_{i=1}^{m}[a_{i1},...,a_{ij},...,a_{in}]\cdot[0,...,1,...,0]\cdot u^i & = & a_{1j}u^1+...+a_{ij}u^i+...+a_{mj}u^m \\
... & & & & & & \\
l_{A,\mathbf{v}}^{\mathbf{u}}(v^n) & = & \sum_{i=1}^{m} R^i(A)\cdot[v^n]_{\mathbf{v}}\cdot u^i & = & \sum_{i=1}^{m}[a_{i1},...,a_{ij},...,a_{in}]\cdot[0,...,0,...,1]\cdot u^i & = & a_{1n}u^1+...+a_{in}u^i+...+a_{mn}u^m
\end{array}$$

(From the above, it is clear why in definition 293 we take the transpose.) Therefore,

$$\left[l_{A,\mathbf{v}}^{\mathbf{u}}\right]_{\mathbf{v}}^{\mathbf{u}} = \begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... & & & & \\ a_{i1} & & a_{ij} & & a_{in} \\ ... & & & & \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{bmatrix} = A,$$

as desired.

The fact that $\dim \mathcal{L}(V,U)$ follows from Proposition 288. ■

Proposition 304 *Let the following objects be given.*

1. Vector spaces V *with basis* $\mathbf{v} = \{v^1, ..., v^j, ..., v^n\}$, U *with basis* $\mathbf{u} = \{u^1, ..., u^i, ..., u^m\}$ *and* W *with basis* $\mathbf{w} = \{w^1, ..., w^k, ..., w^p\}$*;*

2. $l_1 \in \mathcal{L}(V,U)$ *and* $l_2 \in \mathcal{L}(U,W)$*.*

Then

$$[l_2 \circ l_1]_{\mathbf{v}}^{\mathbf{w}} = [l_2]_{\mathbf{u}}^{\mathbf{w}} \cdot [l_1]_{\mathbf{v}}^{\mathbf{u}},$$

or

$$\varphi_{\mathbf{v}}^{\mathbf{w}}(l_2 \circ l_1) = \varphi_{\mathbf{u}}^{\mathbf{w}}(l_2) \cdot \varphi_{\mathbf{v}}^{\mathbf{u}}(l_1).$$

Proof. By definition[1]

$$[l_1]_{\mathbf{v}}^{\mathbf{u}} = \left[\ [l_1(v^1)]_{\mathbf{u}} \quad ... \quad [l_1(v^j)]_{\mathbf{u}} \quad ... \quad [l_1(v^n)]_{\mathbf{u}}\ \right]_{m\times n} :=$$

$$:= \begin{bmatrix} l_1^1(v^1) & ... & l_1^1(v^j) & ... & l_1^1(v^n) \\ ... \\ l_1^i(v^1) & & l_1^i(v^j) & & l_1^i(v^n) \\ ... \\ l_1^m(v^1) & ... & l_1^m(v^j) & ... & l_1^m(v^n) \end{bmatrix} := \begin{bmatrix} l_1^{11} & ... & l_1^{1j} & ... & l_1^{1n} \\ ... \\ l_1^{i1} & & l_1^{ij} & & l_1^{in} \\ ... \\ l_1^{m1} & ... & l_1^{mj} & ... & l_1^{mn} \end{bmatrix} :=$$

$$:= \left[l_1^{ij}\right]_{i\in\{1,...,m\},j\in\{1,...,n\}} := A \in \mathbb{M}(m,n),$$

and therefore $\forall j \in \{1,...,n\}$, $l_1(v^j) = \sum_{i=1}^m l_1^{ij} \cdot u^i$.

Similarly,

$$[l_2]_{\mathbf{u}}^{\mathbf{w}} = \left[\ [l_2(u^1)]_{\mathbf{w}} \quad ... \quad [l_2(u^i)]_{\mathbf{w}} \quad ... \quad [l_2(u^m)]_{\mathbf{w}}\ \right]_{p\times m} :=$$

$$:= \begin{bmatrix} l_2^1(u^1) & ... & l_2^1(u^i) & ... & l_2^1(u^m) \\ ... \\ l_2^k(u^1) & & l_2^k(u^i) & & l_2^k(u^m) \\ ... \\ l_2^p(u^1) & ... & l_2^p(u^i) & ... & l_2^p(u^m) \end{bmatrix} := \begin{bmatrix} l_2^{11} & ... & l_2^{1i} & ... & l_2^{1m} \\ ... \\ l_2^{k1} & & l_2^{ki} & & l_2^{km} \\ ... \\ l_2^{p1} & ... & l_2^{pi} & ... & l_2^{pm} \end{bmatrix} :=$$

$$:= \left[l_2^{ki}\right]_{k\in\{1,...,p\},i\in\{1,...,m\}} := B \in \mathbb{M}(p,m),$$

and therefore $\forall i \in \{1,...,m\}$, $l_2(u^i) = \sum_{k=1}^p l_2^{ki} \cdot w^k$.

Moreover, defined $l := (l_2 \circ l_1)$, we get

$$[l_2 \circ l_1]_{\mathbf{v}}^{\mathbf{w}} = \left[\ [l(v^1)]_{\mathbf{w}} \quad ... \quad [l(v^j)]_{\mathbf{w}} \quad ... \quad [l(v^n)]_{\mathbf{w}}\ \right]_{p\times n} :=$$

$$:= \begin{bmatrix} l^1(v^1) & ... & l^1(v^j) & ... & l^1(v^n) \\ ... \\ l^k(v^1) & & l^k(v^j) & & l^k(v^n) \\ ... \\ l^p(v^1) & ... & l^p(v^j) & ... & l^p(v^n) \end{bmatrix} := \begin{bmatrix} l^{11} & ... & l^{1j} & ... & l^{1n} \\ ... \\ l^{k1} & & l^{kj} & & l^{kn} \\ ... \\ l^{p1} & ... & l^{pj} & ... & l^{pn} \end{bmatrix} :=$$

$$:= \left[l^{kj}\right]_{k\in\{1,...,p\},j\in\{1,...,n\}} := C \in \mathbb{M}(p,n),$$

and therefore $\forall j \in \{1,...,n\}$, $l(v^j) = \sum_{k=1}^p l^{kj} \cdot w^k$.

Now, $\forall j \in \{1,...,n\}$

$$l(v^j) = (l_2 \circ l_1)(v^j) = l_2(l_1(v^j)) = l_2\left(\sum_{i=1}^m l_1^{ij} \cdot u^i\right) =$$
$$\sum_{i=1}^m l_1^{ij} \cdot l_2(u^i) = \sum_{i=1}^m l_1^{ij} \cdot \sum_{k=1}^p l_2^{ki} \cdot w^k = \sum_{k=1}^p \sum_{i=1}^m l_2^{ki} \cdot l_1^{ij} \cdot w^k.$$

[1] $[l_1(v^1)]_{\mathbf{u}}, .., [l_1(v^n)]_{\mathbf{u}}$ .are column vectors.

The above says that $\forall j \in \{1, ..., n\}$, the $j-th$ column of C is

$$\begin{bmatrix} \sum_{i=1}^{m} l_2^{1i} \cdot l_1^{ij} \\ \sum_{i=1}^{m} l_2^{ki} \cdot l_1^{ij} \\ \sum_{i=1}^{m} l_2^{pi} \cdot l_1^{ij} \end{bmatrix}$$

On the other hand, the $j-th$ column of $B \cdot A$ is

$$\begin{bmatrix} [1st \text{ row of } B] \cdot [j-th \text{ column of } A] \\ ... \\ [k-th \text{ row of } B] \cdot [j-th \text{ column of } A] \\ ... \\ [p-th \text{ row of } B] \cdot [j-th \text{ column of } A] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} l_2^{11} & ... & l_2^{1i} & ... & l_2^{1m} \end{bmatrix} \cdot \begin{bmatrix} l_1^{1j} \\ l_1^{ij} \\ l_1^{mj} \end{bmatrix} \\ ... \\ \begin{bmatrix} l_2^{k1} & l_2^{ki} & l_2^{km} \end{bmatrix} \cdot \begin{bmatrix} l_1^{1j} \\ l_1^{ij} \\ l_1^{mj} \end{bmatrix} \\ ... \\ \begin{bmatrix} l_2^{p1} & ... & l_2^{pi} & ... & l_2^{pm} \end{bmatrix} \cdot \begin{bmatrix} l_1^{1j} \\ l_1^{ij} \\ l_1^{mj} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} l_2^{1i} \cdot l_1^{ij} \\ \sum_{i=1}^{m} l_2^{ki} \cdot l_1^{ij} \\ \sum_{i=1}^{m} l_2^{pi} \cdot l_1^{ij} \end{bmatrix}$$

as desired. ■

8.4 Some related properties of a linear function and associated matrix

In this section, the following objects will be given: a vector space V with a basis $\mathbf{v} = \{v^1, ..., v^n\}$; a vector space U with a basis $\mathbf{u} = \{u^1, ..., u^n\}$; $l \in \mathcal{L}(V, U)$ and $A \in \mathbb{M}(m, n)$.

From Remark 147, recall that

$$\text{col span } A = \{z \in \mathbb{R}^m : \exists \;\; x \in \mathbb{R}^n \text{ such that } z = Ax\};$$

Lemma 305 $cr_{\mathbf{u}}(\text{Im}\, l) = \text{col span}\, [l]_{\mathbf{v}}^{\mathbf{u}}$.

Proof. $[\subseteq]$

$y \in cr_{\mathbf{u}}(\text{Im}\, l) \overset{\text{def } cr}{\Rightarrow} \exists u \in \text{Im}\, l$ such that $cr_{\mathbf{u}}(u) = [u]_{\mathbf{u}} = y \overset{\text{def Im}\, l}{\Rightarrow} \exists v \in V$ such that $l(v) = u \Rightarrow \exists v \in V$ such that $[l(v)]_{\mathbf{u}} = y \overset{\text{Prop. } 296}{\Rightarrow} \exists v \in V$ such that $[l]_{\mathbf{v}}^{\mathbf{u}} \cdot [v]_{\mathbf{v}} = y \Rightarrow y \in \text{col span}\, [l]_{\mathbf{v}}^{\mathbf{u}}$.

$[\supseteq]$

We want to show that $y \in \text{col span}\, [l]_{\mathbf{v}}^{\mathbf{u}} \Rightarrow y \in cr_{\mathbf{u}}(\text{Im}\, l)$, i.e., $\exists u \in \text{Im}\, l$ such that $y = [u]_{\mathbf{u}}$.

$y \in \text{col span}\, [l]_{\mathbf{v}}^{\mathbf{u}} \Rightarrow \exists x_y \in \mathbb{R}^n$ such that $[l]_{\mathbf{v}}^{\mathbf{u}} \cdot x_y = y \overset{\text{def } cr}{\Rightarrow} \exists v = \sum_{j=1}^{n} x_{y,j} v^j$ such that $[l]_{\mathbf{v}}^{\mathbf{u}} \cdot [v]_{\mathbf{v}} = y \overset{\text{Prop. } 296}{\Rightarrow} \exists v \in V$ such that $[l(v)]_{\mathbf{u}} = y \overset{u=l(v)}{\Rightarrow} \exists u \in \text{Im}\, l$ such that $y = [u]_{\mathbf{u}}$, as desired. ■

Lemma 306 $\dim \text{Im}\, l = \dim \text{col span}\, [l]_{\mathbf{v}}^{\mathbf{u}} = \text{rank } [l]_{\mathbf{v}}^{\mathbf{u}}$.

Proof. It follows from the above Lemma, the proof of Proposition 283, which says that $cr_{\mathbf{u}}$ is an isomorphism and Proposition 288, which says that isomorphic spaces have the same dimension. ■

Proposition 307 *Given* $l \in \mathcal{L}(V, U)$,

1. *l onto* $\Leftrightarrow \operatorname{rank}[l]_{\mathbf{v}}^{\mathbf{u}} = \dim U$*;*

2. *l one-to-one* $\Leftrightarrow \operatorname{rank}[l]_{\mathbf{v}}^{\mathbf{u}} = \dim V$*;*

3. *l invertible $\Leftrightarrow [l]_{\mathbf{v}}^{\mathbf{u}}$ invertible, and in that case $\left[l^{-1}\right]_{\mathbf{u}}^{\mathbf{v}} = \left[[l]_{\mathbf{v}}^{\mathbf{u}}\right]^{-1}$.*

Proof. Recall that from Remark 277 and Proposition 286, l one-to-one $\Leftrightarrow$ l nonsingular $\Leftrightarrow$ $\ker l = \{0\}$.

1. l onto $\Leftrightarrow \operatorname{Im} l = U \Leftrightarrow \dim \operatorname{Im} l = \dim U \overset{\text{Lemma 306}}{\Leftrightarrow} rank\ [l]_{\mathbf{v}}^{\mathbf{u}} = \dim U$.

2. l one-to-one $\overset{\text{Proposition 274}}{\Leftrightarrow} \dim \operatorname{Im} l = \dim V \overset{\text{Lemma 306}}{\Leftrightarrow} rank\ [l]_{\mathbf{v}}^{\mathbf{u}} = \dim V$.

3. The first part of the statement follows from 1. and 2. above. The second part is proven below. First of all observe that for any vector space W with basis $\mathbf{w}$, $id_W \in \mathcal{L}(W, W)$ and if W has a basis $\mathbf{w} = \left\{w^1, .., w^k\right\}$, we also have that

$$[id_W]_{\mathbf{w}}^{\mathbf{w}} = \left[\left[id_W\left(w^1\right)\right]_{\mathbf{w}}, .., \left[id_W\left(w^k\right)\right]_{\mathbf{w}}\right] = I_k.$$

Moreover, if l is invertible

$$l^{-1} \circ l = id_V$$

and

$$\left[l^{-1} \circ l\right]_{\mathbf{v}}^{\mathbf{v}} = [id_U]_{\mathbf{v}}^{\mathbf{v}} = I_m.$$

Since

$$\left[l^{-1} \circ l\right]_{\mathbf{v}}^{\mathbf{v}} = \left[l^{-1}\right]_{\mathbf{u}}^{\mathbf{v}} \cdot [l]_{\mathbf{v}}^{\mathbf{u}},$$

the desired result follows.

■

Remark 308 *From the definitions of φ and ψ, we have what follows:*

1.

$$l = \psi(\varphi(l)) = \psi\left([l]_{\mathbf{v}}^{\mathbf{u}}\right) = l_{[l]_{\mathbf{v}}^{\mathbf{u}},\mathbf{v}}^{\mathbf{u}}.$$

2.

$$A = \varphi(\psi(A)) = \varphi\left(l_{A,\mathbf{v}}^{\mathbf{u}}\right) = \left[l_{A,\mathbf{v}}^{\mathbf{u}}\right]_{\mathbf{v}}^{\mathbf{u}}.$$

Lemma 309 $cr_{\mathbf{u}}\left(\operatorname{Im} l_{A,\mathbf{v}}^{\mathbf{u}}\right) = \operatorname{col\,span} A$.

Proof. Recall that $\varphi(l) = [l]_{\mathbf{v}}^{\mathbf{u}}$ and $\psi(A) = l_{A,\mathbf{v}}^{\mathbf{u}}$. For any $l \in \mathcal{L}(V, U)$,

$$cr_{\mathbf{u}}(\operatorname{Im} l) \overset{Lemma\ 305}{=} \operatorname{col\,span}[l]_{\mathbf{v}}^{\mathbf{u}} \overset{Def.\ 295}{=} \operatorname{col\,span}\ \varphi(l) \tag{8.4}$$

Take $l = l_{A,\mathbf{v}}^{\mathbf{u}}$. Then from (8.4) ,we have

$$cr_{\mathbf{u}}\left(\operatorname{Im} l_{A,\mathbf{v}}^{\mathbf{u}}\right) = \operatorname{col\,span}\ \varphi\left(l_{A,\mathbf{v}}^{\mathbf{u}}\right) \overset{Rmk.\ 308.2}{=} \operatorname{col\,span}\ A.$$

■

Lemma 310 $\dim \operatorname{Im} l_{A,\mathbf{v}}^{\mathbf{u}} = \dim \operatorname{col\,span} A = \operatorname{rank} A$.

Proof. Since Lemma 306 holds for any $l \in \mathcal{L}(V, U)$ and $l_{A,\mathbf{v}}^{\mathbf{u}} \in \mathcal{L}(V, U)$, we have that

$$\dim \operatorname{Im} l_{A,\mathbf{v}}^{\mathbf{u}} = \dim \operatorname{col\,span}\left[l_{A,\mathbf{v}}^{\mathbf{u}}\right]_{\mathbf{v}}^{\mathbf{u}} \overset{Rmk.\ 308.2}{=} \dim \operatorname{col\,span}\ A \overset{Rmk.\ 242.}{=} \operatorname{rank}\ A.$$

■

Proposition 311 *Let $A \in \mathbb{M}(m, n)$ be given.*

1. $\operatorname{rank} A = m \Leftrightarrow l_{A,\mathbf{v}}^{\mathbf{u}}$ *onto;*

2. $\text{rank} A = n \Leftrightarrow l^{\mathbf{u}}_{A,\mathbf{v}}$ *one-to-one;*

3. A *invertible* $\Leftrightarrow l^{\mathbf{u}}_{A,\mathbf{v}}$ *invertible, and in that case* $l^{\mathbf{v}}_{A^{-1},\mathbf{u}} = \left(l^{\mathbf{u}}_{A,\mathbf{v}}\right)^{-1}$.

Proof. 1. $\text{rank} A = m \overset{\text{Lemma 310}}{\Leftrightarrow} \dim \operatorname{Im} l^{\mathbf{u}}_{A,\mathbf{v}} = m \Leftrightarrow l^{\mathbf{u}}_{A,\mathbf{v}}$ onto;
2. $\text{rank} A = n \overset{(1)}{\Leftrightarrow} \dim \ker l^{\mathbf{u}}_{A,\mathbf{v}} = 0 \overset{\text{Proposition 286}}{\Leftrightarrow} l^{\mathbf{u}}_{A,\mathbf{v}}$ one-to-one,
where (1) the first equivalence follows form the fact that $n = \dim \ker l^{\mathbf{u}}_{A,\mathbf{v}} + \dim \operatorname{Im} l^{\mathbf{u}}_{A,\mathbf{v}}$, and Lemma 310.
3. First statement: A invertible $\overset{\text{Prop. 237}}{\Leftrightarrow} \text{rank} A = m = n \overset{\text{1 and 2 above}}{\Leftrightarrow} l^{\mathbf{u}}_{A,\mathbf{v}}$ invertible.
Second statement: Since $l^{\mathbf{u}}_{A,\mathbf{v}}$ invertible, there exists $\left(l^{\mathbf{u}}_{A,\mathbf{v}}\right)^{-1} : U \to V$ such that

$$id_V = \left(l^{\mathbf{u}}_{A,\mathbf{v}}\right)^{-1} \circ l^{\mathbf{u}}_{A,\mathbf{v}}.$$

Then

$$I = \varphi^{\mathbf{v}}_{\mathbf{v}}(id_V) \overset{\text{Prop. 304}}{=} \varphi^{\mathbf{v}}_{\mathbf{u}}\left(\left(l^{\mathbf{u}}_{A,\mathbf{v}}\right)^{-1}\right) \cdot \varphi^{\mathbf{u}}_{\mathbf{v}}\left(l^{\mathbf{u}}_{A,\mathbf{v}}\right) \overset{\text{Rmk. 308}}{=} \varphi^{\mathbf{v}}_{\mathbf{u}}\left(\left(l^{\mathbf{u}}_{A,\mathbf{v}}\right)^{-1}\right) \cdot A.$$

Then, by definition of inverse matrix,

$$A^{-1} = \varphi^{\mathbf{v}}_{\mathbf{u}}\left(\left(l^{\mathbf{u}}_{A,\mathbf{v}}\right)^{-1}\right)$$

and

$$\psi^{\mathbf{v}}_{\mathbf{u}}\left(A^{-1}\right) = \left(\psi^{\mathbf{v}}_{\mathbf{u}} \circ \varphi^{\mathbf{v}}_{\mathbf{u}}\right)\left(\left(l^{\mathbf{u}}_{A,\mathbf{v}}\right)^{-1}\right) = id_{\mathcal{L}(U,U)}\left(\left(l^{\mathbf{u}}_{A,\mathbf{v}}\right)^{-1}\right) = \left(l^{\mathbf{u}}_{A,\mathbf{v}}\right)^{-1}.$$

Finally, from the definition of $\psi^{\mathbf{v}}_{\mathbf{u}}$, we have

$$\psi^{\mathbf{v}}_{\mathbf{u}}\left(A^{-1}\right) = l^{\mathbf{v}}_{A^{-1},\mathbf{u}},$$

as desired. ■

Remark 312 *Consider* $A \in \mathbb{M}(n,n)$*. Then from Proposition 311,*

$$A \text{ invertible} \Leftrightarrow l^{\mathbf{u}}_{A,\mathbf{v}} \text{ invertible;}$$

from Proposition 237,

$$A \text{ invertible} \Leftrightarrow A \text{ nonsingular;}$$

from Proposition 289,

$$l^{\mathbf{u}}_{A,\mathbf{v}} \text{ invertible} \Leftrightarrow l^{\mathbf{u}}_{A,\mathbf{v}} \text{ nonsingular.}$$

Therefore,

$$A \text{ nonsingular} \Leftrightarrow l^{\mathbf{u}}_{A,\mathbf{v}} \text{ nonsingular.}$$

Symmetrically,

$$[l]^{\mathbf{u}}_{\mathbf{v}} \text{ invertible} \Leftrightarrow [l]^{\mathbf{u}}_{\mathbf{v}} \text{ nonsingular} \Leftrightarrow l \text{ invertible} \Leftrightarrow l \text{ nonsingular.}$$

8.5 Some facts on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

In this Section, we specialize (basically repeat) the content of the previous Section in the important case in which

$$\begin{array}{ll} V = \mathbb{R}^n, & \mathbf{v} = \left(e^j_n\right)_{j=1}^n := \mathbf{e}_n \\ U = \mathbb{R}^m & \mathbf{u} = \left(e^i_m\right)_{i=1}^m := \mathbf{e}_m \\ & v = x \end{array} \tag{8.5}$$

and therefore

$$l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m).$$

From $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ to $\mathbb{M}(m,n)$.

From Definition 293, we have

$$[l]_{\mathbf{e}_n}^{\mathbf{e}_m} = \left[\begin{array}{ccccc} \left[l\left(e_n^1\right)\right]_{\mathbf{e}_m} & \dots & \left[l\left(e_n^j\right)\right]_{\mathbf{e}_m} & \dots & \left[l\left(e_n^n\right)\right]_{\mathbf{e}_m} \end{array}\right] = \\ = \left[\begin{array}{ccccc} l\left(e_n^1\right) & \dots & l\left(e_n^j\right) & \dots & l\left(e_n^n\right) \end{array}\right] := [l]; \tag{8.6}$$

from Definition 295,

$$\varphi := \varphi_{\mathbf{e}_n}^{\mathbf{e}_m} : \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right) \to \mathbb{M}\left(m, n\right), \qquad l \mapsto [l];$$

from Proposition 296,

$$[l] \cdot x = l\left(x\right). \tag{8.7}$$

From $\mathbb{M}(m,n)$ to $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.
From Definition 298,

$$l_A := l_{A,\mathbf{e}_m}^{\mathbf{e}_m} : \mathbb{R}^n \to \mathbb{R}^m, \tag{8.8}$$

$$l_A\left(x\right) = \sum_{i=1}^m \left(R^i\left(A\right) \cdot [x]_{\mathbf{e}_n}\right) \cdot e_m^i = \\ = \left[\begin{array}{c} R^1\left(A\right) \cdot x \\ \dots \\ R^i\left(A\right) \cdot x \\ \dots \\ R^m\left(A\right) \cdot x \end{array}\right] = Ax. \tag{8.9}$$

From Definition 301,

$$\psi := \psi_{\mathbf{e}_n}^{\mathbf{e}_m} : \mathbb{M}\left(m, n\right) \to \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right) \qquad : A \mapsto l_A. \;.$$

From Proposition 303,

$$\mathbb{M}\left(m, n\right) \text{ and } \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right) \text{ are isomorphic.}$$

From Proposition 304, if $l_1 \in \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right)$ and $l_2 \in \mathcal{L}\left(\mathbb{R}^m, \mathbb{R}^p\right)$, then

$$[l_2 \circ l_1] = [l_2] \cdot [l_1]. \tag{8.10}$$

Some related properties.
From Proposition 307, given $l \in \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right)$,
1. l onto $\Leftrightarrow$ rank $[l] = m$;
2. l one-to-one $\Leftrightarrow$ rank $[l] = n$;
3. l invertible $\Leftrightarrow$ $[l]$ invertible, and in that case $\left[l^{-1}\right] = [l]^{-1}$.
From Remark 308,
1.

$$l = \psi\left(\varphi\left(l\right)\right) = \psi\left([l]\right) = l_{[l]}.$$

2.

$$A = \varphi\left(\psi\left(A\right)\right) = \varphi\left(l_A\right) = [l_A].$$

From Proposition 311, given $A \in \mathbb{M}\left(m, n\right)$,
1. rank$A = m \Leftrightarrow l_A$ onto;
2. rank$A = n \Leftrightarrow l_A$ one-to-one;
3. A invertible $\Leftrightarrow l_A$ invertible, and in that case $l_{A^{-1}} = \left(l_A\right)^{-1}$.

Remark 313 *From* (8.7) *and Remark 147,*

$$\operatorname{Im} l := \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ such that } y = [l] \cdot x\} = \operatorname{col span}\ [l]. \tag{8.11}$$

Then, from the above and Remark 242,

$$\dim \operatorname{Im} l = \operatorname{rank}\ [l] = \max \#\ \text{linearly independent columns of } [l]. \tag{8.12}$$

Similarly, from (8.9) *and Remark 242, we get*

$$\operatorname{Im} l_A := \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ such that } y = l_A \cdot x = Ax\} = \operatorname{col\,span} A,$$

and

$$\dim \operatorname{Im} l_A = \operatorname{rank} A = \max \# \text{ linearly independent columns of } A.$$

Remark 314 *Assume that* $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. *The above Remark gives a way of finding a basis of* $\operatorname{Im} l$*: it is enough to consider a number equal to* rank $[l]$ *of linearly independent vectors among the column vectors of* $[l]$. *In a more detailed way, we have what follows.*

1. Compute $[l]$.

2. Compute $\dim \operatorname{Im} l = \operatorname{rank} [l] := r$.

3. To find a basis of $\operatorname{Im} l$*, we have to find r vectors which are a. linearly independent, and b. elements of* $\operatorname{Im} l$. *Indeed, it is enough to r linearly independent columns of* $[l]$.

To get a "simpler" basis, you can make elementary operations on those column. Recall that elementary operations on linearly independent vectors lead to linearly independent vectors, and that elementary operations on vectors lead to vector belonging to the span of the set of starting vectors.

Example 315 *Given* $n \in \mathbb{N}$ *with* $n \geq 3$, *and the linear function*

$$l : \mathbb{R}^n \to \mathbb{R}^3, x := (x_i)_{i=1}^n \mapsto \begin{cases} \sum_{i=1}^n x_i \\ x_2 \\ x_2 + x_3 \end{cases}.$$

find a basis for$\operatorname{Im} l$.

1.

$$[l] = \begin{bmatrix} 1 & 1 & 1 & 1 & ... & 1 \\ 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 1 & 0 & ... & 0 \end{bmatrix}$$

2. Since

$$\operatorname{rank} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} 0 = \operatorname{rank} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 3$$

3. A basis of $\operatorname{Im} l_a$ is given by the column vectors of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Remark 316 *From* (8.7)*, we have that*

$$\ker l = \{x \in \mathbb{R}^n : [l]\, x = 0\},$$

i.e., ker l *is the set, in fact the vector space, of solution to the systems* $[l]\, x = 0$. *In Remark 365, we will describe an algorithm to find a basis of the kernel of an arbitrary linear function.*

Remark 317 *From Remark 313 and Proposition 274 (the Dimension Theorem), given* $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$,

$$\dim \mathbb{R}^n = \dim \ker l + rank \;\; [l],$$

and given $A \in \mathbb{M}(m, n)$,

$$\dim \mathbb{R}^n = \dim \ker l_A + rank \;\; A.$$

8.6 Examples of computation of $[l]_{\mathbf{v}}^{\mathbf{u}}$

1. $id \in \mathcal{L}(V,V)$.

$$[id]_{\mathbf{v}}^{\mathbf{v}} = \left[\left[id\left(v^1\right)\right]_{\mathbf{v}}, ..., \left[id\left(v^j\right)\right]_{\mathbf{v}}, ..., \left[id\left(v^n\right)\right]_{\mathbf{v}}\right] =$$

$$= \left[\left[v^1\right]_{\mathbf{v}}, ..., \left[v^j\right]_{\mathbf{v}}, ..., \left[v^n\right]_{\mathbf{v}}\right] = \left[e_n^1, ..., e_n^j, ..., e_n^n\right] = I_n.$$

2. $0 \in \mathcal{L}(V,U)$.

$$[0]_{\mathbf{v}}^{\mathbf{u}} = \left[[0]_{\mathbf{v}}, ..., [0]_{\mathbf{v}}, ..., [0]_{\mathbf{v}}\right] = 0 \in \mathbb{M}(m,n).$$

3. $l_\alpha \in \mathcal{L}(V,V)$, with $\alpha \in F$.

$$[l_\alpha]_{\mathbf{v}}^{\mathbf{v}} = \left[\left[\alpha \cdot v^1\right]_{\mathbf{v}}, ..., \left[\alpha \cdot v^j\right]_{\mathbf{v}}, ..., \left[\alpha \cdot v^n\right]_{\mathbf{v}}\right] = \left[\alpha \cdot \left[v^1\right]_{\mathbf{v}}, ..., \alpha \cdot \left[v^j\right]_{\mathbf{v}}, ..., \alpha \cdot \left[v^n\right]_{\mathbf{v}}\right] =$$

$$= \left[\alpha \cdot e_n^1, ..., \alpha \cdot e_n^j, ..., \alpha \cdot e_n^n\right] = \alpha \cdot I_n.$$

4. $l_A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, with $A \in \mathbb{M}(m,n)$.

$$[l_A] = \left[A \cdot e_n^1, ..., A \cdot e_n^j, ..., A \cdot e_n^n\right] = A \cdot \left[e_n^1, ..., e_n^j, ..., e_n^n\right] = A \cdot I_n = A.$$

5. (projection function) $proj_{n+k,n} \in \mathcal{L}\left(\mathbb{R}^{n+k}, \mathbb{R}^n\right)$, $proj_{n+k,n} : (x_i)_{i=1}^{n+k} \mapsto (x_i)_{i=1}^{n}$. Defined $proj_{n+k,n} := p$, we have

$$[p] = \left[p\left(e_{n+k}^1\right), ..., p\left(e_{n+k}^n\right), p\left(e_{n+k}^{n+1}\right), ..., p\left(e_{n+k}^{n+k}\right)\right] = \left[e_n^1, ..., e_n^n, 0, ..., 0\right] = [I_n|0],$$

where $0 \in \mathbb{M}(n,k)$.

6. (immersion function) $i_{n,n+k} \in \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^{n+k}\right)$, $i_{n,n+k} : (x_i)_{i=1}^n \mapsto \left((x_i)_{i=1}^n, 0\right)$ with $0 \in \mathbb{R}^k$.Defined $i_{n,n+k} := i$, we have

$$[i] = \left[i\left(e_n^1\right), ..., i\left(e_n^n\right)\right] = \begin{bmatrix} e_n^1 & ... & e_n^n \\ 0 & ... & 0 \end{bmatrix} = \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

where $0 \in \mathbb{M}(k,n)$.

Remark 318 *Point 4. above implies that if $l : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$, then $[l] = A$. In other words, to compute $[l]$ you do not have to take the image of each element in the canonical basis; the first line of $[l]$ is the vector of the coefficient of the first component function of l and so on. For example, if $l : \mathbb{R}^2 \to \mathbb{R}^3$,*

$$x \mapsto \begin{cases} a_{11}x_1 & + & a_{12}x_2 \\ a_{21}x_1 & + & a_{22}x_2 \\ a_{31}x_1 & + & a_{32}x_2 \end{cases},$$

then

$$[l] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

8.7 Change of basis and linear operators

In this section, we answer the following question: how does the matrix representation of a linear function $l \in \mathcal{L}(V,V)$ for given basis $\mathbf{v} = \left\{v^1, .., v^n\right\}$ change when we change the basis?

Definition 319 *Let V be a vector space over a field F. $l \in \mathcal{L}(V,V)$ is called a linear operator on V.*

Proposition 320 *Let P be the change-of-basis matrix from a basis $\mathbf{v}$ to a basis $\mathbf{u}$ for a vector space V. Then, for any $l \in \mathcal{L}(V,V)$,*

$$[l]_{\mathbf{u}}^{\mathbf{u}} = P^{-1}[l]_{\mathbf{v}}^{\mathbf{v}} P,$$

In words, if A is the matrix representing l in the basis v, then $B = P^{-1}AP$ is the matrix which represents l in a new basis $\mathbf{u}$, where P is the change-of-basis matrix from $\mathbf{v}$ to $\mathbf{u}$.

Proof. For any $v \in V$, from Proposition 206, we have that

$$P\,[v]_{\mathbf{u}} = [v]_{\mathbf{v}}\,.$$

Then, premultiplying both sides by $P^{-1}\,[l]_{\mathbf{v}}^{\mathbf{v}}$, we get

$$P^{-1}\,[l]_{\mathbf{v}}^{\mathbf{v}}\,P\,[v]_{\mathbf{u}} = P^{-1}\,[l]_{\mathbf{v}}^{\mathbf{v}}\,[v]_{\mathbf{v}} \overset{\text{Prop. 296}}{=} P^{-1}\,[l\,(v)]_{\mathbf{v}} \overset{\text{Prop. 206}}{=} [l\,(v)]_{\mathbf{u}}\,. \tag{8.13}$$

From Proposition 296,

$$[l]_{\mathbf{u}}^{\mathbf{u}} \cdot [v]_{\mathbf{u}} = [l\,(v)]_{\mathbf{u}}\,. \tag{8.14}$$

Therefore, form (8.13) and (8.14), we get

$$P^{-1}\,[l]_{\mathbf{v}}^{\mathbf{v}}\,P\,[v]_{\mathbf{u}} = [l]_{\mathbf{u}}^{\mathbf{u}} \cdot [v]_{\mathbf{u}}\,. \tag{8.15}$$

Since, from the proof of Proposition 283, $[...]_{\mathbf{u}} := cr_{\mathbf{u}} : V \to \mathbb{R}^n$ is onto, $\forall x \in \mathbb{R}^n$, $\exists v \in V$ such that $[v]_{\mathbf{u}} = x$. Observe that $\forall k \in \{1,..,n\}$, $\left[u^k\right]_{\mathbf{u}} = e_n^k$. Therefore, rewriting (8.15) with respect to k, for any $k \in \{1,..,n\}$, we get

$$P^{-1}\,[l]_{\mathbf{v}}^{\mathbf{v}}\,P I_n = [l]_{\mathbf{u}}^{\mathbf{u}} \cdot I_n,$$

as desired.. ■

Remark 321 *A and B represent l means that there exists basis* $\mathbf{v}$ *and* $\mathbf{u}$ *such that*

$$A = [l]_{\mathbf{v}}^{\mathbf{v}} \;\; \text{and} \quad B = [l]_{\mathbf{u}}^{\mathbf{u}}\,.$$

Moreover, from Definition 203 and Proposition 205, there exists P invertible which is a change-of-basis matrix. The above Proposition 320 says that A and B are similar. Therefore, all the matrix representations of $l \in \mathcal{L}(V,V)$ *form an equivalence class of similar matrices.*

Remark 322 *Now suppose that*

$$f : \mathbb{M}(n,n) \to \mathbb{R}$$

is such that

$$\text{if } A \text{ is similar to } B, \text{ then } f(A) = f(B)\,.$$

Then, given a vector space V of dimension n, we can define

$$F : \mathcal{L}(V,V) \to \mathbb{R}$$

such that for any basis $\mathbf{v}$ *of V,*

$$F(l) = f\left([l]_{\mathbf{v}}^{\mathbf{v}}\right).$$

Indeed, by definition of f and by Remark 321, the definition is well given: for any pair of basis $\mathbf{u}$ *and* $\mathbf{v}$, $f([l]_{\mathbf{u}}) = f([l]_{\mathbf{v}})$.

Remark 323 *For any* $A, P \in \mathbb{M}(n,n)$ *with P invertible,*

$$\text{tr } PAP^{-1} = \text{tr } A,$$

as verified below. $\text{tr } PAP^{-1} = \text{tr } (PA)\,P^{-1} = \text{tr } P^{-1}PA = \text{tr } A$*,where the second equality comes from Property 3 in Proposition 82.*

Remark 322, Proposition 247 and Remark 323 allow to give the following definitions.

Definition 324 *Let* $l \in \mathcal{L}(V,V)$ *and a basis* $\mathbf{v}$ *of V. The definitions of determinant, trace and characteristic polynomial of l are as follows:*

$$\det : \mathcal{L}(V,V) \to \mathbb{R}, \qquad : l \mapsto \det\,[l]_{\mathbf{v}}^{\mathbf{v}}\,,$$

$$\text{tr} : \mathcal{L}(V,V) \to \mathbb{R}, \qquad : l \mapsto \text{tr}\,[l]_{\mathbf{v}}^{\mathbf{v}}\,,$$

$$\Delta_l : \mathcal{L}(V,V) \to \textit{space of polynomials}, \qquad : l \mapsto \Delta_{[l]_{\mathbf{v}}^{\mathbf{v}}}\,.$$

8.8 Diagonalization of linear operators

In this Section, we assume that V is a vector space of dimension n on a field F.

Definition 325 *Given a vector space V of dimension n, $l \in \mathcal{L}(V,V)$ is diagonalizable if there exists a basis $\mathbf{v}$ of V such that $[l]_{\mathbf{v}}^{\mathbf{v}}$ is a diagonal matrix.*

We present below some definitions and results which are analogous to those discussed in Section 6.3. In what follows, V is a vector space of dimension n.

Definition 326 *Let a vector space V on a field F and $l \in \mathcal{L}(V,V)$ be given. A scalar $\lambda \in F$ is called an eigenvalue of l if there exists a nonzero vector $v \in vV$ such that*

$$l(v) = \lambda v.$$

Every vector v satisfying the above relationship is called an eigenvector of l associated with (or belonging to) the eigenvalue λ.

Definition 327 *Let E_λ be the set of eigenvector belonging to λ.*

Proposition 328 *1. There is only one eigenvalue associated with an eigenvector.*
2. The set $E_\lambda \cup \{0\}$ is a subspace of V.

Proof. Similar to the matrix case. ■

Proposition 329 *Let a vector space V on a field F and $l \in \mathcal{L}(V,V)$ be given. Then, the following statements are equivalent.*

1. *$\lambda \in F$ is an eigenvalue of l.*
2. *$\lambda id_V - l$ is singular.*
3. *$\lambda \in F$ is a solution to the characteristic equation $\Delta_l(t)$.*

Proof. Similar to the matrix case. ■

Definition 330 *Let λ be an eigenvalue of l. The algebraic multiplicity of λ is the multiplicity of λ as a solution to the characteristic equation $\Delta_l(t) = 0$. The geometric multiplicity of λ is the dimension of $E_\lambda \cup \{0\}$.*

Proposition 331 *$l \in \mathcal{L}(V,V)$ is diagonalizable $\Leftrightarrow$ there exists a basis of V made up by n (linearly independent) eigenvectors.*
In that case, the elements on the diagonal of D are the associated eigenvalues.

Proof. $[\Rightarrow]$
$[l]_{\mathbf{v}}^{\mathbf{v}}$ =diag $[(\lambda_i)_{i=1}^n] := D \Rightarrow \forall i \in \{1,...,n\}$, $[l(v^i)]_{\mathbf{v}} = [l]_{\mathbf{v}}^{\mathbf{v}} \cdot [v^i]_{\mathbf{v}} = D \cdot e_n^i = \lambda_i e_n^i$. Taking $(cr_{\mathbf{v}})^{-1}$ of both sides, we get

$$\forall i \in \{1,...,n\}, \qquad l(v^i) = \lambda_i v^i.$$

$[\Leftarrow]$
$\forall i \in \{1,...,n\}, l(v^i) = \lambda_i v^i \Rightarrow \forall i \in \{1,...,n\}, [l(v^i)]_{\mathbf{v}} = [l]_{\mathbf{v}}^{\mathbf{v}} \cdot [v^i]_{\mathbf{v}} = D \cdot e_n^i = \lambda_i e_n^i \Rightarrow$

$$[l]_{\mathbf{v}}^{\mathbf{v}} := \left[[l(v^1)]_{\mathbf{v}} |...| [l(v^n)]_{\mathbf{v}}\right] = \left[\lambda_i e_n^1 |...| \lambda_i e_n^n\right] = \text{diag } [(\lambda_i)_{i=1}^n].$$

■

Lemma 332 $\left\{[v^1]_{\mathbf{v}},...,[v^n]_{\mathbf{v}}\right\}$ *are linearly independent* $\Leftrightarrow \left\{v^1,...,v^n\right\}$ *are linearly independent.*

Proof.

$$\sum_{i=1}^n \alpha_i [v^i]_{\mathbf{v}} = 0 \Leftrightarrow (cr_{\mathbf{v}})^{-1}\left(\sum_{i=1}^n \alpha_i [v^i]_{\mathbf{v}}\right) = (cr_{\mathbf{v}})^{-1}(0) \overset{cr^{-1} \text{ is linear}}{\Leftrightarrow}$$

$$\Leftrightarrow \sum_{i=1}^n \alpha_i (cr_{\mathbf{v}})^{-1}\left([v^i]_{\mathbf{v}}\right) = 0 \Leftrightarrow \sum_{i=1}^n \alpha_i v^i = 0.$$

■

Proposition 333 *Let $v^1, ..., v^r$ be eigenvectors of $l \in \mathcal{L}(V, V)$ belonging to pairwise distinct eigenvalues $\lambda_1, ..., \lambda_n$, respectively. Then $v^1, ..., v^n$ are linearly independent.*

Proof. By assumption, $\forall i \in \{1, ..., n\}, l(v^i) = \lambda_i v^i$ and $[l]_{\mathbf{v}}^{\mathbf{v}} \cdot [v^i]_{\mathbf{v}} = \lambda_i \cdot [v^i]_{\mathbf{v}}$. Then, from the analogous theorem for matrices, i.e., Proposition 260,

$$\left\{ [v^1]_{\mathbf{v}}, ..., [v^n]_{\mathbf{v}} \right\} \text{ are linearly independent.}$$

Then, the result follows from Lemma 332. ■

Proof. $l \in \mathcal{L}(V, V)$ is diagonalizable $\Leftrightarrow$ there exists a basis of V made up by n (linearly independent) eigenvectors $\Leftrightarrow$ l has n pairwise distinct eigenvalues. ■

Proposition 334 *λ is an eigenvalue of $l \in \mathcal{L}(V, V) \Leftrightarrow \lambda$ is a root of the characteristic polynomial of l.*

Proof. Similar to the matrix case, i.e., Proposition 254. ■

Proposition 335 *Let λ be an eigenvalue of $l \in \mathcal{L}(V, V)$. The geometric multiplicity of λ is smaller or equal than the algebraic multiplicity of λ.*

Proof. Assume that the geometric multiplicity of λ is r. Then, by assumption, $\dim E_\lambda \cup \{0\} = r$ and therefore there are r linearly independent eigenvectors, $v^1, ..., v^r$, and therefore,

$$\forall i \in \{1, ..., r\}, \quad l(v^i) = \lambda v^i$$

From Lemma 185, there exist $\{w^1, .., w^s\}$ such that $\mathbf{u} = \{v^1, ..., v^r, w^1, .., w^s\}$ is a basis of V. Then, by definition of matrix associated with l with respect to that basis, we have

$$[l]_{\mathbf{u}}^{\mathbf{u}} = \left[[l(v^1)]_{\mathbf{u}} |...| [l(v^r)]_{\mathbf{u}} | [l(w^1)]_{\mathbf{u}} |...| [l(w^s)_{\mathbf{u}}] \right] =$$

$$= \begin{bmatrix} \lambda & 0 & ... & 0 & | & a_{11} & a_{12} & ... & a_{1s} \\ 0 & \lambda & ... & 0 & | & a_{21} & a_{22} & ... & a_{2s} \\ ... & ... & ... & ... & | & ... & ... & ... & ... \\ 0 & 0 & ... & \lambda & | & a_{r1} & a_{r2} & ... & a_{rs} \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & ... & 0 & | & b_{11} & b_{12} & ... & b_{1s} \\ 0 & 0 & ... & 0 & | & b_{21} & b_{22} & ... & b_{2s} \\ ... & ... & ... & ... & | & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & | & b_{s1} & b_{s2} & ... & b_{ss} \end{bmatrix} := \begin{bmatrix} \lambda I_r & A \\ 0 & B \end{bmatrix},$$

for well chosen values of the entries of A and B. Then the characteristic polynomial of $[l]_{\mathbf{u}}^{\mathbf{u}}$ is

$$\det \left(tI - \begin{bmatrix} \lambda I_r & A \\ 0 & B \end{bmatrix} \right) = \det \left(\begin{bmatrix} (t - \lambda) I_r & A \\ 0 & tI_s - B \end{bmatrix} \right) = (t - \lambda)^r \cdot \det (tI_s - B).$$

Therefore, the algebraic multiplicity of λ is greater or equal than r. ■

8.9 Appendices.

8.9.1 The dual and double dual space of a vector space

The dual space of a vector space

Definition 336 *The[2] dual space of a vector space V is*

$$V^* := \mathcal{L}(V, \mathbb{R}).$$

Remark 337 *From Proposition 303, we have that if $\dim V$ is finite, then $\dim V^* = \dim V$.*

[2] Here I follow pages 98 and 99 in Hoffman and Kunze (1971).

Definition 338 *The Kronecker delta is a function defined as follows*

$$\delta : \mathbb{N}^2 \to \{0,1\}, \qquad (i,j) \mapsto \delta_{ij} := \begin{cases} 1 & if \quad i=j \\ \\ 0 & \quad otherwise. \end{cases}$$

Proposition 339 *Let V be a finite dimensional vector space over a field F and let $\mathcal{V} = \left\{v^i\right\}_{i=1}^n$ be a basis for V. Then,*

1. there exists a unique basis $\mathcal{V}^* = \{l_i\}_{i=1}^n$ for V^* such that $\forall i,j \in \{1,...,n\}$, $l_i\left(v^j\right) = \delta_{ij}$, i.e., $\left(l_i\left(v^j\right)\right)_{i=1}^n = e_n^j$;

2.

$$\forall f \in V^*, \quad f = \sum_{i=1}^n f\left(v^i\right) \cdot l_i, \text{ i.e., } [f]_{\mathcal{V}^*} = \left(f\left(v^i\right)\right)_{i=1}^n,$$

i.e., the coordinates of f with respect to $\mathcal{V}^*$ are the images via f of the elements of the basis $\mathcal{V}$, and

3.

$$\forall v \in V, \; v = \sum_{i=1}^n l_i(v) \cdot v^i, \text{ i.e., } [v]_{\mathcal{V}} = (l_i(v))_{i=1}^n,$$

i.e., the coordinates of v with respect to $\mathcal{V}$ are the images of v via the elements of the basis $\mathcal{V}$.

Proof. 1. From Proposition 284, we have that given a basis $\mathcal{V} = \left\{v^i\right\}_{i=1}^n$ for V,

$$\forall i \in \{1,...,n\}, \; \exists\,!\; l_i \in V^* \text{ such that } \forall j \in \{1,...,n\}, \;\; l_i\left(v^j\right) = \delta_{ij}.$$

We are left with showing that $\{l_i\}_{i=1}^n$ is a linearly independent and the therefore a basis of the n dimensional vector space V^*. We want to show that if $\sum_{i=1}^n \alpha_i l_i = 0$, then $(\alpha_i)_{i=1}^n = 0$, i.e., if $\forall v \in V$, $\sum_{i=1}^n \alpha_i l_i(v) = 0$, then $(\alpha_i)_{i=1}^n = 0$. Then, $\forall j \in \{1,...,n\}$, $0 = \sum_{i=1}^n \alpha_i l_i\left(v^j\right) = \alpha_j$, as desired.

2. Take $f \in V^*$; then , since $\mathcal{V}^* = \{l_i\}_{i=1}^n$ is a basis for V^*, $\exists\; (\beta_i)_{i=1}^n \in \mathbb{R}^n$ such that $f = \sum_{i=1}^n \beta_i l_i$. Then,

$$\forall v^j \in \mathcal{V}, \quad f\left(v^j\right) = \sum_{i=1}^n \beta_i l_i\left(v^j\right) = \beta_j,$$

as desired.

3. Take $v \in V$; then , since $\mathcal{V} = \left\{v^i\right\}_{i=1}^n$ is a basis for V, $\exists\; (\gamma_i)_{i=1}^n \in \mathbb{R}^n$ such that $v = \sum_{i=1}^n \gamma_i v^i$. Then,

$$\forall l^j \in \mathcal{V}^*, \quad l_j(v) = \sum_{i=1}^n \gamma_i l_j\left(v^i\right) = \gamma_j,$$

as desired. ■

Definition 340 *Let a basis $\mathcal{V} = \left\{v^i\right\}_{i=1}^n$ for V be given. The unique basis $\mathcal{V}^* = \{l_i\}_{i=1}^n$ for V^* such that $\forall i,j \in \{1,...,n\}$, $l_i\left(v^j\right) = \delta_{ij}$ is called the dual basis of $\mathcal{V}$.*

Corollary 341 *If $\mathcal{V}^*$ is the unique dual basis of $\mathcal{V} = \left\{v^i\right\}_{i=1}^n$, defined $l : V \to \mathbb{R}^n$, $v \mapsto (l_i(v))_{i=1}^n$, we have that*

$$[l]_{\mathcal{V}}^{\mathcal{E}_n} = I \in \mathbb{M}(n,n).$$

Proof. $[l]_{\mathcal{V}}^{\mathcal{E}_n} := \left[l\left(v^1\right), ..., l\left(v^n\right)\right] = I.$ ■

The double dual space of a vector space

Question. Take[3] a basis $\mathcal{V}^*$ of V^*. Is $\mathcal{V}^*$ the dual basis of some basis $\mathcal{V}$ of V? Corollary 347 below will answer positively to that question.

Definition 342 *The double dual V^{**} of V is the dual space of V^*, i.e.,*

$$V^{**} := (V^*)^* := \mathcal{L}(V^*, \mathbb{R}).$$

[3] Here I follow pages 107 and 108 in Hoffman and Kunze (1971).

Remark 343 *If V has finite dimension, then*

$$\dim V = \dim V^* = \dim V^{**}.$$

Proposition 344 *For any $v \in V$, the function*

$$L_v : V^* \to \mathbb{R}, \qquad f \mapsto f(v)$$

is linear, i.e., $L_v \in V^{**}$.

Proof. For any $\alpha, \beta \in F$, $f_1, f_2 \in V^*$,

$$L_v(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(v) = \alpha f_1(v) + \beta f_2(v) = \alpha L_{f_1} + \beta L_{f_2}.$$

■

Proposition 345 *Let V be a finite dimensional vector space. Then*

$$\varphi : V \to V^{**}, \qquad v \mapsto L_v$$

is an isomorphism.

Proof. Step 1. φ is linear.
For any $\alpha, \beta \in F$, $v_1, v_2 \in V$,by definition of φ,

$$\varphi(\alpha v_1 + \beta v_2) = L_{\alpha v_1 + \beta v_2},$$

and

$$\alpha\varphi(v_1) + \beta\varphi(v_2) = \alpha L_{v_1} + \beta L_{v_2}.$$

Moreover, $\forall f \in V^*$,

$$L_{\alpha v_1 + \beta v_2}(f) := f(\alpha v_1 + \beta v_2) \overset{f \in V^*}{=} \alpha f(v_1) + \beta f(v_2) \overset{\text{def. } L_v}{=} \alpha L_{v_1} + \beta L_{v_2}.$$

Step 2. φ is nonsingular.
We want to show that $\ker\varphi = \{0\}$, i.e., $\langle v \neq 0 \Rightarrow \varphi(v) := L_v \neq 0\rangle$, i.e., $\forall v \neq 0$, $\exists f \in V^*$ such that $f(v) \neq 0$.Take a basis $\mathcal{V} = \left\{v, v^i\right\}_{i=2}^n$of V. The existence of such a basis is insured by Proposition 185 . Then, from Proposition 339, there exists a unique dual basis $\mathcal{V}^* = (l_i)_{i=1}^n$ associated with $\mathcal{V}$, and $l_1(v) = 1 \neq 0$.
Step 3. Desired result.
From Remark 337, $n = \dim V = dimV^* = \dim V^{**}$. Then, from Proposition ??? 281 ?? my EUI notes, since φ is nonsingular, it is invertible and therefore an isomorphism. ■

Corollary 346 *Let V be a finite dimensional vector space over a field F. If $L \in V^{**}$, then $\exists\,!$ $v \in V$ such that $\forall f \in V^*$, $L(f) = f(v)$.*

Proof. Such v is just $\varphi(L)$. ■

Corollary 347 *Let V be a finite dimensional vector space over a field F. Any basis of V^* is the unique dual basis of some basis of V.*

Proof. Take a basis $\{l_i\}_{i=1}^n$ of V^*. From Proposition 339, there exists a unique basis $\{L_i\}_{i=1}^n$ of V^{**}such that

$$\forall i, j \in \{1, ..., n\}, L_i(f_j) = \delta_{ij}. \tag{8.16}$$

Then, from the previous Corollary,

$$\forall i \in \{1, ..., n\}, \exists\,! v^i \in V \text{ such that } \forall f \in V^*, \quad L_i(f) = f(v_i). \tag{8.17}$$

Now, from the definition of L_{v^i}, we have

$$\forall f \in V^*, \quad L_{v^i}(f) = f(v_i). \tag{8.18}$$

Then, from (8.17) and (8.18) ,we get

$$L_i = L_{v^i}. \tag{8.19}$$

Then,

$$\varphi^{-1}(L_i) \overset{(8.19)}{=} \varphi^{-1}(L_{v^i}) = v^i,$$

where the last equality follows from Proposition 345. Since $\{L_i\}_{i=1}^n$ is a basis of V^{**} and φ is an isomorphism, then from Proposition 279, $\{v^i\}_{i=1}^n$ is a basis of V.Moreover, from (8.17), $f_i\left(v^j = L_i(f_j)\right) \overset{(8.16)}{=} \delta_{ij}$, and then, by definition , $\{l_i\}_{i=1}^n$ is the unique dual basis associated with $\{v_i\}_{i=1}^n$. ■

A sometimes useful result is presented below.

Proposition 348 *If V is a vector space of dimension bigger than n, for any $i \in \{1, ..., n\}$, $l_i \in \mathcal{L}(V, \mathbb{R})$,[4] and $\{l_i\}_{i=1}^n$ is a linearly independent set, then $l : V \to \mathbb{R}^n, v \mapsto (l_i(v))_{i=1}^n$ is onto.*

Proof. Take an n dimensional vector subspace V' of V. From Corollary 347, there exists a basis $\mathcal{V} = \{v^i\}_{i=1}^n$ of V' such that $\{l_i\}_{i=1}^n$ is its unique associated dual basis. Then, by definition of dual basis, for any $i \in \{1, ..., n\}$, $l(v^i) = e_n^i$. Therefore, for any $x \in \mathbb{R}^n$,

$$l\left(\sum_{i=1}^n x_i v^i\right) = \sum_{i=1}^n x_i e_n^i = x,$$

as desired. ■

8.9.2 Vector spaces as Images or Kernels of well chosen linear functions

This section is preliminary and incomplete.

Proposition 349 *If L is a vector subspace of $\mathbb{R}^S$ of dimension A, then*

1. *a. $\exists l_1 \in L\left(\mathbb{R}^S, \mathbb{R}^{S-A}\right)$ such that $\ker l_1 = L$;*

   b. $\exists \sigma \in \Sigma$ and $\exists E \in M(S-A, A)$ such that

   $$[l_1] = [I_{S-A}|E] \cdot P_\sigma \in \mathbb{M}(S-A, S);$$

   c. $E = \psi_\sigma(L)$ where ψ_σ is a chart of an atlas of $G_{S,A}$.

2. *a. $\exists l_2 \in L\left(\mathbb{R}^A, \mathbb{R}^S\right)$ such that $Iml_2 = L$;*

   b. [5]

   $$[l_2] = P_{\sigma^{-1}} \begin{bmatrix} -E \\ I_{A\times A} \end{bmatrix}$$

   and, therefore, if $\sigma = id$,

   $$[l_2] = \begin{bmatrix} -E \\ I_A \end{bmatrix}_{S\times A}$$

3. $$L = [I_{S-A}|E] \cdot P_\sigma = ImP_{\sigma^{-1}} \begin{bmatrix} -E \\ I_{A\times A} \end{bmatrix}$$

Moreover,

4. *Let $M, M' \in M^f(S-A, S)$ be given. Then*

   $$\ker M = \ker M' \Leftrightarrow \text{ there exists } B \in \mathbb{M}^f(S-A, S-A) \text{ such that } M' = BM.$$

[4] $\mathcal{L}(V, \mathbb{R})$ is the vector space of linear functions from V to $\mathbb{R}$.

[5] Recall that $\forall \sigma \in \Sigma$,

$$P_{\sigma^{-1}} = P_\sigma^{-1} = P_\sigma^T.$$

5. Let $Y, Y' \in M^f(S,A)$ be given. Then

$$\mathrm{Im}Y = \mathrm{Im}Y' \Leftrightarrow \text{ there exists a unique } C \in \mathbb{M}^f(A,A) \text{ such that } Y' = YC.$$

Proof. 1.
Take a basis $\{l^1, ...l^A\} \subseteq \mathbb{R}^S$ of L. Define

$$C = \begin{bmatrix} l^1 \\ ... \\ l^A \end{bmatrix}_{S\times A},$$

where the vectors are taken to be row vectors. By definition of basis, rank $C = A$. From the Dimension Theorem, $\dim \mathbb{R}^S = \dim Iml_C + \dim \ker l_C$ and $S = A + \dim \ker l_C$, or $\dim \ker l_C = S - A$. Let $\{k^1, ..., k^{S-A}\} \subseteq \mathbb{R}^S$ be a basis of $\ker l_C$. Then, $\forall s \in \{A, ..., S-A\}$, $Ck^s = 0$, and

$$\forall a \in 1, ..., A, \forall s \in \{1, ..., S-A\}, \ l^a \cdot k^s = 0,$$

i.e.,[6]

$$L = (\ker l_C)^{\perp}. \tag{8.20}$$

Define

$$M = \begin{bmatrix} k^1 \\ ... \\ k^{S-A} \end{bmatrix}_{(S-A)\times S}.$$

By definition of basis, rank $M = S - A$. Moreover, from the fact that $\{k^1, ..., k^{S-A}\} \subseteq \mathbb{R}^S$ be a basis of $\ker l_C$ and from (8.20), we have that

$$L = \{x \in \mathbb{R}^S : \forall s \in \{1, ..., S-A\}, \ k^s \cdot x = 0\} = \{x \in \mathbb{R}^S : M \cdot x = 0\} = \ker l_M.$$

2.
a. Let $\{v^a\}_{a=1}^A$ be a basis of $L \subseteq \mathbb{R}^S$. Take $l_2 \in \mathcal{L}(\mathbb{R}^A, \mathbb{R}^S)$ such that

$$\forall a \in \{1, ..., A\}, l_2(e_A^a) = v^a,$$

where e_A^a is the a–th element in the canonical basis in $\mathbb{R}^A$. From a standard Proposition in linear algebra[7], such function does exists and, in fact, it is unique. Then, from the (linear algebra) dimension theorem

$$\dim \mathrm{Im}l_2 = A - \dim \ker l_2 \le A.$$

Moreover, $L = \mathrm{span}\ \{v^a\}_{a=1}^A \subseteq \mathrm{Im}l_2$ and $\dim L = A$. Therefore $\dim \mathrm{Im}l_2 = A$,and $L = \mathrm{Im}l_2$, as desired.
b. From Step 2a above, and by definition of $[l_2]$, we have that

$$[l_2] = [v^1, ..., v^a, ..., v^A] \in \mathbb{M}(S,A),$$

where $\{v^a\}_{a=1}^A$ is a basis of L and the vectors v^a are written as column vectors. Therefore, we are left with finding a basis of $L = \ker [I_{S-A}|E] \cdot P_\sigma$, which we claim is given by the A column vectors of

$$P_{\sigma^{-1}} \begin{bmatrix} -E \\ I_{A\times A} \end{bmatrix},$$

a fact which is proved below:
i. The vectors belong to $\ker [I_{S-A}|E] \cdot P_\sigma$:

$$[I|E] \cdot P_\sigma \cdot P_{\sigma^{-1}} \begin{bmatrix} -E \\ I \end{bmatrix} = [I|E] \cdot \begin{bmatrix} -E \\ I \end{bmatrix} = 0.$$

[6] We are using the obvious fact that:
Two vector spaces V and W are orthogonal iff each each element in a basis of V is orthogonal to each element in a basis of W.

[7] Let V and U be finite dimensional vectors spaces such that $S = \{v^1, ..., v^n\}$ is a basis of V and $\{u^1, ..., u^n\}$ is a set of arbitrary vectors in U. Then there exists a unique linear function $l : V \to U$ such that $\forall i \in \{1, ..., n\}$, $l(v^i) = u^i$ - see my math class notes, Proposition 273, page 82.

ii. the vectors are linearly independent:

$$\operatorname{rank} P_{\sigma^{-1}} \begin{bmatrix} -E \\ I \end{bmatrix} \overset{(1)}{=} \operatorname{rank} \begin{bmatrix} -E \\ I \end{bmatrix} = A,$$

where (1) follows from Corollary 7, page 10 in the 4-author book.
3.
It is an immediate consequence of 2. and 3. above.
4.
See Villanacci and others (2002), Proposition 39, page 388, or observe what follows
[⇒]
From, say, Theorem on page 41, in Ostaszewski (1990)[8], we have that

$$(\ker Y)^{\perp} = \operatorname{Im} Y^T,$$

(this result is proved, along other things, in my handwritten file some-facts-on-Stiefel-manifolds-2011-10-12.pdf
and therefore, using the assumption

$$\ker Y^T = (\operatorname{Im} Y)^{\perp} = (\operatorname{Im} Y')^{\perp} = \ker (Y')^T.$$

Then, from part 3 in the present Proposition,

$$\text{there exists } \widehat{B} \in \mathbb{M}^f (A, A) \text{ such that } Y'^T = \widehat{B} Y^T.$$

and

$$Y^{'} = Y \widehat{B}^T,$$

and it is then enough to take $C = \widehat{B}^T$.
[⇐]
$\operatorname{Im} Y \subseteq \operatorname{Im} Y'$: $w \in \operatorname{Im} Y \Rightarrow \exists z \in \mathbb{R}^A$ such that $w = Yz = Y'Cz \Rightarrow \exists z' = Cz \in \mathbb{R}^A$ such that $w = Y'z' \Leftrightarrow w \in \operatorname{Im} Y'$.
$\operatorname{Im} Y' \subseteq \operatorname{Im} Y$: $w' \in \operatorname{Im} Y' \Rightarrow \exists z' \in \mathbb{R}^A$ such that $w' = Y'z' = Y'CC^{-1}z' = YC^{-1}z' \Rightarrow \exists z = C^{-1}z' \in \mathbb{R}^A$ such that $w' = Yz \Leftrightarrow w' \in \operatorname{Im} Y$.
Uniqueness. Since $Y \in \mathbb{M}^f (S, A)$, there exists $\sigma \in \Sigma(S)$ such that $P_\sigma Y = \begin{bmatrix} Y^* \\ \widehat{Y} \end{bmatrix}$, with $Y^* \in \mathbb{M}^f (A, A)$. Define also $P_\sigma Y' = \begin{bmatrix} Y'^* \\ \widehat{Y}' \end{bmatrix}$. Then, premultiplying $Y' = YC$ by P_σ, we get $Y'^* = Y^* C$ and $C = Y^{*-1} Y'^*$. ■

[8] Ostaszewski, A., (1990), *Advanced mathematical methods,* London School of Economics Mathematics Series, Cambridge University Press, Cambridge, UK

Chapter 9

Solutions to systems of linear equations

9.1 Some preliminary basic facts

Let's recall some basic definition from Section 1.6.

Definition 350 *Consider the following linear system with m equations and n unknowns*

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

which can be rewritten as

$$Ax = b$$

$A_{m\times n}$ is called matrix of the coefficients (or coefficient matrix) associated with the system and $M_{m\times(n+1)} = \begin{bmatrix} A & \mid & b \end{bmatrix}$ is called augmented matrix associated with the system.

Recall the following definition.

Definition 351 *Two linear system are said to be equivalent if they have the same solutions.*

Let's recall some basic facts we discussed in previous chapters.

Remark 352 *It is well known that the following operations applied to a system of linear equations lead to an equivalent system:*
I) interchange two equations;
II) multiply both sides of an equation by a nonzero real number;
III) add left and right hand side of an equation to the left and right hand side of another equation;
IV) change the place of the unknowns.
The transformations I), II), III) and IV) are said elementary transformations*, and, as it is well known, they do not change the solution set of the system they are applied to.*
Those transformations correspond to elementary operations on rows of M or columns of A in the way described below
I) interchange two rows of M;
II) multiply a row of M by a nonzero real number;
III) sum a row of M to another row of M ;
IV) interchange two columns of A.
The above described operations do not change the rank of A and they do not change the rank of M - see Proposition 229.

Homogenous linear system.

Definition 353 *A linear system for which* $b = 0$*, i.e., of the type*

$$Ax = 0$$

with $A \in \mathbb{M}(m, n)$*, is called homogenous system.*

Remark 354 *Obviously,* 0 *is a solution of the homogenous system. The set of solution of a homogeneous system is* $\ker l_A$*. From Remark 317,*

$$\dim \ker l_A = n - \text{rank } A.$$

9.2 A solution method: Rouchè-Capelli's and Cramer's theorems

The solution method presented in this section is based on two basic theorems.

1. Rouchè-Capelli's Theorem, which gives necessary and sufficient condition for the existence of solutions;

2. Cramer's Theorem, which gives a method to compute solutions - if they exist.

Theorem 355 $(Rouch\grave{e} - Capelli)$ *A system with* m *equations and* n *unknowns*

$$A_{m \times n} x = b \tag{9.1}$$

has solutions

$\Leftrightarrow$

$$\text{rank } A = \text{rank } \begin{bmatrix} A & | & b \end{bmatrix}$$

Proof. $[\Rightarrow]$

Let x^* be a solution to 9.1. Then, b is a linear combination, via the solution x^* of the columns of A. Then, from Proposition 229,

$$\text{rank } \begin{bmatrix} A & | & b \end{bmatrix} = \text{rank } \begin{bmatrix} A & | & 0 \end{bmatrix} = \text{rank } \begin{bmatrix} A \end{bmatrix}.$$

$[\Leftarrow]$

1st proof.

We want to show that

$$\exists x \in \mathbb{R}^n \text{ such that } Ax^* = b, \text{ i.e., } b = \sum_{j=1}^{n} x_j^* \cdot C^j(A).$$

By assumption, rank A = rank $\begin{bmatrix} A & | & b \end{bmatrix} := r$. Since rank $A = r$, there are r linearly independent column vectors of A, say $\left\{C^j(A)\right\}_{j \in R}$, where $R \subseteq \{1, ..., n\}$ and $\#R = r$.

Since rank $\begin{bmatrix} A & | & b \end{bmatrix} = r$, $\left\{C^j(A)\right\}_{j \in R} \cup \{b\}$ is a linearly dependent set and from Lemma 190, b is a linear combinations of the vectors in $\left\{C^j(A)\right\}_{j \in R}$, i.e., $\exists \left(x_j\right)_{j \in R}$ such that $b = \sum_{j \in R} x_j \cdot C^j(A)$ and

$$b = \sum_{j \in R} x_j \cdot C^j(A) + \sum_{j' \in \{1,...,n\} \setminus R} 0 \cdot C_{j'}(A).$$

Then, $x^* = \left(x_j^*\right)_{j=1}^n$ such that

$$x_j^* = \begin{cases} x_j & if \quad j \in R \\ \\ 0 & if \quad j' \in \{1, ..., n\} \setminus R \end{cases}$$

is a solution to $Ax = b$.

Second proof.

Since rank $A := r \leq \min\{m, n\}$, by definition of rank, there exists a rank r square submatrix A^* of A. From Remark 352, reordering columns of A and rows of $\begin{bmatrix} A & | & b \end{bmatrix}$ does not change the rank of A or $\begin{bmatrix} A & | & b \end{bmatrix}$ and leads to the following system, which is equivalent to $Ax = b$:

$$\begin{bmatrix} A^* & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} b' \\ b'' \end{bmatrix},$$

where $A_{12} \in \mathbb{M}(r, n-r), A_{21} \in \mathbb{M}(m-r, r), A_{22} \in \mathbb{M}(m-r, n-r), x^1 \in \mathbb{R}^r, x^2 \in \mathbb{R}^{n-r}, b' \in \mathbb{R}^r, b'' \in \mathbb{R}^{m-r}$,

(x^1, x^2) has been obtained from x performing on it the same permutations performed on the columns of A,

(b', b'') has been obtained from b performing on it the same permutations performed on the rows of A.

Since

$$\text{rank} \begin{bmatrix} A^* & A_{12} & b' \end{bmatrix} = \text{rank } A^* = r,$$

the r rows of $\begin{bmatrix} A^* & A_{12} & b' \end{bmatrix}$ are linearly independent. Since

$$\text{rank} \begin{bmatrix} A^* & A_{12} & b' \\ A_{21} & A_{22} & b'' \end{bmatrix} = r,$$

the rows of that matrix are linearly dependent and from Lemma 190, the last $m - r$ rows of $[A|b]$ are linear combinations of the first r rows. Therefore, using again Remark 352, we have that $Ax = b$ is equivalent to

$$\begin{bmatrix} A^* & A_{12} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = b'$$

or, using Remark 74.2,

$$A^* x^1 + A_{12} x_2 = b'$$

and

$$x^1 = (A^*)^{-1} \left(b' - A_{12} x^2\right) \in \mathbb{R}^r$$

while x_2 can be chosen arbitrarily; more precisely

$$\left\{ \left(x^1, x^2\right) \in \mathbb{R}^n : x_1 = (A^*)^{-1} \left(b' - A_{12} x^2\right) \in \mathbb{R}^r \text{ and } x_2 \in \mathbb{R}^{n-r} \right\}$$

is the nonempty set of solution to the system $Ax = b$. ■

Theorem 356 *(Cramer) A system with n equations and n unknowns*

$$A_{n \times n} x = b$$

with $\det A \neq 0$*, has a unique solution* $x = (x_1, ..., x_i, ..., x_n)$ *where for* $i \in \{1, ..., n\}$,

$$x_i = \frac{\det A_i}{\det A}$$

and A_i is the matrix obtained from A substituting the column vector b in the place of the $i-$th column.

Proof. since $\det A \neq 0$, A^{-1} exists and it is unique. Moreover, from $Ax = b$, we get $A^{-1}Ax = A^{-1}b$ and

$$x = A^{-1} b$$

Moreover

$$A^{-1} b = \frac{1}{\det A} Adj\ A \cdot b$$

It is then enough to verify that

$$Adj\ A \cdot b = \begin{bmatrix} \det A_1 \\ ... \\ \det A_i \\ ... \\ \det A_n \end{bmatrix}$$

which we omit (see Exercise 7.34, page 268, in Lipschutz (1991). ■

The combinations of Rouche-Capelli and Cramer's Theorem allow to give a method to solve any linear system - apart from computational difficulties.

Remark 357 ***Rouche'-Capelli and Cramer's Theorem based method.***
Let the following system with m equations and n unknowns be given.

$$A_{m\times n}x = b$$

1. Compute rank A *and* rank $\begin{bmatrix} A & \mid & b \end{bmatrix}$.

i. If

$$\text{rank } A \neq \text{rank } \begin{bmatrix} A & \mid & b \end{bmatrix},$$

then the system has no solution.

ii. If

$$\text{rank } A = \text{rank } \begin{bmatrix} A & \mid & b \end{bmatrix} := r,$$

then the system has solutions which can be computed as follows.

2. Extract a square r-dimensional invertible submatrix A_r from A.

i. Discard the equations, if any, whose corresponding rows are not part of A_r.

ii. In the remaining equations, bring on the right hand side the terms containing unknowns whose coefficients are not part of the matrix A_r, if any.

iii. You then get a system to which Cramer's Theorem can be applied, treating as constant the expressions on the right hand side and which contain $n-r$ unknowns. Those unknowns can be chosen arbitrarily. Sometimes it is said that then the system has "∞^{n-r}" solutions or that the system admits $n-r$ degrees of freedom. More formally, we can say what follows.

Definition 358 *Given $S, T \subseteq \mathbb{R}^n$, we define the sum of the sets S and T, denoted by $S+T$, as follows*

$$\{x \in \mathbb{R}^n : \exists s \in S, \exists t \in T \text{ such that } x = s+t\}.$$

Proposition 359 *Assume that the set S of solutions to the system $Ax=b$ is nonempty and let $x^* \in S$. Then*

$$S = \{x^*\} + \ker l_A := \left\{x \in \mathbb{R}^n : \exists x^0 \in \ker l_A \textit{ such that } x = x^* + x^0\right\}$$

Proof. [⊆]

Take $x \in S$. We want to find $x^0 \in \ker A$ such that $x = x^* + x^0$. Take $x^0 = x - x^*$. Clearly $x = x^* + (x - x^*)$. Moreover,

$$Ax_0 = A(x - x^*) \overset{(1)}{=} b - b = 0 \tag{9.2}$$

where (1) follows from the fact that $x, x^* \in S$.

[⊇]

Take $x = x^* + x^0$ with $x^* \in S$ and $x^0 \in \ker A$. Then

$$Ax = Ax^* + Ax_0 = b + 0 = b.$$

■

Remark 360 *The above proposition implies that a linear system either has no solutions, or has a unique solution, or has infinite solutions.*

Definition 361 *V is an affine subspace of $\mathbb{R}^n$ if there exists a vector subspace W of $\mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$ such that*

$$V = \{x\} + W$$

We say that the[1] *dimension of the affine subspace V is* $\dim W$.

[1] If W' and W'' are vector subspaces of $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $V := \{x\} + W' = \{x\} + W''$, then $W' = W''$.
Take $w \in W'$, then $x + w \in V = \{x\} + W''$. Then there exists $x \in \{x\}$ and $\widehat{w} \in W''$ such that $x + w = x + \widehat{w}$. Then $w = \widehat{w} \in W''$, and $W' \subseteq W''$. Similar proof applies for the opposite inclusion.

Remark 362 *Let $a \in \mathbb{R}$ be given. First of all observe that*

$$\left\{(x, y) \in \mathbb{R}^2 : y = ax\right\} = \ker l,$$

where $l \in \mathcal{L}\left(\mathbb{R}^2, \mathbb{R}\right)$ and $l(x, y) = ax - y$. Let's present a geometric description of Proposition 359. We want to verify that the following two sets are equal.

$$S := \{(x_0, y_0)\} + \left\{(x, y) \in \mathbb{R}^2 : y = ax\right\},$$

$$T := \left\{(x, y) \in \mathbb{R}^2 : y = a(x - x_0) + y_0\right\}.$$

In words, we want to verify that the affine space "$\{(x_0, y_0)\}$ plus $\ker l$" is nothing but the set of points belonging to the line with slope a and going trrough the point (x_0, y_0).

$S \subseteq T$. Take $(x', y') \in S$; then $\exists x'' \in \mathbb{R}$ such that $x' = x_0 + x''$ and $y' = y_0 + ax''$. We have tho check that $y' = a(x' - x_0) + y_0$.Indeed, $a(x' - x_0) + y_0 = a(x_0 + x'' - x_0 + y) = ax'' + y_0$.

$T \subseteq S$. Take (x', y') such that $y' = a(x' - x_0) + y_0$. Take $x'' = x' + x_0$.Then $x' = x_0 + x''$ and $y' = y_0 + ax''$, as desired.

Remark 363 *Since $\dim \ker l_A = n - \text{rank } A$, the above Proposition and Definition say that if a nonhomogeneous systems has solutions, then the set of solutions is an affine space of dimension $n - \text{rank } A$.*

Exercise 364 *Apply the algorithm described in Remark 357 to solve the following linear system.*

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 4x_1 + 5x_2 + 6x_3 = 2 \\ 5x_1 + 7x_2 + 9x_3 = 3 \end{cases}$$

The associated matrix $\left[\begin{array}{ccc} A & | & b \end{array}\right]$ is

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & | & 1 \\ 4 & 5 & 6 & | & 2 \\ 5 & 7 & 9 & | & 3 \end{array}\right]$$

1. Since the third row of the matrix $\left[\begin{array}{ccc} A & | & b \end{array}\right]$ is equal to the sum of the first two rows, and since

$$\det \left[\begin{array}{cc} 1 & 2 \\ 4 & 5 \end{array}\right] = 5 - 8 = -3,$$

we have that

$$\text{rank } A = \text{rank } \left[\begin{array}{ccc} A & | & b \end{array}\right] = 2,$$

and the system has solutions.

2. Define

$$A_2 = \left[\begin{array}{cc} 1 & 2 \\ 4 & 5 \end{array}\right]$$

i. Discarding the equations, whose corresponding rows are not part of $A_2$and, in the remaining equations, bringing on the right hand side the terms containing unknowns whose coefficients are not part of the matrix A_2, we get

$$\begin{cases} x_1 + 2x_2 = 1 - 3x_3 \\ 4x_1 + 5x_2 = 2 - 6x_3 \\ 5x_1 + 7x_2 = 3 - 9x_3 \end{cases}$$

iii. Then, using Cramer's Theorem, recalling that $\det A_2 = -3$,we get

$$x_1 = \frac{\det \left[\begin{array}{cc} 1 - 3x_3 & 2 \\ 2 - 6x_3 & 5 \end{array}\right]}{-3} = x_3 - \frac{1}{3},$$

$$x_2 = \frac{\det \left[\begin{array}{cc} 1 & 1 - 3x_3 \\ 4 & 2 - 6x_3 \end{array}\right]}{-3} = \frac{2}{3} - 2x_3,$$

and the solution set is

$$\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_3 - \frac{1}{3}, x_2 = \frac{2}{3} - 2x_3\right\}.$$

Remark 365 ***How to find a basis of*** $\ker$.

Let $A \in \mathbb{M}(m,n)$ *be given and* $rankA = r \leq \min\{m,n\}$. *Then, from the second proof of Rouche'-Capelli's Theorem, we have that system*

$$Ax = 0$$

admits the following set of solutions

$$\left\{ \left(x^1, x^2\right) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : x^1 = \left(A^*\right)^{-1} \left(-A_{12} \cdot x^2\right) \right\} \tag{9.3}$$

Observe that $\dim \ker l_A = n - r := p$. *Then, a basis of* $\ker l_A$ *is*

$$\mathcal{B} = \left\{ \begin{bmatrix} -\left[A^*\right]^{-1} A_{12} \cdot e_p^1 \\ e_p^1 \end{bmatrix}, ..., \begin{bmatrix} -\left[A^*\right]^{-1} A_{12} \cdot e_p^p \\ e_p^p \end{bmatrix} \right\}.$$

To check the above statement, we check that 1. $\mathcal{B} \subseteq \ker l_A$, *and 2.* $\mathcal{B}$ *is linearly independent*[2].

1. It follows from (9.3)*;*

2. It follows from the fact that $\det\left[e_p^1, ..., e_p^p\right] = \det I = 1$.

Example 366

$$\begin{cases} x_1 + x_2 + x_3 + 2x_4 = 0 \\ x_1 - x_2 + x_3 + 2x_4 = 0 \end{cases}$$

Defined

$$A^* = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

the starting system can be rewritten as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}.$$

Then a basis of $\ker$ *is*

$$\mathcal{B} = \left\{ \begin{bmatrix} -\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} -\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} \right\} =$$

$$= \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

An algorithm to find eigenvalues and eigenvectors of $A_{n \times n}$ **and to show if** A **is diagonalizable.**

Step 1. Find the characteristic polynomial $\Delta(t)$ of A.

Step 2. Find the roots of $\Delta(t)$ to obtain the eigenvalues of A.

Step 3. Repeat (a) and (b) below for each eigenvalue λ of A:

(a) Form $M = A - \lambda I$;

(b) Find a basis of $\ker M$. These basis vectors are linearly independent eigenvectors of A belonging to λ.

Step 4. Consider the set $S = \{v_1, v_2, ..., v_m\}$ of all eigenvectors obtained in Step 3:

(a) if $m \neq n$, then A is not diagonalizable.

(b) If $m = n$, let P be the matrix whose columns are the eigenvectors $\{v_1, v_2, ..., v_m\}$. Then,

$$D = P^{-1}AP = \text{diag } (\lambda_i)',$$

where for $i \in \{1, ..., n\}$, λ_i is the eigenvalue corresponding to the eigenvector v^i.

[2] Observe that $\mathcal{B}$ is made up by p vectors and $\dim \ker l_A = p$.

Example 367 *Apply the above algorithm to*

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

1.

$$\det [tI - A] = \det \begin{bmatrix} t-1 & -4 \\ -2 & t-3 \end{bmatrix} = t^2 - 4t - 5.$$

2. $t^2 - 4t - 5 = 0$ *has solutions* $\lambda_1 = -1, \lambda_2 = 5.$
3. $\lambda_1 = -1.$
(a)

$$M = \begin{bmatrix} -1-1 & -4 \\ -2 & -1-3 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix}$$

(b) $\ker A = \left\{(x,y) \in \mathbb{R}^2 : x = -\frac{1}{2}y\right\} = \left\{(x,y) \in \mathbb{R}^2 : 2x = -y\right\}$. *Therefore a basis of that one dimensional space is* $\{(2,-1)\}$.
3. $\lambda_2 = 5.$
(a)

$$M = \begin{bmatrix} 5-1 & -4 \\ -2 & 5-3 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$$

(b) $\ker A = \left\{(x,y) \in \mathbb{R}^2 : x = y\right\}$. *Therefore a basis of that one dimensional space is* $\{(1,1)\}$.
4. A is diagonalizable.

$$P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

and

$$P^{-1}AP = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

Example 368 *Discuss the following system (i.e., say if admits solutions).*

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + x_2 + 2x_3 = 8 \\ 2x_1 + 2x_2 + 3x_3 = 12 \end{cases}$$

The augmented matrix $[A|b]$ *is:*

$$\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 1 & 1 & 2 & | & 8 \\ 2 & 2 & 3 & | & 12 \end{bmatrix}$$

$$\text{rank}\,[A|b] = \text{rank} \begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 1 & 1 & 2 & | & 8 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} = 2 = \text{rank } A$$

From Step 2 of Rouche'-Capelli and Cramer's method, we can consider the system

$$\begin{cases} x_2 + x_3 = 4 - x_1 \\ x_2 + 2x_3 = 8 - x_1 \end{cases}$$

Therefore, x_1 *can be chosen arbitrarily and since* $\det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1$,

$$x_2 = \det \begin{bmatrix} 4-x_1 & 1 \\ 8-x_1 & 2 \end{bmatrix} = -x_1$$

$$x_3 = \det \begin{bmatrix} 1 & 4-x_1 \\ 1 & 8-x_1 \end{bmatrix} = 4$$

Therefore, the set of solution is

$$\left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_2 = -x_1, x_3 = 4\right\}$$

Example 369 *Discuss the following system*

$$\begin{cases} x_1 & + & x_2 & = & 2 \\ x_1 & - & x_2 & = & 0 \end{cases}$$

The augmented matrix $[A|b]$ *is:*

$$\left[\begin{array}{cccc} 1 & 1 & | & 2 \\ 1 & -1 & | & 0 \end{array}\right]$$

Since $\det A = -1 - 1 = -2 \neq 0,$

$$\text{rank}\,[A|b] = \text{rank}A = 2$$

and the system has a unique solution:

$$x_1 = \frac{\det \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}}{-2} = \frac{-2}{-2} = 1$$

$$x_2 = \frac{\det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}}{-2} = \frac{-2}{-2} = 1$$

Therefore, the set of solution is

$$\{(1,1)\}$$

Example 370 *Discuss the following system*

$$\begin{cases} x_1 & + & x_2 & = & 2 \\ x_1 & + & x_2 & = & 0 \end{cases}$$

The augmented matrix $[A|b]$ *is:*

$$\begin{cases} 1 & 1 & | & 2 \\ 1 & 1 & | & 0 \end{cases}$$

Since $\det A = 1 - 1 = 0,$ *and* $\det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = -2 \neq 0,$ *we have that*

$$\text{rank}\,[A|b] = 2 \neq 1 = \text{rank}A$$

and the system has no solutions. Therefore, the set of solution is $\varnothing$.

Example 371 *Discuss the following system*

$$\begin{cases} x_1 & + & x_2 & = & 2 \\ 2x_1 & + & 2x_2 & = & 4 \end{cases}$$

The augmented matrix $[A|b]$ *is:*

$$\left[\begin{array}{cccc} 1 & 1 & | & 2 \\ 2 & 2 & | & 4 \end{array}\right]$$

From rank properties,

$$\text{rank} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 1$$

$$\text{rank} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = 1$$

Recall that elementary operations of ***rows*** *on the augmented matrix do not change the rank of either the augmented or coefficient matrices.*

Therefore

$$\text{rank}\,[A|b] = 1 = \text{rank}A$$

and the system has infinite solutions. More precisely, the set of solutions is

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 2 - x_2\right\}$$

Example 372 *Say for which value of the parameter $k \in \mathbb{R}$ the following system has one, infinite or no solutions:*

$$\begin{cases} (k-1)\,x + (k+2)\,y = 1 \\ -x + ky = 1 \\ x - 2y = 1 \end{cases}$$

$$[A|b] = \begin{bmatrix} k-1 & k+2 & | & 1 \\ -1 & k & | & 1 \\ 1 & -2 & | & 1 \end{bmatrix}$$

$$\det[A|b] = \det \begin{bmatrix} k-1 & k+2 & 1 \\ -1 & k & 1 \\ 1 & -2 & 1 \end{bmatrix} = \det \begin{bmatrix} -1 & k \\ 1 & -2 \end{bmatrix} - \det \begin{bmatrix} k-1 & k+2 \\ 1 & -2 \end{bmatrix} + \det \begin{bmatrix} k-1 & k+2 \\ -1 & k \end{bmatrix} =$$

$$(2-k) - (-2k+2-k-2) + \left(k^2 - k + k + 2\right) = 2 - k + 2k + k + k^2 + 2 = 2k + k^2 + 4$$

$\Delta = -1 - 17 < 0$. *Therefore, the determinant is never equal to zero and* rank $[A|b] = 3$. *Since* rank $A_{3\times 2} \leq 2$, *the solution set of the system is empty of each value of k.*

Remark 373 *To solve a parametric linear system $Ax = b$, where $A \in \mathbb{M}(m,n) \setminus \{0\}$, it is convenient to proceed as follows.*

1. *Perform "easy" row operations on $[A|b]$;*
2. *Compute* $\min\{m, n+1\} := k$ *and consider the $k \times k$ submatrices of the matrix $[A|b]$.*

   There are two possibilities.

   Case 1. There exists a $k \times k$ submatrix whose determinant is different from zero for some values of the parameters;

   Case 2. All $k \times k$ submatrices have zero determinant for each value of the parameters.

   If Case 2 occurs, then at least one row of $[A|b]$ is a linear combinations of other rows; therefore you can eliminate it. We can therefore assume that we are in Case 1. In that case, proceed as described below.
3. *among those $k \times k$ submatrices, choose a matrix A^* which is a submatrix of A, if possible; if you have more than one matrix to choose among, choose "the easiest one" from a computational viewpoint, i.e., that one with highest number of zeros, the lowest number of times a parameters appear, ... ;*
4. *compute* $\det A^*$, *a function of the parameter;*
5. *analyze the cases* $\det A^* \neq 0$ *and possibly* $\det A^* = 0$.

Example 374 *Say for which value of the parameter $a \in \mathbb{R}$ the following system has one, infinite or no solutions:*

$$\begin{cases} ax_1 + x_2 + x_3 & = & 2 \\ x_1 - ax_2 & = & 0 \end{cases}$$

Example 375 *Say for which value of the parameter $a \in \mathbb{R}$ the following system has one, infinite or no solutions:*

$$\begin{cases} ax_1 + x_2 + x_3 & = & 2 \\ x_1 - ax_2 & = & 0 \\ 2ax_1 + ax_2 & = & 4 \end{cases}$$

$$[A|b] = \begin{bmatrix} a & 1 & 1 & 2 \\ 1 & -a & 0 & 0 \\ 2a & a & 0 & 4 \end{bmatrix}$$

$$\det \begin{bmatrix} a & 1 & 1 \\ 1 & -a & 0 \\ 2a & a & 0 \end{bmatrix} = a + 2a^2 = 0 \qquad \textit{if} \qquad a = 0, -\frac{1}{2}.$$

Therefore, if $a \neq 0, -\frac{1}{2}$,

$$rank\,[A|b] = rangoA = 3$$

and the system has a unique solution.

If $a = 0$,

$$[A|b] = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Since $\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$, *rank* $A = 2$. *On the other hand,*

$$\det \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = -4 \neq 0$$

and therefore rank $[A|b] = 3$, *and*

$$rank\,[A|b] = 3 \neq 2 = rangoA$$

and the system has no solutions.

If $a = -\frac{1}{2}$,

$$[A|b] = \begin{bmatrix} -\frac{1}{2} & 1 & 1 & 2 \\ 1 & \frac{1}{2} & 0 & 0 \\ -1 & -\frac{1}{2} & 0 & 4 \end{bmatrix}$$

Since

$$\det \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} = -\frac{1}{2},$$

$$rangoA = 2.$$

Since

$$\det \begin{bmatrix} -\frac{1}{2} & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 4 \end{bmatrix} = -\det \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = -4$$

and therefore

$$rank\,[A|b] = 3,$$

$$rank\,[A|b] = 3 \neq 2 = rangoA$$

and the system has no solutions.

Example 376 *Say for which value of the parameter* $a \in \mathbb{R}$ *the following system has one, infinite or no solutions:*

$$\begin{cases} (a+1)\,x_1 & + & (-2)\,x_2 & + & 2ax_3 & = & a \\ ax_1 & + & (-a)\,x_2 & + & x_3 & = & -2 \\ 2 & + & (2a-4)\,x_2 & + & (4a-2)\,x_3 & = & 2a+4 \end{cases}$$

Then,

$$[A|b] = \begin{bmatrix} a+1 & -2 & 2a & | & a \\ a & -a & 1 & | & -2 \\ 2 & 2a-4 & 4a-2 & | & 2a+4 \end{bmatrix}.$$

It is easy to see that we are in Case 2 described in Remark 373: all 3×3 *submatrices of* $[A|b]$ *have determinant equal to zero - indeed, the last row is equal to* 2 *times the first row plus* (-2) *times the second row. We can then erase the third equation/row to get the following system and matrix.*

$$\begin{cases} (a+1)\,x_1 & + & (-2)\,x_2 & + & 2ax_3 & = & a \\ ax_1 & + & (-a)\,x_2 & + & x_3 & = & -2 \end{cases}$$

$$[A|b] = \begin{bmatrix} a+1 & -2 & 2a & | & a \\ a & -a & 1 & | & -2 \end{bmatrix}$$

$$\det\begin{bmatrix} -2 & 2a \\ -a & 1 \end{bmatrix} = 2a^2 - 2 = 0,$$

whose solutions are $-1, 1$. *Therefore, if* $a \in \mathbb{R}\setminus\{-1,1\}$,

$$rank\,[A|b] = 2 = rango A$$

and the system has infinite solutions. Let's study the system for $a \in \{-1, 1\}$.
If $a = -1$, *we get*

$$[A|b] = \begin{bmatrix} 0 & -2 & -2 & | & -1 \\ -1 & 1 & 1 & | & -2 \end{bmatrix}$$

and since

$$\det\begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix} = 2 \neq 0$$

we have again

$$rank\,[A|b] = 2 = rango A$$

If $a = 1$,*we have*

$$[A|b] = \begin{bmatrix} 2 & -2 & 2 & | & 1 \\ 1 & -1 & 1 & | & -2 \end{bmatrix}$$

and

$$rank\,[A|b] = 2 > 1 = rango A$$

and the system has no solution..

Example 377 *Say for which value of the parameter* $a \in \mathbb{R}$ *the following system has one, infinite or no solutions:*

$$\begin{cases} ax_1 & + & x_2 & = & 1 \\ x_1 & + & x_2 & = & a \\ 2x_1 & + & x_2 & = & 3a \\ 3x_1 & + & 2x_2 & = & a \end{cases}$$

$$[A|b] = \begin{bmatrix} a & 1 & | & 1 \\ 1 & 1 & | & a \\ 2 & 1 & | & 3a \\ 3 & 2 & | & a \end{bmatrix}$$

Observe that

$$rank\ A_{4\times 2} \leq 2.$$

$$\det\begin{bmatrix} 1 & 1 & a \\ 2 & 1 & 3a \\ 3 & 2 & a \end{bmatrix} = 3a = 0 \qquad \text{if} \qquad a = 0.$$

Therefore, if $a \in \mathbb{R}\setminus\{0\}$,

$$rank\,[A|b] = 3 > 2 \geq rank\ A_{4\times 2}$$

If $a = 0$,

$$[A|b] = \begin{bmatrix} 0 & 1 & | & 1 \\ 1 & 1 & | & 0 \\ 2 & 1 & | & 0 \\ 3 & 2 & | & 0 \end{bmatrix}$$

and since

$$\det\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} = -1$$

the system has no solution for $a = 0$.
Summarizing, $\forall a \in \mathbb{R}$, *the system has no solutions.*

Chapter 10

Appendix. Basic results on complex numbers

A[1] simple motivation to introduce complex numbers is to observe that the equation

$$x^2 + 1 = 0 \tag{10.1}$$

does not have any solution in $\mathbb{R}$. The symbol $\sqrt{-1}$, denoted by i, was introduced to "solve the equation anyway" and was called the imaginary unit. It was also said that $\pm i$ are solutions to the equation (10.1). Numbers as $3 + 4i$ were called *complex numbers.* In what follows, we present a rigorous description of what said above.

10.1 Definition of the set $\mathbb{C}$ of complex numbers

Definition 378 *The set of complex numbers is the set* $\mathbb{R}^2$ *with equality, addition and multiplication defined respectively as follows:* $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$,

a. *Equality:*

$$(x_1, x_2) = (y_1, y_2) \text{ if } x_1 = y_1 \text{ and } x_2 = y_2;$$

b. *Addition:*

$$(x_1, x_2) \boxplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2);$$

c. *Product:*

$$(x_1, x_2) \boxdot (y_1, y_2) = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1),$$

where addition and multiplication on real numbers are the standard ones. The set of complex number is denoted by $\mathbb{C}$. *For any* $x = (x_1, x_2) \in \mathbb{C}$, x_1 *is called the real part of* x *and* x_2 *the imaginary part of* x.

Remark 379 *For simplicity of notation we will use the symbols* $+$ *and* $\cdot$ *in the place of* $\boxplus$ *and* $\boxdot$, *respectively.*

Remark 380 *"Note that the symbol* $i = \sqrt{-1}$ *does not appear anywhere in this definition. Presently, we shall introduce* i *as a particular complex number which has all the algebraic properties ascribed to the fictitious symbol* $\sqrt{-1}$ *by the early mathematicians. However, before we do this, we will discuss the basic properties of the operations just defined." (Apostol (1967), page 359).*

Proposition 381 $\mathbb{C}$ *is a field.*

Proof. We have to verify that the operations presented in Definition 378 satisfy properties of the definition of field (see Definition 129. We will show only some of them.

Property 2 for multiplication.

[1] The present Chapter follows very closely Chapter 9 in Apostol (1967).

Given $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$, we have

$$(xy) \cdot z = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1)(z_1, z_2) =$$

$$= ((x_1y_1 - x_2y_2)z_1 - (x_1y_2 + x_2y_1)z_2, \quad (x_1y_1 - x_2y_2)z_2 + (x_1y_2 + x_2y_1)z_1 \ ) =$$

$$= (x_1y_1z_1 - x_2y_2z_1 - x_1y_2z_2 - x_2y_1z_2, \quad x_1y_1z_2 - x_2y_2z_2 + x_1y_2z_1 + x_2y_1z_1 \ )$$

and

$$x \cdot (y \cdot z) = (x_1, x_2)(y_1z_1 - y_2z_2, y_1z_2 + y_2z_1) =$$

$$= (x_1(y_1z_1 - y_2z_2) - x_2(y_1z_2 + y_2z_1), \quad x_1(y_1z_2 + y_2z_1) + x_2(y_1z_1 - y_2z_2)) =$$

$$= (x_1y_1z_1 - x_1y_2z_2 - x_2y_1z_2 - x_2y_2z_1, \quad x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1 - x_2y_2z_2)$$

Property 4.

The null element with respect to the sum is $(0, 0)$. The null element with respect to the multiplication is $(1, 0)$:

$\forall (x_1, x_2) \in \mathbb{C}$, $(1, 0)(x_1, x_2) = (1x_1 - 0x_2, 1x_2 + 0x_1) = (x_1, x_2)$.

Property 5. The negative element of (x_1, x_2) is simply $(-x_1, -x_2)$ defined $-(x_1, x_2)$.

Property 6. $\forall (x_1, x_2) \in \mathbb{C} \setminus \{0\}$, we want to find its inverse element $(x_1, x_2)^{-1} := (a, b)$. We must then have

$$(x_1, x_2)(a, b) = (1, 0)$$

or

$$\begin{cases} x_1 \cdot a - x_2 \cdot b &= 1 \\ x_2 \cdot a + x_1 \cdot b &= 0 \end{cases}$$

which admits a unique solution iff $(x_1)^2 + (x_2)^2 \neq 0$,i.e., $(x_1, x_2) \neq 0$, as we assumed. Then

$$\begin{cases} a = \frac{x_1}{(x_1)^2 + (x_2)^2} \\ b = \frac{-x_2}{(x_1)^2 + (x_2)^2} \end{cases}$$

Summarizing,

$$(x_1, x_2)^{-1} = \left(\frac{x_1}{(x_1)^2 + (x_2)^2}, \frac{-x_2}{(x_1)^2 + (x_2)^2} \right).$$

■

Remark 382 *As said in the above proof,*

$0 \in \mathbb{C}$ *means* $0 = (0, 0)$;

$1 \in \mathbb{C}$ *means* $1 = (1, 0)$;

for any $x = (x_1, x_2) \in \mathbb{C}$, *we define* $-x = (-x_1, -x_2)$.

Remark 383 *Obviously all the laws of algebra deducible from the field axioms also hold for complex numbers.*

10.2 The complex numbers as an extension of the real numbers

Definition 384

$$\mathbb{C}_0 = \{(x_1, x_2) \in \mathbb{C} : x_2 = 0\}.$$

Definition 385 *A subset G of a field F is a subfield of F if G is a field with respect to the operations defining F as field.*

Proposition 386 $\mathbb{C}_0$ *is a subfield of* $\mathbb{C}$.

Proof. Properties 1., 2. and 3. follow trivially. Properties 4., 5. and 6. follows from what said below.

The null element with respect to addition is $(0,0) \in \mathbb{C}_0$ and the null element with respect to product is $(1,0) \in \mathbb{C}_0$; the negative element of any $(x,0) \in \mathbb{C}_0$ is $(-x,0) \in \mathbb{C}_0$. The inverse of any element in $\mathbb{C}_0 \setminus \{0\}$ is $\left(\frac{x_1}{(x_1)^2}, 0\right) = \left(\frac{1}{x_1}, 0\right) \in \mathbb{C}_0$. ■

Definition 387 *Given two fields F_1 and F_2, if there exists a function $f : F_1 \to F_2$ which is invertible and preserves operations, then the function is called a (field) isomorphism and the fields are said isomorphic.*

Remark 388 *In the case described in the above definition, each elements in F_1 can be "identified" with its isomorphic image.*

Proposition 389 *$\mathbb{R}$ is field isomorphic to $\mathbb{C}_0$, the isomorphism being*

$$f : \mathbb{R} \to \mathbb{C}_0, \qquad a \mapsto (a,0).$$

Proof. 1. f is invertible.

Its inverse is

$$f^{-1} : \mathbb{C}_0 \to \mathbb{R}, \qquad (a,0) \mapsto a.$$

$\left(f \circ f^{-1}\right)((a,0)) = f\left(f^{-1}(a,0)\right) = f(a) = (a,0)$.

$\left(f^{-1} \circ f\right)(a) = f^{-1}(f(a)) = f^{-1}(a,0) = a$.

2. $f(a+b) = f(a) + f(b)$.

$f(a+b) = (a+b,0)$; $f(a) + f(b) = (a,0) + (b,0) = (a+b,0)$.

3. $f(ab) = f(a) f(b)$.

$f(ab) = (ab,0)$; $\quad f(a) f(b) = (a,0)(b,0) = (ab - 0, 0a + 0b) = (ab,0)$. ■

Definition 390 *A field F is an extension of a field G if F contains a subfield F' isomorphic to G.*

Remark 391 *From the two above Propositions 386 and 389, $\mathbb{C}_0$ is a subfield of $\mathbb{C}$ isomorphic to $\mathbb{R}$. Then $\mathbb{C}$ is an extension of $\mathbb{R}$, and using what said in Remark 388,*

$$\forall a \in \mathbb{R}, \text{ we could identify } a \in \mathbb{R} \text{ with } (a,0) \in \mathbb{C}_0. \tag{10.2}$$

To be precise using the notation of Proposition 389,

$$f(a) = (a,0)$$

and also

$$f(\mathbb{R}) = \mathbb{C}_0 \subseteq \mathbb{C}. \tag{10.3}$$

Very often, consistently with (10.2) *and* (10.3), *authors write*

$$\mathbb{R} \subseteq \mathbb{C}.$$

10.3 The imaginary unit i

Definition 392 *For any $x \in \mathbb{C}$, $x^2 = x \cdot x = (x_1, x_2)(x_1, x_2) = \left((x_1)^2 - (x_2)^2, 2x_1x_2\right)$.*

Proposition 393 *The equation*

$$x^2 + 1 = 0 \tag{10.4}$$

has solutions $x^* = (0,1) \in \mathbb{C}$ and $x^{**} = -(0,1) \in \mathbb{C}$.

Proof. We show that x^* is a solution. Observe that in the above equation $1 \in \mathbb{C}$, i.e., we want to show that

$$(0,1)(0,1) + (1,0) = (0,0).$$

$$(0,1)(0,1) + (1,0) = (0-1,0) + (1,0) = (0,0),$$

as desired. ■

On the base of the above proposition, we can give the following definition.

Definition 394

$$i := (0,1)\,.$$

Remark 395 *We then have that* $-i = (0,-1)$ *and then both* i *and* $-i$ *are solutions to equation* (10.4) *and that*

$$i^2 = (-i)^2 = (0,1)(0,1) = (0-1,0) = (-1,0) = -(1,0) = -1,$$

where 1 *is the complex unit.*

The following simple Proposition makes sense of the definition of $a+bi$ as a complex number.

Proposition 396 $\forall\,(a,b) \in \mathbb{C}$,

$$(a,b) = (a,0) + (b,0)\,i.$$

Proof. $(a,0)+(b,0)\,i = (a,0)+(b,0)\,(0,1) = (a,0)+(b\cdot 0 - 0\cdot 1, b+0) = (a,0)+(0,b) = (a,b).$ ■

Remark 397 *From Proposition 396, and using* (10.2)*, we could write*

$$(a,b) = a + bi,$$

or more precisely, using Proposition 389,

$$(a,b) = f(a) + i f(b).$$

Remark 398 *The use of the identification of* (a,b) *with* $a+bi$ *is mainly mnemonic. In fact, treating* $a+bi$ *as "made up by real numbers" and using the fact that* $i^2 = -1$*, we get back the definition of product and inverse as follows:*

$$(a+bi)\,(c+di) = ac + adi + bci - bd = (ac-bd) + (ad+bc)\,i,$$

and if $a+bi \neq 0$*,*

$$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)\,(a-bi)} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i.$$

We will sometime use the mnemonic notation.

10.4 Geometric interpretation: modulus and argument

Since a complex number is an ordered pair of real numbers, it may be represented geometrically by a point in the plane. The x-axis is called real axis and the y-axis is called the imaginary axis. Therefore, if $z = (a,b) \in \mathbb{C}\backslash\{(0,0)\}$, we can express a and b in polar coordinates, finding the unique $r \in \mathbb{R}_{++}$ and the unique $\theta \in (-\pi, \pi]$ such that

$$\begin{aligned} a &= r\cos\theta \\ b &= r\sin\theta \end{aligned}$$

Therefore,

$$z = r(\cos\theta + i\sin\theta) \tag{10.5}$$

Remark 399 $r \in \mathbb{R}_{++}$ *is called the modulus or absolute value of* $z = (a,b)$ *and we write*

$$r = |z| = \sqrt{a^2+b^2}.$$

$\theta \in (-\pi,\pi]$ *is called the principal argument of* $z = (a,b)$*, and we write*

$$\theta = \arg z.$$

Remark 400 *We let* θ *belong to the half-open interval* $(-\pi,\pi]$ *of length* 2π *in order to assign a unique argument* θ *to a complex number. For a given* $z = a+ib$ *the angle* θ *is determined only up to multiples of* 2π*, i.e. if* θ *is an argument of* z*,* $\forall K \in \mathbb{Z}$*,* $\theta + 2K\pi$ *is an argument as well.*

Remark 401 *Taken* $r = 0, \forall \theta \in \mathbb{R}$

$$\mathbb{C} \ni 0 = 0(\cos\theta + i\sin\theta)$$

Definition 402 *The complex conjugate of* $z = (a, b)$ *is* $\overline{z} = z = (a, -b)$.

Remark 403 *The following properties follow from the above definitions. For any* $z_1, z_2, z \in \mathbb{C}$,

1.
$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2};$$

2.
$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2};$$

3.
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}};$$

4.
$$z \cdot \overline{z} = |z|^2 .$$

Remark 404 *For a discussion of the solution of a quadratic equation, see Apostol (1967), pages 362 and bottom of page 364.*

Remark 405 ***Some trigonometric identities.*** *The following convenient trigonometric identities hold true*[2] *for any* $\alpha, \beta \in \mathbb{R}$*:*

1. (fundamental formula) $\cos^2\alpha + \sin^2\alpha = 1$;
2. summation formulas:

$$\text{a.} \quad \cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta;$$

$$\text{b.} \quad \sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta;$$

and then

$$\text{a.} \quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha;$$

$$\text{b.} \quad \sin 2\alpha = 2\sin\alpha\cos\alpha;$$

10.5 Complex exponentials

Now we extend the definition of e^a to $\mathbb{C}$, i.e., we replace $a \in \mathbb{R}$ with any $z \in \mathbb{C}$ and we verify that in this extension :

P1 e^z agrees with the usual exponential when $z \in \mathbb{R}$, or, more precisely, if $z = (x, 0) \in \mathbb{C}_0$, then $e^{(x,0)} = (e^x, 0) = f(e^x)$.

P2 $\forall x, y \in \mathbb{C}$, $e^x e^y = e^{x+y}$, i.e. the law of exponents will be valid for all complex numbers.

Definition 406 *Take*[3] $z = (a, b)$, e^z *is the complex number given by the equation*

$$e^{a+ib} = e^z = e^a(\cos b + i\sin b) \tag{10.6}$$

i.e.,

$$e^z = e^a(\cos b, \sin b)$$

Remark 407 *Observe that for* $b = 0$, $e^z = (e^x, 0)$ *and P1 is satisfied.*

Now we show that P2 is satisfied.

[2] See, for example, Simon and Blume (1994), Appendix A2.
[3] For a motivation of the presented definition see Apostol (1967) page 366.

Theorem 408 $\forall x, y \in \mathbb{C}$,

$$e^x e^y = e^{x+y}$$

Proof. Writing $x = (a, b)$ and $y = (u, v)$, we have

$$e^x = e^a(\cos b,\ \sin b) \qquad\qquad e^y = e^u(\cos v,\ \sin v),$$

so

$$e^x e^y = e^a e^u \left[(\cos b \cos v,\ -\sin b \sin v) + i(\cos b \sin v,\ \sin b \cos v)\right].$$

Using addition formulas(2) and the law of exponents for real exponentials, we obtain

$$e^x e^y = e^{a+u} \left[\cos(b+v),\ \sin(b+v)\right]$$

Since $x + y = (a + u,\ b + v)$ and from (10.6), we have

$$e^{x+y} = e^{(a+u,\ b+v)} = e^{(a+u)} \left[\cos(b+v),\ \sin(b+v)\right]$$

as desired. ■

Theorem 409 *Let $z \in \mathbb{C} \backslash \{(0,0)\}$ be given. Then*

$$z = re^{\theta i}, \tag{10.7}$$

where $r := |z|$ and $\theta = \arg z$.

Proof. If $z = (a, b)$, from (10.5) we have

$$z = r(\cos\theta,\quad \sin\theta)$$

If we take $x = 0$ and $y = \theta$ in(10.6), we obtain

$$e^{\theta i} = e^{\theta(0,1)} = e^{(0,\theta)} = e^0(\cos\theta,\quad \sin\theta) = (\cos\theta,\quad \sin\theta),$$

as desired. ■

Theorem 410 *From* (10.7) *and* (10.5) *,if $z \in \mathbb{C} \backslash \{(0,0)\}$, then*

$$z = r(\cos\theta,\quad \sin\theta) = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{10.8}$$

Proposition 411 *For any $z = r(\cos\theta,\ i\sin\theta)$, any $w = s(\cos\rho,\ \sin\rho)$ and $n \in \mathbb{N}$, where $r = |z|, \theta = \arg(z), s = |w|$ and $\rho = \arg(w)$, we have*

$$1. \quad zw = rs\left(\cos(\theta+\rho),\quad \sin(\theta+\rho)\right),$$

$$2. \quad z^n = r^n\left(\cos n\theta,\quad \sin n\theta\right).$$

Proof. 1. From (10.7) and (10.8),

$$zw = re^{i\theta}se^{i\rho} = rse^{i(\theta+\rho)} = rs(\cos(\theta+\rho) + i\sin(\theta+\rho))$$

2. We can prove the result by induction.
For $n = 1$, $z = r(\cos\theta + i\sin\theta)$ follows from (10.5).
For arbitrary $n \in \mathbb{N}$,

$$z^n = z^{n-1}z = r^{n-1}(\cos(n-1)\theta + i\sin(n-1)\theta)r(\cos\theta + i\sin\theta) =$$

$$= r^n\left[\cos(n-1)\theta\cos\theta + i(\sin\theta\cos(n-1)\theta + \sin(n-1)\theta\cos\theta) - \sin(n-1)\theta\sin\theta\right]$$

$$= r^n(\cos n\theta + i\sin n\theta),$$

where last equality follows from 2a in Remark 405. ■

Remark 412 *Expression 2. in Proposition 411 is also valid for negative integers n if we define z^{-m} to be $(z^{-1})^m$, where m is a positive integer. Moreover, for any $z_1, z_2 \in \mathbb{C}, \backslash \{(0,0)\}$, with $r_1, r_2 \in \mathbb{R}_{++}$ and $\theta, \phi \in \mathbb{R}$, modulus and argument, respectively, we have*

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta}}{r_2 e^{i\phi}} = \frac{r_1}{r_2}e^{i(\theta-\phi)},$$

and $\frac{r_1}{r_2}$ is the modulus of $\frac{z_1}{z_2}$ and $(\theta - \phi)$ is an admissible argument of $\frac{z_1}{z_2}$.

10.6 Complex-valued functions

Definition 413 *A function $f : S \subseteq \mathbb{R} \to \mathbb{C}$ is called a complex-valued function of a real variable.*

A function $f : S \subseteq \mathbb{C} \to \mathbb{C}$ is called a complex-valued function of a complex variable, or a function of a complex variable.

Remark 414 *Some examples of functions of a complex variable are the logarithm, the trigonometric functions and the exponential functions defined in $\mathbb{C}$; new properties are revealed in this context, as the example in the following Proposition does show.*

Proposition 415 *The exponential function*

$$f : \mathbb{C} \to \mathbb{C}, \qquad f(z) = e^z$$

is periodic with period $2\pi i$.

Proof. Taken $z = (a, b)$ and $n \in \mathbb{Z}$, from(10.6) we have

$$e^{z+2n\pi i} = e^a [\cos(b + 2n\pi) + i \sin(b + 2n\pi)] = e^a [\cos b + i \sin b] = e^z,$$

or

$$e^{z+(2n\pi,0)(0,1)} = (e^a, 0) [\cos(b + 2n\pi), \quad \sin(b + 2n\pi)] = (e^a, 0) (\cos b, \quad \sin b) = e^z,$$

i.e.,

$$f(z + 2n\pi i) = f(z).$$

■

Below, we present some simple results on complex-valued functions of a real variable.

Definition 416 *Given*

$$f : S \subseteq \mathbb{R} \to \mathbb{C}, \qquad x \mapsto u(x) + iv(x)$$

or

$$f : S \subseteq \mathbb{R} \to \mathbb{C}, \qquad x \mapsto (u(x), \quad v(x))$$

$u : S \subseteq \mathbb{R} \to \mathbb{R}$, $u : x \mapsto u(x)$ is a real-valued function called real part of f, and
$v : S \subseteq \mathbb{R} \to \mathbb{R}$, $v : x \mapsto v(x)$ is a real-valued function called imaginary part of f.

Now we define continuity, differentiation and integration of f in terms of the corresponding concepts for the functions u and v.

Definition 417 *Given*

$$f : S \subseteq \mathbb{R} \to \mathbb{C}, \qquad x \mapsto u(x) + iv(x),$$

or

$$f : S \subseteq \mathbb{R} \to \mathbb{C}, \qquad x \mapsto (u(x), \quad v(x))$$

1. *f is continuous at a point if both u and v are continuous at that point;*
2. *if both derivatives of u and v, u' and v' exist, the derivative of f is $f'(x) = u'(x) + iv'(x)$ for any $x \in S$ or $f'(x) = (u'(x), \ v'(x))$ for any $x \in S$*
3. *if both integrals of u and v exist, the integral of f on $[a, b]$ is $\int_b^a f(x)dx = \int_b^a u(x)dx + i \int_b^a v(x)dx$. or $\int_b^a f(x)dx = \left(\int_b^a u(x)dx, \ \int_b^a v(x)dx. \right)$*

Many of the theorems of differential and integral calculus are also valid for complex-valued functions. In order to show an example, we consider the zero-derivative theorem.

Theorem 418 *Let $I \subseteq \mathbb{R}$ be an open interval. Given $f : I \subseteq \mathbb{R} \to \mathbb{C}$, $f : x \mapsto f(x) = u(x) + iv(x)$, if $\forall x \in I$, $f'(x) = 0$, then*

$$f \text{ is constant on } I.$$

Proof. Since $f'(x) = u'(x) + iv'(x)$, the statement $f'(x) = 0$ means that $\forall x \in I$ both $u'(x) = 0$ and $v'(x) = 0$.

Then, by zero-derivative theorem for real-valued functions, both u and v are constant on I. Therefore f is constant on I. ■

We now discuss some examples of differentiation and integration formulas of a complex-valued function of a real variable.

Theorem 419 *Let $t \in \mathbb{C}$ and $f : \mathbb{R} \to \mathbb{C}$, $f(x) = e^{tx}$ be given, then*

$$f'(x) = te^{tx}$$

Proof. Given $t = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$,we have

$$f(x) = e^{tx} = e^{\alpha x + i\beta x} \overset{(10.6)}{=} e^{\alpha x} \cos \beta x + ie^{\alpha x} \sin \beta x.$$

Therefore, the real and imaginary parts of $f(x)$ are given by

$$\begin{array}{ll} u : \mathbb{R} \to \mathbb{R}, & u : x \mapsto e^{\alpha x} \cos \beta x \\ v : \mathbb{R} \to \mathbb{R} & v : x \mapsto e^{\alpha x} \sin \beta x \end{array}$$

and u and v are differentiable . Then, we have

$$u'(x) = \alpha e^{\alpha x} \cos \beta x - \beta e^{\alpha x} \sin \beta x$$

and

$$v'(x) = \alpha e^{\alpha x} \sin \beta x + \beta e^{\alpha x} \cos \beta x.$$

Since $f'(x) = u'(x) + iv'(x)$,we also have

$$\begin{aligned} f'(x) &= (\alpha e^{\alpha x} \cos \beta x - \beta e^{\alpha x} \sin \beta x) + i\left(\alpha e^{\alpha x} \sin \beta x + \beta e^{\alpha x} \cos \beta x\right) &= \\ &= \alpha e^{\alpha x}(\cos \beta x + i \sin \beta x) + i\beta e^{\alpha x}(\cos \beta x + i \sin \beta x) = (\alpha + i\beta)e^{(\alpha + i\beta)x} &= \\ &= te^{tx} \end{aligned}$$

■

Remark 420 *Theorem 419 implies that, for $t \neq 0$,*

$$\int e^{tx} dx = \frac{e^{tx}}{t} \tag{10.9}$$

If we let $t = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}, \alpha \neq 0$ and $\beta \neq 0$, and equate the real and imaginary parts of equation (10.9)*, we obtain the integration formulas*

$$\int e^{\alpha x} \cos \beta x \; dx = \tfrac{e^{\alpha x}(\alpha \cos \beta x + \beta \sin \beta x)}{\alpha^2 + \beta^2} \quad \text{and} \quad \int e^{\alpha x} \sin \beta x dx = \tfrac{e^{\alpha x}(\alpha \sin \beta x - \beta \cos \beta x)}{\alpha^2 + \beta^2},$$

as shown in detail below,

$$\int e^{tx} dx = \int e^{\alpha x + i\beta x} dx \overset{(10.6)}{=} \int \left((e^{\alpha x} \cos \beta x) + i \left(e^{\alpha x} \sin \beta x \right) \right) dx \overset{by\ def.}{=}$$

$$= \int \big((e^{\alpha x} \cos \beta x) \, dx + i \int (e^{\alpha x} \sin \beta x) \big) \, dx$$

$$\frac{e^{tx}}{t} = \frac{e^{\alpha x + i\beta x}}{\alpha + i\beta} \overset{(10.6),\ Rmk\ 398}{=} \frac{e^{\alpha x}\left(\cos \beta x + i \sin \beta x\right) \cdot (\alpha - i\beta)}{\alpha^2 + \beta^2} =$$

$$= \frac{e^{\alpha x}\left[(\alpha \sin \beta x + \beta \cos \beta x) + i(\alpha \sin \beta x - \beta \sin \beta x)\right]}{\alpha^2 + \beta^2}.$$

Part II

Some topology in metric spaces

Chapter 11

Metric spaces

11.1 Definitions and examples

Definition 421 *Let X be a nonempty set. A metric or distance on X is a function $d : X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$*

1. (a.) $d(x,y) \geq 0$, and (b.) $d(x,y) = 0 \Leftrightarrow x = y$,

2. $d(x,y) = d(y,x)$,

3. $d(x,z) \leq d(x,y) + d(y,z)$ (Triangle inequality).

(X,d) is called a metric space.

Remark 422 *Observe that the definition requires that $\forall x, y \in X$, it must be the case that $d(x,y) \in \mathbb{R}$.*

Example 423 *n-dimensional Euclidean space with Euclidean metric.*

Given $n \in \mathbb{N} \setminus \{0\}$, take $X = \mathbb{R}^n$, and

$$d_{2,n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x,y) \mapsto \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}.$$

$(X, d_{2,n})$ was shown to be a metric space in Proposition 58, Section 2.3. $d_{2,n}$ is called the Euclidean distance in $\mathbb{R}^n$. In what follows, unless needed, we write simply d_2 in the place of $d_{2,n}$.

Proposition 424 *(Discrete metric space) Given a nonempty set X and the function*

$$d : X^2 \to \mathbb{R}, \quad d(x,y) = \begin{cases} 0 & if \quad x = y \\ 1 & if \quad x \neq y, \end{cases}$$

(X,d) is a metric space, called discrete metric space.

Proof. 1a. $0, 1 \geq 0$.

1b. From the definition, $d(x,y) = 0 \Leftrightarrow x = y$.

2. It follows from the fact that $x = y \Leftrightarrow y = x$ and $x \neq y \Leftrightarrow y \neq x$.

3. If $x = z$, the result follows. If $x \neq z$, then it cannot be $x = y$ and $y = z$, and again the result follows. ■

Proposition 425 *Given $n \in \mathbb{N} \setminus \{0\}, p \in [1, +\infty), \ X = \mathbb{R}^n$,*

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x,y) \mapsto \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}},$$

(X,d) is a metric space.

Proof. 1a. It follows from the definition of absolute value.
1b. [$\Leftarrow$]Obvious.
[$\Rightarrow$] $\left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} = 0 \Leftrightarrow \sum_{i=1}^{n} |x_i - y_i|^p = 0 \Rightarrow$ for any i , $|x_i - y_i| = 0 \Rightarrow$for any i , $x_i - y_i = 0$.
2. It follows from the fact $|x_i - y_i| = |y_i - x_i|$.
3. First of all observe that

$$d(x,z) = \left(\sum_{i=1}^{n} |(x_i - y_i) + (y_i - z_i)|^p\right)^{\frac{1}{p}}.$$

Then, it is enough to show that

$$\left(\sum_{i=1}^{n} |(x_i - y_i) + (y_i - z_i)|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |(y_i - z_i)|^p\right)^{\frac{1}{p}}$$

which is a consequence of Proposition 426 below. ■

Proposition 426 *Taken* $n \in \mathbb{N}\setminus\{0\}, p \in [1, +\infty), X = \mathbb{R}^n, a, b \in \mathbb{R}^n$

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}}$$

Proof. It follows from the proof of the Proposition 429 below. ■

Definition 427 *Let* $\mathbb{R}^\infty$ *be the set of sequences in* $\mathbb{R}$.

Definition 428 *For any* $p \in [1, +\infty)$*, define*[1]

$$l^p = \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty : \sum_{n=1}^{+\infty} |x_n|^p < +\infty \right\},$$

i.e., roughly speaking, l^p *is the set of sequences whose associated series are absolutely convergent.*

Proposition 429 *(Minkowski inequality).* $\forall (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in l^p, \forall p \in [1, +\infty)$,

$$\left(\sum_{n=1}^{+\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{+\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{+\infty} |y_n|^p\right)^{\frac{1}{p}}. \tag{11.1}$$

Proof. If either $(x_n)_{n\in\mathbb{N}}$ or $(y_n)_{n\in\mathbb{N}}$ are such that $\forall n \in \mathbb{N}$, $x_n = 0$ or $\forall n \in \mathbb{N}$, $y_n = 0$, i.e., if either sequence is the constant sequence of zeros, then (11.1) is trivially true.
Then, we can consider the case in which

$$\exists \alpha, \beta \in \mathbb{R}_{++} \text{ such that } \left(\sum_{n=1}^{+\infty} |x_n|^p\right)^{\frac{1}{p}} = \alpha \text{ and } \left(\sum_{n=1}^{+\infty} |y_n|^p\right)^{\frac{1}{p}} = \beta. \tag{11.2}$$

Define

$$\forall n \in \mathbb{N}, \quad \widehat{x}_n = \left(\frac{|x_n|}{\alpha}\right)^p \quad \text{and} \quad \widehat{y}_n = \left(\frac{|y_n|}{\beta}\right)^p. \tag{11.3}$$

Then

$$\sum_{n=1}^{+\infty} \widehat{x}_n = \sum_{n=1}^{+\infty} \widehat{y}_n = 1. \tag{11.4}$$

For any $n \in \mathbb{N}$, from the triangle inequality for the absolute value, we have

$$|x_n + y_n| \leq |x_n| + |y_n|;$$

[1] For basic results on series, see, for example, Section 10.5 in Apostol (1967).

since $\forall p \in [1, +\infty)$, $f_p : \mathbb{R}_+ \to \mathbb{R}$, $f_p(t) = t^p$ is an increasing function, we have

$$|x_n + y_n|^p \leq (|x_n| + |y_n|)^p . \tag{11.5}$$

Moreover, from (11.3),

$$(|x_n| + |y_n|)^p = \left(\alpha |\widehat{x}_n|^{\frac{1}{p}} + \beta |\widehat{y}_n|^{\frac{1}{p}}\right)^p = (\alpha+\beta)^p \left(\left(\frac{\alpha}{\alpha+\beta} |\widehat{x}_n|^{\frac{1}{p}} + \frac{\beta}{\alpha+\beta} |\widehat{y}_n|^{\frac{1}{p}}\right)^p\right). \tag{11.6}$$

Since $\forall p \in [1, +\infty)$, f_p is convex (just observe that $f_p''(t) = p(p-1)t^{p-2} \geq 0$), we get

$$\left(\frac{\alpha}{\alpha+\beta} |\widehat{x}_n| + \frac{\beta}{\alpha+\beta} |\widehat{y}_n|\right)^p \leq \frac{\alpha}{\alpha+\beta} |\widehat{x}_n| + \frac{\beta}{\alpha+\beta} |\widehat{y}_n| \tag{11.7}$$

From $(11.5), (11.6)$ and (11.7), we get

$$|x_n + y_n|^p \leq (\alpha+\beta) \cdot \left(\frac{\alpha}{\alpha+\beta} |\widehat{x}_n|^p + \frac{\beta}{\alpha+\beta} |\widehat{y}_n|\right).$$

From the above inequalities and basic properties of the series, we then get

$$\sum_{n=1}^{+\infty} |x_n + y_n|^p \leq (\alpha+\beta)^p \left(\frac{\alpha}{\alpha+\beta} \sum_{n=1}^{+\infty} |\widehat{x}_n| + \frac{\beta}{\alpha+\beta} \sum_{n=1}^{+\infty} |\widehat{y}_n|\right) \overset{(11.4)}{=} (\alpha+\beta)^p \left(\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}\right) = (\alpha+\beta)^p .$$

Therefore, using (11.2), we get

$$\sum_{n=1}^{+\infty} |x_n + y_n|^p \leq \left(\left(\sum_{n=1}^{+\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{+\infty} |y_n|^p\right)^{\frac{1}{p}}\right)^p ,$$

and therefore the desired result. ■

Proposition 430 *(l^p, d_p) with*

$$d_p : l^p \times l^p \to \mathbb{R}, \quad d_p\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\right) = \left(\sum_{n=1}^{+\infty} |x_n - y_n|^p\right)^{\frac{1}{p}} .$$

is a metric space.

Proof. We first of all have to check that $d_p(x, y) \in \mathbb{R}$, i.e., that $\left(\sum_{n=1}^{+\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$ converges.

$$\left(\sum_{n=1}^{+\infty} |x_n - y_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{+\infty} |x_n + (-y_n)|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{+\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{+\infty} |y_n|^p\right)^{\frac{1}{p}} < +\infty,$$

where the first inequality follows from Minkowski inequality and the second inequality from the assumption that we are considering sequences in l^p.

Properties 1 and 2 of the distance follow easily from the definition. Property 3 is again a consequence of Minkowski inequality:

$$d_p(x, z) = \left(\sum_{n=1}^{+\infty} |(x_n - y_n) + (y_n - z_n)|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{+\infty} |x_n - y_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{+\infty} |(y_n - z_n)|^p\right)^{\frac{1}{p}} := d_p(x, y) + d_p(y, z) .$$

■

Definition 431 *Let T be a non empty set. $\mathcal{B}(T)$ is the set of all bounded real functions defined on T, i.e.,*

$$\mathcal{B}(T) := \{f : T \to \mathbb{R} \quad : \quad \sup\{|f(x)| : x \in T\} < +\infty\} ,$$

and[2]

$$d_\infty : \mathcal{B}(T) \times \mathcal{B}(T) \to \mathbb{R}, \quad d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in T\}$$

Definition 433

$$l^\infty = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty : \sup\{|x_n| : n \in \mathbb{N}\} < +\infty\right\}$$

is called the set of bounded real sequences, and, still using the symbol of the previous definition,

$$d_\infty : l^\infty \times l^\infty \to \mathbb{R}, \quad d_\infty\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\right) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}$$

Proposition 434 *$(\mathcal{B}(T), d_\infty)$and (l^∞, d_∞) are metric spaces, and d_∞ is called the sup metric.*

Proof. We show that $(\mathcal{B}(T), d_\infty)$ is a metric space. As usual, the difficult part is to show property 3 of d_∞, which is done below.
$\forall f, g, h \in \mathcal{B}(T), \forall x \in T,$

$$|f(x) - g(x)| \le \ |f(x) - h(x)| + \ |h(x) - g(x)| \le$$

$$\le \sup\{|f(x) - h(x)| : x \in T\} + \sup\{|h(x) - g(x)| : x \in T\} =$$

$$= d_\infty(f,h) + d_\infty(h,g).$$

Then,$\forall x \in T$,

$$d_\infty(f,g) := \sup\ |f(x) - g(x)| \le d_\infty(f,g) + d_\infty(h,g).$$

■

Exercise 435 *If (X,d) is a metric space, then*

$$\left(X, \frac{d}{1+d}\right)$$

is a metric space.

Proposition 436 *Given a metric space (X,d) and a set Y such that $\varnothing \ne Y \subseteq X$, then $\left(Y, d_{|Y\times Y}\right)$ is a metric space.*

Proof. By definition. ■

Definition 437 *Given a metric space (X,d) and a set Y such that $\varnothing \ne Y \subseteq X$, then $\left(Y, d_{|Y\times Y}\right)$, or simply, (Y,d) is called a metric subspace of X.*

Example 438 *1. Given $\mathbb{R}$ with the (Euclidean) distance $d_{2,1}$, $([0,1), d_{2,1})$ is a metric subspace of $(\mathbb{R}, d_{2,1})$.*
2. Given $\mathbb{R}^2$ with the (Euclidean) distance $d_{2,2}$, $(\{0\} \times \mathbb{R}, d_{2,2})$ is a metric subspace of $\left(\mathbb{R}^2, d_{2,2}\right)$.

Exercise 439 *Let $C([0,1])$ be the set of continuous functions from $[0,1]$ to $\mathbb{R}$. Show that a metric on that set is defined by*

$$d(f,g) = \int_0^1 |f(x) - g(x)|\ dx,$$

where $f, g \in C([0,1])$.

Example 440 *Let X be the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and consider $d(f,g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)|$. (X,d) is **not** a metric space because d is not a function from X^2 to $\mathbb{R}$: it can be $\sup_{x\in\mathbb{R}} |f(x) - g(x)| = +\infty$.*

---

[2]

Definition 432 *Observe that $d_\infty(f,g) \in \mathbb{R}$:*

$$d_\infty(f,g) := \sup\{|f(x) - g(x)| : x \in T\} \le \sup\{|f(x)| : x \in T\} + \sup\{|g(x)| : x \in T\} < +\infty.$$

Example 441 *Let* $X = \{a, b, c\}$ *and* $d : X^2 \to \mathbb{R}$ *such that*

$$d(a,b) = d(b,a) = 2$$

$$d(a,c) = d(c,a) = 0$$

$$d(b,c) = d(c,b) = 1.$$

Since $d(a,b) = 2 > 0 + 1 = d(a,c) + d(b,c)$, *then* (X, d) *is* ***not*** *a metric space.*

Example 442 *Given* $n \in \mathbb{N} \setminus \{0\}, p \in (0,1)$, $X = \mathbb{R}^2$, *define*

$$d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \left(\sum_{i=1}^{2} |x_i - y_i|^p \right)^{\frac{1}{p}},$$

(X, d) *is* ***not*** *a metric space, as shown below. Take* $x = (0,1), y = (1,0)$ *and* $z = (0,0)$. *Then*

$$d(x,y) = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}},$$

$$d(x,z) = (0^p + 1^p)^{\frac{1}{p}} = 1$$

$$d(z,y) = 1.$$

Then, $d(x,y) - (d(x,z) + d(z,y)) = 2^{\frac{1}{p}} - 2 > 0$.

11.2 Open and closed sets

Definition 443 *Let* (X, d) *be a metric space.* $\forall x_0 \in X$ *and* $\forall r \in \mathbb{R}_{++}$, *the open* r*-ball of* x_0 *in* (X, d) *is the set*

$$B_{(X,d)}(x_0, r) = \{x \in X : d(x, x_0) < r\}.$$

If there is no ambiguity about the metric space (X, d) *we are considering, we use the lighter notation* $B(x_0, r)$ *in the place of* $B_{(X,d)}(x_0, r)$.

Example 444 *1.*

$$B_{(\mathbb{R},d_2)}(x_0, r) = (x_0 - r, x_0 + r)$$

is the open interval of radius r *centered in* x_0.

2.

$$B_{(\mathbb{R},d_2)}(x_0, r) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \sqrt{(x_1 - x_{01})^2 + (x_2 - x_{02})^2} < r \right\}$$

is the open disk of radius r *centered in* x_0.

3. In $\mathbb{R}^2$ *with the metric* d *given by*

$$d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1, |x_2 - y_2||\}$$

the open ball $B(0,1)$ *can be pictured as done below:*

`a square around zero.`

Definition 445 *Let* (X, d) *be a metric space.* x *is an interior point of* $S \subseteq X$ *if there exists an open ball centered in* x *and contained in* S, *i.e.,* $\exists\, r \in \mathbb{R}_{++}$ *such that* $B(x, r) \subseteq S$.

Definition 446 *The set of all interior points of* S *is called the Interior of* S *and it is denoted by* $Int_{(X,d)}\, S$ *or simply by* $Int\, S$.

Remark 447 *Int* $S \subseteq S$, *simply because* $x \in Int\ S \Rightarrow x \in B(x,r) \subseteq S$, *where the first inclusion follows from the definition of open ball and the second one from the definition of Interior. In other words, to find interior points of* S, *we can limit our search to points belonging to* S.

It is not true that $\forall S \subseteq X,\ S \subseteq Int\ S$, *as shown below. We want to prove that*

$$\neg(\forall S \subseteq X, \forall x \in S, \quad x \in S \Rightarrow x \in IntS),$$

i.e.,

$$(\exists S \subseteq X \text{ and } x \in S \text{ such that } x \notin Int\ S).$$

Take $(X,d) = (\mathbb{R}, d_2)$, $S = \{1\}$ *and* $x = 1$. *Then, clearly* $1 \in \{1\}$, *but* $1 \notin Int\ S : \forall r \in \mathbb{R}_{++}$, $(1-r, 1+r) \nsubseteq \{1\}$.

Remark 448 *To understand the following example, recall that* $\forall a, b \in \mathbb{R}$ *such that* $a < b$, $\exists c \in \mathbb{Q}$ *and* $d \in \mathbb{R}\backslash\mathbb{Q}$ *such that* $c, d \in (a,b)$ *- see, for example, Apostol (1967).*

Example 449 *Let* $(\mathbb{R}, d_2)$ *be given.*

1. $Int\ \mathbb{N} = Int\ \mathbb{Q} = \varnothing$.
2. $\forall a, b \in \mathbb{R}$, $a < b$, $Int\ [a,b] = Int\ [a,b) = Int\ \ (a,b] = Int\ \ (a,b) = (a,b)$.
3. $Int\ \mathbb{R} = \mathbb{R}$.
4. $Int\ \varnothing = \varnothing$.

Definition 450 *Let* (X,d) *be a metric space. A set* $S \subseteq X$ *is open in* (X,d), *or* (X,d)*-open, or open with respect to the metric space* (X,d), *if* $S \subseteq Int\ S$, *i.e.,* $S = Int\ S$, *i.e.,*

$$\forall x \in S,\ \exists r \in \mathbb{R}_{++} \text{ such that } B_{(X,d)}(x,r) := \{y \in X : d(y,x) < r\} \subseteq S.$$

Remark 451 *Let* $(\mathbb{R}, d_2)$ *be given. From Example 449, it follows that*

$\mathbb{N}, \mathbb{Q}, [a,b], [a,b), (a,b]$ *are not open sets, and* $(a,b), \mathbb{R}$ *and* $\varnothing$ *are open sets. In particular, open interval are open sets, but there are open sets which are not open interval. Take for example* $S = (0,1) \cup (2,3)$.

Exercise 452 $\forall n \in \mathbb{N}, \forall i \in \{1, ..., n\}, \forall a_i, b_i \in \mathbb{R}$ *with* $a_i < b_i$,

$$\times_{i=1}^{n} (a_i, b_i)$$

is $(\mathbb{R}^n, d_2)$ *open.*

Proposition 453 *Let* (X,d) *be a metric space. An open ball is an open set.*

Proof. Take $y \in B(x_0, r)$. Define

$$\delta = r - d(x_0, y). \tag{11.8}$$

First of all, observe that, since $y \in B(x_0, r)$, $d(x_0, y) < r$ and then $\delta \in \mathbb{R}_{++}$. It is then enough to show that $B(y, \delta) \subseteq B(x_0, r)$, i.e., we assume that

$$d(z, y) < \delta \tag{11.9}$$

and we want to show that $d(z, x_0) < r$. From the triangle inequality

$$d(z, x_0) \leq d(z, y) + d(y, x_0) \overset{(11.9),(11.8)}{<} < \delta + (r - \delta) = r,$$

as desired. ■

Example 454 *In a discrete metric space* (X,d), $\forall x \in X, \forall r \in (0,1]$, $B(x,r) := \{y \in X : d(x,y) < r\} = \{x\}$ *and* $\forall r > 1$, $B(x,r) := \{y \in X : d(x,y) < r\} = X$. *Then, it is easy to show that any subset of a discrete metric space is open, as verified below. Let* (X,d) *be a discrete metric space and* $S \subseteq X$. *For any* $x \in S$, *take* $\varepsilon = \frac{1}{2}$; *then* $B\left(x, \frac{1}{2}\right) = \{x\} \subseteq S$.

Definition 455 *Let a metric space (X,d) be given. A set $T \subseteq X$ is closed in (X,d) if its complement in X, i.e., $X \backslash T$ is open in (X,d).*

If no ambiguity arises, we simply say that T is closed in X, or even, T is closed; we also write T^C in the place of $X \backslash T$.

Remark 456 *S is open $\Leftrightarrow S^C$ is closed, simply because S^C closed $\Leftrightarrow \left(S^C\right)^C = S$ is open.*

Example 457 *The following sets are closed in $(\mathbb{R}, d_2)$: $\mathbb{R}; \mathbb{N}; \varnothing; \forall a, b \in \mathbb{R}$, $a < b$, $\{a\}$ and $[a,b]$.*

Remark 458 ***It is false that:***

$$S \text{ is not open} \Rightarrow S \text{ is closed}$$

(and therefore that S is not closed $\Rightarrow$ S is open), i.e., there exist sets which are not open and not closed, for example $(0,1]$ in $(\mathbb{R}, d_2)$. There are also two sets which are both open and closed: $\varnothing$ and $\mathbb{R}^n$ in $(\mathbb{R}^n, d_2)$.

Proposition 459 *Let a metric space (X,d) be given.*
1. $\varnothing$ and X are open sets.
2. The union of any (finite or infinite) collection of open sets is an open set.
3. The intersection of any finite collection of open sets is an open set.

Proof. 1.
$\forall x \in X$, $\forall r \in \mathbb{R}_{++}$, $B(x,r) \subseteq X$. $\varnothing$ is open because it contains no elements.
2.
Let $\mathcal{I}$ be a collection of open sets and $S = \cup_{A \in \mathcal{I}} A$. Assume that $x \in S$. Then there exists $A \in \mathcal{I}$ such that $x \in A$. Then, for some $r \in \mathbb{R}_{++}$

$$x \in B(x,r) \subseteq A \subseteq S$$

where the first inclusion follows from fact that A is open and the second one from the definition of S.
3.
Let $\mathcal{F}$ be a collection of open sets, i.e., $\mathcal{F} = \{A_n\}_{n \in N}$, where $N \subseteq \mathbb{N}$, $\#N$ is finite and $\forall n \in N$, A_n is an open set. Take $S = \cap_{n \in N} A_n$. If $S = \varnothing$, we are done. Assume that $S \neq \varnothing$ and that $x \in S$. Then from the fact that each set A is open and from the definition of S as the intersection of sets

$$\forall n \in N, \exists r_n \in \mathbb{R}_{++} \text{ such that } x \in B(x, r_n) \subseteq A_n$$

Since N is a finite set, there exists a positive $r^* = \min\{r_n : n \in N\} > 0$. Then

$$\forall n \in N, x \in B(x, r^*) \subseteq B(x, r_n) \subseteq A_n$$

and from the very definition of intersections

$$x \in B(x, r^*) \subseteq \cap_{n \in N} B(x, r_n) \subseteq \cap_{n \in N} A_n = S.$$

■

Remark 460 *The assumption that $\#N$ is finite cannot be dispensed with:*

$$\cap_{n=1}^{+\infty} B\left(0, \frac{1}{n}\right) = \cap_{n=1}^{+\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not open.

Remark 461 *A generalization of metric spaces is the concept of topological spaces. In fact, we have the following definition which " assumes the previous Proposition".*
Let X be a nonempty set. A collection $\mathcal{T}$ of subsets of X is said to be a topology on X if
1. $\varnothing$ and X belong to $\mathcal{T}$,
2. The union of any (finite or infinite) collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$,
3. The intersection of any finite collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.
$(X, \mathcal{T})$ is called a topological space.
The members of $\mathcal{T}$ are said to be open set with respect to the topology $\mathcal{T}$, or $(X, \mathcal{T})$ open.

Proposition 462 *Let a metric space (X, d) be given.*
1. $\varnothing$ and X are closed sets.
2. The intersection of any (finite or infinite) collection of closed sets is a closed set.
3. The union of any finite collection of closed sets is a closed set.

Proof. 1
It follows from the definition of closed set, the fact that $\varnothing^C = X$, $X^C = \varnothing$ and Proposition 459.
2.
Let $\mathcal{I}$ be a collection of closed sets and $S = \cap_{B \in \mathcal{I}} B$. Then, from de Morgan's laws,

$$S^C = (\cap_{B \in \mathcal{I}} B)^C = \cup_{B \in \mathcal{I}} B^C$$

Then from Remark 456, $\forall B \in \mathcal{I}$, B^C is open and from Proposition 459.1, $\cup_{B \in \mathcal{I}} B^C$ is open as well.
2.
Let $\mathcal{F}$ be a collection of closed sets, i.e., $\mathcal{F} = \{B_n\}_{n \in N}$, where $N \subseteq \mathbb{N}$, $\#N$ is finite and $\forall n \in N$, B_n is an open set. Take $S = \cup_{n \in N} B_n$. Then, from de Morgan's laws,

$$S^C = (\cup_{n \in N} B_n)^C = \cap_{n \in N} B_n^C$$

Then from Remark 456, $\forall n \in N$, B^C is open and from Proposition 459.2, $\cap_{B \in \mathcal{I}} B^C$ is open as well.
■

Remark 463 *The assumption that $\#N$ is finite cannot be dispensed with:*

$$\left(\cap_{n=1}^{+\infty} B\left(0, \frac{1}{n}\right)\right)^C = \cup_{n=1}^{+\infty} B\left(0, \frac{1}{n}\right)^C = \cup_{n=1}^{+\infty}\left(\left(-\infty, -\frac{1}{n},\right] \cup \left[\frac{1}{n}, +\infty\right)\right) = \mathbb{R}\backslash\{0\}.$$

is not closed.

Definition 464 *If S is both closed and open in (X, d), S is called clopen in (X, d).*

Remark 465 *In any metric space (X, d), X and $\varnothing$ are clopen.*

Proposition 466 *In any metric space (X, d), $\{x\}$ is closed.*

Proof. We want to show that $X \backslash \{x\}$ is open. If $X = \{x\}$, then $X \backslash \{x\} = \varnothing$, and we are done. If $X \neq \{x\}$, take $y \in X$, where $y \neq x$. Taken

$$r = d(y, x) \tag{11.10}$$

with $r > 0$, because $x \neq y$. We are left with showing that $B(y, r) \subseteq X \backslash \{x\}$, which is true because of the following argument. Suppose otherwise; then $x \in B(y, r)$, i.e., $r \overset{(11.10)}{=} d(y, x) < r$, a contradiction. ■

Remark 467 *From Example 454, any set in any discrete metric space is open. Therefore, the complement of each set is open, and therefore each set is then clopen.*

Definition 468 *Let a metric space (X, d) and a set $S \subseteq X$ be given. x is an boundary point of S if*
any open ball centered in x intersects both S and its complement in X, i.e.,

$$\forall r \in \mathbb{R}_{++}, \quad B(x, r) \cap S \neq \varnothing \qquad \wedge \qquad B(x, r) \cap S^C \neq \varnothing.$$

Definition 469 *The set of all boundary points of S is called the Boundary of S and it is denoted by $\mathcal{F}(S)$.*

Exercise 470 $\mathcal{F}(S) = \mathcal{F}(S^C)$.

Exercise 471 *$\mathcal{F}(S)$ is a closed set.*

Definition 472 *The closure of S, denoted by* $\text{Cl}\ (S)$ *is the intersection of all closed sets containing S, i.e.,* $\text{Cl}\ (S) = \cap_{S' \in \mathcal{S}} S'$ *where* $\mathcal{S} := \{S' \subseteq X : S' \text{ is closed and } S' \supseteq S\}$.

Proposition 473 *1.* $\text{Cl}\ (S)$*is a closed set;*
2. S is closed $\Leftrightarrow S = \text{Cl}\ (S)$.

Proof. 1.
It follows from the definition and Proposition 462.
2.
[$\Leftarrow$]
It follows from 1. above.
[$\Rightarrow$]
Since S is closed, then $S \in \mathcal{S}$. Therefore, $\text{Cl}\ (S) = S \cap (\cap_{S' \in \mathcal{S}} S') = S$. ■

Definition 474 *$x \in X$ is an accumulation point for $S \subseteq X$ if any open ball centered at x contains points of S different from x, i.e., if*

$$\forall r \in \mathbb{R}_{++}, \quad (S \backslash \{x\}) \cap B(x, r) \neq \varnothing$$

The set of accumulation points of S is denoted by $D(S)$ and it is called the Derived set of S.

Definition 475 *$x \in X$ is an isolated point for $S \subseteq X$ if $x \in S$ and it is not an accumulation point for S, i.e.,*

$$x \in S \text{ and } \exists r \in \mathbb{R}_{++} \text{ such that } (S \backslash \{x\}) \cap B(x, r) = \varnothing,$$

or

$$\exists r \in \mathbb{R}_{++} \text{ such that } S \cap B(x, r) = \{x\}.$$

The set of isolated points of S is denoted by $Is(S)$.

Proposition 476 $D(S) = \{x \in \mathbb{R}^n : \forall r \in \mathbb{R}_{++}, \ S \cap B(x, r) \text{ has an infinite cardinality}\}$.

Proof. [$\subseteq$]
Suppose otherwise, i.e., x is an accumulation point of S and $\exists r \in \mathbb{R}_{++}$ such that $S \cap B(x, r) = \{x_1, ..., x_n\}$.Then defined $\delta := \min \{d(x, x_i) : i \in \{1, ..., n\}\}$, $(S \backslash \{x\}) \cap B\left(x, \frac{\delta}{2}\right) = \varnothing$, a contradiction.
[$\supseteq$]
Since $S \cap B(x, r)$ has an infinite cardinality, then $(S \backslash \{x\}) \cap B(x, r) \neq \varnothing$. ■

11.2.1 Sets which are open or closed in metric subspaces.

Remark 477 *1.* $[0, 1)$ *is* $([0, 1), d_2)$ *open.*
2. $[0, 1)$ *is not* $(\mathbb{R}, d_2)$ *open. We want to show*

$$\neg \left\langle \forall x_0 \in [0, 1), \exists r \in \mathbb{R}_{++} \text{ such that } B_{(\mathbb{R}, d_2)}(x_0, r) \right\rangle = (x_0 - r, x_0 + r) \subseteq [0, 1), \tag{11.11}$$

i.e.,

$$\exists x_0 \in [0, 1) \text{ such that } \forall r \in \mathbb{R}_{++}, \ \exists x' \in \mathbb{R} \text{ such that } x' \in (x_0 - r, x_0 + r) \text{ and } x' \notin [0, 1).$$

It is enough to take $x_0 = 0$ and $x' = -\frac{r}{2}$.

3. Let $([0, +\infty), d_2)$ be given. $[0, 1)$ is open, as shown below. By definition of open set, - go back to Definition 450 and read it again - we have that, given the metric space $((0, +\infty), d_2)$, $[0, 1)$ is open if

$$\forall x_0 \in [0, 1), \ \exists r \in \mathbb{R}_{++} \text{ such that } B_{([0,+\infty), d_2)}(x_0, r) := \left\{ \underline{x \in [0, +\infty)} : d(x_0, x) < r \right\} \subseteq ([0, 1)).$$

If $x_0 \in (0, 1)$, then take $r = \min\{x_0, 1 - x_0\} > 0$.
If $x_0 = 0$, then take $r = \frac{1}{2}$. Therefore, we have $B_{(\mathbb{R}_+, d_2)}\left(0, \frac{1}{2}\right) = \left\{x \in \mathbb{R}_+ : |x - 0| < \frac{1}{2}\right\} = \left[0, \frac{1}{2}\right) \subseteq [0, 1)$.

Remark 478 *1.* $(0,1)$ *is* $((0,1), d_2)$ *closed.*
2. $(0,1]$ *is* $((0,+\infty), d_2)$ *closed, simply because* $(1,+\infty)$ *is open.*

Proposition 479 *Let a metric space* (X,d)*, a metric subspace* (Y,d) *of* (X,d) *and a set* $S \subseteq Y$ *be given.*

$$S \text{ is open in } (Y,d) \Leftrightarrow \text{there exists a set } O \text{ open in } (X,d) \text{ such that } S = Y \cap O.$$

Proof. Preliminary remark.

$\forall x_0 \in Y, \forall r \in \mathbb{R}_{++}$,

$$B_{(Y,d)}(x_0,r) := \{x \in Y : d(x_0,x) < r\} = Y \cap \{x \in X : d(x_0,x) < r\} = Y \cap B_{(X,d)}(x_0,r). \tag{11.12}$$

[$\Rightarrow$]
Taken $x_0 \in S$, by assumption $\exists r_{x_0} \in \mathbb{R}_{++}$ such that $B_{(Y,d)}(x_0,r) \subseteq S \subseteq Y$. Then

$$S = \cup_{x_0 \in S} B_{(Y,d)}(x_0,r) \overset{(11.12)}{=} \cup_{x_0 \in S}\left(Y \cap B_{(X,d)}(x_0,r)\right) \overset{\text{distributive laws}}{=} Y \cap \left(\cup_{x_0 \in S} B_{(X,d)}(x_0,r)\right),$$

and the it is enough to take $O = \cup_{x_0 \in S} B_{(X,d)}(x_0,r)$ to get the desired result.
[$\Leftarrow$]
Take $x_0 \in S$. then, $x_0 \in O$, and, since, by assumption, O is open in (X,d), $\exists r \in \mathbb{R}_{++}$ such that $B_{(X,d)}(x_0,r) \subseteq O$. Then

$$B_{(Y,d)}(x_0,r) \overset{(11.12)}{=} Y \cap B_{(X,d)}(x_0,r) \subseteq O \cap Y = S,$$

where the last equality follows from the assumption. Summarizing, $\forall x_0 \in S, \exists r \in \mathbb{R}_{++}$ such that $B_{(Y,d)}(x_0,r) \subseteq S$, as desired. ■

Corollary 480 *Let a metric space* (X,d)*, a metric subspace* (Y,d) *of* (X,d) *and a set* $S \subseteq Y$ *be given.*
1.

$$\langle S \text{ closed in } (Y,d)\rangle \Leftrightarrow \langle \text{there exists a set } C \text{ closed in } (X,d) \text{ such that } S = Y \cap C.\rangle.$$

2.

$$\langle S \text{ open (respectively, closed) in } (X,d)\rangle \underset{\not\Leftarrow}{\Rightarrow} \langle S \text{ open (respectively, closed) in } (Y,d)\rangle.$$

3. If Y is open (respectively, closed) in X,

$$\langle S \text{ open (respectively, closed) in } (X,d)\rangle \Leftarrow \langle S \text{ open (respectively, closed) in } (Y,d)\rangle.$$

i.e., "the implication $\Leftarrow$ *in the above statement 2. does hold true".*

Proof. 1.
$\langle S \text{ closed in } (Y,d)\rangle \overset{\text{def.}}{\Leftrightarrow} \langle Y \backslash S \text{ open in } (Y,d)\rangle \overset{\text{Prop. 479}}{\Leftrightarrow}$
$\Leftrightarrow \langle \text{there exists an open set } S'' \text{ in } (X,d) \text{ such that } Y \backslash S = S'' \cap Y\rangle \Leftrightarrow$
$\Leftrightarrow \langle \text{there exists a closed set } S' \text{ in } (X,d) \text{ such that } S = S' \cap Y\rangle$,
where the last equivalence is proved below;
[$\Leftarrow$]
Take $S'' = X \backslash S'$, open in (X,d) by definition. We want to show that
if $S'' = X \backslash S$, $S = S' \cap Y$ and $Y \subseteq X$, then $Y \backslash S = S'' \cap Y$:

$$\begin{array}{lll} x \in Y \backslash S & iff & x \in Y \ \wedge \ x \notin S \\ & & x \in Y \ \wedge \ (x \notin S' \cap Y) \\ & & x \in Y \ \wedge (\neg \ (x \in S' \cap Y)) \\ & & x \in Y \ \wedge (\neg \ (x \in S' \wedge x \in Y)) \\ & & x \in Y \ \wedge ((x \notin S' \vee x \notin Y)) \\ & & (x \in Y \ \wedge x \notin S') \ \vee ((x \in Y \wedge x \notin Y)) \\ & & x \in Y \ \wedge x \notin S' \end{array}$$

$$\begin{array}{lll} x \in S'' \cap Y & iff & x \in Y \ \wedge \ x \in S'' \\ & & x \in Y \ \wedge \ (x \in X \wedge x \notin S') \\ & & (x \in Y \ \wedge \ x \in X) \wedge x \notin S' \\ & & x \in Y \wedge x \notin S' \end{array}$$

[⇒]
Take $S' = X \backslash S$.Then S' is closed in (X, d). We want to show that
if $T' = X \backslash S''$, $Y \backslash S = S'' \cap Y, Y \subseteq X$, then $S = S' \cap Y$.

Observe that we want to show that $Y \backslash S = Y \backslash (S' \cap Y)$,or from the assumptions, we want to show that

$$S'' \cap Y = Y \backslash ((X \backslash S'') \cap Y).$$

$$\begin{array}{lll} x \in Y \backslash ((X \backslash S'') \cap Y) & iff & x \in Y \ \wedge \ (\neg (x \in X \backslash S'' \wedge x \in Y)) \\ & & x \in Y \ \wedge \ (x \notin X \backslash S'' \vee x \notin Y) \\ & & x \in Y \ \wedge (x \in S'' \vee x \notin Y) \\ & & (x \in Y \wedge x \in S'') \ \vee (x \in Y \wedge x \notin Y) \\ & & x \in Y \wedge x \in S'' \\ & & x \in S'' \cap Y \end{array}$$

2. and 3.
Exercises. ■

11.3 Sequences

Unless otherwise specified, up to the end of the chapter, we assume that

X is a metric space with metric d,

and

$\mathbb{R}^n$ is the metric space with Euclidean metric.

Definition 481 *A sequence in X is a function $x : \mathbb{N} \to X$.*

Usually, for any $n \in \mathbb{N}$, the value $x(n)$ is denoted by x_n,which is called the n-th term of the sequence; the sequence is denoted by $(x_n)_{n \in \mathbb{N}}$.

Definition 482 *Given a nonempty set X, X^∞ is the set of sequences $(x_n)_{n \in \mathbb{N}}$ such that $\forall n \in \mathbb{N}$, $x_n \in X$.*

Definition 483 *A strictly increasing sequence of natural numbers is a sequence $(k_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$ such*

$$1 < k_1 < k_2 < \ ... < k_n < \ ...$$

Definition 484 *A subsequence of a sequence $(x_n)_{n \in \mathbb{N}}$ is a sequence $(y_n)_{n \in \mathbb{N}}$ such that there exists a strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that $\forall n \in \mathbb{N}$, $y_n = x_{k_n}$.*

Definition 485 *A sequence $(x_n)_{n \in \mathbb{N}} \in X^\infty$ is said to be (X, d) convergent to $x_0 \in X$ (or convergent to $x_0 \in X$ with respect to the metric space (X, d)) if*

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, \quad d(x_n, x_0) < \varepsilon \tag{11.13}$$

x_0 is called the limit of the sequence $(x_n)_{n \in \mathbb{N}}$ and we write

$$\lim_{n \to +\infty} x_n = x_0, \text{ or } x_n \overset{n}{\to} x_0. \tag{11.14}$$

$(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) is convergent if there exist $x_0 \in X$ such that (11.13) *holds. In that case, we say that the sequence converges to x_0 and x_0 is the limit of the sequence.*[3]

Remark 486 *A more precise, and heavy, notation for* (11.14) *would be*

$$\lim_{\substack{n \to +\infty \\ (X,d)}} x_n = x_0 \quad or \quad x_n \underset{(X,d)}{\overset{n}{\to}} x_0$$

[3] For the last sentence in the Definition, see, for example, Morris (2007), page 121.

Remark 487 *Observe that* $\left(\frac{1}{n}\right)_{n\in\mathbb{N}_+}$ *converges with respect to* $(\mathbb{R}, d_2)$ *and it does not converge with respect to* $(\mathbb{R}_{++}, d_2)$ *.*

Proposition 488 $\lim_{n\to+\infty} x_n = x_0 \Leftrightarrow \lim_{n\to+\infty} d(x_n, x_0) = 0.$

Proof. Observe that we can define the sequence $(d(x_n, x_0))_{n\in\mathbb{N}}$ in $\mathbb{R}$. Then from definition 485, we have that $\lim_{n\to+\infty} d(x_n, x_0) = 0$ means that

$$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, \quad |d(x_n, x_0) - 0| < \varepsilon.$$

■

Remark 489 *Since* $(d(x_n, x_0))_{n\in\mathbb{N}}$ *is a sequence in* $\mathbb{R}$*, all well known results hold for that sequence. Some of those results are listed below.*

Proposition 490 *(Some properties of sequences in* $\mathbb{R}$*).*

All the following statements concern sequences in $\mathbb{R}$*.*

1. Every convergent sequence is bounded.

2. Every increasing (decreasing) sequence that is bounded above (below) converges to its sup *(*inf*).*

3. Every sequence has a monotone subsequence.

4. (Bolzano-Weierstrass 1) Every bounded sequence has a convergent subsequence.

5. (Bolzano-Weierstrass 2) Every sequence contained in a closed and bounded set has a convergent subsequence in the set.

Moreover, suppose that $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *are* sequences in $\mathbb{R}$ *and* $\lim_{n\to\infty} x_n = x_0$ *and* $\lim_{n\to+\infty} y_n = y_0$*. Then*

6. $\lim_{n\to+\infty} (x_n + y_n) = x_0 + y_0$*;*

7. $\lim_{n\to+\infty} x_n \cdot y_n = x_0 \cdot y_0$*;*

8. if $\forall n \in \mathbb{N}$, $x_n \neq 0$ *and* $x_0 \neq 0$, $\lim_{n\to+\infty} \frac{1}{x_n} = \frac{1}{x_0}$*;*

9. if $\forall n \in \mathbb{N}$, $x_n \leq y_n$*, then* $x_0 \leq y_0$*;*

10. Let $(z_n)_{n\in\mathbb{N}}$ *be a sequence such that* $\forall n \in \mathbb{N}$, $x_n \leq z_n \leq y_n$*, and assume that* $x_0 = y_0$*. Then* $\lim_{n\to+\infty} z_n = x_0$*.*

Proof. Most of the above results can be found in Chapter 12 in Simon and Blume (1994), Ok page 50 on, Morris pages 126-128, apostol; `put a reference to apostol.` ■

Proposition 491 *If* $(x_n)_{n\in\mathbb{N}}$ *converges to* x_0 *and* $(y_n)_{n\in\mathbb{N}}$ *is a subsequence of* $(x_n)_{n\in\mathbb{N}}$*, then* $(y_n)_{n\in\mathbb{N}}$ *converges to* x_0*.*

Proof. By definition of subsequence, there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers, i.e., $1 < k_1 < k_2 < \ ... < k_n < \ ...$, such that $\forall n \in \mathbb{N}$, $y_n = x_{k_n}$.

If $n \to +\infty$, then $k_n \to +\infty$. Moreover, $\forall n$, $\exists k_n$ such that

$$d(x_0, x_{k_n}) = d(x_0, y_n)$$

Taking limits of both sides for $n \to +\infty$, we get the desired result. ■

Proposition 492 *A sequence in* (X, d) *converges at most to one element in* X*.*

Proof. Assume that $x_n \overset{n}{\to} p$ and $x_n \overset{n}{\to} q$; we want to show that $p = q$. From the Triangle inequality,

$$\forall n \in \mathbb{N}, \qquad 0 \leq d(p, q) \leq d(p, x_n) + d(x_n, q) \tag{11.15}$$

Since $d(p, x_n) \to 0$ and $d(x_n, q) \to 0$, Proposition 490.10 and (11.15) imply that $d(p, q) = 0$ and therefore $p = q$. ■

Proposition 493 *Given a sequence* $(x_n)_{n\in\mathbb{N}} = \left(\left(x_n^i\right)_{i=1}^k\right)_{n\in\mathbb{N}}$ *in* $\mathbb{R}^k$*,*

$$\left\langle (x_n)_{n\in\mathbb{N}} \text{ converges to } x\right\rangle \Leftrightarrow \left\langle \forall i \in \{1, ..., k\}, \left(x_n^i\right)_{n\in\mathbb{N}} \text{ converges to } x^i\right\rangle,$$

and

$$\lim_{n\to+\infty} x_n = \left(\lim_{n\to+\infty} x_n^i\right)_{i=1}^n.$$

Proof. [⇒]
Observe that

$$\left|x_n^i - x^i\right| = \sqrt{\left(x_n^i - x^i\right)^2} \le d\left(x_n x\right).$$

Then, the result follows.
[⇐]
By assumption, $\forall \varepsilon > 0$ and $\forall i \in \{1, ..., k\}$, there exists n_0 such that $\forall n > n_0$, we have $\left|x_n^i - x^i\right| < \frac{\varepsilon}{\sqrt{k}}$. Then $\forall n > n_0$,

$$d\left(x_n x\right) = \left(\sum_{i=1}^{k} \left|x_n^i - x^i\right|^2\right)^{\frac{1}{2}} < \left(\sum_{i=1}^{k} \left|\frac{\varepsilon}{\sqrt{k}}\right|^2\right)^{\frac{1}{2}} = \left(\varepsilon^2 \sum_{i=1}^{k} \frac{1}{k}\right)^{\frac{1}{2}} = \varepsilon.$$

■

Proposition 494 *Suppose that* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *are sequences in* $\mathbb{R}^k$ *and* $\lim_{n\to\infty} x_n = x_0$ *and* $\lim_{n\to+\infty} y_n = y_0$. *Then*

1. $\lim_{n\to+\infty} (x_n + y_n) = x_0 + y_0$;
2. $\forall c \in \mathbb{R}$, $\lim_{n\to+\infty} c \cdot x_n = c \cdot x_0$;
3. $\lim_{n\to+\infty} x_n \cdot y_n = x_0 \cdot y_0$.

Proof. It follows from Propositions 490 and 493. ■

Example 495 *In Proposition 434 , we have seen that* $(\mathcal{B}([0,1]), d_\infty)$ *is a metric space. Observe that defined* $\forall n \in \mathbb{N}\setminus\{0\}$,

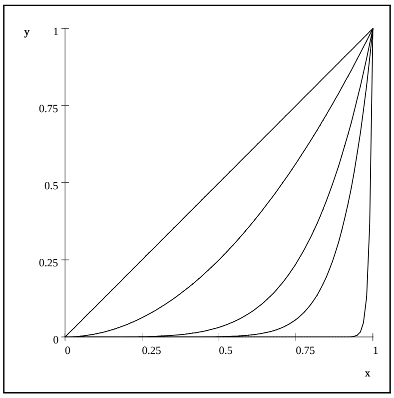
$$f_n : [0,1] \to \mathbb{R}, \qquad t \mapsto t^n,$$

we have that $(f_n)_n \in \mathcal{B}([0,1])^\infty$. *Moreover,* $\forall \bar{t} \in [0,1]$, $\left(f_n\left(\bar{t}\right)\right)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ *and it converges in* $(\mathbb{R}, d_2)$. *In fact,*

$$\lim_{n\to+\infty} \bar{t}^n = \begin{cases} 0 & if \quad \bar{t} \in [0,1) \\ 1 & if \quad \bar{t} = 1. \end{cases}$$

Define

$$f : [0,1] \to \mathbb{R}, \qquad t \mapsto \begin{cases} 0 & if \quad t \in [0,1) \\ 1 & if \quad t = 1. \end{cases}.$$



We want to check that ***it is false that***

$$f_m \underset{(\mathcal{B}([0,1]),d_\infty)}{\longrightarrow} f,$$

i.e., it is false that $d_\infty(f_m, f) \underset{m}{\to} 0$. *Then, we have to check*

$$\neg\left\langle \ \forall\varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \text{ such that } \forall n > N_\varepsilon,\ d_\infty(f_n, f) < \varepsilon \ \right\rangle,$$

i.e.,

$$\exists \varepsilon > 0 \quad \text{such that } \forall N_\varepsilon \in \mathbb{N}, \exists n > N_\varepsilon \text{ such that } d_\infty(f_n, f) \geq \varepsilon \ .$$

Then, taken $\varepsilon = \frac{1}{4}$, *it suffice to show that*

$$\left\langle \ \forall m \in \mathbb{N}, \exists \bar{t} \in (0,1) \text{ such that } \left| f_m(\bar{t}) - f(\bar{t}) \right| \geq \tfrac{1}{4} \ \right\rangle .$$

It is then enough to take $\bar{t} = \left(\frac{1}{2}\right)^m$.

Exercise 496 *For any metric space* (X, d) *and* $(x_n)_{n\in\mathbb{N}} \in X^\infty$,

$$\left\langle x_n \underset{(X,d)}{\rightarrow} x \right\rangle \Leftrightarrow \left\langle x_n \underset{\left(X, \frac{1}{1+d}\right)}{\rightarrow} x \right\rangle .$$

11.4 Sequential characterization of closed sets

Proposition 497 *Let* (X, d) *be a metric space and* $S \subseteq X$.[4]

$$\langle \ S \text{ is closed } \rangle \ \Leftrightarrow \ \langle \ \text{any } (X,d) \text{ convergent sequence } (x_n)_{n\in\mathbb{N}} \in S^\infty \text{ converges to an element of } S \ \rangle .$$

Proof. We want to show that

S is closed $\Leftrightarrow$

$$\Leftrightarrow \left\langle \left\langle \ (x_n)_{n\in\mathbb{N}} \text{ is such that } \begin{array}{l} 1.\ \forall n \in \mathbb{N}, x_n \in S, \quad \text{and} \\ 2.\ x_n \to x_0 \end{array} \right\rangle \Rightarrow 3.\ x_0 \in S \right\rangle .$$

[⇒]

We are going to show that if S is closed, 2. and not 3. hold, then 1. does not hold. Taken a sequence converging to $x_0 \in X \backslash S$. Since S is closed, $X \backslash S$ is open and therefore $\exists r \in \mathbb{R}_{++}$ such that $B(x_0, r) \subseteq X \backslash S$. Since $x_n \to x_0$, $\exists M \in \mathbb{N}$ such that $\forall n > M$, $x_n \in B(x_0, r) \subseteq X \backslash S$, contradicting Assumption 1 above.

[⇐]

Suppose otherwise, i.e., S is not closed. Then, $X \backslash S$ is not open. Then, $\exists\ \overline{x} \in X \backslash S$ such that $\forall n \in \mathbb{N}$, $\exists x_n \in X$ such that $x_n \in B\left(\overline{x}, \frac{1}{n}\right) \cap S$, i.e.,

i. $\overline{x} \in X \backslash S$

ii. $\forall n \in \mathbb{N}$, $x_n \in S$,

iii. $d(x_n, \overline{x}) < \frac{1}{n}$, and therefore $x_n \to \overline{x}$,

and i., ii. and iii. contradict the assumption. ■

Remark 498 *The Appendix to this chapter contains some other characterizations of closed sets and summarizes all the presented characterizations of open and closed sets.*

11.5 Compactness

Definition 499 *Let* (X, d) *be a metric space,* S *a subset of* X, *and* Γ *be a set of arbitrary cardinality. A family* $\mathcal{S} = \{S_\gamma\}_{\gamma\in\Gamma}$ *such that* $\forall \gamma \in \Gamma$, S_γ *is* (X, d) *open, is said to be an open cover of* S *if* $S \subseteq \cup_{\gamma\in\Gamma} S_\gamma$.

A subfamily $\mathcal{S}'$ *of* $\mathcal{S}$ *is called a subcover of* S *if* $S \subseteq \cup_{S'\in\mathcal{S}'} S'$.

Definition 500 *A metric space* (X, d) *is compact if every open cover of* X *has a finite subcover.*

A set $S \subseteq X$ *is compact in* X *if every open cover of* S *has a finite subcover of* S.

Example 501 *Any finite set in any metric space is compact.*

Take $S = \{x_i\}_{i=1}^n$ *in* (X, d) *and an open cover* $\mathcal{S}$ *of* S. *For any* $i \in \{1, ..., n\}$, *take an open set in* $\mathcal{S}$ *which contains* x_i; *call it* S_i. *Then* $\mathcal{S}' = \{S_i : i \in \{1, ..., n\}\}$ *is the desired open subcover of* $\mathcal{S}$.

[4] Proposition 559 in Appendix 11.8.1 presents a different proof of the result below.

Example 502 *1.* $(0,1)$ *is not compact in* $(\mathbb{R}, d_2)$.
We want to show that the following statement is true:

$$\neg \langle \forall \mathcal{S} \text{ such that } \cup_{S\in\mathcal{S}} S \supseteq (0,1)\,,\ \exists \mathcal{S}' \subseteq \mathcal{S} \text{ such that } \#\mathcal{S}' \text{ is finite and } \cup_{S\in\mathcal{S}'} S \supseteq (0,1)\rangle\,,$$

i.e.,

$$\exists \mathcal{S} \text{ such that } \cup_{S\in\mathcal{S}} S \supseteq (0,1) \text{ and } \forall \mathcal{S}' \subseteq \mathcal{S} \text{ either } \#\mathcal{S}' \text{ is infinite or } \cup_{S\in\mathcal{S}'} S \not\supseteq (0,1)\,.$$

Take $\mathcal{S} = \left(\left(\frac{1}{n}, 1\right)\right)_{n\in\mathbb{N}\setminus\{0,1\}}$ *and* $\mathcal{S}'$ *any finite subcover of* $\mathcal{S}$.*Then there exists a finite set* N *such that* $\mathcal{S}' = \left(\left(\frac{1}{n}, 1\right)\right)_{n\in N}$. *Take* $n^* = \max\{n \in N\}$. *Then,* $\cup_{S\in\mathcal{S}'} S = \cup_{n\in N}\left(\frac{1}{n}, 1\right) = \left(\frac{1}{n^*}, 1\right)$ *and* $\left(\frac{1}{n^*}, 1\right) \not\supseteq (0,1)$.

2. $(0,1]$ *is not compact in* $((0,+\infty), d_2)$. *Take* $\mathcal{S} = \left(\left(\frac{1}{n}, 1+\frac{1}{n}\right)\right)_{n\in\mathbb{N}\setminus\{0\}}$ *and* $\mathcal{S}'$ *any finite subcover of* $\mathcal{S}$.*Then there exists a finite set* N *such that* $\mathcal{S}' = \left(\left(\frac{1}{n}, 1+\frac{1}{n}\right)\right)_{n\in N}$. *Take* $n^* = \max\{n \in N\}$ *and* $n^{**} = \min\{n \in N\}$. *Then,* $\cup_{S\in\mathcal{S}'} S = \cup_{n\in N}\left(\frac{1}{n}, 1+\frac{1}{n}\right) = \left(\frac{1}{n^*}, 1+\frac{1}{n^{**}}\right)$ *and* $\left(\frac{1}{n^*}, 1+\frac{1}{n^{**}}\right) \not\supseteq (0,1]$.

Proposition 503 *Let* (X,d) *be a metric space.*

$$X \text{ compact and } C \subseteq X \text{ closed} \quad\Rightarrow\quad \langle C \text{ compact}\rangle\,.$$

Proof. Take an open cover $\mathcal{S}$ of C. Then $\mathcal{S} \cup (X \backslash C)$ is an open cover of X. Since X is compact, then there exists an open covers $\mathcal{S}'$of $\mathcal{S} \cup (X \backslash C)$ which cover X. Then $\mathcal{S}' \backslash \{X \backslash C\}$ is a finite subcover of $\mathcal{S}$ which covers C. ■

11.5.1 Compactness and bounded, closed sets

Definition 504 *Let* (X,d) *be a metric space and a nonempty subset* S *of* X. S *is bounded in* (X,d) *if* $\exists r \in \mathbb{R}_{++}$ *such that* $\forall x, y \in S$, $d(x,y) < r$.

Proposition 505 *Given a metric space* (X,d) *and a nonempty subset* S *of* X, *then*

$$S \text{ is bounded} \Leftrightarrow \exists r^* \in \mathbb{R}_{++} \text{ and } \exists \overline{z} \in X \text{ such that } S \subseteq B(\overline{z}, r^*).$$

Proof. $[\Rightarrow]$ Take $r^* = r$ and an arbitrary point $\overline{z}$ in $S \subseteq X$.
$[\Rightarrow]$ Take $x, y \in S$. Then

$$d(x,y) \le d(x,\overline{z}) + d(\overline{z},y) < 2r.$$

Then it is enough to take $r^* = 2r$. ■

Proposition 506 *The finite union of bounded set is bounded.*

Proof. Take $n \in \mathbb{N}$ and $\{S_i\}_{i=1}^n$ such that $\forall i \in \{1,...,n\}$, S_i is bounded. Then, $\forall i \in \{1,...,n\}$, $\exists r \in \mathbb{R}_{++}$ such that $S_i \subseteq B(\overline{z}, r_i)$. Take $r = \max\{r_i\}_{i=1}^n$. Then $\cup_{i=1}^n S_i \subseteq \cup_{i=1}^n B(\overline{z}, r_i) \subseteq B(\overline{z}, r)$. ■

Proposition 507 *Let* (X,d) *be a metric space and* S *a subset of* X.

$$S \text{ compact} \quad\Rightarrow\quad S \text{ bounded.}$$

Proof. If $S = \varnothing$, we are done. Assume then that $S \neq \varnothing$, and take $\overline{x} \in S$ and $\mathcal{B} = \{B(\overline{x}, n)\}_{n\in\mathbb{N}\setminus\{0\}}$. $\mathcal{B}$ is an open cover of X and therefore of S. Then, there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that

$$\mathcal{B}' = \{B(\overline{x}, n_i)\}_{i\in N}\,,$$

where N is a finite set and $\mathcal{B}'$ covers S.
Then takes $n^* = \max_{i\in N} n_i$, we get $S \subseteq B(\overline{x}, n^*)$ as desired. ■

Proposition 508 *Let* (X,d) *be a metric space and* S *a subset of* X.

$$S \text{ compact} \quad\Rightarrow\quad S \text{ closed.}$$

Proof. If $S = X$, we are done by Proposition 462. Assume that $S \neq X$: we want to show that $X \backslash S$ is open. Take $y \in S$ and $x \in X \backslash S$. Then, taken $r_y \in \mathbb{R}$ such that

$$0 < r_y < \frac{1}{2} d(x, y),$$

we have

$$B(y, r_y) \cap B(x, r_y) = \varnothing.$$

Now, $\mathcal{S} = \{B(y, r_y) : y \in S\}$ is an open cover of S, and since S is compact, there exists a finite subcover $\mathcal{S}'$ of $\mathcal{S}$ which covers S, say

$$\mathcal{S}' = \{B(y_n, r_n)\}_{n \in N},$$

such that N is a finite set. Take

$$r^* = \min_{n \in N} r_n,$$

and therefore $r^* > 0$. Then $\forall n \in N$,

$$B(y_n, r_n) \cap B(x, r_n) = \varnothing,$$

$$B(y_n, r_n) \cap B(x, r^*) = \varnothing,$$

and

$$(\cup_{n \in \mathbb{N}} B(y_n, r_n)) \cap B(x, r^*) = \varnothing.$$

Since $\{B(y_n, r_n)\}_{n \in N}$ covers S, we then have

$$S \cap B(x, r^*) = \varnothing,$$

or

$$B(x, r^*) \subseteq X \backslash S.$$

Therefore, we have shown that

$$\forall x \in X \backslash S, \exists r^* \in \mathbb{R}_{++} \text{ such that } B(x, r^*) \subseteq X \backslash S,$$

i.e., $X \backslash S$ is open and S is closed. ■

Remark 509 *Summarizing, we have seen that in any metric space*

$$S \text{ compact } \Rightarrow S \text{ bounded and closed.}$$

The opposite implication is false. In fact, the following sets are bounded, closed and not compact.

1. Let the metric space $((0, +\infty), d_2)$. $(0, 1]$ is closed from Remark 478, it is clearly bounded and it is not compact from Example 502.2 .

2. (X, d) where X is an infinite set and d is the discrete metric.

X is closed, from Remark 467 .

X is bounded: take $x \in X$ and $r = 2$.

X is not compact. Take $\mathcal{S} = \{B(x, 1)\}_{x \in X}$. Then $\forall x \in X$ there exists a unique element S_x in $\mathcal{S}$ such that $x \in S_x$. [5]

Remark 510 *In next section we are going to show that if (X, d) is an Euclidean space with the Euclidean distance and $S \subseteq X$, then*

$$S \text{ compact } \Leftarrow S \text{ bounded and closed.}$$

[5] For other examples, see among others, page 155, Ok (2007).

11.5.2 Sequential compactness

Definition 511 *Let a metric space (X, d) be given. $S \subseteq X$ is sequentially compact if every sequence of elements of S has a subsequence which converges to an element of S, i.e.,*

$$\left\langle (x_n)_{n\in\mathbb{N}} \text{ is a sequence in } S \right\rangle \Rightarrow \left\langle \exists \text{ a subsequence } (y_n)_{n\in\mathbb{N}} \text{ of } (x_n)_{n\in\mathbb{N}} \text{ such that } y_n \to x \in S \right\rangle.$$

In what follows, we want to prove that in metric spaces, compactness is equivalent to sequential compactness. To do that requires some work and the introduction of some, useful in itself, concepts.

Proposition 512 *(Nested intervals) For every $n \in \mathbb{N}$, define $I_n = [a_n, b_n] \subseteq \mathbb{R}$ such that $I_{n+1} \subseteq I_n$. Then $\cap_{n\in\mathbb{N}} I_n \neq \varnothing$.*

Proof. By assumption,

$$a_1 \leq a_2 \leq ... \leq a_n \leq ... \tag{11.16}$$

and

$$...b_n \leq b_{n-1} \leq ... \leq b_1 \tag{11.17}$$

Then,

$$\forall m, n \in \mathbb{N}, \quad a_m < b_n$$

simply because, if $m > n$, then $a_m < b_m \leq b_n$,where the first inequality follows from the definition of interval I_m and the second one from (11.17), and if $m \leq n$, then $a_m \leq a_n \leq b_n$,where the first inequality follows from (11.16) and the second one from the definition of interval I_n.

Then $A := \{a_n : n \in \mathbb{N}\}$ is nonempty and bounded above by b_n for any n.Then $\sup A := s$ exists.

Since $\forall n \in \mathbb{N}$, b_n is an upper bound for A,

$$\forall n \in \mathbb{N}, \quad s \leq b_n$$

and from the definition of sup,

$$\forall n \in \mathbb{N}, \quad a_n \leq s$$

Then

$$\forall n \in \mathbb{N}, \quad a_n \leq s \leq b_n$$

and

$$\forall n \in \mathbb{N}, \quad I_n \neq \varnothing.$$

■

Remark 513 *The statement in the above Proposition is false if instead of taking closed bounded intervals we take either open or unbounded intervals. To see that consider $I_n = \left(0, \frac{1}{n}\right)$ and $I_n = [n, +\infty]$.*

Proposition 514 *(Bolzano- Weirstrass) If $S \subseteq \mathbb{R}^n$ has infinite cardinality and is bounded, then S admits at least an accumulation point, i.e., $D(S) \neq \varnothing$.*

Proof. Step 1. $n = 1$.

Since S is bounded, $\exists a_0, b_0 \in \mathbb{R}$ such that $S \subseteq [a_0, b_0] := B_0$.Divide B_0 in two subinterval of equal length:

$$\left[a_0, \frac{a_0 + b_0}{2}\right] \text{ and } \left[\frac{a_0 + b_0}{2}, b_0\right]$$

Choose an interval which contains an infinite number of points in S. Call $B_1 = [a_1, b_1]$ that interval. Proceed as above for B_1. We therefore obtain a family of intervals

$$B_0 \supseteq B_1 \supseteq .... \supseteq B_n \supseteq ...$$

Observe that *lenght* $B_0 := b_0 - a_0$ and

$$\forall n \in \mathbb{N}, \textit{lenght } B_n = \frac{b_0 - a_0}{2^n}.$$

Therefore, $\forall \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall n > N$, *lenght* $B_n < \varepsilon$.

From Proposition 512, it follows that

$$\exists x \in \cap_{n=0}^{+\infty} B_n$$

We are now left with showing that x is an accumulation point for S

$$\forall r \in \mathbb{R}_{++}, B(x,r) \text{ contains an infinite number of points.}$$

By construction, $\forall n \in \mathbb{N}$, B_n contains an infinite number of points; it is therefore enough to show that

$$\forall r \in \mathbb{R}_{++}, \exists n \in \mathbb{N} \text{ such that } B(x,r) \supseteq B_n.$$

Observe that

$$B(x,r) \supseteq B_n \Leftrightarrow (x-r, x+r) \supseteq [a_n, b_n] \Leftrightarrow x - r < a_n < b_n < x + r \Leftrightarrow \max\{x - a_n, b_n - x\} < r$$

Moreover, since $x \in [a_n, b_n]$,

$$\max\{x - a_n, b_n - x\} < b_n - a_n = \textit{lenght } B_n = \frac{b_0 - a_0}{2^n}$$

Therefore, it suffices to show that

$$\forall r \in \mathbb{R}_{++}, \exists n \in \mathbb{N} \text{ such that } \frac{b_0 - a_0}{2^n} < r$$

i.e., $n \in \mathbb{N}$ and $n > \log_2 (b_0 - a_0)$.

Step 2. Omitted (See Ok (2007)). ■

Remark 515 *The above Proposition does not say that there exists an accumulation point which belongs to S. To see that, consider $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.*

Proposition 516 *Let a metric space (X, d) be given and consider the following statements.*

1. *S is compact set;*
2. *Every infinite subset of S has an accumulation point which belongs to S, i.e.,*

$$\langle T \subseteq S\ \wedge \#T \textit{ is infinite}\rangle \Rightarrow \langle D(T) \cap S \neq \varnothing\rangle,$$

3. *S is sequentially compact*
4. *S is closed and bounded.*

Then

$$1. \Leftrightarrow 2. \Leftrightarrow 3 \Rightarrow 4.$$

If $X = \mathbb{R}^n$, $d = d_2$, then we also have that

$$3 \Leftarrow 4.$$

Proof. $(1) \Rightarrow (2)$[6]

Take an infinite subset $T \subseteq S$ and suppose otherwise. Then, no point in S is an accumulation point of T, i.e., $\forall x \in S\ \exists r_x > 0$ such that

$$B(x, r_x) \cap T \backslash \{x\} = \varnothing.$$

Then[7]

$$B(x, r_x) \cap T \subseteq \{x\}. \tag{11.19}$$

[6] Proofs of $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are taken from Aliprantis and Burkinshaw (1990), pages 38-39.

[7] In general,

$$A \backslash B = C\ \Rightarrow A \subseteq C \cup B, \tag{11.18}$$

as shown below.

Since

$$S \subseteq \cup_{x \in S} B(x, r_x)$$

and S is compact, $\exists x_1, ..., x_n$ such that

$$S \subseteq \cup_{i=1}^{n} B(x_i, r_i)$$

Then, since $T \subseteq S$,

$$T = S \cap T \subseteq (\cup_{i=1}^{n} B(x_i, r_i)) \cap T = \cup_{i=1}^{n} (B(x_i, r_i) \cap T) \subseteq \{x_1, ..., x_n\}$$

where the last inclusion follows from (11.19). But then $\#T \leq n$, a contradiction.

(2) $\Rightarrow$ (3)

Take a sequence $(x_n)_{n \in \mathbb{N}}$ of elements in S.

If $\#\{x_n : n \in \mathbb{N}\}$ is finite, then $\exists x_{n^*}$ such that $x_j = x_{n^*}$ for j in an infinite subset of $\mathbb{N}$, and $(x_{n^*}, ..., x_{n^*}, ...)$ is the required convergent subsequence - converging to $x_{n^*} \in S$.

If $\#\{x_n : n \in \mathbb{N}\}$ is infinite, then there exists a subsequence $(y_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ with an infinite amount distinct values, i.e., such that $\forall n, m \in \mathbb{N}, n \neq m$, we have $y_n \neq y_m$. To construct the subsequence $(y_n)_{n \in \mathbb{N}}$, proceed as follows.

$y_1 = x_1 := x_{k_1}$,
$y_2 = x_{k_2} \notin \{x_{k_1}\}$,
$y_3 = x_{k_3} \notin \{x_{k_1}, x_{k_2}\}$,
...
$y_n = x_{k_n} \notin \{x_{k_1}, x_{k_2}, ... x_{k_{n-1}}\}$,
...

Since $T := \{y_n : n \in \mathbb{N}\}$ is an infinite subset of S, by assumption it does have an accumulation point x in S; moreover, we can redefine $(y_n)_{n \in \mathbb{N}}$ in order to have $\forall n \in \mathbb{N}$, $y_n \neq x$ [8], as follows. If $\exists k$ such that $y_k = x$, take the (sub)sequence $(y_{k+1}, y_{k+2}, ...) = (y_{k+n})_{n \in \mathbb{N}}$. With some abuse of notation, call still $(y_n)_{n \in \mathbb{N}}$ the sequence so obtained. Now take a further subsequence as follows, using the fact that x is an accumulation point of $\{y_n : n \in \mathbb{N}\} := T$,

$y_{m_1} \in T$ such that $d(y_{m1}, x) < \frac{1}{1}$,
$y_{m_2} \in T$ such that $d(y_{m2}, x) < \min\left\{\frac{1}{2}, (d(y_m, x))_{m \leq m_1}\right\}$,
$y_{m_3} \in T$ such that $d(y_{m3}, x) < \min\left\{\frac{1}{3}, (d(y_m, x))_{m \leq m_2}\right\}$,
...
$y_{m_n} \in T$ such that $d(y_{m_n}, x) < \min\left\{\frac{1}{n}, (d(y_m, x))_{m \leq m_{n-1}}\right\}$,

Observe that since $\forall n, d(y_{m_n}, x) < \min\left\{(d(y_m, x))_{m \leq m_{n-1}}\right\}$, we have that $\forall n$, $m_n > m_{n-1}$ and therefore $(y_{m_n})_{n \in \mathbb{N}}$ is a subsequence of $(y_n)_{n \in \mathbb{N}}$ and therefore of $(x_n)_{n \in \mathbb{N}}$. Finally, since

$$\lim_{n \to +\infty} d(y_{m_n}, x) < \lim_{n \to +\infty} \frac{1}{n} = 0$$

we also have that

$$\lim_{n \to +\infty} y_{m_n} = x$$

as desired.

Since $A \backslash B = A \cap B^C$, by assumption, we have

$$\left(A \cap B^C\right) \cup B = C \cup B$$

Moreover,

$$\left(A \cap B^C\right) \cup B = (A \cup B) \cap \left(B^C \cup B\right) = A \cup B \supseteq A$$

Observe that the inclusion in (11.18) can be strict, i.e., it can be

$$A \backslash B = C \ \wedge A \subset C \cup B;$$

just take $A = \{1\}, B = \{2\}$ and $C = \{1\}$:

$$A \backslash B = \{1\} = C \ \wedge A = \{1\} \subset C \cup B = \{1, 2\} .;$$

[8] Below we need to have $d(y_n, x) > 0$.

(3) $\Rightarrow$ (1)
It is the content of Proposition 525 below.
(1) $\Rightarrow$ (4)
It is the content of Remark 509.
If $X = \mathbb{R}^n$, (4) $\Rightarrow$ (2)
Take an infinite subset $T \subseteq S$. Since S is bounded T is bounded as well. Then from Bolzano-Weirestrass theorem, i.e., Proposition 514, $D(T) \neq \varnothing$. Since $T \subseteq S$, from Proposition 551, $D(T) \subseteq D(S)$ and since S is closed, $D(S) \subseteq S$. Then, summarizing $\varnothing \neq D(T) \subseteq S$ and therefore $D(T) \cap S = D(T) \neq \varnothing$, as desired. ■

To complete the proof of the above Theorem it suffices to show sequential compactness implies compactness, which is done below, and it requires some preliminary results.

Definition 517 *Let (X, d) be a metric space and S a subset of X. S is totally bounded if $\forall \varepsilon > 0, \exists$ a finite set $T \subseteq S$ such that $S \subseteq \cup_{x \in T} B(x, \varepsilon)$.*

Proposition 518 *Let (X, d) be a metric space and S a subset of X.*

$$S \text{ totally bounded} \quad \Rightarrow \quad S \text{ bounded.}$$

Proof. It follows from the definition of totally bounded sets and from Proposition 506. ■

Remark 519 *In the previous Proposition, the opposite implication does not hold true.*

Example 520 *Take (X, d) where X is an infinite set and d is the discrete metric. Then, if $\varepsilon = \frac{1}{2}$, a ball is needed to "take care of each element in X" . Similar situation arises in the following probably more interesting example.*

Example 521 *Consider the metric space (l^2, d_2) - see Proposition 430. Recall that*

$$l^2 = \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^\infty : \sum_{n=1}^{+\infty} |x_n|^2 < +\infty \right\}$$

and

$$d_2 : l^2 \times l^2 \to \mathbb{R}_+, \qquad \left((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \right) \mapsto \left(\sum_{n=1}^{+\infty} |x_n - y_n|^2 \right)^{\frac{1}{2}}.$$

Define $e_m = (e_{m,n})_{n \in \mathbb{N}}$ such that

$$e_{m,n} = \begin{cases} 1 & if \quad n = m \\ 0 & if \quad n \neq m, \end{cases}$$

and $S = \{e_m : m \in \mathbb{N}\}$. In other words, $S =$

$$\{(1, 0, 0, ..., 0, ...), (0, 1, 0, ..., 0, ...), (0, 0, 1, ..., 0, ...), ...\}.$$

Observe that $\forall m \in \mathbb{N}$, $\sum_{n=1}^{+\infty} |e_{m,n}|^2 = 1$ and therefore $S \subseteq l^2$. We now want to check that S is bounded, but not totally bounded. The main ingredient of the argument below is that

$$\forall m, p \in \mathbb{N} \text{ such that } m \neq p, \quad d(e_m, e_p) = \sqrt{2}. \tag{11.20}$$

1. S is bounded. For any $m \in \mathbb{N}$, $d(e_1, e_m) = \left(\sum_{n=1}^{+\infty} |e_{1,n}, e_{m,n}|^2 \right)^{\frac{1}{2}} = 2^{\frac{1}{2}}$.

2. S is not totally bounded. We want to show that $\exists \varepsilon > 0$ such that for any finite subset T of S there exists $x \in S$ such that $x \notin \cup_{x \in T} B(x, \varepsilon)$. Take $\varepsilon = 1$and let $T = \{e_k : k \in N\}$ with N arbitrary finite subset of $\mathbb{N}$. Then, for $k' \in \mathbb{N} \backslash N$, $e_{k'} \in S$ and from (11.20) , for any $k' \in \mathbb{N} \backslash N$ and $k \in N$, $d(e_k, e_{k'}) = \sqrt{2} > 1$. Therefore, for $k' \in \mathbb{N} \backslash N$, $e_{k'} \notin \cup_{k \in N} B(e_k, 1)$.

Remark 522 *In $(\mathbb{R}^n, d_2)$, S bounded $\Rightarrow$ S totally bounded.*

Lemma 523 *Let (X, d) be a metric space and S a subset of X.*

$$S \text{ sequentially compact } \Rightarrow S \text{ totally bounded.}$$

Proof. Suppose otherwise, i.e., $\exists \varepsilon > 0$ such that for any finite set $T \subseteq S$, $S \nsubseteq \cup_{x \in T} B(x, \varepsilon)$. We are now going to construct a sequence in S which does not admit any convergent subsequence, contradicting sequential compactness.

Take an arbitrary

$$x_1 \in S.$$

Then, by assumption $S \nsubseteq B(x_1, \varepsilon)$. Then take $x_2 \in S \backslash B(x_1, \varepsilon)$, i.e.,

$$x_2 \in S \text{ and } d(x_1, x_2) > \varepsilon.$$

By assumption, $S \nsubseteq B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$. Then, take $x_3 \in S \backslash (B(x_1, \varepsilon) \cup B(x_2, \varepsilon))$, i.e.,

$$x_3 \in S \text{ and for } i \in \{1, 2\}, \quad d(x_3, x_i) > \varepsilon.$$

By the axiom of choice, we get that

$$\forall n \in \mathbb{N}, \quad x_n \in S \text{ and for } i \in \{1, ..., n-1\}, \quad d(x_n, x_i) > \varepsilon.$$

Therefore, we have constructed a sequence $(x_n)_{n \in \mathbb{N}} \in S^\infty$ such that

$$\forall i, j \in \mathbb{N}, \text{ if } i \neq j, \text{ then } d(x_i, x_j) > \varepsilon. \tag{11.21}$$

But, then it is easy to check that $(x_n)_{n \in \mathbb{N}}$ does not have any convergent subsequence in S, as verified below. Suppose otherwise, then $(x_n)_{n \in \mathbb{N}}$ would admit a subsequence $(x_m)_{m \in \mathbb{N}} \in S^\infty$ such that $x_m \to x \in S$. But, by definition of convergence, $\exists N \in \mathbb{N}$ such that $\forall m > N$, $d(x_m, x) < \frac{\varepsilon}{2}$, and therefore

$$d(x_m, x_{m+1}) \leq d(x_m, x) + d(x_{m+1}, x) < \varepsilon,$$

contradicting (11.21). ■

Lemma 524 *Let (X, d) be a metric space and S a subset of X.*

$$\left\langle \begin{array}{l} S \text{ sequentially compact} \\ \mathcal{S} \text{ is an open cover of } S \end{array} \right\rangle \Rightarrow \left\langle \exists \varepsilon > 0 \text{ such that } \forall x \in S, \exists O_x \in \mathcal{S} \text{ such that } B(x, \varepsilon) \subseteq O_x \right\rangle.$$

Proof. Suppose otherwise; then

$$\forall n \in \mathbb{N}_+, \ \exists x_n \in S \text{ such that } \forall O \in \mathcal{S}, \quad B\left(x_n, \frac{1}{n}\right) \nsubseteq O. \tag{11.22}$$

By sequential compactness, the sequence $(x_n)_{n \in \mathbb{N}} \in S^\infty$ admits a subsequence, without loss of generality the sequence itself, $(x_n)_{n \in \mathbb{N}} \in S^\infty$ such that $x_n \to x \in S$. Since $\mathcal{S}$ is an open cover of S, $\exists \ O \in \mathcal{S}$ such that $x \in O$ and, since O is open, $\exists \varepsilon > 0$ such that

$$B(x, \varepsilon) \subseteq O. \tag{11.23}$$

Since $x_n \to x$, $\exists M \in \mathbb{N}$ such that $\{x_{M+i}, i \in \mathbb{N}\} \subseteq B\left(x, \frac{\varepsilon}{2}\right)$. Now, take $n > \max\left\{M, \frac{2}{\varepsilon}\right\}$. Then,

$$B\left(x_n, \frac{1}{n}\right) \subseteq B(x, \varepsilon). \tag{11.24}$$

i.e., $d(y, x_n) < \frac{1}{n} \Rightarrow d(y, x) < \varepsilon$, as shown below.

$$d(y, x) \leq d(y, x_n) + d(x_n, x) < \frac{1}{n} + \frac{\varepsilon}{2} < \varepsilon.$$

From (11.23) and (11.24), we get $B\left(x_n, \frac{1}{n}\right) \subseteq O \in \mathcal{S}$, contradicting (11.22). ■

Proposition 525 *Let (X, d) be a metric space and S a subset of X.*

$$S \text{ sequentially compact } \Rightarrow S \text{ compact.}$$

Proof. Take an open cover $\mathcal{S}$ of S. Since S is sequentially compact, from Lemma 524,

$$\exists \overline{\varepsilon} > 0 \text{ such that } \forall x \in S \ \exists O_x \in \mathcal{S} \text{ such that } B(x, \overline{\varepsilon}) \subseteq O_x.$$

Moreover, from Lemma 523 and the definition of total boundedness, there exists a finite set $T \subseteq S$ such that $S \subseteq \cup_{x \in T} B(x, \overline{\varepsilon}) \subseteq \cup_{x \in T} O_x$. But then $\{O_x : x \in T\}$ is the required subcover of $\mathcal{S}$ which covers S. ■

We conclude our discussion on compactness with some results we hope will clarify the concept of compactness in $\mathbb{R}^n$.

Proposition 526 *Let X be a proper subset of $\mathbb{R}^n$, and C a subset of X.*

$$\begin{array}{c} C \text{ is bounded and } (\mathbb{R}^n, d_2) \text{ closed} \\ (\Updownarrow 1) \\ C \text{ is } (\mathbb{R}^n, d_2) \text{ compact} \\ (\Updownarrow 2) \\ C \text{ is } (X, d_2) \text{ compact} \\ \Downarrow (not \Uparrow) \ 3 \\ C \text{ is bounded and } (X, d_2) \text{ closed} \end{array}$$

Proof. [1 $\Updownarrow$]

It is the content of (Propositions 507, 508 and last part of) Proposition 516.

To show the other result, observe preliminarily that

$$(X \cap S_\alpha) \cup (X \cap S_\beta) = X \cap (S_\alpha \cup S_\beta)$$

[2 $\Downarrow$]

Take $\mathcal{T} := \{T_\alpha\}_{\alpha \in A}$ such that $\forall \alpha \in A$, T_α is (X, d) open and $C \subseteq \cup_{\alpha \in A} T_\alpha$. From Proposition 479,

$$\forall \alpha \in A, \ \ \exists \ S_\alpha \text{ such that } S_\alpha \text{ is } (\mathbb{R}^n, d_2) \text{ open and } T_\alpha = X \cap S_\alpha.$$

Then

$$C \subseteq \cup_{\alpha \in A} T_\alpha \subseteq \cup_{\alpha \in A} (X \cap S_\alpha) = X \cap (\cup_{\alpha \in A} S_\alpha).$$

We then have that

$$C \subseteq \cup_{\alpha \in A} S_\alpha,$$

i.e., $\mathcal{S} := \{S_\alpha\}_{\alpha \in A}$ is a $(\mathbb{R}^n, d_2)$ open cover of C and since C is $(\mathbb{R}^n, d_2)$ compact, then there exists a finite subcover $\{S_i\}_{i \in N}$ of $\mathcal{S}$ such that

$$C \subseteq \cup_{i \in N} S_i.$$

Since $C \subseteq X$, we then have

$$C \subseteq (\cup_{i \in N} S_i) \cap X = \cup_{i \in N} (S_i \cap X) = \cup_{i \in N} T_i,$$

i.e., $\{T_i\}_{i \in N}$ is a (X, d) open subcover of $\{T_\alpha\}_{\alpha \in A}$ which covers C, as required.

[2 $\Uparrow$]

Take $\mathcal{S} := \{S_\alpha\}_{\alpha \in A}$ such that $\forall \alpha \in A$, S_α is $(\mathbb{R}^n, d_2)$ open and $C \subseteq \cup_{\alpha \in A} S_\alpha$. From Proposition 479,

$$\forall \alpha \in A, \ \ T_\alpha := X \cap S_\alpha \text{ is } (X, d) \text{ open}.$$

Since $C \subseteq X$, we then have

$$C \subseteq (\cup_{\alpha \in A} S_\alpha) \cap X = \cup_{\alpha \in A} (S_\alpha \cap X) = \cup_{\alpha \in A} T_\alpha.$$

Then, by assumption, there exists $\{T_i\}_{i \in N}$ is an open subcover of $\{T_\alpha\}_{\alpha \in A}$ which covers C, and therefore there exists a set N with finite cardinality such that

$$C \subseteq \cup_{i \in N} T_i = \cup_{i \in N} (S_i \cap X) = (\cup_{i \in N} S_i) \cap X \subseteq (\cup_{i \in N} S_i),$$

i.e., $\{S_i\}_{i \in N}$ is a $(\mathbb{R}^n, d_2)$ open subcover of $\{S_\alpha\}_{\alpha \in A}$ which covers C, as required.

[3 $\Downarrow$]

It is the content of Propositions 507, 508.

[3 not $\Uparrow$]

See Remark 509.1. ■

Remark 527 *The proof of part [2 $\Updownarrow$] above can be used to show the following result.*
Given a metric space (X,d), a metric subspace (Y,d) a set $C \subseteq Y$, then

$$C \text{ is } (Y,d) \text{ compact} \\ \Updownarrow \\ C \text{ is } (X,d) \text{ compact}$$

In other words, (X',d) compactness of $C \subseteq X' \subseteq X$ is an intrinsic property of C: it does not depend by the subspace X'you are considering. On the other hand, as we have seen, closedness and openness are ***not*** *an intrinsic property of the set.*

Remark 528 *Observe also that to define "anyway" compact sets as closed and bounded sets would not be a good choice. The conclusion of the extreme value theorem (see Theorem 621) would not hold in that case. That theorem basically says that a continuous real valued function on a compact set admits a global maximum. It is not the case that a continuous real valued function on a closed and bounded set admits a global maximum: consider the continuous function*

$$f : (0,1] \to \mathbb{R}, \qquad f(x) = \frac{1}{x}.$$

The set $(0,1]$ is bounded and closed (in $((0,+\infty), d_2)$ and f has no maximum on $(0,1]$.

11.6 Completeness

11.6.1 Cauchy sequences

Definition 529 *Let (X,d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ is a Cauchy sequence if*

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \text{ such that } \forall l,m > N, \quad d(x_l, x_m) < \varepsilon.$$

Proposition 530 *Let a metric space (X,d) and a sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ be given.*

1. *$(x_n)_{n\in\mathbb{N}}$ is convergent $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is Cauchy, but not vice-versa;*
2. *$(x_n)_{n\in\mathbb{N}}$ is Cauchy $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is bounded;*
3. *$(x_n)_{n\in\mathbb{N}}$ is Cauchy and it has a subsequence converging to $x \in X$ $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is convergent to $x \in X$.*

Proof. 1.
[$\Rightarrow$] Since $(x_n)_{n\in\mathbb{N}}$ is convergent, by definition, $\exists x \in X$ such that $x_n \to x$, $\exists N \in \mathbb{N}$ such that $\forall l,m > N$, $d(x,x_l) < \frac{\varepsilon}{2}$ and $d(x,x_m) < \frac{\varepsilon}{2}$. But then $d(x_l,x_m) \le d(x_l,x) + d(x,x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
[$\not\Leftarrow$]
Take $X = (0,1)$, $d =$ absolute value, $(x_n)_{n\in\mathbb{N}\setminus\{0\}} \in (0,1)^\infty$ such that $\forall n \in \mathbb{N}\setminus\{0\}$, $x_n = \frac{1}{n}$.
$(x_n)_{n\in\mathbb{N}\setminus\{0\}}$ is Cauchy:

$$\forall \varepsilon > 0, \quad d\left(\frac{1}{l}, \frac{1}{m}\right) = \left|\frac{1}{l} - \frac{1}{m}\right| < \left|\frac{1}{l}\right| + \left|\frac{1}{m}\right| = \frac{1}{l} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where the last inequality is true if $\frac{1}{l} < \frac{\varepsilon}{2}$ and $\frac{1}{m} < \frac{\varepsilon}{2}$, i.e., if $l > \frac{2}{\varepsilon}$ and $m > \frac{2}{\varepsilon}$. Then, it is enough to take $N > \frac{2}{\varepsilon}$ and $N \in \mathbb{N}$, to get the desired result.
$(x_n)_{n\in\mathbb{N}\setminus\{0\}}$ is not convergent to any point in $(0,1)$:
take any $\overline{x} \in (0,1)$. We want to show that

$$\exists \varepsilon > 0 \text{ such that } \forall N \in \mathbb{N}\ \exists n > N \text{ such that } d(x_n, \overline{x}) > \varepsilon.$$

Take $\varepsilon = \frac{\overline{x}}{2} > 0$ and $\forall N \in \mathbb{N}$, take $n^* \in \mathbb{N}$ such that $\frac{1}{n^*} < \min\left\{\frac{1}{N}, \frac{\overline{x}}{2}\right\}$. Then, $n^* > N$, and

$$\left|\frac{1}{n^*} - \overline{x}\right| = \overline{x} - \frac{1}{n^*} > \overline{x} - \frac{\overline{x}}{2} = \frac{\overline{x}}{2} = \varepsilon.$$

2.
Take $\varepsilon = 1$. Then $\exists N \in \mathbb{N}$ such that $\forall l, m > N, \quad d(x_l, x_m) < 1$. If $N = 1$, we are done. If $N \geq 1$, define

$$r = \max\{1, d(x_1, x_N), ..., d(x_{N-1}, x_N)\}.$$

Then

$$\{x_n : n \in \mathbb{N}\} \subseteq B(x_N, r).$$

3.
Let $(x_{n_k})_{k\in\mathbb{N}}$ be a convergent subsequence to $x \in X$. Then,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x).$$

Since $d(x_n, x_{n_k}) \to 0$, because the sequence is Cauchy, and $d(x_{n_k n}, x) \to 0$, because the subsequence is convergent, the desired result follows. ■

11.6.2 Complete metric spaces

Definition 531 *A metric space (X, d) is complete if every Cauchy sequence is a convergent sequence.*

Remark 532 *If a metric space is complete, to show convergence you do not need to guess the limit of the sequence: it is enough to show that the sequence is Cauchy.*

Example 533 $((0,1),$ *absolute value) is not a complete metric space; it is enough to consider* $\left(\frac{1}{n}\right)_{n\in\mathbb{N}\setminus\{0\}}$.

Example 534 *Let (X, d) be a discrete metric space. Then, it is complete. Take a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$. Then, we claim that $\exists N \in \mathbb{N}$ and $\overline{x} \in X$ such that $\forall n > N, \quad x_n = \overline{x}$. Suppose otherwise:*

$$\forall N \in \mathbb{N}, \exists m, m' > N \text{ such that } x_m \neq x_{m'},$$

but then $d(x_m, x_{m'}) = 1$, contradicting the fact that the sequence is Cauchy.

Example 535 $(\mathbb{Q}, d_2)$ *is not a complete metric space. Since $\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall x \in \mathbb{R}\setminus\mathbb{Q}$, we can find $(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}^\infty$ such that $x_n \to x$.*

Proposition 536 $\left(\mathbb{R}^k, d_2\right)$ *is complete.*

Proof. 1. $(\mathbb{R}, d_2)$ is complete.
Take a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$. Then, from Proposition 530.2, it is bounded. Then from Bolzanol-Weierstrass Theorem (i.e., Proposition 490.4), $(x_n)_{n\in\mathbb{N}}$ does have a convergent subsequence - i.e., $\exists (y_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ which is a subsequence of $(x_n)_{n\in\mathbb{N}}$ and such that $y_n \to a \in \mathbb{R}$. Then from Proposition 530.3.
2. For any $k \geq 2$, $\left(\mathbb{R}^k, d_2\right)$ is complete.
Take a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \in \left(\mathbb{R}^k\right)^\infty$ For $i \in \{1, ..., k\}$, consider $\left(x_n^i\right)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$. Then, for any $n, m \in \mathbb{N}$,

$$\left|x_n^i - x_m^i\right| < \|x_n - x_m\|.$$

Then, $\forall i \in \{1, ..., k\}$, $\left(x_n^i\right)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ is Cauchy and therefore from 1. above, $\forall i \in \{1, ..., k\}$, $\left(x_n^i\right)_{n\in\mathbb{N}}$ is convergent. Finally, from Proposition 493, the desired result follows. ■

Example 537 *For any nonempty set T, $(\mathcal{B}(T), d_\infty)$ is a complete metric space.*
Let $(f_n)_{n\in\mathbb{N}} \in (\mathcal{B}(T))^\infty$ be a Cauchy sequence. For any $\overline{x} \in T$, $(f_n(\overline{x}))_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ is a Cauchy sequence, and since $\mathbb{R}$ is complete, it has a convergent subsequence, without loss of generality, $(f_n(\overline{x}))_{n\in\mathbb{N}}$ itself converging say to $f_{\overline{x}} \in \mathbb{R}$. Define

$$f : T \to \mathbb{R}, \qquad : \overline{x} \mapsto f_{\overline{x}}.$$

We are going to show that (i). $f \in \mathcal{B}(T)$, and (ii) $f_n \to f$.

(i). Since $(f_n)_n$ *is Cauchy,*

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \text{ such that } \forall l, m > N, \quad d_\infty (f_l, f_m) := \sup_{x \in T} |f_l (x) - f_m (x)| < \varepsilon.$$

Then,

$$\forall x \in T, \quad |f_l (x) - f_m (x)| \leq \sup_{x \in T} |f_l (x) - f_m (x)| = d_\infty (f_l, f_m) < \varepsilon. \tag{11.25}$$

Taking limits of both sides of (11.25) *for* $l \to +\infty$*, and using the continuity of the absolute value function, we have that*

$$\forall x \in T, \quad \lim_{l \to +\infty} |f_l (x) - f_m (x)| = |f (x) - f_m (x)| < \varepsilon. \tag{11.26}$$

Since[9]

$$\forall x \in T, | \quad |f (x)| - |f_m (x)| \quad | \leq \ | \ f (x) - f_m (x) \ | < \varepsilon,$$

and therefore,

$$\forall x \in T, |f (x)| \leq \ f_m (x) + \varepsilon.$$

Since $f_l \in \mathcal{B}(T)$*,* $f \in \mathcal{B}(T)$ *as well.*
(ii) From (11.26)*, we also have that*

$$\forall x \in T, |f (x) - f_m (x)| < \varepsilon,$$

and by definition of sup

$$d_\infty (f_m, f) := \sup_{x \in T} |f_m (x) - f (x)| < \varepsilon,$$

i.e., $d_\infty (f_m, f) \to 0$.

For future use, we also show the following result.

Proposition 538

$$\mathcal{BC}(X) := \{f : X \to \mathbb{R} : f \text{ is bounded and continuous}\}$$

endowed with the metric $d (f, g) = \sup_{x \in X} |f (x) - g (x)|$ *is a complete metric space.*

Proof. See Stokey and Lucas (1989), page 47. ■

11.6.3 Completeness and closedness

Proposition 539 *Let a metric space* (X, d) *and a metric subspace* (Y, d) *of* (X, d) *be given.*

1. Y *complete* $\Rightarrow Y$ *closed;*

2. Y *complete* $\Leftarrow Y$ *closed* *and* X *complete.*

Proof. 1.
Take $(x_n)_{n \in \mathbb{N}} \in Y^\infty$ such that $x_n \to x$. From Proposition 497, it is enough to show that $x \in Y$. Since $(x_n)_{n \in \mathbb{N}}$ is convergent in X, then it is Cauchy. Since Y is complete, by definition, $x_n \to x \in Y$.
2.
Take a Cauchy sequence $(x_n)_{n \in \mathbb{N}} \in Y^\infty$. We want to show that $x_n \to x \in Y$. Since $Y \subseteq X$, $(x_n)_{n \in \mathbb{N}}$ is Cauchy in X, and since X is complete, $x_n \to x \in X$. But since Y is closed, $x \in Y$. ■

Remark 540 *An example of a metric subspace* (Y, d) *of* (X, d) *which is closed and not complete is the following one.* $(X, d) = (\mathbb{R}_{++}, d_2)$*,* $(Y, d) = ((0, 1], d_2)$ *and* $(x_n)_{n \in \mathbb{N} \setminus \{0\}} = \left(\frac{1}{n}\right)_{n \in \mathbb{N} \setminus \{0\}}$.

Corollary 541 *Let a complete metric space* (X, d) *and a metric subspace* (Y, d) *of* (X, d) *be given. Then,*

$$Y \text{ complete} \Leftrightarrow Y \text{ closed.}$$

[9] See, for example, page 37 in Ok (2007).

11.7 Fixed point theorem: contractions

Definition 542 *Let (X, d) be a metric space. A function $\varphi : X \to X$ is said to be a contraction if*

$$\exists k \in (0,1) \text{ such that } \forall x, y \in X, \quad d(\phi(x), \phi(y)) \leq k \cdot d(x, y).$$

The inf *of the set of k satisfying the above condition is called contraction coefficient of ϕ.*

Example 543 *1. Given $(\mathbb{R}, d_2)$,*

$$f_\alpha : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \alpha x$$

is a contraction iff $|\alpha| < 1$; in that case $|\alpha|$ is the contraction coefficient of f_α.
2. Let S be a nonempty open subset of $\mathbb{R}$ and $f : S \to S$ a differentiable function. If

$$\sup_{x \in S} |f'(x)| < 1,$$

then f is a contraction.

Definition 544 *For any $f, g \in X \subseteq \mathcal{B}(T)$, we say that $f \leq g$ if $\forall x \in T$, $f(x) \leq g(x)$.*

Proposition 545 *(Blackwell) Let the following objects be given:*
1. a nonempty set T;
2. X is a nonempty subset of the set $\mathcal{B}(T)$ such that $\forall f \in X$, $\forall \alpha \in \mathbb{R}_+$, $f + \alpha \in X$;
3. $\phi : X \to X$ is increasing, i.e., $f \leq g \Rightarrow \phi(f) \leq \phi(g)$;
4. $\exists \delta \in (0,1)$ such that $\forall f \in X, \forall \alpha \in \mathbb{R}_+, \phi(f + \alpha) \leq \phi(f) + \delta\alpha$.
Then ϕ is a contraction with contraction coefficient δ.

Proof. $\forall f, g \in X$, $\forall x \in T$

$$f(x) - g(x) \leq |f(x) - g(x)| \leq \sup_{x \in T} |f(x) - g(x)| = d_\infty(f, g).$$

Therefore, $f \leq g + d_\infty(f, g)$, and from Assumption 3,

$$\phi(f) \leq \phi(g + d_\infty(f, g)).$$

Then, from Assumption 4,

$$\exists \delta \in (0,1) \text{ such that } \phi(g + d_\infty(f, g)) \leq \phi(g) + \delta d_\infty(f, g),$$

and therefore

$$\phi(f) \leq \phi(g) + \delta d_\infty(f, g). \tag{11.27}$$

Since the argument above is symmetric with respect to f and g, we also have

$$\phi(g) \leq \phi(f) + \delta d_\infty(f, g). \tag{11.28}$$

From (11.27) and (11.28) and the definition of absolute value, we have

$$|\phi(f) - \phi(g)| \leq \delta d_\infty(f, g),$$

as desired. ■

Proposition 546 *(Banach fixed point theorem) Let (X, d) be a complete metric space. If $\phi : X \to X$ is a contraction with coefficient k, then*

$$\exists!\ x^* \in X \text{ such that } x^* = \phi(x^*). \tag{11.29}$$

and

$$\forall x_0 \in X \text{ and } \forall n \in \mathbb{N} \setminus \{0\}, \quad d(\phi^n(x_0), x^*) \leq k^n \cdot d(x_0, x^*), \tag{11.30}$$

where $\phi^n := (\overset{1}{\phi} \circ \overset{2}{\phi} \circ \ldots \circ \overset{n}{\phi})$.
Proof. *(11.29) holds true.*

Take any $x_0 \in X$ *and define the sequence*

$$(x_n)_{n\in\mathbb{N}} \in X^{\infty}, \text{ with } \forall n \in \mathbb{N}, x_{n+1} = \phi(x_n).$$

We want to show that 1. that $(x_n)_{n\in\mathbb{N}}$ *is Cauchy, 2. its limit is a fixed point for* ϕ*, and 3. that fixed point is unique.*

1. First of all observe that

$$\forall n \in \mathbb{N}\backslash\{0\}, d(x_{n+1}, x_n) \leq k^n d(x_1, x_0), \tag{11.31}$$

where k *is the contraction coefficient of* ϕ*, as shown by induction below.*

Step 1: $\mathcal{P}(1)$ *is true:*

$$d(x_2, x_1) = d(\phi(x_1), \phi(x_0)) \leq kd(x_1, x_0)$$

from the definition of the chosen sequence and the assumption that ϕ *is a contraction.*

Step 2. $\mathcal{P}(n-1) \Rightarrow \mathcal{P}(n)$:

$$d(x_{n+1}, x_n) = d(\phi(x_n), \phi(x_{n-1})) \leq kd(x_n, x_{n-1}) \leq k^n d(x_1, x_0)$$

from the definition of the chosen sequence, the assumption that ϕ *is a contraction and the assumption of the induction step.*

Now, for any $m, l \in \mathbb{N}$ *with* $m > l$,

$$d(x_m, x_l) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{l+1}, x_l) \leq$$

$$\leq \left(k^{m-1} + k^{m-2} + ... + k^l\right) d(x_1, x_0) \leq k^l \tfrac{1-k^{m-l}}{1-k} d(x_1, x_0),$$

where the first inequality follows from the triangle inequality, the third one from the following computation[10]*:*

$$k^{m-1} + k^{m-2} + ... + k^l = k^l\left(1 + k + ... + k^{m-l+1}\right) = k^l \frac{1-k^{m-l}}{1-k}.$$

Finally, since $k \in (0,1)$ *,we get*

$$d(x_m, x_l) \leq \frac{k^l}{1-k} d(x_1, x_0). \tag{11.32}$$

If $x_1 = x_0$*, then for any* $m, l \in \mathbb{N}$ *with* $m > l, d(x_m, x_l) = 0$ *and* $\forall n \in \mathbb{N}$*,* $x_n = x_0$ *and the sequence is converging and therefore it is Cauchy. Therefore, consider the case* $x_1 \neq x_0$*.From (11.32) it follows that* $(x_n)_{n\in\mathbb{N}} \in X^{\infty}$ *is Cauchy:* $\forall \varepsilon > 0$ *choose* $N \in \mathbb{N}$ *such that* $\frac{k^N}{1-k} d(x_1, x_0) < \varepsilon$*, i.e.,* $k^N < \frac{\varepsilon(1-k)}{d(x_1,x_0)}$ *and* $N > \frac{\log \frac{\varepsilon(1-k)}{d(x_1,x_0)}}{\log \kappa}$.

2. Since (X,d) *is a complete metric space,* $(x_n)_{n\in\mathbb{N}} \in X^{\infty}$ *does converge say to* $x^* \in X$*, and, in fact, we want to show that* $\phi(x^*) = x^*$*. Then,* $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ *such that* $\forall n > N$,

$$d(\phi(x^*), x^*) \leq d(\phi(x^*), x_{n+1}) + d\left(x^{n+1}, x^*\right) \leq$$

$$\leq d(\phi(x^*), \phi(x_n)) + d\left(x^{n+1}, x^*\right) \leq kd(x^*, x_n) + d\left(x^{n+1}, x^*\right) \leq$$

$$\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

[10] We are also using the basic facat used to study geometrical series. Define

$$s_n \equiv 1 + a + a^2 + ... + a^n;$$

:

Multiply both sides of the above equality by$(1-a)$:

$$(1-a)s_n \equiv (1-a)\left(1 + a + a^2 + ... + a^n\right)$$

$$(1-a)s_n \equiv \left(1 + a + a^2 + ... + a^n\right) - \left(a + a^2 + ... + a^{n+1}\right) = 1 - a^{n+1}$$

Divide both sides by $(1-a)$:

$$s_n \equiv \left(1 + a + a^2 + ... + a^n\right) - \left(a + a^2 + ... + a^{n+1}\right) = \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a} - \frac{a^{n+1}}{1-a}$$

where the first equality comes from the triangle inequality, the second one from the construction of the sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$*, the third one from the assumption that* ϕ *is a contraction and the last one from the fact that* $(x_n)_{n\in\mathbb{N}}$ *converges to* x^**. Since* ε *is arbitrary,* $d(\phi(x^*), x^*) = 0$*, as desired.*

3. Suppose that $\widehat{x}$ *is another fixed point for* ϕ *- beside* x^**. Then,*

$$d(\widehat{x}, x^*) = d(\phi(\widehat{x}), \phi(x^*)) \le kd(\widehat{x}, x^*)$$

and assuming $\widehat{x} \ne x^*$ *would imply* $1 \le k$*, a contradiction of the fact that* ϕ *is a contraction with contraction coefficient* k*.*

(11.30) hods true.

We show the claim by induction on $n \in \mathbb{N}\setminus\{0\}$*.*

$\mathcal{P}(1)$ *is true.*

$$d(\phi(x_0), x^*) = d(\phi(x_0), \phi(x^*)) \le k \cdot d(x_0, x^*),$$

where the equality follows from the fact that x^* *is a fixed point for* ϕ*, and the inequality by the fact that* ϕ *is a contraction.*

$\mathcal{P}(n-1)$ *is true implies that* $\mathcal{P}(n)$ *is true.*

$$d(\phi^n(x_0), x^*) = d(\phi^n(x_0), \phi(x^*)) = d\left(\phi\left(\phi^{n-1}(x_0)\right), \phi(x^*)\right) \le$$

$$\le k \cdot d\left(\phi^{n-1}(x_0), x^*\right) \le k \cdot k^{n-1} \cdot d(x_0, x^*) = k^n \cdot d(x_0, x^*).$$

■

11.8 Appendices.

11.8.1 Some characterizations of open and closed sets

Remark 547 *From basic set theory, we have* $A^C \cap B = \varnothing \Leftrightarrow B \subseteq A$*, as verified below.*

$$\neg\left\langle \exists x : x \in A^C \wedge x \in B\right\rangle = \langle \forall x : x \in A \vee \neg(x \in B)\rangle =$$

$$= \langle \forall x : \neg(x \in B) \vee x \in A\rangle \overset{(*)}{=} \langle \forall x : x \in B \Rightarrow x \in A\rangle,$$

where $(*)$ *follows from the fact that* $\langle p \Rightarrow q\rangle = \langle(\neg p) \vee q\rangle$*.*

Proposition 548 S *is open* $\Leftrightarrow S \cap \mathcal{F}(S) = \varnothing$*.*

Proof. [⇒]

Suppose otherwise, i.e., $\exists x \in S \cap \mathcal{F}(S)$. Since $x \in \mathcal{F}(S)$, $\forall r \in \mathbb{R}_{++}$, $B(x,r) \cap S^C \ne \varnothing$. Then, from Remark 547, $\forall r \in \mathbb{R}_{++}$, it is false that $B(x,r) \subseteq S$, contradicting the assumption that S is open.

[⇐]

Suppose otherwise, i.e., $\exists x \in S$ such that

$$\forall r \in \mathbb{R}_{++}, B(x,r) \cap S^C \ne \varnothing \tag{11.33}$$

Moreover

$$x \in B(x,r) \cap S \ne \varnothing \tag{11.34}$$

But (11.33) and (11.34) imply $x \in \mathcal{F}(S)$. Since $x \in S$, we would have $S \cap \mathcal{F}(S) \ne \varnothing$, contradicting the assumption. ■

Proposition 549 S *is closed* $\Leftrightarrow \mathcal{F}(S) \subseteq S$*.*

Proof.

$$S \text{ closed} \Leftrightarrow S^C \text{ open} \overset{(1)}{\Leftrightarrow} S^C \cap \mathcal{F}\left(S^C\right) = \varnothing \overset{(2)}{\Leftrightarrow} S^C \cap \mathcal{F}(S) = \varnothing \overset{(3)}{\Leftrightarrow} \mathcal{F}(S) \subseteq S$$

where

(1) follows from Proposition 548;

(2) follows from Remark 470

(3) follows Remark 547. ■

Proposition 550 *S is closed $\Leftrightarrow D(S) \subseteq S$.*

Proof. We are going to use Proposition 549, i.e., S is closed $\Leftrightarrow \mathcal{F}(S) \subseteq S$.
[$\Rightarrow$]
Suppose otherwise, i.e.,

$$\exists x \notin S \text{ such that } \forall r \in \mathbb{R}_{++}, \ (S \setminus \{x\}) \cap B(x,r) \neq \varnothing$$

and since $x \notin S$, it is also true that

$$\forall r \in \mathbb{R}_{++}, \ S \cap B(x,r) \neq \varnothing \tag{11.35}$$

and

$$\forall r \in \mathbb{R}_{++}, \ S^C \cap B(x,r) \neq \varnothing \tag{11.36}$$

From (11.35) and (11.36), it follows that $x \in \mathcal{F}(S)$, while $x \notin S$, which from Proposition 549 contradicts the assumption that S is closed.
[$\Leftarrow$]
Suppose otherwise, i.e., using Proposition 549,

$$\exists x \in \mathcal{F}(S) \text{ such that } x \notin S$$

Then, by definition of $\mathcal{F}(S)$,

$$\forall r \in \mathbb{R}_{++}, \quad B(x,r) \cap S \neq \varnothing.$$

Since $x \notin S$, we also have

$$\forall r \in \mathbb{R}_{++}, \quad B(x,r) \cap (S \setminus \{x\}) \neq \varnothing,$$

i.e., $x \in D(S)$ and $x \notin S$, a contradiction. ■

Proposition 551 *$\forall S, T \subseteq X$, $S \subseteq T \Rightarrow D(S) \subseteq D(T)$.*

Proof. Take $x \in D(S)$. Then

$$\forall r \in \mathbb{R}_{++}, \quad (S \setminus \{x\}) \cap B(x,r) \neq \varnothing. \tag{11.37}$$

Since $S \subseteq T$, we also have

$$(T \setminus \{x\}) \cap B(x,r) \supseteq (S \setminus \{x\}) \cap B(x,r). \tag{11.38}$$

From (11.37) and (11.38), we get $x \in D(T)$. ■

Proposition 552 *$S \cup D(S)$ is a closed set.*

Proof. Take $x \in (S \cup D(S))^C$ i.e., $x \notin S$ and $x \notin D(S)$. We want to show that

$$\exists r \in \mathbb{R}_{++} \text{ such that } \ B(x,r) \cap (S \cup D(S)) = \varnothing,$$

i.e.,

$$\exists r \in \mathbb{R}_{++} \text{ such that } \ (B(x,r) \cap S) \cup (B(x,r) \cap D(S)) = \varnothing,$$

Since $x \notin D(S)$, $\exists r \in \mathbb{R}_{++}$ such that $B(x,r) \cap (S \setminus \{x\}) = \varnothing$. Since $x \notin S$, we also have that

$$\exists r \in \mathbb{R}_{++} \text{ such that } \ S \cap B(x,r) = \varnothing. \tag{11.39}$$

We are then left with showing that $B(x,r) \cap D(S) = \varnothing$. If $y \in B(x,r)$, then from (11.39), $y \notin S$ and $B(x,r) \cap S \setminus \{y\} = \varnothing$, i.e., $y \notin D(S)$,i.e., $B(x,r) \cap D(S) = \varnothing$, as desired. ■

Proposition 553 Cl $(S) = S \cup D(S)$.

Proof. [$\supseteq$]
Since
$$S \subseteq \text{Cl }(S) \tag{11.40}$$
from Proposition 551,
$$D(S) \subseteq D(\text{Cl }(S)). \tag{11.41}$$
Since Cl (S) is closed, from Proposition 550,
$$D(\text{Cl }(S)) \subseteq \text{Cl }(S) \tag{11.42}$$
From (11.40) , and (11.41) , (11.42), we get
$$S \cup D(S) \subseteq \text{Cl }(S)$$
[$\subseteq$]
Since, from Proposition 552, $S \cup D(S)$ is closed and contains S, then by definition of Cl (S),
$$\text{Cl }(S) \subseteq S \cup D(S).$$
■

To proceed in our analysis, we need the following result.

Lemma 554 *For any metric space (X, d) and any $S \subseteq X$, we have that*
1. $X = Int\ \ S \cup \mathcal{F}(S) \cup Int\ S^C$, and
2. $(Int\ \ S \cup \mathcal{F}(S))^C = Int\ S^C$.

Proof. If either $S = \varnothing$ or $S = X$, the results are trivial. Otherwise, observe that either $x \in S$ or $x \in X \setminus S$.
1. If $x \in S$,then
either $\exists r \in \mathbb{R}_{++}$ such that $B(x, r) \subseteq S$ and then $x \in Int\ S$,
or $\forall r \in \mathbb{R}_{++}$, $B(x, r) \cap S^C \neq \varnothing$ and then $x \in \mathcal{F}(S)$.
Similarly, if $x \in X \setminus S$,then
either $\exists r' \in \mathbb{R}_{++}$ such that $B(x, r') \subseteq X \setminus S$ and then $x \in Int\ (X \setminus S)$,
or $\forall r' \in \mathbb{R}_{++}$, $B(x, r') \cap S \neq \varnothing$ and then $x \in \mathcal{F}(S)$.
2. By definition of Interior and Boundary of a set, $(Int\ \ S \cup \mathcal{F}(S)) \cap Int\ S^C = \varnothing$.
Now, for arbitrary sets $A, B \subseteq X$ such that $A \cup B = X$ and $A \cap B = \varnothing$, we have what follow:
$A \cup B = X \Leftrightarrow (A \cup B)^C = X^C \Leftrightarrow A^C \cap B^C = \varnothing$, and from Remark 547, $B^C \subseteq A$;
$A \cap B = \varnothing \Leftrightarrow A \cap (B^C)^C = \varnothing \Rightarrow A \subseteq B^C$.
Therefore we can the desired result. ■

Proposition 555 Cl $(S) = Int\ S \cup \mathcal{F}(S)$.

Proof. From Lemma 554, it is enough to show that
$$(\text{Cl }(S))^C = Int\ S^C.$$
[$\supseteq$]
Take $x \in Int\ S^C$. Then, $\exists r \in \mathbb{R}_{++}$ such that $B(X, r) \subseteq S^C$ and therefore $B(x, r) \cap S = \varnothing$ and, since $x \notin S$,
$$B(x, r) \cap (S \backslash \{x\}) = \varnothing.$$
Then $x \notin S$ and $x \notin D(S)$, i.e.,
$$x \notin S \cup D(S) = \text{Cl }(S)$$
where last equality follows from Proposition 553. In other words, $x \in (\text{Cl }(S))^C$.
[$\subseteq$]
Take $x \in (\text{Cl }(S))^C = (D(S) \cup S)^C$. Since $x \notin D(S)$,
$$\exists r \in \mathbb{R}_{++} \text{ such that } \ (S \backslash \{x\}) \cap B(x, r) = \varnothing \tag{11.43}$$

Since $x \notin S$,

$$\exists r \in \mathbb{R}_{++} \text{ such that } \ S \cap B(x, r) = \varnothing \tag{11.44}$$

i.e.,

$$\exists r \in \mathbb{R}_{++} \text{ such that } B(x, r) \subseteq S^C \tag{11.45}$$

and $x \in Int\ S^C$. ■

Definition 556 *$x \in X$ is an adherent point for S if $\forall r \in \mathbb{R}_{++}, B(x, r) \cap S \neq \varnothing$ and*

$$Ad(S) := \{x \in X : \forall r \in \mathbb{R}_{++}, B(x, r) \cap S \neq \varnothing\}$$

Corollary 557 *1.* Cl $(S) = Ad(S)$.
2.A set S is closed $\Leftrightarrow Ad(S) = S$.

Proof. 1.
[$\subseteq$]
$x \in$ Cl $(S) \Rightarrow \langle x \in IntS$ or $\mathcal{F}(S)\rangle$ and in both cases the desired conclusion is insured.
[$\supseteq$]
If $x \in S$, then, by definition of closure, $x \in$ Cl (S). If $x \notin S$,then $S = S \backslash \{x\}$ and, from the assumption, $\forall r \in \mathbb{R}_{++}, B(x, r) \cap (S \backslash \{x\}) \neq \varnothing$, i.e., $x \in D(S)$ which is contained in Cl $(S)$from Proposition 553.
2. It follows from 1. above and Proposition 473.2. ■

Proposition 558 *$x \in$ Cl $(S) \Leftrightarrow \exists (x_n)_{n \in \mathbb{N}}$ in S converging to x.*

Proof. [$\Rightarrow$]
From Corollary 557, if $x \in$ Cl (S) then $\forall n \in \mathbb{N}$, we can take $x_n \in B\left(x, \frac{1}{n}\right) \cap S$. Then $d(x, x_n) < \frac{1}{n}$ and $\lim_{n \to +\infty} d(x, x_n) = 0$.
[$\Leftarrow$]
By definition of convergence,

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ such that } \forall n > n_\varepsilon, \quad d(x_n, x) < \varepsilon \text{ or } x_n \in B(x, \varepsilon)$$

or

$$\forall \varepsilon > 0, \ B(x, \varepsilon) \cap S \supseteq \{x_n : n > n_\varepsilon\}$$

and

$$\forall \varepsilon > 0, \ B(x, \varepsilon) \cap S \neq \varnothing$$

i.e., $x \in Ad(S)$, and from the Corollary 557.1, the desired result follows. ■

Proposition 559 *S is closed $\Leftrightarrow$ any convergent sequence $(x_n)_{n \in \mathbb{N}}$ with elements in S converges to an element of S.*

Proof. We are going to show that S is closed using Proposition 550, i.e., S is closed $\Leftrightarrow D(S) \subseteq S$. We want to show that

$$\langle D(S) \subseteq S \rangle \Leftrightarrow \left\langle \left\langle \begin{array}{ccc} (x_n)_{n \in \mathbb{N}} \text{ is such that} & 1.\ \forall n \in \mathbb{N}, x_n \in S, & \text{and} \\ & 2.\ x_n \to x_0 & \end{array} \right\rangle \Rightarrow x_0 \in S \right\rangle,$$

[$\Rightarrow$]
Suppose otherwise, i.e., there exists $(x_n)_{n \in \mathbb{N}}$ such that 1. $\forall n \in \mathbb{N}, x_n \in S$. and 2.$x_n \to x_0$, but $x_0 \notin S$.
By definition of convergent sequence, we have

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, \quad d(x_n, x_0) < \varepsilon$$

and, since $\forall n \in \mathbb{N}, x_n \in S$,

$$\{x_n : n > n_0\} \subseteq B(x_0, \varepsilon) \cap (S \backslash \{x_0\})$$

Then,

$$\forall \varepsilon > 0, B(x_0, \varepsilon) \cap (S \backslash \{x_0\}) \neq \varnothing$$

and therefore $x_0 \in D(S)$ while $x_0 \notin S$, contradicting the fact that S is closed.
[$\Leftarrow$]
Suppose otherwise, i.e., $\exists\ x_0 \in D(S)$ and $x_0 \notin S$. We are going to construct a convergent sequence $(x_n)_{n\in\mathbb{N}}$ with elements in S which converges to x_0 (a point not belonging to S).
From the definition of accumulation point,

$$\forall n \in \mathbb{N}, \quad (S \backslash \{x_0\}) \cap B\left(x, \frac{1}{n}\right) \neq \varnothing.$$

Then, we can take $x_n \in (S \backslash \{x\}) \cap B\left(x, \frac{1}{n}\right)$, and since $d(x_n, x_0) < \frac{1}{n}$, we have that $d(x_n, x_0) \to 0$. ■
Summarizing, the following statements are equivalent:

1. S is open (i.e., $S \subseteq Int\ S$)
2. S^C is closed,
3. $S \cap \mathcal{F}(S) = \varnothing$,

and the following statements are equivalent:

1. S is closed,
2. S^C is open,
3. $\mathcal{F}(S) \subseteq S$,
4. $S = \mathrm{Cl}\ (S)$.
5. $D(S) \subseteq S$,
6. $Ad(S) = S$,
7. any convergent sequence $(x_n)_{n\in\mathbb{N}}$ with elements in S converges to an element of S.

11.8.2 The set of extended real numbers

Definition 560 $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ *is called set of extended real numbers.*[11]

We extend the standard order $\geq$ on $\mathbb{R}$ letting

$$+\infty > -\infty$$

and

$$\forall t \in \mathbb{R}, \qquad +\infty > t > -\infty.$$

Then the relation $>$ on $\overline{\mathbb{R}}$ is transitive, reflexive, antisymmetric and complete, i.e., $\left(\overline{\mathbb{R}}, >\right)$ is a complete partial order.

Definition 561 *Given* $A \subseteq \overline{\mathbb{R}}$, *then*

1. $\sup A = a$ *with* $a \in \mathbb{R}$ *means (i)* $\forall x \in A$, $a \geq x$, *and (ii)* $\forall \varepsilon > 0$, $\exists x^* \in A$ *such that* $a - \varepsilon < x^*$;
2. $\sup A = +\infty$ *means that* A *is unbounded above, i.e.,* $\exists (a_n)_{n\in\mathbb{N}} \in A^\infty$ *such that* $a_n \to +\infty$;
3. $\sup A = -\infty$ *means* $\forall a \in A$, $a = -\infty$, *i.e.,* $A = \{-\infty\}$.

[11] In this Appendix, I follow Ok (2007), page 45, 122 and 158.

The notation on interval used on $\mathbb{R}$ extends to $\overline{\mathbb{R}}$ as follows. For example

$$\forall a > -\infty, \quad [-\infty, a] := \left\{t \in \overline{\mathbb{R}} : -\infty \leq t \leq a\right\}.$$

We also extend addition and multiplication as follows. $\forall t \in \mathbb{R}$

$$t + (+\infty) := (+\infty) + t = +\infty,$$

$$t + (-\infty) := (-\infty) + t = -\infty$$

$$(+\infty) + (+\infty) = +\infty,$$

$$(-\infty) + (-\infty) = -\infty,$$

$$t \cdot (+\infty) := (+\infty) \cdot t = \begin{cases} +\infty & \text{if} \quad t \in (0, +\infty] \\ -\infty & \text{if} \quad t \in [-\infty, 0) \end{cases}$$

$$t \cdot (-\infty) := (-\infty) \cdot t = \begin{cases} -\infty & \text{if} \quad t \in (0, +\infty] \\ +\infty & \text{if} \quad t \in [-\infty, 0) \end{cases}$$

The following expressions are not defined

$$(+\infty) + (-\infty)$$

$$(-\infty) + (+\infty)$$

$$(+\infty) \cdot 0$$

$$0 \cdot (+\infty).$$

Therefore $\overline{\mathbb{R}}$ is not a field. We want now to put a metric on $\overline{\mathbb{R}}$.

Definition 562

$$f : \overline{\mathbb{R}} \to [-1, +1], \quad f(t) = \begin{cases} -1 & \textit{if} \quad t = -\infty \\ \frac{t}{1+|t|} & \textit{if} \quad t \in \mathbb{R} \\ +1 & \textit{if} \quad t = +\infty \end{cases}$$

$$d^* : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \mathbb{R}, \quad d(x, y) = |f(x) - f(y)|.$$

Exercise 563 $\left(\overline{\mathbb{R}}, d^*\right)$ *is a metric space and* d^* *is a bounded metric.*

Exercise 564 *Show that for any* $a \in \overline{\mathbb{R}}$, *if* $\forall n \in \mathbb{N}$, $a_n = a$, *then* $\lim_{n\to+\infty} a_n = a$.

Exercise 565 *Discuss the following statements:*
$\left(\overline{\mathbb{R}}, d^*\right)$ *is homeomorphic to* $[-1, +1]$ *, i.e., there exists a continuous invertible function* $g : \overline{\mathbb{R}} \to [-1, +1]$ *with continuous inverse.*
$\left(\overline{\mathbb{R}}, d^*\right)$ *is a compact metric space.*

11.8.3 Limsup and Liminf

The[12] following two simple propositions give some motivation for the introduction of the concepts of lim sup and lim inf of a real sequence.

Definition 566 *Given a sequence* $(s_n)_n \in \mathbb{R}^\infty$, *for any* $N \in \mathbb{N}$, *define*

$$u_N := \inf\{s_n : n > N\} \in \overline{\mathbb{R}},$$

and

$$v_N := \sup\{s_n : n > N\} \in \overline{\mathbb{R}}.$$

[12] In this Section, I follow Ross (1980), pages 44-60 and Francini (2004).

Proposition 567 *The sequences* $(u_N)_N$ *is increasing and the sequence* $(v_N)_N$ *is decreasing.*

Proof. If $N' < N''$, then $\{s_n : n > N'\} \supseteq \{s_n : n > N''\}$ and we have that

$$N' < N'' \Rightarrow u_{N'} := \inf\{s_n : n > N'\} \leq \inf\{s_n : n > N''\} := u_{N''}$$

or

$$u_1 \leq u_2 \leq ... \leq u_n \leq ...$$

and

$$N' < N'' \Rightarrow v_{N'} := \sup\{s_n : n > N'\} \geq \sup\{s_n : n > N''\} := v_{N''},$$

or

$$v_1 \geq v_2 \geq ... \geq v_n \geq ...$$

as desired. ■

Proposition 568 $\lim u_n$ *and* $\lim v_n$ *exist (finite or infinite) and*

$$\lim_{N\to+\infty} \inf\{s_n : n > N\} \leq \lim_{N\to+\infty} \sup\{s_n : n > N\}.$$

Proof. $\lim u_n$ and $\lim v_n$ exist (finite or infinite) from the above Proposition 567 and `Corollary 10.5 page 43 in Ross (1980)` . Moreover, since

$$\inf\{s_n : n > N\} \leq \sup\{s_n : n > N\},$$

"taking limits of both sides", we get the desired results (in fact, we are using `second corollary, page 262, in Marcellini and Sbordone)`. ■

Thanks to Proposition 568, we can then give the following

Definition 569 *Given a sequence* $(s_n)_n \in \mathbb{R}^\infty$*, define*

$$u := \lim_{N\to+\infty} u_N \qquad \text{and} \qquad \lim_{N\to+\infty} v_N := v.$$

Proposition 570 *Given a sequence* $(s_n)_n \in \mathbb{R}^\infty$*, assume that*

$$\lim_{n\to+\infty} s_n = s \in \overline{\mathbb{R}}. \tag{11.46}$$

Then,
1. for any $N \in \mathbb{N}$*,*

$$u_N \leq s \leq v_N,$$

2.

$$u \leq s \leq v,$$

or

$$\lim_{N\to+\infty} \inf\{s_n : n > N\} \leq \lim_{n\to+\infty} s_n = s \leq \lim_{N\to+\infty} \sup\{s_n : n > N\}.$$

Proof. 1.
For any $n \in \mathbb{N}$, define[13] $s'_n := s_{N+n}$ and the sequence $(s'_n)_n$, i.e., the subsequence of $(s_n)_n$ "starting from N" . From (11.46) and Proposition `see my proposition (cr: sub-seq) - compare my proof with that one of Proposition 11.2 in Ross or in any other Italian calculus 1 book - or just give the proof`

$$\lim_{n\to+\infty} s'_n = s \in \overline{\mathbb{R}}. \tag{11.47}$$

From the definitions of inf and sup and the fact that $s'_n := s_{N+n} \in \{s_n : n > N\}$

$$\forall n \in \mathbb{N}, \quad u_N \leq s'_n \leq u_N. \tag{11.48}$$

Taking limits of all sides of (11.48) with respect to n, using (11.47), and Proposition 490, we get

$$u_N \leq s \leq u_N, \tag{11.49}$$

as desired.
2. Taking limits of all sides of (11.49) with respect to N, we get the desired result. ■

[13] Ross defines $\mathbb{N} := \{1, ..., n, ...\}$.

Definition 571 ***(Definition of limsup and liminf)*** *Given a sequence* $(s_n)_n \in \mathbb{R}^\infty$,

$$\lim_{n\to+\infty} \sup s_n := \begin{cases} \lim_{N\to+\infty} \sup\{s_n : n > N\} & if \quad (s_n)_n \text{ is bounded above} \\ +\infty & if \quad (s_n)_n \text{ is unbounded above} \end{cases}$$

$$\lim_{n\to+\infty} \inf s_n := \begin{cases} \lim_{N\to+\infty} \inf\{s_n : n > N\} & if \quad (s_n)_n \text{ is bounded below} \\ -\infty & if \quad (s_n)_n \text{ is unbounded below} \end{cases}$$

Remark 572 *1. In the above definitions, we have to distinguish two cases because if, say,* $(s_n)_n$ *is unbounded above, then* $\sup\{s_n : n > N\} = +\infty$ *and the limit in the first line of the definition of* $\lim\sup$ *is not defined.*

2. If $(s_n)_n$ *is bounded above, then* $\lim_{n\to+\infty} \sup s_n \in \mathbb{R}$*: by assumption* $\exists k \in \mathbb{R}$ *such that* $\forall n \in \mathbb{N}$, $s_n \le k$. *Then,*

$$\forall n \in \mathbb{N}, \qquad \sup\{s_n : n > N\} \le k \in \mathbb{R},$$

and taking limit of both sides with respect to N*, we get*

$$\lim_{N\to+\infty} \sup\{s_n : n > N\} \le k < +\infty.$$

Similarly, if $(s_n)_n$ *is bounded below, then* $\lim_{n\to+\infty} \inf s_n \in \mathbb{R}$.

Proposition 573 *Given a sequence* $(s_n)_n \in \mathbb{R}^\infty$,

$$\lim_{n\to+\infty} \inf s_n \le \lim_{n\to+\infty} \sup s_n.$$

Proof. Let's summarize the possible cases consistently with Definition 571 and the proof of the desired result as follows.

		$\lim_{n\to+\infty} \inf s_n$	$\lim_{n\to+\infty} \sup s_n$	desired result
$(s_n)_n$ is bdd below,	$(s_n)_n$ is bdd above	s	s'	Proposition 568
$(s_n)_n$ is bdd below,	$(s_n)_n$ is unbdd above	s	$+\infty$	Definition of $(\overline{\mathbb{R}}, \ge)$
$(s_n)_n$ is unbdd below,	$(s_n)_n$ is bdd above	$-\infty$	s'	Definition of $(\overline{\mathbb{R}}, \ge)$
$(s_n)_n$ is unbdd below,	$(s_n)_n$ is unbdd above	$-\infty$	$+\infty$	Definition of $(\overline{\mathbb{R}}, \ge)$

■

Proposition 574 *Given a sequence* $(s_n)_n \in \mathbb{R}^\infty$,

$$\left\langle \lim_{n\to+\infty} s_n = s \in \overline{\mathbb{R}} \right\rangle \Leftrightarrow \left\langle \lim_{n\to+\infty} \sup s_n = \lim_{n\to+\infty} \inf s_n := s \in \overline{\mathbb{R}} \right\rangle.$$

Proof. $[\Rightarrow]$

Case 1. $\lim_{n\to+\infty} s_n = +\infty$.

By assumption,

$$\forall M \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that } \forall n > N, \quad s_n > M. \tag{11.50}$$

Then, by definition of inf,

$$u_N := \inf\{s_n : n > N\} \ge M,$$

and from Proposition 567,

$$\forall n > N, \; u_n \ge u_N.$$

Summarizing,

$$\forall M \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that } \forall n > N, \; u_n \ge M,$$

i.e.,

$$\lim_{n\to+\infty} \inf s_n = +\infty. \tag{11.51}$$

We have now to prove that $\lim\sup s_n = +\infty$.

First proof. It follows from Proposition 573 and (11.51).

Second proof. Again from (11.50), we get

$$v_N := \sup\{s_n : n > N\} \geq M,$$

and

$$\forall N' > N, \quad s_{N'+1} \in \{s_n : n > N'\} \text{ and } s_{N'+1} > M.$$

Therefore,

$$\forall N' > N, \; v_{N'} = \sup\{s_n : n > N'\} \geq M.$$

Summarizing,

$$\forall M \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that } \forall N' > N, \; v_{N'} \geq M,$$

i.e.,

$$\lim_{n\to+\infty} \sup s_n = +\infty.$$

Case 2. $\lim_{n\to+\infty} s_n = -\infty$.

Similar to Case 1.

Case 3. $\lim_{n\to+\infty} s_n = s \in \mathbb{R}$.

By assumption,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n > N, \; s_n < s + \varepsilon.$$

Then

$$v_N := \sup\{s_n : n > N\} \leq s + \varepsilon.$$

From, Proposition 567,

$$\forall n > N, \quad v_n \leq v_N \leq s + \varepsilon.$$

Taking limits of both sides, we get

$$\forall \varepsilon > 0, \quad \lim_{n\to+\infty} \sup s_n \leq s + \varepsilon,$$

and therefore

$$\lim_{n\to+\infty} \sup s_n \leq s. \tag{11.52}$$

Similarly, by assumption,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n > N, \; s_n > s + \varepsilon.$$

Then

$$u_N := \inf\{s_n : n > N\} \geq s + \varepsilon.$$

From, Proposition 567,

$$\forall n > N, \quad u_n \geq u_N \geq s + \varepsilon.$$

Taking limits of both sides, we get

$$\forall \varepsilon > 0, \quad \lim_{n\to+\infty} \inf s_n \geq s + \varepsilon,$$

and therefore

$$\lim_{n\to+\infty} \inf s_n \geq s. \tag{11.53}$$

From $(11.52), (11.53)$ and Remark 573, the desired result follows.

[$\Leftarrow$]

Case 1. $\lim_{n\to+\infty} \sup s_n = \lim_{n\to+\infty} \inf s_n = +\infty$.

$\lim_{n\to+\infty} \inf s_n = +\infty$ means that

$$\forall M \in \mathbb{N} \quad \exists N' \in \mathbb{N} \text{ such that } \forall N > N', \quad \inf\{s_n : n > N\} > M$$

and therefore
$$\forall M \in \mathbb{N} \quad \exists N' \in \mathbb{N} \text{ such that } \forall n > N', \quad s_n > M,$$
i.e., the desired result.
Case 2. $\lim_{n\to+\infty} \sup s_n = \lim_{n\to+\infty} \inf s_n = -\infty$.
Similar to the proof in Case 1, using the fact that $\lim_{n\to+\infty} \sup s_n = -\infty$.
Case 3. $\lim_{n\to+\infty} \sup s_n = \lim_{n\to+\infty} \inf s_n = s \in \mathbb{R}$.
$\lim_{n\to+\infty} \sup s_n = s \in \mathbb{R}$ implies that

$$\forall \varepsilon > 0, \exists N_1 \in \mathbb{N} \text{ such that } \forall N > N_1, \ \sup \{s_n : n > N\} < s + \varepsilon$$

and, by definition of sup,

$$\forall \varepsilon > 0, \exists N_1 \in \mathbb{N} \text{ such that } \forall n > N_1, \ s_n < s + \varepsilon.$$

$\lim_{n\to+\infty} \inf s_n = s \in \mathbb{R}$ implies that

$$\forall \varepsilon > 0, \exists N_2 \in \mathbb{N} \text{ such that } \forall N > N_2, \ \inf \{s_n : n > N\} > s - \varepsilon$$

and, by definition of inf,

$$\forall \varepsilon > 0, \exists N_2 \in \mathbb{N} \text{ such that } \forall n > N_2, \ s_n > s - \varepsilon.$$

Summarizing,

$$\forall \varepsilon > 0, \exists N^* = \max \{N_1, N_2\} \text{ such that } \forall n > N^*, \quad s - \varepsilon < s < s + \varepsilon,$$

as desired. ∎

Proposition 575 *Every sequence has a monotone subsequence.*

Proof. Given a sequence $(s_n)_n \in \mathbb{R}^\infty$, we say that s_m is a dominant term if $\forall n > m$, $s_m > s_n$. We then distinguish two cases.
Case 1. The set of dominant terms in $(s_n)_n$ has infinite cardinality.
Consider the subsequence $(s_{n_k})_k$ whose terms are dominant terms. Then, $\forall k \in \mathbb{N}$, $s_{n_k} > s_{n_k+1}$, i.e., $(s_{n_k})_k$ is a *decreasing* subsequence.
Case 2. The set of dominant terms in $(s_n)_n$ has finite cardinality.
Let that set be $\{m_1, ..., m_K\}$. Then,

$$\forall n_1 > m_K, \ \exists \ n_2 > n_1 \text{ such that } s_{n_2} \geq s_{n_1} \tag{11.54}$$

Since s_{n_2} is not a dominant term, we also have that

$$\exists \ n_3 > n_2 \text{ such that } s_{n_3} \geq s_{n_2}.$$

Iterating this procedure we can construct a sequence of natural number

$$n_1 < n_2 < ... < n_k < ...$$

and of associated terms in the starting sequence

$$s_{n_1} \leq s_{n_2} \leq ... \leq s_{n_k} \leq ...,$$

i.e., we constructed a *nondecreasing* subsequence of $(s_n)_n$. ∎

Lemma 576 *If $(s_n)_n$ is nonincreasing and $\lim_{n\to+\infty} s_n = s$, then $\forall n \in \mathbb{N}$, $s_n \geq s$.*

Proof. Suppose otherwise, i.e., $\exists n^* \in \mathbb{N}$ such that

$$s_{n^*} < s.$$

Since $(s_n)_n$ is nonincreasing, then $\forall n > n^*$, we have that $s_n \leq s_{n_*}$. Then taking limits of both sides for $n \to +\infty$, we get
$$\lim_{n\to+\infty} s_n = s \leq s_{n^*},$$
a contradiction. ∎

Proposition 577 *Let $(s_n)_n \in \mathbb{R}^\infty$ be given. There exist monotonic subsequence $(s_{n_k})_k$ and $(s_{n_l})_l$ such that*

$$1. \quad \lim_{k\to+\infty} s_{n_k} = \lim_{n\to+\infty} \sup s_n$$

and

$$2. \quad \lim_{l\to+\infty} s_{n_l} = \lim_{n\to+\infty} \inf s_n.$$

Proof. 1.

We are going to consider Cases 1 and 2 in Proposition 575. In Case 1, the required here subsequence is just the sequence constructed there; in Case 2, the required here subsequence is a subsequence of the subsequence constructed there.

Case 1. The set of dominant terms in $(s_n)_n$ has infinite cardinality.

Consider the decreasing subsequence of dominant terms $(s_{n_k})_k = (s_{n_1}, s_{n_2}, ..., s_{n_k}, s_{n_{k+1}}, ...)$ constructed in Proposition 575. Then

$$s_{n_k} = \sup\left\{s_{n_{k'}} : k' \geq k\right\} \overset{(1)}{=} \sup\{s_n : n \geq n_k\} = \sup\{s_n : n > n_k - 1\} := v_{n_k - 1}$$

where (1) follows from the fact that any s_{n_k} is a dominant term. Taking limits for $k \to +\infty$, we get

$$\lim_{k\to+\infty} s_{n_k} = \lim_{k\to+\infty} v_{n_k-1} = \lim_{N\to+\infty} v_N = \lim_{n\to+\infty} \sup s_n.$$

Case 2. The set of dominant terms in $(s_n)_n$ has finite cardinality.

Subcase 2a. $v = -\infty$.

From Proposition 574, part [$\Leftarrow$], Case 2 the result follows - the subsequence of $(s_n)_n$ being the sequence itself.

Subcase 2a. $v \neq -\infty$.

Construct a sequence $(t_n)_n = \mathbb{R}^\infty$ as follows: $\forall n \in \mathbb{N}$,

$$t_n = \begin{cases} v - \frac{1}{n} & if \quad v \neq +\infty \\ n & if \quad v = +\infty \end{cases}.$$

By construction the above sequence has the following properties: it is increasing, $\forall n \in \mathbb{N}$, $t_n < v$ and $\lim_{n\to+\infty} t_n = v$. We then have

$$\forall N \in \mathbb{N}, \; t_N < v \leq v_N := \sup\{s_n : n > N\}, \tag{11.55}$$

where the last inequality comes from Proposition 567 and Lemma 576. Then from definition of sup,

$$\forall N \in \mathbb{N}, \forall n > N, s_n \geq t_N. \tag{11.56}$$

Then, consider the nondecreasing subsequence $(s_{n_k})_k$ constructed in Proposition 575, we have

$$\forall n_1 > K, \exists \; n_2 > n_1 \text{ such that } s_{n_2} \geq s_{n_1} \geq t_{n_1 - 1},$$

and from Therefore, we can choose a subsequence $(s_{n_k})_k$ of $(s_n)_n$ such that

$$s_{n_1} \leq s_{n_2} \leq ... \leq s_{n_k} \leq ...,$$

and, from (11.56),

$$\forall k \in \mathbb{N}, \quad s_{n_k} \geq t_{n_{k-1}} \tag{11.57}$$

Now,

$$v_{n_{k+1}-1} := \sup\{s_n : n > n_{k+1} - 1\} = \sup\{s_n : n \geq n_{k+1}\} = \sup\left\{s_{n_{k+1}}, s_{n_{k+1}+1}, ...\right\} \geq s_{n_{k+1}} \overset{(11.57)}{\geq} t_{n_k}$$

Therefore

$$v = \lim_{N\to+\infty} t_N = \lim_{k\to+\infty} t_{n_k} \leq \lim_{k\to+\infty} s_{n_{k+1}} = \lim_{n\to+\infty} s_n \leq \lim_{k\to+\infty} v_{n_{k+1}-1} = \lim_{N\to+\infty} v_N = v,$$

as desired.

2. The proof is similar to that one above, and it can also obtained directly from it using `Exercise 11.8 in Ross, i.e.,` Lemma 582 below and the Second characterization of liminf and limsup. ■

Proposition 578 *Every bounded sequence has a convergent subsequence.*

Proof. Given an arbitrary sequence $(s_n)_n$, from Proposition 577, it does have a monotonic subsequence. That subsequence is a bounded monotone sequence and therefore, from `Theorem 10.2 Ross,` it does converge. ∎

Definition 579 *A subsequential limit of a sequence $(s_n)_n$ is an element of $\overline{\mathbb{R}}$ which is the limit of a subsequence of $(s_n)_n$.*

Proposition 580 ***(First characterization of limsup and liminf)*** *Let S be the set of subsequential limits of a sequence $(s_n)_n$. Then,*

1. $S \neq \varnothing$,

2.

$$\sup S = \limsup s_n \text{ and } \inf S = \liminf s_n,$$

3. $\lim_{n\to+\infty} s_n$ exists $\Leftrightarrow \#S = 1$.

Proof. 1.

It follows from Proposition 577.

2.

Take $t \in S$; then there exists a subsequence $(s_{n_k})_k$ of $(s_n)_n$ such that $\lim_{k\to+\infty} s_{n_k} = t$. From Proposition 574, we have that

$$t = \lim_{k\to+\infty} \inf s_{n_k} = \lim_{k\to+\infty} \sup s_{n_k}. \tag{11.58}$$

Since

$$\forall N \in \mathbb{N}, \quad \{s_{n_k} : k > N\} \subseteq \{s_n : n > N\},$$

we have

$$\inf \{s_n : n > N\} \leq \inf \{s_{n_k} : k > N\} \leq \sup \{s_{n_k} : k > N\} \leq \sup \{s_n : n > N\},$$

and taking limits for $N \to +\infty$, we get

$$\lim_{n\to\infty} \inf s_n \leq \lim_{k\to\infty} \inf s_{n_k} \leq \lim_{k\to\infty} \sup s_{n_k} \leq \lim_{n\to\infty} \sup s_n.$$

Then, from (11.58) and given the arbitrary choice of t, we have

$$\forall t \in S, \ \lim_{n\to\infty} \inf s_n \leq \lim_{k\to\infty} \inf s_{n_k} = t = \lim_{k\to\infty} \sup s_{n_k} \leq \lim_{n\to\infty} \sup s_n.$$

By definition of inf and sup, we then have

$$t_1 := \lim_{n\to\infty} \inf s_n \leq \inf S \leq \sup S \leq \lim_{n\to\infty} \sup s_n := t_2$$

From Proposition 577, we have $t_1, t_2 \in S$ and therefore

$$\lim_{n\to\infty} \inf s_n = \inf S \quad \text{ and } \sup S = \lim_{n\to\infty} \sup s_n.$$

3.

$\lim_{n\to+\infty} s_n$ exists $\Leftrightarrow \#S = 1$.

[⇒]

Since $(s_n)_n$ has a limit, each subsequence has the same limit, from Proposition ...." `my proposition and compare my proof with that one of Proposition 11.2 in Ross` ".

[⇐]

Let t^* be the unique element in S, i.e., any subsequence of (s_n) converges to t^*. Then, from Proposition 577,

$$t^* = \lim_{n\to+\infty} \sup s_n = \lim_{n\to+\infty} \inf s_n,$$

and from Proposition 574, the desired result follows. ∎

Proposition 581 ***(Second characterization of limsup and liminf)***
1.

$$\lim_{n\to+\infty}\inf s_n = \sup_{N\in\mathbb{N}}\inf\{s_n : n > N\};$$

2.

$$\lim_{n\to+\infty}\sup s_n = \inf_{N\in\mathbb{N}}\sup\{s_n : n > N\}.$$

Proof. 1.
Case a. $(u_N)_n$ is bounded above.
Since $(u_N)_n$ is increasing and bounded above, from `Theorem 10.2 in Ross, with my pencil specification,`

$$\lim_{N\to+\infty} u_N = \sup\{s_n : n > N\},$$

as desired.
Case b. $(u_N)_n$ is unbounded above.
Since $(u_N)_n$ is unbounded above, then $\sup_{n\in\mathbb{N}} u_n = +\infty$. Since $(u_N)_n$ is increasing and bounded above, from `Theorem 10.4(i) in Ross,`the desired result follows.
2.
Similar to 1. ■

Lemma 582

$$\lim_{n\to+\infty}\inf s_n = -\lim_{n\to+\infty}\sup(-s_n)$$

Proof. Observe preliminarily that, given $S \subseteq \overline{\mathbb{R}}$,

$$\sup S = -\inf(-S), \tag{11.59}$$

and

$$\inf S = -\sup(-S). \tag{11.60}$$

Then, from (11.59) ,we have

$$v_N := \sup\{s_n : n > N\} = -\inf\{-s_n : n > N\} \tag{11.61}$$

Moreover,

$$\inf\{v_N : n\in\mathbb{N}\} = \inf_{N\in\mathbb{N}}\sup\{s_n : n > N\} \overset{\text{Prop. 581}}{=} \lim_{n\to+\infty}\sup s_n,$$

and

$$\inf\{v_N : n\in\mathbb{N}\} \overset{11.60}{=} -\sup\{-v_N : n\in\mathbb{N}\} \overset{11.61}{=} -\sup\inf\{-s_n : n > N\} \overset{\text{Prop. 581}}{=} -\lim_{n\to+\infty}\inf(-s_n),$$

as desired. ■

Proposition 583 ***(Third characterization of limsup and liminf, if they are finite).*** *Let $s\in\mathbb{R}$ be given.*[14]
1.

$$\lim_{n\to+\infty}\sup s_n = s \quad\Leftrightarrow\quad \forall\varepsilon>0, \\ \cdot \quad \exists M\in\mathbb{N} \text{ such that } \forall n>M, \quad s_n \le s+\varepsilon, \quad \text{and} \\ \cdot \quad \forall n\in\mathbb{N},\ \exists n_N > N \text{ such that } \quad s_{n_N} \ge s-\varepsilon.$$

2.

$$\lim_{n\to+\infty}\inf s_n = s \quad\Leftrightarrow\quad \forall\varepsilon>0, \\ \cdot \quad \exists M\in\mathbb{N} \text{ such that } \forall n>M, \quad s_n \ge s-\varepsilon, \quad \text{and} \\ \cdot \quad \forall n\in\mathbb{N},\ \exists n_N > N \text{ such that } \quad s_{n_N} \le s+\varepsilon.$$

[14] For a wordy interpretation of the result below, see Ok (2007), page 54.

Proof. 1.
[⇒]
By assumption, $\forall \varepsilon > 0$, $\exists M' \in \mathbb{N}$ such that $\forall N > M'$, $v_N \leq s + \varepsilon$. Moreover, by definition of sup, $\forall n > M := M' + 1$, $s_n \leq \sup \{s_{n'} : n' > M\} := v_M \leq s + \varepsilon$, where the last inequality follows from the fact that $M := M' + 1$. Summarizing, $\forall \varepsilon > 0, \exists M \in \mathbb{N}$ such that $\forall n > M$, $s_n \leq s + \varepsilon$, as desired.
Moreover, from the Second characterization of limsup, i.e., Proposition 581, we have that

$$\forall N \in \mathbb{N}, \quad s = \inf_{n \in \mathbb{N}} \sup \{s_n : n > N\} \leq v_N,$$

and, since $v_N := \sup \{s_n : n > N\}$, $\exists\, s \in \{s_n : n > N\}$, say s_{n_N} with $n_N > N$ such that

$$\forall N \in \mathbb{N}, \quad s - \varepsilon \leq v_N - \varepsilon \leq s_{n_N}.$$

Summarizing,

$$\forall N \in \mathbb{N}, \ \exists n_N > N \ \text{ such that } s - \varepsilon \leq s_{n_N},$$

as desired.
[⇐]
From the first part of the assumption, $\exists M \in \mathbb{N}$ such that $\forall s \in \{s_n : n > M\}$, $s \leq s + \varepsilon$. Then $v_M := \sup \{s_n : n > M\} \leq s + \varepsilon$. Since, $(v_N)_N$ is decreasing, we have that

$$\exists M \in \mathbb{N} \text{ such that } \forall N > M, \quad v_N \leq s + \varepsilon. \tag{11.62}$$

From the second part of the assumption, $\forall N \in \mathbb{N}$, $\exists n_N > N$ such that $s_{n_N} \geq s - \varepsilon$, and therefore $v_N := \sup \{s_n : n > N\} \geq s_{n_N} \geq s - \varepsilon$. Therefore,

$$\forall N \in \mathbb{N}, \ v_N \geq s - \varepsilon. \tag{11.63}$$

From (11.62) and (11.63), the desired result follows.
2.
Similar to 1. ∎

Remark 584 *For some other results on limsup and liminf, see Ross (1980), pages 57-60 and Ok (2007), page 56. Among those results, let's state the following one:*
For any bounded sequences $(s_n)_n, (t_n)_n \in \mathbb{R}^\infty$, we have

$$\liminf s_n + \liminf t_n \leq \liminf (s_n + t_n) \leq \limsup (s_n + t_n) \leq \limsup s_n + \limsup t_n.$$

11.8.4 Norms and metrics

In these notes, a field K can be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 585 *A norm on a vector space E on a field K is a function*

$$|| \cdot || : E \to \mathbb{R}, \quad x \mapsto ||x||$$

which satisfies the following properties: $\forall x, y \in E, \forall \lambda \in K$,

1. $||x|| \geq 0$ *(non negativity),*
2. $||x + y|| \leq ||x|| + ||y||$ *(triangle inequality),*
3. $||\lambda x|| = |\lambda| \cdot ||x||$ *(homogeneity),* [15]
4. $||x|| = 0 \Rightarrow x = 0$ *(separation).*

Proposition 586 *Given a norm $|| \cdot ||$ on E, $\forall x, y \in E$*

1. $x = 0 \Rightarrow ||x|| = 0$

[15] If $\lambda \in \mathbb{R}$, $|\lambda|$ is the absolute value of λ; if $\lambda = (a, b) \in \mathbb{C}$, then $|\lambda| = \sqrt{a^2 + b^2}$.

2. $||x - y|| = ||y - x||$

3. $|(||x|| - ||y||)| \leq ||x - y||$.

Proof.

1. Since E is a vector space, $\forall x \in E$, $0x = 0$ (from Proposition 135 in Villanacci (2012)). Then

$$||0|| = ||0x|| \overset{(a)}{=} |0| \cdot ||x|| \overset{(b)}{=} 0||x|| = 0$$

where (a) follows from property 3 of norm and (b) from the definition of absolute value.

2. $||x - y|| = || - y + x|| =$ [16]$|| - y - (-x)|| \overset{(c)}{=} ||(-1)y + (-1)(-x)|| = ||(-1)(y - x)|| = | - 1| \cdot ||y - x|| = ||y - x||$
where (c) follows from Proposition 135.4 in Villanacci (2012).

3. From the definition of absolute value, we want to show that

$$-||x - y|| \leq ||x|| - ||y|| \leq ||x - y||.$$

Indeed,

$$||x|| = ||x - y + y|| \leq ||x - y|| + ||y||,$$

i.e., $||x - y|| \geq ||x|| - ||y||$, and

$$||y|| = ||y - x + x|| \leq ||y - x|| + ||x|| = ||x - y|| + ||x||,$$

i.e., $-||x - y|| \leq ||x|| - ||y||$, as desired.

■

Proposition 587 *If properties 2. and 3. in Definition 585 hold true, then property 1. in the same definition holds true.*

Proof. We want to show that

$$\langle ||x + y|| \leq ||x|| + ||y|| \wedge ||\lambda x|| = |\lambda| \cdot ||x|| \rangle$$

$$\Rightarrow \langle \forall x \in E, ||x|| \geq 0 \rangle$$

Observe that if $x = 0$, from Proposition 586.2 (which uses only property 3 of Definition 585) we have $||x|| = 0$. Then

$$\begin{array}{rcl} ||0|| & \leq & ||x - x|| \leq ||x|| + || - x|| \text{ and therefore} \\ ||x|| & \geq & -||x|| = -| - 1| \cdot ||x|| = -||x||. \end{array}$$

Now, if $||x|| < 0$, we would have a negative number strictly larger than a positive number, which is a contradiction. ■

Definition 588 *The pair $(E, ||\cdot||)$, where E is a vector space and $||\cdot||$ is a norm, is called a normed vector space.*

Remark 589 *Normed spaces are, by definition, vector space.*

Definition 590 *A seminorm is a function satisfying properties 1, 2 and 3 in Definition 585.*

Definition 591 *Given a non-empty set X, a function $d : X \times X \to \mathbb{R}$ is called a metric or a distance on X if $\forall x, y, z \in X$,*

1. $d(x, y) \geq 0$ *(non negativity),*

2. $d(x, y) = 0 \Leftrightarrow x = y$ *(coincidence),*

[16]$\forall v \in V$ $(-1)v = -v$ and $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$.

3. $d(x,y) = d(y,x)$ *(symmetry),*

4. $d(x,z) \le d(x,y) + d(y,z)$ *(triangle inequality),*

(X,d) *is called a metric space.*

Definition 592 *Given a normed vector space* $(E, ||\cdot||)$*, the metric*

$$d : E^2 \to \mathbb{R}, (x,y) \mapsto ||x-y||$$

is called the metric induced by the norm $||\cdot||$*.*

Proposition 593 *Given a normed vector space* $(E, ||\cdot||)$*,*

$$d : E \times E \to \mathbb{R}, \quad (x,y) \mapsto ||x-y||$$

is a metric and (E,d) *is a metric space.*

Proof.

1. It follows from the fact that $x, y \in E \Rightarrow x - y \in E$ and property 1 of the norm.
2. It follows from property 1 of the norm and Proposition 586.1.
3. It follows from Proposition 586.2.
4. $d(x,z) = ||x-z|| = ||(x-y)+(y-z)|| \le ||x-y|| + ||y-z|| = d(x,y) + d(y,z)$.

■

Proposition 594 *If* $||\cdot||$ *is a norm on a vector space* E *and*

$$d : E \times E \to \mathbb{R}, \quad (x,y) \mapsto ||x-y||$$

then $\forall x, y, z \in E, \forall \lambda \in K$

a. $d(x,0) = ||x||$

b. $d(x+z, y+z) = d(x,y)$ *(translation invariance)*

c. $d(\lambda x, \lambda y) = |\lambda| d(x,y)$ *(homogeneity).*

Proof.

a. $d(x,0) = ||x-0|| = ||x||$

b. $d(x+z, y+z) = ||(x+z)-(y+z)|| = ||x-y|| = d(x,y)$

c. $d(\lambda x, \lambda y) = ||\lambda x - \lambda y|| = ||\lambda(x-y)|| = |\lambda| \cdot ||x-y|| = |\lambda| d(x,y)$.

■

Proposition 595 *Let* (E,d) *be a metric space such that* d *satisfies translation invariance and homogeneity. Then*

$$n : E \to \mathbb{R}, \quad x \mapsto d(x,0)$$

is a norm and $\forall x, y \in E$*,* $n(y-x) = d(x,y)$*.*

Proof.

1. $n(x) = d(x,0) \ge 0$, where the inequality follows from property 1 in Definition 4,
2. $$n(x+y) = d(x+y,0) \overset{(a)}{=} d(x+y-y, 0-y) = d(x,-y) \overset{(b)}{\le} d(x,0) + d(0,-y) =$$
$$\overset{(c)}{=} d(x,0) + d(-y,0) \overset{(d)}{=} d(0,y) + d(0,x) = n(y) + n(x),$$
   where (a) follows from translation invariance, (b) from triangle inequality in Definition 591, (c) from symmetry in Definition 591 and (d) from homogeneity.

3.
$$n(\lambda x) = d(\lambda x, 0) = |\lambda| d(x, 0) = |\lambda| n(x),$$

4.
$$n(x) = 0 \Rightarrow d(x, 0) = 0 \Rightarrow x = 0.$$

It follows that

$$n(y - x) = d(y - x, 0) = d(y - x + x, 0 + x) = d(y, x) = d(x, y).$$

■

Remark 596 *The above Proposition suggests that the following statement is* ***false:***

Given a metric space (E, d), then $n_d : E \to \mathbb{R}, : x \mapsto d(x, 0)$ is a norm on E.

The fact that the above statement is false is verified below. Take an arbitrary vector space E with the discrete metric d,

$$d : E \times E \to \mathbb{R}, \quad d(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

First of all, let's verify that d does not satisfy (translation invariance and homogeneity), otherwise from Proposition 595, we would contradict the desired result. Indeed homogeneity fails.

Take $x \neq y$ and $\lambda = 2$ then

$$d(x, y) = 1 \quad \text{and} \quad d(\lambda x, \lambda y) = 1$$

$$|\lambda| d(x, y) = 2 \neq 1.$$

Let's now show that in the case of the discrete metric

$$n : E \to \mathbb{R}, \quad x \mapsto d(x, 0)$$

is not a norm. Take $x \neq 0$ and $\lambda = 2$ then

$$||\lambda x|| = d(\lambda x, 0) = 1$$
$$|\lambda| d(x, 0) = 2.$$

Chapter 12

Functions

12.1 Limits of functions

In what follows we take for given metric spaces (X, d) and (X', d') and sets $S \subseteq X$ and $T \subseteq X'$.

Definition 597 *Given $x_0 \in D(S)$, i.e., given an accumulation point x_0 for S, and $f : S \to T$, we write*

$$\lim_{x \to x_0} f(x) = l$$

if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in \left(B_{(X,d)}(x_0, \delta) \cap S\right) \setminus \{x_0\} \;\Rightarrow f(x) \in B_{(X',d')}(l, \varepsilon)$$

or

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that} \langle\; x \in S \quad \wedge \quad 0 < d(x, x_0) < \delta \rangle \Rightarrow d'(f(x), l) < \varepsilon$$

Proposition 598 *Given $x_0 \in D(S)$ and $f : S \to T$,*

$$\langle \lim_{x \to x_0} f(x) = l \rangle$$
$$\Leftrightarrow$$
$$\left\langle \begin{array}{l} \text{for any sequence } (x_n)_{n \in \mathbb{N}} \text{ in } S \text{ such that } \forall n \in \mathbb{N},\;\; x_n \neq x_0 \text{ and } \lim_{n \to +\infty} x_n = x_0, \\ \lim_{n \to +\infty} f(x_n) = l. \end{array} \right\rangle$$

Proof. `for the following proof see also Proposition 6.2.4, page 123 in Morris.`
[⇒]
Take

$$\text{a sequence } (x_n)_{n \in \mathbb{N}} \text{ in } S \text{ such that } \forall n \in \mathbb{N}, \;\; x_n \neq x_0 \text{ and } \lim_{n \to +\infty} x_n = x_0$$

We want to show that $\lim_{n \to +\infty} f(x_n) = l$, i.e.,

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, \;\; d(f(x_n), l) < \varepsilon$$

Since $\lim_{x \to x_0} f(x) = l$,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in S \;\; \wedge \;\; 0 < d(x, x_0) < \delta \Rightarrow d(f(x), l) < \varepsilon$$

Since $\lim_{n \to +\infty} x_n = x$

$$\forall \delta > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, \;\; 0 \overset{(*)}{<} d(x_n, x_0) < \delta$$

where $(*)$ follows from the fact that $\forall n \in \mathbb{N}$, $x_n \neq x_0$.
Therefore, combining the above results, we get

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, d(f(x_n), l) < \varepsilon$$

as desired.

[⇐]
Suppose otherwise, then

$$\begin{aligned} &\exists \varepsilon > 0 \text{ such that } \forall \delta_n = \tfrac{1}{n}, \text{ i.e., } \forall\ n \in \mathbb{N},\ \exists x_n \in S \text{ such that} \\ &x_n \in S\ \wedge 0 < d(x_n, x_0) < \tfrac{1}{n} \text{and } d(f(x), l) \geq \varepsilon. \end{aligned} \tag{12.1}$$

Consider $(x_n)_{n\in\mathbb{N}}$; then, from the above and from Proposition 488, $x_n \to x_0$, and from the above (specifically the fact that $0 < d(x_n, x_0)$), we also have that $\forall n \in \mathbb{N},\ \ x_n \neq x_0$. Then by assumption, $\lim_{n\to+\infty} f(x_n) = l$, i.e., by definition of limit,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that if } n > N, \text{ then } |f(x_n - l)| < \varepsilon,$$

contradicting (12.1). ■

Proposition 599 *(uniqueness) Given $x_0 \in D(S)$ and $f : S \to T$,*

$$\left\langle \lim_{x\to x_0} f(x) = l_1 \textit{ and } \lim_{x\to x_0} f(x) = l_2 \right\rangle \Rightarrow \langle l_1 = l_2 \rangle$$

Proof. It follows from Proposition 598 and Proposition 492. ■

Proposition 600 *Given $S \subseteq X$, $x_0 \in D(S)$ and $f, g : S \to \mathbb{R}$, , and*

$$\lim_{x\to x_0} f(x) = l \textit{ and } \lim_{x\to x_0} g(x) = m$$

1. $\lim_{x\to x_0} f(x) + g(x) = l + m$;
2. $\lim_{x\to x_0} f(x) \cdot g(x) = l \cdot m$;
3. *if* $m \neq 0$ *and* $\forall x \in S$, $g(x) \neq 0$, $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{l}{m}$.

Proof. It follows from Proposition 598 and Proposition 490. ■

12.2 Continuous Functions

Definition 601 *Given a metric space (X, d) and a set $V \subseteq X$, an open neighborhood of V is an open set containing V.*

Remark 602 *Sometimes, an open neighborhood is simply called a neighborhood.*

Definition 603 *Take $x_0 \in S$ and $f : S \to T$. Then, f is continuous at x_0 if*

$$\forall \varepsilon > 0, \exists \delta > 0 \textit{ such that } x \in \left(B_{(X.d)}(x_0, \delta) \cap S\right) \Rightarrow f(x) \in B_{(X',d')}(f(x_0), \varepsilon),$$

i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0 \textit{ such that } x \in S\ \wedge d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \varepsilon,$$

i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0,\ f\left(B_{(X,d)}(x_0, \delta) \cap S\right) \subseteq B_{(X',d')}(f(x_0), \varepsilon),$$

i.e.,

for any open neighborhood V of $f(x_0)$,
there exists an open neighborhood U of x_0 such that $f(U \cap S) \subseteq V$.

If f is continuous at x_0 for every x_0 in S, f is continuous on S.

Remark 604 *If x_0 is an isolated point of S, f is continuos at x_0. If x_0 is an accumulation point for S, f is continuous at x_0 if and only if $\lim_{x\to x_0} f(x) = f(x_0)$.*

Proposition 605 *Suppose that $Z \subseteq X''$, where (X''', d'') is a metric space and*

$$f : S \to T, \; g : W \supseteq f(S) \to Z$$

$$h : S \to Z, \; h(x) = g(f(x))$$

If f is continuous at $x_0 \in S$ and g is continuous at $f(x_0)$, then h is continuous at x_0.

Proof. Exercise (see Apostol (1974), page 79) or Ok, page 206. ■

Proposition 606 *Take $f, g : S \subseteq \mathbb{R}^n \to \mathbb{R}$. If f and g are continuous, then*

1. *$f + g$ is continuous;*
2. *$f \cdot g$ is continuous;*
3. *if $\forall x \in S$, $g(x) \neq 0$, $\frac{f}{g}$ is continuous.*

Proof. If x_0 is an isolated point of S, from Remark 604, we are done. If x_0 is an accumulation point for S, the result follows from Remark 604 and Proposition 600. ■

Proposition 607 *Let $f : S \subseteq X \to \mathbb{R}^m$, and for any $j \in \{1, ..., m\}$ $f_j : S \to \mathbb{R}$ be such that $\forall x \in S$,*

$$f(x) = (f_j(x))_{j=1}^m$$

Then,

$$\langle f \text{ is continuous} \rangle \Leftrightarrow \langle \forall j \in \{1, ..., m\}, \; f_j \text{ is continuous} \rangle$$

Proof. The proof follows the strategy used in Proposition 493. ■

Definition 608 *Given for any $i \in \{1, ..., n\}$, $S_i \subseteq \mathbb{R}$, $f : \times_{i=1}^n S_i \to \mathbb{R}$ is continuous in each variable separately if $\forall i \in \{1, ..., n\}$ and $\forall x_i^0 \in S_i$,*

$$f_{x_i^0} : \times_{k \neq i} S_k \to \mathbb{R}, \qquad f_{x_i^0}\left((x_k)_{k \neq i}\right) = f\left(x_1, .., x_{i-1}, x_i^0, x_{i+1}, ..., x_n\right)$$

is continuous.

Proposition 609 *Given for any $i \in \{1, ..., n\}$,*

$$f : \times_{i=1}^n S_i \to \mathbb{R} \text{ is continuous} \Rightarrow f \text{ is continuous in each variable separately}$$

Proof. Exercise. ■

Remark 610 *It is false that*

$$f \text{ is continuous in each variable separately} \Rightarrow f \text{ is continuous}$$

To see that consider $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & if \quad (x, y) \neq 0 \\ 0 & if \quad (x, y) = 0 \end{cases}$$

The following Proposition is useful to show continuity of functions using the results about continuity of functions from $\mathbb{R}$ to $\mathbb{R}$.

Proposition 611 *For any $k \in \{1, ..., n\}$, take $S_k \subseteq X$, and define $S := \times_{k=1}^n S_k \subseteq X^n$. Moreover, take $i \in \{1, ..., n\}$ and let*

$$g : S_i \to Y, \qquad : x_i \mapsto g(x_i)$$

be a continuous function and

$$f : S \to Y, \qquad (x_k)_{k=1}^n \mapsto g(x_i).$$

Then f is continuous.

Example 612 *An example of the objects described in the above Proposition is the following one.*

$$g : [0, \pi] \to \mathbb{R}, \quad g(x) = \sin x,$$

$$f : [0, \pi] \times [-\pi, 0] \to \mathbb{R}, \quad f(x, y) = \sin x.$$

Proof. of Proposition 611. We want to show that

$$\forall x_0 \in S, \forall \varepsilon > 0 \;\; \exists \delta > 0 \text{ such that } d(x, x_0) < \delta \wedge x \in S \Rightarrow d(f(x), f(x_0)) < \varepsilon$$

We know that

$$\forall x_{i0} \in S_i, \forall \varepsilon > 0 \;\; \exists \delta' > 0 \text{ such that } d(x_i, x_{i0}) < \delta' \wedge x_i \in S \Rightarrow d(g(x_i), g(x_{i0})) < \varepsilon$$

Take $\delta = \delta'$. Then $d(x, x_0) < \delta \wedge x \in S \Rightarrow d(x_i, x_{i0}) < \delta' \wedge x_i \in S$ and $\varepsilon > d(g(x_i), g(x_{i0})) = d(f(x), f(x_0))$, as desired. ■

Exercise 613 *Show that the following function is continuous.*

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x_1 x_2) = \begin{pmatrix} e^{x_1} + \cos(x_1 \cdot x_2) \\ \frac{\sin^2 x_1}{e^{x_2}} \\ x_1 + x_2 \end{pmatrix}$$

From Proposition 607, it suffices to show that each component function is continuous. We are going to show that $f_1 : \mathbb{R}^2 \to \mathbb{R}$,

$$f_1(x_1 x_2) = e^{x_1} + \cos(x_1 \cdot x_2)$$

is continuous, leaving the proof of the continuity of the other component functions to the reader.
1. $f_{11} : \mathbb{R}^2 \to \mathbb{R}, \;\; f_{11}(x_1, x_2) = e^{x_1}$ *is continuous from Proposition 611 and "Calculus 1";*
2. $h_1 : \mathbb{R}^2 \to \mathbb{R}, \;\; h_1(x_1, x_2) = x_1$ *is continuous from Proposition 611 and "Calculus 1",*
$h_2 : \mathbb{R}^2 \to \mathbb{R}, \;\; h_2(x_1, x_2) = x_2$ *is continuous from Proposition 611 and "Calculus 1",*
$g : \mathbb{R}^2 \to \mathbb{R}, \;\; g(x_1, x_2) = h_1(x_1, x_2) \cdot h_2(x_1, x_2) = x_1 \cdot x_2$ *is continuous from Proposition 606.2,*
$\phi : \mathbb{R} \to \mathbb{R}, \; \phi(x) = \cos x$ *is continuous from "Calculus 1",*
$f_{12} : \mathbb{R}^2 \to \mathbb{R}, \;\; f_{12}(x_1, x_2) = (\phi \circ g)(x_1, x_2) = \cos(x_1 \cdot x_2)$ *is continuous from Proposition 605 (continuity of composition).*
3. $f_1 = f_{11} + f_{12}$ *is continuous from Proposition 606.1.*

The following Proposition is useful in the proofs of several results.

Proposition 614 *Let* S, T *be arbitrary sets,* $f : S \to T$, $\{A_i\}_{i=1}^n$ *a family of subsets of* S *and* $\{B_i\}_{i=1}^n$ *a family of subsets of* T*.Then*

1. " inverse image preserves inclusions, unions, intersections and set differences", i.e.,

   a. $B_1 \subseteq B_2 \Rightarrow f^{-1}(B)_1 \subseteq f\left({}^{-1}B_2\right)$,
   b. $f^{-1}(\cup_{i=1}^n B_i) = \cup_{i=1}^n f^{-1}(B_i)$,
   c. $f^{-1}(\cap_{i=1}^n B_i) = \cap_{i=1}^n f^{-1}(B_i)$,
   d. $f^{-1}(B_1 \backslash B_2) = f^{-1}(B_1) \backslash f^{-1}(B_2)$,

2. " image preserves inclusions, unions, only", i.e.,

   e. $A_1 \subseteq A_2 \Rightarrow f(A)_1 \subseteq f(A_2)$,
   f. $f(\cup_{i=1}^n A_i) = \cup_{i=1}^n f(A_i)$,
   g. $f(\cap_{i=1}^n A_i) \subseteq \cap_{i=1}^n f(A_i)$, and
   if f is one-to-one, then $f(\cap_{i=1}^n A_i) = \cap_{i=1}^n f(A_i)$,
   h. $f(A_1 \backslash A_2) \supseteq f(A_1) \backslash f(A_2)$,and
   if f is one-to-one and onto, then $f(A_1 \backslash A_2) = f(A_1) \backslash f(A_2)$,

3. "relationship between image and inverse image"

i. $A_1 \subseteq f^{-1}(f(A_1))$, and
if f is one-to-one, then $A_1 = f^{-1}(f(A_i))$,
l. $B_1 \supseteq f\left(f^{-1}(B_1)\right)$, and
if f is onto, then $B_1 = f\left(f^{-1}(B_1)\right)$.

Proof.
...
g.
(i). $y \in f(A_1 \cap A_2) \Leftrightarrow \exists x \in A_1 \cap A_2$ such that $f(x) = y$;
(ii). $y \in f(A_1) \cap f(A_2) \Leftrightarrow y \in f(A_1) \wedge y \in f(A_2) \Leftrightarrow (\exists x_1 \in A_1$ such that $f(x_1) = y) \wedge (\exists x_2 \in A_2$ such that $f(x_2) = y)$
To show that $(i) \Rightarrow (ii)$ it is enough to take $x_1 = x$ and $x_2 = x$.
... ■

Proposition 615 *$f : X \to Y$ is continuous $\Leftrightarrow$*

$$V \subseteq Y \text{ is open} \Rightarrow \quad f^{-1}(V) \subseteq X \text{ is open.}$$

Proof. [$\Rightarrow$]
Take a point $x_0 \in f^{-1}(V)$. We want to show that

$$\exists r > 0 \text{ such that } B(x_0, r) \subseteq f^{-1}(V)$$

Define $y_0 = f(x_0) \in V$. Since $V \subseteq Y$ is open,

$$\exists \varepsilon > 0 \text{ such that } B(y_0, \varepsilon) \subseteq V \tag{12.2}$$

Since f is continuous,

$$\forall \varepsilon > 0, \exists \delta > 0, f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon) = B(y_0, \varepsilon) \tag{12.3}$$

Then, taken $r = \delta$, we have

$$B(x_0, r) = \ B(x_0, \delta) \overset{(1)}{\subseteq} f^{-1}(f(B(x_0, \delta))) \overset{(2)}{\subseteq} f^{-1}(B(y_0, \varepsilon)) \overset{(3)}{\subseteq} f^{-1}(V)$$

where (1) follows from 3.i in Proposition 614,
(2) follows from 1.a in Proposition 614 and (12.3)
(3) follows from 1.a in Proposition 614 and (12.2).
[$\Leftarrow$]
Take $x_0 \in X$ and define $y_0 = f(x_0)$; we want to show that f is continuous at x_0.
Take $\varepsilon > 0$, then $B(y_0, \varepsilon)$ is open and, by assumption,

$$f^{-1}(B(y_0, \varepsilon)) \subseteq X \text{ is open.} \tag{12.4}$$

Moreover, by definition of y_0,

$$x_0 \in f^{-1}(B(y_0, \varepsilon)) \tag{12.5}$$

(12.4) and (12.5) imply that

$$\exists \delta > 0 \text{ such that } B(x_0, \delta) \subseteq f^{-1}(B(y_0, \varepsilon)) \tag{12.6}$$

Then

$$f(B(x_0, \delta)) \overset{(1)}{\subseteq} f\left(f^{-1}(B(y_0, \varepsilon))\right) \overset{(2)}{\subseteq} (B(y_0, \varepsilon))$$

where
(1) follows from 2.e in Proposition 614 and (12.6),
(2) follows from 2.l in Proposition 614 ■

Proposition 616 *$f : X \to Y$ is continuous $\Leftrightarrow$*

$$V \subseteq Y \text{ closed} \Rightarrow \quad f^{-1}(V) \subseteq X \text{ closed.}$$

Proof. [$\Rightarrow$]
V closed in $Y \Rightarrow Y\backslash V$ open. Then

$$f^{-1}(Y\backslash V) = f^{-1}(Y) \ \backslash \ f^{-1}(V) = X\backslash f^{-1}(V) \tag{12.7}$$

where the first equality follows from 1.d in Proposition 614.
Since f is continuous and $Y\backslash V$ open,then from (12.7) $X\backslash f^{-1}(V) \subseteq X$ is open and therefore $f^{-1}(V)$ is closed.
[$\Leftarrow$]
We want to show that for every open set V in Y, $f^{-1}(V)$ is open.

$$V \text{ open} \Rightarrow Y\backslash V \text{ closed} \Rightarrow f^{-1}(Y\backslash V) \text{ closed} \Leftrightarrow X\backslash f^{-1}(V) \text{ closed} \Leftrightarrow f^{-1}(V) \text{ open.}$$

■

Definition 617 *A function $f : X \to Y$ is open if*

$$S \subseteq X \text{ open } \Rightarrow f(S) \text{ open;}$$

it is closed if

$$S \subseteq X \text{ closed } \Rightarrow f(S) \text{ closed.}$$

Exercise 618 *Through simple examples show the relationship between open,closed and continuous functions.*

We can summarize our discussion on continuous function in the following Proposition.

Proposition 619 *Let f be a function between metric spaces (X, d) and (Y, d'). Then the following statements are equivalent:*

1. *f is continuous;*
2. *$V \subseteq Y$ is open $\Rightarrow$ $f^{-1}(V) \subseteq X$ is open;*
3. *$V \subseteq Y$ closed $\Rightarrow$ $f^{-1}(V) \subseteq X$ closed;*
4. *$\forall x_0 \in X$, $\forall (x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $\lim_{n\to+\infty} x_n = x_0$, $\lim_{n\to+\infty} f(x_n) = f(x_0)$.*

12.3 Continuous functions on compact sets

Proposition 620 *Given $f : X \to Y$, if S is a compact subset of X and f is continuous, then $f(S)$ is a compact subset (of Y).*

Proof. Let $\mathcal{F}$ be an open covering of $f(S)$, so that

$$f(S) \subseteq \cup_{A\in\mathcal{F}} A. \tag{12.8}$$

We want to show that $\mathcal{F}$ admits an open subcover which covers $f(S)$. Since f is continuous,

$$\forall A \in \mathcal{F}, \quad f^{-1}(A) \text{ is open in } X$$

Moreover,

$$S \overset{(1)}{\subseteq} f^{-1}(f(S)) \overset{(2)}{\subseteq} f^{-1}(\cup_{A\in\mathcal{F}} A) \overset{(3)}{=} \cup_{A\in\mathcal{F}} f^{-1}(A)$$

where
(1) follows from 3.i in Proposition 614,
(2) follows from 1.a in Proposition 614 and (12.8),
(3) follows from 1.b in Proposition 614.
In other words $\left\{ f^{-1}(A)\right\}_{A\in\mathcal{F}}$ is an open cover of S. Since S is compact there exists $A_1, .., A_n \in F$ such that

$$S \subseteq \cup_{i=1}^n f^{-1}(A).$$

Then

$$f(S) \overset{(1)}{\subseteq} f\left(\cup_{i=1}^{n}\left(f^{-1}(A)\right)\right) \overset{(2)}{=} \cup_{i=1}^{n} f\left(f^{-1}(A_i)\right) \overset{(3)}{\subseteq} \cup_{i=1}^{n} A_i$$

where
(1) follows from 1.a in Proposition 614,
(2) follows from 2.f in Proposition 614,
(3) follows from 3.l in Proposition 614. ■

Proposition 621 ***(Extreme Value Theorem)*** *If S a nonempty, compact subset of X and $f: S \to \mathbb{R}$ is continuous, then f admits global maximum and minimum on S, i.e.,*

$$\exists x_{\min}, x_{\max} \in S \text{ such that } \forall x \in S, \quad f(x_{\min}) \le f(x) \le f(x_{\max}).$$

Proof. From the previous Proposition $f(S)$ is closed and bounded. Therefore, since $f(S)$ is bounded, there exists $M = \sup f(S)$. By definition of sup,

$$\forall \varepsilon > 0, \; B(M, \varepsilon) \cap f(S) \neq \varnothing$$

Then[1], $\forall n \in \mathbb{N}$,take

$$\alpha_n \in B\left(M, \frac{1}{n}\right) \cap f(S).$$

Then, $(\alpha_n)_{n\in\mathbb{N}}$ is such that $\forall n \in \mathbb{N}$, $\alpha_n \in f(S)$ and $0 < d(\alpha_n, M) < \frac{1}{n}$. Therefore, $\alpha_n \to M$,and since $f(S)$ is closed, $M \in f(S)$. But $M \in f(S)$ means that $\exists x_{\max} \in S$ such that $f(x_{\max}) = M$ and the fact that $M = \sup f(S)$ implies that $\forall x \in S, \quad f(x) \le f(x_{\max})$. Similar reasoning holds for $x_{\min}$. ■

We conclude the section showing a result useful in itself and needed to show the inverse function theorem - see Section 17.3.

Proposition 622 *Let $f : X \to Y$ be a function from a metric space (X, d) to another metric space (Y, d'). Assume that f is one-to-one and onto. If X is compact and f is continuous, then the inverse function f^{-1} is continuous.*

Proof. Exercise. ■

We are going to use the above result to show that a " well behaved" consumer problem does have a solution.

Let the following objects be given.

Price vector $p \in \mathbb{R}^n_{++}$, consumption vector $x \in \mathbb{R}^n$, consumer's wealth $w \in \mathbb{R}_{++}$, continuous utility function $u : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto u(x)$. The consumer solves the following problem. For given, $p \in \mathbb{R}^n_{++}$,$w \in \mathbb{R}_{++}$, find x which gives the maximum value to the utility function u under the constraint $x \in C(p, w)$ defined as

$$\{x = (x_i)_{i=1}^n \in \mathbb{R}^n : \forall i \in \{1, ..., n\}, x_i \ge 0 \text{ and } px \le w\}.$$

As an application of Propositions 621 and 516, we have to show that for any $p \in \mathbb{R}^n_{++}$,$w \in \mathbb{R}_{++}$,
1. $C(p, w) \neq \varnothing$,
2. $C(p, w)$ is bounded, and
3. $C(p, w)$ is closed,
1. $0 \in C(p, w)$.
2. Clearly if $S \subseteq \mathbb{R}^n$,then S is bounded iff

S is bounded below, i.e., $\exists \underline{x} = (\underline{x}_i)_{i=1}^n \in \mathbb{R}^n$such that $\forall x = (x_i)_{i=1}^n \in S$, we have that $\forall i \in \{1, ..., n\}$, $x_i \ge \underline{x}_i$, and

S is bounded above, i.e., $\exists \overline{x} = (\overline{x}_i)_{i=1}^n \in \mathbb{R}^n$such that $\forall x = (x_i)_{i=1}^n \in S$, we have that $\forall i \in \{1, ..., n\}$, $x_i \le \overline{x}_i$.

$C(p, w)$ is bounded below by zero, i.e., we can take $\underline{x} = 0$. $C(p, w)$ is bounded above because for every $i \in \{1, ..., n\}$,

$$x_i \le \frac{w - \sum_{i' \neq i} p_{i'} x_{i'}}{p_i} \le \frac{w}{p_i},$$

[1] The fact that $M \in f(S)$ can be also proved as follows: from Proposition 557 , $M \in \text{Cl}\, f(S) = f(S)$, where the last equality follows from the fact that $f(S)$ is closed.

where the first inequality comes from the fact that $px \leq w$, and the second inequality from the fact that $p \in \mathbb{R}^n_{++}$ and $x \in \mathbb{R}^n_+$.Then we can take $\overline{x} = (m, m, ..., m)$, where $m = \max \left\{ \frac{w}{p_i} \right\}_{i=1}^n$.

3. Define

$$\text{for } i \in \{1, ..., n\}, \; g_i : \mathbb{R}^n \to \mathbb{R}, \quad x = (x_i)_{i=1}^n \mapsto x_i,$$

and

$$h : \mathbb{R}^n \to \mathbb{R}, \quad x = (x_i)_{i=1}^n \mapsto w - px.$$

All the above functions are continuous and clearly,

$$C(p, w) = \{x = (x_i)_{i=1}^n \in \mathbb{R}^n : \forall i \in \{1, ..., n\}, \; g_i(x) \geq 0 \text{ and } h(x) \geq 0\}.$$

Moreover,

$$C(p, w) = \{x = (x_i)_{i=1}^n \in \mathbb{R}^n : \forall i \in \{1, ..., n\}, \; g_i(x) \in [0, +\infty) \text{ and } h(x) \in [0, +\infty)\} =$$

$$= \cap_{i=1}^n g_i^{-1}([0, +\infty)) \cap h^{-1}([0, +\infty)).$$

$[0, +\infty)$ is a closed set and since for any g_i is continuous and h is continuous, from Proposition 619.3,the following sets are closed

$$\forall i \in \{1, ..., n\}, \; g_i^{-1}([0, +\infty)) \text{ and } h^{-1}([0, +\infty)).$$

Then the desired result follows from the fact that intersection of closed set is closed.

Chapter 13

Correspondence and fixed point theorems

13.1 Continuous Correspondences

Definition 623 *Consider*[1] *two metric spaces* (X, d_X) *and* (Y, d_Y). *A correspondence* φ *from* X *to* Y *is a rule which associates a subset of* Y *with each element of* X, *and it is described by the notation*

$$\varphi : X \to\to Y, \ \varphi : x \mapsto\mapsto \varphi(x).$$

Remark 624 *In other words, a correspondence* $\varphi : X \to\to Y$ *can be identified with a function from* X *to* 2^Y *(the set of all subsets of* Y*). If we identify* x *with* $\{x\}$, *a function from* X *to* Y *can be thought as a particular correspondence.*

Remark 625 *Some authors make part of the Definition of correspondence the fact that* φ *is not empty valued, i.e., that* $\forall x \in X$, $\varphi(x) \neq \varnothing$.

In what follows, unless otherwise stated, (X, d_X) and (Y, d_Y) are assumed to be metric spaces and are denoted by X and Y, respectively.

Definition 626 *Given* $U \subseteq X$, $\varphi(U) = \cup_{x \in U} \varphi(x) = \{y \in Y : \exists x \in U \text{ such that } y \in \varphi(x)\}$.

Definition 627 *The graph of* $\varphi : X \to\to Y$ *is*

$$\text{graph } \varphi := \{(x, y) \in X \times Y : y \in \varphi(x)\}.$$

Definition 628 *Consider* $\varphi : X \to\to Y$. φ *is Upper Hemi-Continuous (UHC) at* $x \in X$ *if* $\varphi(x) \neq \varnothing$ *and for every open neighborhood* V *of* $\varphi(x)$, *there exists an open neighborhood* U *of* x *such that for every* $x' \in U$, $\varphi(x') \subseteq V$ *(or* $\varphi(U) \subseteq V$*).*

φ *is UHC if it is UHC at every* $x \in X$.

Definition 629 *Consider* $\varphi : X \to\to Y$. φ *is Lower Hemi-Continuous (LHC) at* $x \in X$ *if* $\varphi(x) \neq \varnothing$ *and for any open set* V *in* Y *such that* $\varphi(x) \cap V \neq \varnothing$, *there exists an open neighborhood* U *of* x *such that for every* $x' \in U$, $\varphi(x') \cap V \neq \varnothing$.

φ *is LHC if it is LHC at every* $x \in X$.

Example 630 *Consider* $X = \mathbb{R}_+$ *and* $Y = [0, 1]$, *and*

$$\varphi_1(x) = \begin{cases} [0,1] & if \quad x = 0 \\ \{0\} & if \quad x > 0. \end{cases}$$

$$\varphi_2(x) = \begin{cases} \{0\} & if \quad x = 0 \\ [0,1] & if \quad x > 0. \end{cases}$$

φ_1 *is UHC and not LHC;* φ_2 *is LHC and not UHC.*

[1] This chapter is based mainly on McLean (1985) and Hildebrand (1974).

Some (partial) intuition about the above definitions can be given as follows.

Upper Hemi-Continuity does not allow "explosions". In other words, φ *is not UHC at* x if there exists a small enough open neighborhood of x such that φ does "explode", i.e., it becomes much bigger in that neighborhood.

Lower Hemi-Continuity does not allow "implosions". In other words, φ *is not LHC at* x if there exists a small enough open neighborhood of x such that φ does "implode", i.e., it becomes much smaller in that neighborhood.

In other words, "UHC $\Rightarrow$ no explosion" and "LHC $\Rightarrow$ no implosion"(or "explosion $\Rightarrow$ not UHC" and "implosion $\Rightarrow$ not LHC)". On the other hand, opposite implications are false, i.e.,

it is false that "explosion $\Leftarrow$ not UHC" and "implosion $\Leftarrow$ not LHC", or, in an equivalent manner,

it is false that "no explosion $\Rightarrow$ UHC" and "no implosion $\Rightarrow$ LHC".

An example of a correspondence which neither explodes nor explodes and which is not UHC and not LHC is presented below.

$$\varphi : \mathbb{R}_+ \rightarrow\rightarrow \mathbb{R}, \qquad \varphi : x \mapsto\mapsto \begin{cases} [1,2] & if \quad x \in [0,1) \\ [3,4] & if \quad x \in [1,+\infty) \end{cases}$$

φ does not implode or explode if you move away from 1 (in a small open neighborhood of 1): on the right of 1, φ does not change; on the left, it changes completely. Clearly, φ is neither UHC nor LHC (in 1).

The following correspondence is both UHC and LHC:

$$\varphi : \mathbb{R}_+ \rightarrow\rightarrow \mathbb{R}, \qquad \varphi : x \mapsto\mapsto [x, x+1]$$

A, maybe disturbing, example is the following one

$$\varphi : \mathbb{R}_+ \rightarrow\rightarrow \mathbb{R}, \qquad \varphi : x \mapsto\mapsto (x, x+1) .$$

Observe that the graph of the correspondence under consideration "does not implode, does not explode, does not jump". In fact, the above correspondence is LHC, but it is not UHC in any $x \in \mathbb{R}_+$, as verified below. We want to show that

$$\textit{not} \left\langle \begin{array}{l} \text{for every neighborhood } V \text{ of } \varphi(x), \\ \\ \text{there exists a neighborhood } U \text{ of } x \text{ such that for every } x' \in U,\ \varphi(x') \subseteq V \end{array} \right\rangle$$

i.e.,

$$\left\langle \begin{array}{l} \text{there exists a neighborhood } V^* \text{ of } \varphi(x) \text{ such that.} \\ \\ \text{for every neighborhood } U \text{ of } x \text{ there exists } x' \in U \text{ such that } \varphi(x') \nsubseteq V \end{array} \right\rangle$$

Just take $V = \varphi(x) = (x, x+1)$; then for any open neighborhood U of x and, in fact, $\forall x' \in U \backslash \{x\}$, $\varphi(x') \nsubseteq V$.

Example 631 *The correspondence below*

$$\varphi : \mathbb{R}_+ \rightarrow\rightarrow \mathbb{R}, \qquad \varphi : x \mapsto\mapsto \begin{cases} [1,2] & if \quad x \in [0,1] \\ [3,4] & if \quad x \in [1,+\infty) \end{cases}$$

is UHC, but not LHC.

Remark 632 *Summarizing the above results, we can maybe say that a correspondence which is both UHC and LHC, in fact a continuous correspondence, is a correspondence which agrees with our intuition of a graph without explosions, implosions or jumps.*

Proposition 633 *1. If $\varphi : X \rightarrow\rightarrow Y$ is either UHC or LHC and it is a function, then it is a continuous function.*

2. If $\varphi : X \rightarrow\rightarrow Y$ is a continuous function, then it is a UHC and LHC correspondence.

Proof.
1.
Case 1. φ is UHC.
First proof. Use the fourth characterization of continuous function in Definition 603.
Second proof. Recall that a function $f : X \to Y$ is continuous iff $[V \text{ open in } Y] \Rightarrow \left[f^{-1}(V) \text{ open in } X\right]$. Take V open in Y. Consider $x \in f^{-1}(V)$, i.e., x such that $f(x) \in V$. By assumption f is UHC and therefore $\exists$ an open neighborhood U of x such that $f(U) \subseteq V$. Then, $U \subseteq f^{-1} \circ f(U) \subseteq f^{-1}(V)$. Then, for any $x \in f^{-1}(V)$, we have found an open set U which contains x and is contained in $f^{-1}(V)$, i.e., $f^{-1}(V)$ is open.
Case 2. φ is LHC.
See Remark 637 below.
2.
The results follows from the definitions and again from Remark 637 below.
■

Definition 634 *$\varphi : X \to\to Y$ is a continuous correspondence if it is both UHC and LHC.*

Very often, checking if a correspondence is UHC or LHC is not easy. We present some *related* concepts which are more convenient to use.

Definition 635 *$\varphi : X \to\to Y$ is "sequentially LHC" at $x \in X$ if*
for every sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $x_n \to x$, and for every $y \in \varphi(x)$,
there exists a sequence $(y_n)_{n\in\mathbb{N}} \in Y^\infty$ such that $\forall\, n \in \mathbb{N}$, $y_n \in \varphi(x_n)$ and $y_n \to y$,
φ is "sequentially LHC" if it is "sequentially LHC" at every $x \in X$.

Proposition 636 *Consider $\varphi : X \to\to Y$. φ is LHC at $x \in X \Leftrightarrow \varphi$ is LHC in terms of sequences at $x \in X$.*

Proof.
$[\Rightarrow]$
Consider an arbitrary sequence $(x_n)_{n\in\mathbb{N}} \subseteq X^\infty$ such that $x_n \to x$ and an arbitrary $y \in \varphi(x)$. For every $r \in \mathbb{N}\setminus\{0\}$, consider $B\left(y, \frac{1}{r}\right)$. Clearly $B\left(y, \frac{1}{r}\right) \cap \varphi(x) \neq \varnothing$, since y belongs to both sets. From the fact that φ is LHC, we have that

$$\forall r \in \mathbb{N}\setminus\{0\},\ \exists \text{ a neighborhood } U_r\ of\ x \text{ such that } \forall z \in U_r,\ \varphi(z) \cap B\left(y, \frac{1}{r}\right) \neq \varnothing. \quad (1)$$

Since $x_n \to x$,

$$\forall r \in \mathbb{N}\setminus\{0\}, \quad \exists\, n_r \in \mathbb{N} \text{ such that } n \geq n_r \Rightarrow x_n \in U_r \quad (2).$$

Consider $\{n_1, ..., n_r, ...\}$. For any $r \in \mathbb{N}\setminus\{0\}$, if $n_r < n_{r+1}$, define $n'_r := n_r$ and $n'_{r+1} := n_{r+1}$, i.e., "just add a $'$ to the name of those indices". If $n_r \geq n_{r+1}$, define $n'_r := n_r$ and $n'_{r+1} := n_r + 1$. Then, $\forall r \in \mathbb{N}\setminus\{0\}$, $n'_{r+1} > n'_r$ and condition (2) still hold, i.e.,

$$\forall r,\ \exists\, n'_r \text{ such that } n \geq n'_r \Rightarrow x_n \in U_r \text{ and } n'_{r+1} > n'_r. \quad (3)$$

We can now define the desired sequence $(y_n)_{n\in\mathbb{N}}$. For any $n \in \left[n'_r, n'_{r+1}\right)$, observe that from (3), $x_n \in U_r$, and, then, from (1), $\varphi(x_n) \cap B\left(y, \frac{1}{r}\right) \neq \varnothing$. Then,

$$\text{for any } r, \text{ for any } n \in \left[n'_r, n'_{r+1}\right) \cap \mathbb{N}, \text{ choose } y_n \in \varphi(x_n) \cap B\left(y, \frac{1}{r}\right) \quad (4).$$

We are left with showing that $y_n \to y$, i.e., $\forall \varepsilon > 0$, $\exists\, \overline{m}$ such that $n \geq \overline{m} \Rightarrow y_n \in B(y, \varepsilon)$. Observe that (4) just says that
for any $n \in [n'_1, n'_2)$, $y_n \in \varphi(x_n) \cap B\left(y, \frac{1}{1}\right)$,
for any $n \in [n'_2, n'_3)$, $y_n \in \varphi(x_n) \cap B\left(y, \frac{1}{2}\right) \subseteq B(y, 1)$
...
for any $n \in \left[n'_r, n'_{r+1}\right)$, $y_n \in \varphi(x_n) \cap B\left(y, \frac{1}{r}\right) \subseteq B\left(y, \frac{1}{r-1}\right)$,

and so on. Then, for any $\varepsilon > 0$, choose $r > \frac{1}{\varepsilon}$ (so that $\frac{1}{r} < \varepsilon$) and $\overline{m} = n'_r$, then from the above observations, $n \in [\overline{m}, n'_{r+1}) \Rightarrow y_n \in B\left(y, \frac{1}{r}\right) \subseteq B(y, \varepsilon)$,and for $n > n'_{r+1}$, a fortiori, $y_n \in B\left(y, \frac{1}{r}\right)$.

[$\Leftarrow$]

Assume otherwise, i.e., $\exists$ an open set V such that

$$\varphi(x) \cap V \neq \varnothing \quad (5)$$

and such that $\forall$ open neighborhood U of x $\exists\, x_U \in U$ such that $\varphi(x_U) \cap V = \varnothing$.

Consider the following family of open neighborhood of x:

$$\left\{B\left(x, \frac{1}{n}\right) : n \in \mathbb{N} \setminus \{0\}\right\}.$$

Then $\forall n \in \mathbb{N} \setminus \{0\}$, $\exists\, x_n \in B\left(x, \frac{1}{n}\right)$, and therefore $x_n \to x$, such that

$$\varphi(x_n) \cap V = \varnothing \qquad (6).$$

From (5), we can take $y \in \varphi(x) \cap V$. By assumption, we know that there exists a sequence $(y_n)_{n \in \mathbb{N}} \in Y^\infty$ such that $\forall n \in \mathbb{N} \setminus \{0\}$, $y_n \in \varphi(x_n)$ and $y_n \to y$. Since V is open and $y \in V$, $\exists\, \overline{n}$ such that $n > \overline{n} \Rightarrow y_n \in V$. Therefore,

$$y \in \varphi(x_n) \cap V \ \ (7).$$

But (7) contradicts (6).

■

Thanks to the above Proposition from now on we talk simply of Lower Hemi-Continuous correspondences.

Remark 637 *If $\varphi : X \to\to Y$ is LHC and it is a function, then it is a continuous function. The result follows from the characterization of Lower Hemi-Continuity in terms of sequences and from the characterization of continuous functions presented in Proposition 619.*

Definition 638 *$\varphi : X \to\to Y$ is closed, or "sequentially UHC", at $x \in X$ if*

for every sequence $(x_n)_{n \in \mathbb{N}} \in X^\infty$ such that $x_n \to x$, and for every sequence $(y_n)_{n \in \mathbb{N}} \in Y^\infty$ such that $y_n \in \varphi(x_n)$ and $y_n \to y$,

it is the case that $y \in \varphi(x)$.

φ is closed if it is closed at every $x \in X$.

Proposition 639 *φ is closed $\Leftrightarrow$* graph φ *is a closed set in $X \times Y$.* [2]

Proof. An equivalent way of stating of the above Definition is the following one: for every sequence $(x_n, y_n)_{n \in \mathbb{N}} \in (X \times Y)^\infty$ such that $\forall n \in \mathbb{N}$, $(x_n, y_n) \in$ graph φ and $(x_n, y_n) \to (x, y)$, it is the case that $(x, y) \in$ graph φ. Then, from the characterization of closed sets in terms of sequences, i.e., Proposition 497, the desired result follows. ■

Remark 640 *Because of the above result, many author use the expression "φ has closed graph" in the place of "φ is closed".*

Remark 641 *The definition of closed correspondence does NOT reduce to continuity in the case of functions, as the following example shows.*

$$\varphi_3 : \mathbb{R}_+ \to\to \mathbb{R},\ \varphi_3(x) = \begin{cases} \{0\} & if \quad x = 0 \\ \left\{\frac{1}{x}\right\} & if \quad x > 0. \end{cases}$$

φ is a closed correspondence, but it is not a continuous function.

Definition 642 *$\varphi : X \to\to Y$ is closed (non-empty, convex, compact ...) valued if for every $x \in X$, $\varphi(x)$ is a closed (non-empty, convex, compact ...) set.*

Proposition 643 *Consider $\varphi : X \to\to Y$. φ closed $\overset{\Rightarrow}{\not\Leftarrow}$ φ closed valued.*

[2] $((X \times Y), d^*)$ with $d^*((x, x'), (y, y')) = d(x, x') + d'(y, y')$ is a metric space.

Proof.

[$\Rightarrow$]

We want to show that every sequence in $\varphi(x)$ which converges, in fact, converges in $\varphi(x)$. Choose $x \in X$ and a sequence $(y_n)_{n \in \mathbb{N}}$ such that $\{y_n : n \in \mathbb{N}\} \subseteq \varphi(x)$ and such that $y_n \to y$. Then setting for any $n \in \mathbb{N}$, $x_n = x$, we get $x_n \to x, \quad y_n \in \varphi(x_n), \quad y_n \to y$. Then, since φ is closed, $y \in \varphi(x)$. This shows that $\varphi(x)$ is a closed set.

[$\not\Leftarrow$]

φ_2 in Example 630 is closed valued, but not closed.

■

Remark 644 *Consider $\varphi : X \rightarrow\rightarrow Y$. φ is UHC* $\overset{\not\Rightarrow}{\not\Leftarrow}$ *φ is closed.*

[$\not\Rightarrow$]

$\varphi_4 : \mathbb{R}_+ \rightarrow\rightarrow \mathbb{R}$, $\varphi_4(x) = [0,1)$ *for every* $x \in \mathbb{R}$

is UHC and not closed.

[$\not\Leftarrow$]

φ_3 in Remark 641 is closed and not UHC, simply because it is not a continuous "function".

Proposition 645 *Consider $\varphi : X \rightarrow\rightarrow Y$. If φ is UHC (at x) and closed valued (at x), then φ is closed (at x).*

Proof.

Take an arbitrary $x \in X$. We want to show that φ is closed at x, i.e., assume that $x_n \to x, \quad y_n \in \varphi(x_n), \quad y_n \to y$; we want to show that $y \in \varphi(x)$. Since $\varphi(x)$ is a closed set, it suffices to show that $y \in Cl\phi(x)$, i.e.,[3] $\forall \varepsilon > 0$, $B(y, \varepsilon) \cap \varphi(x) \neq \varnothing$.

Consider $\left\{B\left(z, \frac{\varepsilon}{2}\right) : z \in \varphi(x)\right\}$. Then, $\cup_{z \in \varphi(x)} B\left(z, \frac{\varepsilon}{2}\right) := V$ is open and contains $\varphi(x)$. Since φ is UHC at x, there exists an open neighborhood U of x such that

$$\varphi(U) \subseteq V. \qquad (1)$$

Since $x_n \to x \in U$, $\exists \widehat{n} \in \mathbb{N}$ such that $\forall n > \widehat{n}, \quad x_n \in U$, and, from (1), $\varphi(x_n) \subseteq V$. Since $y_n \in \varphi(x_n)$,

$$\forall n > \widehat{n}, \quad y_n \in V := \cup_{z \in \varphi(x)} B\left(z, \frac{\varepsilon}{2}\right). \qquad (2)$$

From (2), $\forall n > \widehat{n}$, $\exists z_n^* \in \varphi(x)$ such that $y_n \in B\left(z_n^*, \frac{\varepsilon}{2}\right)$ and then

$$d(y_n, z^*) < \frac{\varepsilon}{2}. \qquad (3)$$

Since $y_n \to y$, $\exists n^*$ such that $\forall n > n^*$,

$$d(y_n, y) < \frac{\varepsilon}{2} \qquad (4)$$

From (3) and (4), $\forall n > \max\{\widehat{n}, n^*\}$, $z_n^* \in \varphi(x)$ and $d(y, z^*) \leq d(y, y_n) + d(y_n, z^*) < \varepsilon$, i.e., $z_n^* \in B(y, \varepsilon) \cap \varphi(x) \neq \varnothing$.

■

Proposition 646 *Consider $\varphi : X \rightarrow\rightarrow Y$. If φ is closed and there exists a compact set $K \subseteq Y$ such that $\varphi(X) \subseteq K$, then φ is UHC.*

Therefore, in simpler terms, if φ is closed (at x)and Y is compact, then φ is UHC (at x).

Proof.

Assume that there exists $x \in X$ such that φ is not UHC at $x \in X$, i.e., there exist an open neighborhood V of $\varphi(x)$ such that for every open neighborhood U_x of x, $\quad \varphi(U_x) \cap V^C \neq \varnothing$. In particular, $\forall n \in \mathbb{N} \setminus \{0\}$, $\quad \varphi\left(B\left(x, \frac{1}{n}\right)\right) \cap V^C \neq \varnothing$. Therefore, we can construct a sequence $(x_n)_{n \in \mathbb{N}} \in X^\infty$ such that $x_n \to x$ and $\varphi(x_n) \cap V^C \neq \varnothing$. Now, take $y_n \in \varphi(x_n) \cap V^C$. Since $y_n \in \varphi(X) \subseteq K$ and K is compact, and therefore sequentially compact, up to a subsequence, $y_n \to y \in K$. Moreover, since $\forall n \in \mathbb{N} \setminus \{0\}$, $y_n \in V^C$ and V^C is closed,

$$y \in V^C \quad (1).$$

[3] See Corollary 557.

Since φ is closed and $x_n \to x,\quad y_n \in \varphi(x_n),\quad y_n \to y$, we have that $y \in \varphi(x)$. Since, by assumption, $\varphi(x) \subseteq V$, we have that

$$y \in V \quad (2).$$

But (2) contradicts (1).

■

None of the Assumptions of the above Proposition can be dispensed of. All the examples below show correspondences which are not UHC.

Example 647 *1.*

$$\varphi : \mathbb{R}_+ \to\to \mathbb{R},\ \varphi(x) = \begin{cases} \left\{\frac{1}{2}\right\} & if \quad x \in [0,2] \\ \{1\} & if \quad x > 2. \end{cases}$$

$Y = [0,1]$, but φ is not closed.
2.

$$\varphi : \mathbb{R}_+ \to\to \mathbb{R},\ \varphi(x) = \begin{cases} \{0\} & if \quad x = 0 \\ \left\{\frac{1}{x}\right\} & if \quad x > 0. \end{cases}$$

φ is closed, but $\varphi(X) = \mathbb{R}_+$, which is closed, but not bounded.
3.

$$\varphi : [0,1] \to\to [0,1),\ \varphi(x) = \begin{cases} \{0\} & if \quad x \in [0,1) \\ \left\{\frac{1}{2}\right\} & if \quad x = 1. \end{cases}$$

φ is closed (in Y), but $Y = [0,1)$ is not compact. Observe that if you consider

$$\varphi : [0,1] \to\to [0,\overset{\downarrow}{1]},\ \varphi(x) = \begin{cases} \{0\} & if \quad x \in [0,1) \\ \left\{\frac{1}{2}\right\} & if \quad x = 1. \end{cases},$$

then φ is not closed.

Definition 648 *Consider $\varphi : X \to\to Y$, $V \subseteq Y$.*
The strong inverse image of V via φ is

$${}^{s}\varphi^{-1}(V) := \{x \in X : \varphi(x) \subseteq V\};$$

.
The weak inverse image of V via φ is

$${}^{w}\varphi^{-1}(V) := \{x \in X : \varphi(x) \cap V \neq \varnothing\}.$$

Remark 649 *1. $\forall\, V \subseteq Y$, ${}^{s}\varphi^{-1}(V) \subseteq {}^{w}\varphi^{-1}(V)$.*
2. If φ is a function, the usual definition of inverse image coincides with both above definitions.

Proposition 650 *Consider $\varphi : X \to\to Y$.*
1.1. φ is UHC $\Leftrightarrow$ for every open set V in Y, ${}^{s}\varphi^{-1}(V)$ is open in X;
1.2. φ is UHC $\Leftrightarrow$ for every closed set V in Y, ${}^{w}\varphi^{-1}(V)$ is closed in X;
2.1. φ is LHC $\Leftrightarrow$ for every open set V in Y, ${}^{w}\varphi^{-1}(V)$ is open in X;
2.2. φ is LHC $\Leftrightarrow$ for every closed set V in Y, ${}^{s}\varphi^{-1}(V)$ is closed in X.[4]

Proof.

[1.1., ⇒] Consider V open in Y. Take $x_0 \in {}^{s}\varphi^{-1}(V)$; by definition of ${}^{s}\varphi^{-1}$, $\varphi(x_0) \subseteq V$. By definition of UHC correspondence, $\exists$ an open neighborhood U of x_0 such that $\forall x \in U,\ \varphi(x) \in V$. Then $x_0 \in U \subseteq {}^{s}\varphi^{-1}(V)$.

[1.1., ⇐] Take an arbitrary $x_0 \in X$ and an open neighborhood V of $\varphi(x_0)$. Then $x_0 \in {}^{s}\varphi^{-1}(V)$ and ${}^{s}\varphi^{-1}(V)$ is open by assumption. Therefore (just identifying U with ${}^{s}\varphi^{-1}(V)$), we have proved that φ is UHC.

To show 1.2, preliminarily, observe that

$$\left({}^{w}\varphi^{-1}(V)\right)^{C} = {}^{s}\varphi^{-1}\left(V^{C}\right). \tag{13.1}$$

[4] Part 2.2 of the Proposition will be used in the proof of the Maximum Theorem.

(To see that, simply observe that $\left({}^{w}\varphi^{-1}(V)\right)^{C} := \{x \in X : \varphi(x) \cap V = \varnothing\}$ and ${}^{s}\varphi^{-1}\left(V^{C}\right) := \left\{x \in X : \varphi(x) \subseteq V^{C}\right\}$)

[1.2., $\Rightarrow$] V closed $\Leftrightarrow V^{C}$ open $\overset{Assum.,\,(1.1)}{\Leftrightarrow}$ ${}^{s}\varphi^{-1}\left(V^{C}\right) \overset{(13.1)}{=} \left({}^{w}\varphi^{-1}(V)\right)^{C}$ open $\Leftrightarrow {}^{w}\varphi^{-1}(V)$ closed.

[1.2., $\Leftarrow$]

From (1.1.), it suffices to show that $\forall$ open set V in Y, ${}^{s}\varphi^{-1}(V)$ is open in X. Then,

V open $\Leftrightarrow V^{C}$ closed $\overset{Assum.^{w}}{\Leftrightarrow}$ $\varphi^{-1}\left(V^{C}\right)$ closed $\Leftrightarrow \left({}^{w}\varphi^{-1}\left(V^{C}\right)\right)^{C} \overset{(13.1)}{=} {}^{s}\varphi^{-1}(V)$ open.

The proofs of parts 2.1. and 2.2. are similar to the above ones.

■

Remark 651 *Observe that φ is UHC* $\overset{\nRightarrow}{\nLeftarrow}$ *for every closed set V in Y, ${}^{s}\varphi^{-1}(V)$ is closed in X.*

[$\nRightarrow$]

Consider

$$\varphi : \mathbb{R}_{+} \to\to \mathbb{R}, \quad \varphi(x) = \begin{cases} [0,2] & if \quad x \in [0,1] \\ [0,1] & if \quad x > 1. \end{cases}$$

φ is UHC and $[0,1]$ is closed, but ${}^{s}\varphi^{-1}([0,1]) := \{x \in \mathbb{R}_{+} : \varphi(x) \subseteq [0,1]\} = (1,+\infty)$ is not closed.

[$\nLeftarrow$]

Consider

$$\varphi : \mathbb{R}_{+} \to\to \mathbb{R}_{+}, \quad \varphi(x) = \begin{cases} \left[0,\frac{1}{2}\right] \cup \{1\} & if \quad x = 0 \\ [0,1] & if \quad x > 0. \end{cases}$$

For any closed set in $Y := \mathbb{R}_{+}$, ${}^{s}\varphi^{-1}(V)$ can be only one of the following set, and each of them is closed: $\{0\}, \mathbb{R}_{+}, \varnothing$. On the other hand, φ is not UHC in 0.

Definition 652 *Let the vector spaces (X, d_X), (Y, d_Y) and (Z, d_Z) and the correspondences $\varphi : X \to\to Y, \psi : Y \to\to Z$ be given. The composition of φ with ψ is*

$\psi \circ \varphi : X \to\to Z,$

$$(\psi \circ \varphi)(x) := \cup_{y \in \varphi(x)} \psi(y) = \{z \in Z : \exists x \in X \ such\ that\ z \in \psi(\varphi(x))\}$$

.

Proposition 653 *Consider $\varphi : X \to\to Y, \psi : Y \to\to Z$. If φ and ψ are UHC, then $\psi \circ \varphi$ is UHC.*

Proof.

Step 1. ${}^{s}(\psi \circ \varphi)^{-1}(V) = {}^{s}\varphi^{-1}\left({}^{s}\psi^{-1}(V)\right)$.

$${}^{s}(\psi \circ \varphi)^{-1}(V) = \{x \in X : \psi(\varphi(x)) \subseteq V\} = \{x \in X : \forall y \in \varphi(x), \psi(y) \subseteq V\} =$$
$$= \left\{x \in X : \forall y \in \varphi(x), y \in {}^{s}\psi^{-1}(V)\right\} = \left\{x \in X : \varphi(x) \subseteq {}^{s}\psi^{-1}(V)\right\} = {}^{s}\varphi^{-1}\left({}^{s}\psi^{-1}(V)\right).$$

Step 2. Desired result.

Take V open in Z. From Theorem 650, we want to show that ${}^{s}(\psi \circ \varphi)(V)$ is open in X. From step 1, we have that ${}^{s}(\varphi \circ \psi)^{-1}(V) = {}^{s}\varphi^{-1}\left({}^{s}\psi^{-1}(V)\right)$. Now, ${}^{s}\psi^{-1}(V)$ is open because ψ is UHC, and ${}^{s}\varphi^{-1}\left({}^{s}\psi^{-1}(V)\right)$ is open because φ is UHC.

■

Proposition 654 *Consider $\varphi : X \to\to Y$. If φ is UHC and compact valued, and $A \subseteq X$ is a compact set, then $\varphi(A)$ is compact.*

Proof.

Consider an arbitrary open cover $\{C_\alpha\}_{\alpha \in I}$ for $\varphi(A)$. Since $\varphi(A) := \cup_{x \in A} \varphi(x)$ and φ is compact valued, there exists a finite set $N_x \subseteq I$ such that

$$\varphi(x) \subseteq \cup_{\alpha \in N_x} C_\alpha := G_x. \tag{13.2}$$

Since for every $\alpha \in N_x$, C_α is open, then G_x is open. Since φ is UHC, ${}^{s}\varphi^{-1}(G_x)$ is open. Moreover, $x \in {}^{s}\varphi^{-1}(G_x)$: this is the case because, by definition, $x \in {}^{s}\varphi^{-1}(G_x)$ iff $\varphi(x) \subseteq G_x$, which is just (13.2). Therefore, $\left\{ {}^{s}\varphi^{-1}(G_x) \right\}_{x\in A}$ is an open cover of A. Since, by assumption, A is compact, there exists a finite set $\{x_i\}_{i=1}^m \subseteq A$ such that $A \subseteq \cup_{i=1}^m \left({}^{s}\varphi^{-1}(G_{x_i}) \right)$. Finally,

$$\varphi(A) \subseteq \varphi\left(\cup_{i=1}^m \left({}^{s}\varphi^{-1}(G_{x_i}) \right)\right) \overset{(1)}{\subseteq} \cup_{i=1}^m \varphi\left({}^{s}\varphi^{-1}(G_{x_i}) \right) \overset{(2)}{\subseteq} \cup_{i=1}^m G_{x_i} = \cup_{i=1}^m \cup_{\alpha\in N_{x_i}} C_\alpha,$$

and $\left\{ \{C_\alpha\}_{\alpha\in N_{x_i}} \right\}_{i=1}^m$ is a finite subcover of $\{C_\alpha\}_{\alpha\in I}$. We are left with showing (1) and (2) above.

(1). In general, it is the case that $\varphi(\cup_{i=1}^m S_i) \subseteq \cup_{i=1}^m \varphi(S_i)$.

$y \in \varphi(\cup_{i=1}^m S_i) \Leftrightarrow \exists x \in \cup_{i=1}^m S_i$ such that $y \in \varphi(x) \Rightarrow \exists i$ such that $y \in \varphi(x) \subseteq \varphi(S_i) \Rightarrow y \in \cup_{i=1}^m \varphi(S_i)$.

(2). In general, it is the case that $\varphi\left({}^{s}\varphi^{-1}(A) \right) \subseteq A$.

$y \in \varphi\left({}^{s}\varphi^{-1}(A) \right) \Rightarrow \exists x \in {}^{s}\varphi^{-1}(A)$ such that $y \in \varphi(x)$. But, by definition of ${}^{s}\varphi^{-1}(A)$, and since $x \in {}^{s}\varphi^{-1}(A)$, it follows that $\varphi(x) \subseteq A$ and therefore $y \in A$.

■

Remark 655 *Observe that the assumptions in the above Proposition cannot be dispensed of, as verified below.*

Consider $\varphi : \mathbb{R}_+ \to\to \mathbb{R}, \varphi(x) = [0,1)$. *Observe that* φ *is UHC and bounded valued but not closed valued , and* $\varphi([0,1]) = [0,1)$ *is not compact.*

Consider $\varphi : \mathbb{R}_+ \to\to \mathbb{R}, \varphi(x) = \mathbb{R}_+$. *Observe that* φ *is UHC and closed valued, but not bounded valued, and* $\varphi([0,1]) = \mathbb{R}_+$ *is not compact.*

Consider $\varphi : \mathbb{R}_+ \to\to \mathbb{R}_+, \quad \varphi(x) = \begin{cases} \{x\} & if \quad x \neq 1 \\ \{0\} & if \quad x = 1. \end{cases}$ *Observe that* φ *is not UHC and* $\varphi([0,1]) = [0,1)$ *is not compact.*

`Add Proposition 5, page 25 and Proposition 6, page 26, from Hildebrand (1974) ... maybe as exercises ...`

Remark 656 *Below, we summarize some facts we showed in the present Section, in a somehow informal manner.*

$$\langle(\text{if } \varphi \text{ is a fcn., it is cnt.}\rangle \Leftrightarrow \langle\varphi \text{ is UHC}\rangle \underset{\not\Leftarrow}{\not\Rightarrow} \langle\varphi \text{ is sequentially UHC,i.e., closed}\rangle \underset{\Leftarrow}{\not\Rightarrow} \langle(\text{if } \varphi \text{ is a fcn., it is cnt.}\rangle$$

$$\langle(\text{if } \varphi \text{ is a fcn, it is continuous}\rangle \Leftrightarrow \langle\varphi \text{ is LHC}\rangle \Leftrightarrow \langle\varphi \text{ is sequentially LHC}\rangle$$

$$\langle\varphi \text{ UHC and closed valued at } x\rangle \Rightarrow \langle\varphi \text{ is closed at } x\rangle$$
$$\langle\varphi \text{ UHC at } x\ \rangle \Leftarrow \langle\varphi \text{ is closed at } x \text{ and } \operatorname{Im}\varphi \text{ compact}\rangle$$

13.2 The Maximum Theorem

Theorem 657 *(Maximum Theorem) Let the metric spaces* (Π, d_Π), (X, d_X), *the correspondence* $\beta : \Pi \to\to X$ *and a function* $u : X \times \Pi \to \mathbb{R}$ *be given.*[5] *Define*

$$\xi : \Pi \to\to X,$$
$$\xi(\pi) = \{z \in \beta(\pi) : \forall x \in \beta(\pi), u(z,\pi) \geq u(x,\pi)\} = \arg\max_{x\in\beta(\pi)} u(x,\pi),$$

Assume that
β *is non-empty valued, compact valued and continuous,*
u *is continuous.*
Then
1. ξ *is non-empty valued, compact valued, UHC and closed,and*
2.

$$v : \Pi \to \mathbb{R}, \qquad v : \pi \mapsto \max_{x\in\beta(\pi)} u(x,\pi).$$

is continuous.

[5] Obviously, β stands for "budget correspondence" and u for "utility function".

Proof.

ξ is non-empty valued.

It is a consequence of the fact that β is non-empty valued and compact valued and of the Extreme Value Theorem - see Proposition 621.

ξ is compact valued.

We are going to show that for any $\pi \in \Pi$, $\xi(\pi)$ is a sequentially compact set. Consider a sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $\{x_n : n \in \mathbb{N}\} \subseteq \xi(\pi)$. Since $\xi(\pi) \subseteq \beta(\pi)$ and $\beta(\pi)$ is compact by assumption, without loss of generality, up to a subsequence, $x_n \to x_0 \in \beta(\pi)$. We are left with showing that $x_0 \in \xi(\pi)$. Take an arbitrary $z \in \beta(\pi)$. Since $\{x_n : n \in \mathbb{N}\} \subseteq \xi(\pi)$, we have that $u(x_n, \pi) \geq u(z, \pi)$. By continuity of u, taking limits with respect to n of both sides, we get $u(x_0, \pi) \geq u(z, \pi)$, i.e., $x_0 \in \xi(\pi)$, as desired.

ξ is UHC.

From Proposition 650, it suffices to show that given an arbitrary closed set V in X, ${}^w\xi^{-1}(V) := \{\pi \in \Pi : \xi(\pi) \cap V \neq \varnothing\}$ is closed in Π. Consider an arbitrary sequence $(\pi_n)_{n\in\mathbb{N}}$ such that $\{\pi_n : n \in \mathbb{N}\} \subseteq {}^w\xi^{-1}(V)$ and such that $\pi_n \to \pi_0$. We have to show that $\pi_0 \in {}^w\xi^{-1}(V)$.

Take a sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that for every n, $x_n \in \xi(\pi_n) \cap V \neq \varnothing$. Since $\xi(\pi_n) \subseteq \beta(\pi_n)$, it follows that $x_n \in \beta(\pi_n)$. We can now show the following

Claim. There exists a subsequence $(x_{n_k})_{n_k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \to x_0$ and $x_0 \in \beta(\pi_0)$.

Proof of the Claim.

Since $\{\pi_n : n \in \mathbb{N}\} \cup \{\pi_0\}$ is a compact set (Show it), and since, by assumption, β is UHC and compact valued, from Proposition 654, $\beta(\{\pi_n : n \in \mathbb{N}\} \cup \{\pi_0\})$ is compact. Since $\{x_n\}_n \subseteq \beta(\{\pi_n\} \cup \{\pi_0\})$, there exists a subsequence $(x_{n_k})_{n_k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which converges to some x_0. Since β is compact valued, it is closed valued, too. Then, β is UHC and closed valued and from Proposition 645, β is closed. Since

$$\pi_{n_k} \to \pi_0, \quad x_{n_k} \in \beta(\pi_{n_k}), \quad x_{n_k} \to x_0,$$

the fact that β is closed implies that $x_0 \in \beta(\pi_0)$.

End of the Proof of the Claim.

Choose an arbitrary element z_0 such that $z_0 \in \beta(\pi_0)$. Since we assumed that $\pi_n \to \pi_0$ and since β is LHC, there exists a sequence $(z_n)_{n\in\mathbb{N}} \in X^\infty$ such that $z_n \in \beta(\pi_n)$ and $z_n \to z_0$.

Summarizing, and taking the subsequences of $(\pi_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ corresponding to $(x_{n_k})_{n_k\in\mathbb{N}}$, we have for any n_k,

$$\begin{array}{l} \pi_{n_k} \to \pi_0, \\ x_{n_k} \to x_0, \quad x_{n_k} \in \xi(\pi_{n_k}), \quad x_0 \in \beta(\pi_0), \\ z_{n_k} \to z_0, \quad z_{n_k} \in \beta(\pi_{n_k}), \quad z_0 \in \beta(\pi_0). \end{array}$$

Then for any n_k, we have that $u(x_{n_k}, \pi_{n_k}) \geq u(z_{n_k}, \pi_{n_k})$. Since u is continuous, taking limits, we get that $u(x_0, \pi_0) \geq u(z_0, \pi_0)$ .Since the choice of z_0 in $\beta(\pi_0)$ was arbitrary, we have then $x_0 \in \xi(\pi_0)$.

Finally, since $(x_{n_k})_{n_k\in\mathbb{N}} \in V^\infty$, $x_{n_k} \to x_0$ and V is closed, $x_0 \in V$. Then $x_0 \in \xi(\pi_0) \cap V$ and $\pi_0 \in \{\pi \in \Pi : \xi(\pi) \cap V \neq \varnothing\} := {}^w\xi^{-1}(V)$, which was the desired result.

ξ is closed.

ξ is UHC and compact valued, and therefore closed valued. Then, from Proposition 645, it is closed, too.

v is a continuous function.

The basic idea of the proof is that v is a function and "it is equal to" the composition of UHC correspondences; therefore, it is a continuous function. A precise argument goes as follows.

Let the following correspondences be given:

$$(\xi, id) : \Pi \to\to X \times \Pi, \qquad \pi \mapsto \xi(\pi) \times \{\pi\},$$

$$\beta : X \times \Pi \rightarrow\rightarrow \mathbb{R}, \qquad (x,\pi) \mapsto \{u(x,\pi)\}.$$

Then, from Definition 652,

$$(\beta \circ (\xi, id))(\pi) = \cup_{(x,\pi)\in\xi(\pi)\times\{\pi\}} \{u(x,\pi)\}.$$

By definition of ξ,

$$\forall \pi \in \Pi, \forall \overline{x} \in \xi(\pi), \qquad \cup_{(x,\pi)\in\xi(\pi)\times\{\pi\}} \{u(x,\pi)\} = \{u(\overline{x},\pi)\},$$

and

$$\forall \pi \in \Pi, \qquad (\beta \circ (\xi, id))(\pi) = \{u(\overline{x},\pi)\} = \{v(\pi)\}. \tag{13.3}$$

Now, (ξ, id) is UHC, and since u is a continuous function, β is UHC as well. From Proposition 653, $\beta \circ (\xi, id)$ is UHC and, from 13.3, v is a continuous function.

■

A sometimes more useful version of the Maximum Theorem is one which does not use the fact that β is UHC.

Theorem 658 *(Maximum Theorem) Consider the correspondence $\beta : \Pi \rightarrow\rightarrow X$ and the function $u : X \times \Pi \rightarrow \mathbb{R}$ defined in Theorem 657 and Π, X Euclidean spaces.*

Assume that

β is non-empty valued, compact valued, convex valued, closed and LHC.

u is continuous.

Then

1. ξ is a non-empty valued, compact valued, closed and UHC correspondence;

2. v is a continuous function.

Proof.

The desired result follows from next Proposition.

■

Proposition 659 *Consider the correspondence $\beta : \Pi \rightarrow\rightarrow X$, with Π and X Euclidean spaces.*

Assume that β is non-empty valued, compact valued, convex valued, closed and LHC. Then β is UHC.

Proof.

See Hildenbrand (1974) Lemma 1 page 33. The proof requires also Theorem 1 in Hildenbrand (1974).

■

The following result allows to substitute the requirement "β is LHC" with the easier to check requirement "$Cl\beta$ is LHC".

Proposition 660 *Consider the correspondence $\varphi : \Pi \rightarrow\rightarrow X$. φ is LHC $\Leftrightarrow$ $Cl\phi$ is LHC.*

Proof.

Preliminary Claim.

V open set, , $Cl\phi(\pi) \cap V \neq \varnothing \Rightarrow \varphi(\pi) \cap V \neq \varnothing$.

Proof of the Preliminary Claim.

Take $z \in Cl\phi(\pi) \cap V \neq \varnothing$. Since V is open, $\exists \varepsilon > 0$ such that $B(z,\varepsilon) \subseteq V$. Since $z \in Cl\phi(\pi)$, $\exists \{z_n\} \subseteq \varphi(\pi)$ such that $z_n \rightarrow z$. But then $\exists n_\varepsilon$ such that $n > n_\varepsilon \Rightarrow z_n \in B(z,\varepsilon) \subseteq V$. But $z_n \in V$ and $z_n \in \varphi(\pi)$ implies that $\varphi(\pi) \cap V \neq \varnothing$.

End of the Proof of the Preliminary Claim.

[$\Rightarrow$]

Take an open set V such that $Cl\phi(\pi) \cap V \neq \varnothing$. We want to show that there exists an open set U^* such that $\pi \in U^*$ and $\forall \xi \in U^*$, $Cl\phi(\xi) \cap V \neq \varnothing$. From the preliminary remark, it must be the case that $\varphi(\pi) \cap V \neq \varnothing$. Then, since φ is LHC, there exists an open set U such that $\pi \in U$ and $\forall \xi \in U^*$, $\varphi(\pi) \cap V \neq \varnothing$. Since $Cl\phi(\pi) \supseteq \varphi(\pi)$, we also have $Cl\phi(\pi) \cap V \neq \varnothing$. Choosing $U^* = U$, we are done.

[$\Leftarrow$]

Since $\varphi(\pi) \cap V \neq \varnothing$, then $Cl\phi(\pi) \cap V \neq \varnothing$, and, by assumption, $\exists$ open set U' such that $\pi \in U'$ and $\forall \xi \in U'$, $Cl\phi(\xi) \cap V \neq \varnothing$. Then, from the preliminary remark, it must be the case that $\varphi(\pi) \cap V \neq \varnothing$.

■

Remark 661 *In some economic models, a convenient strategy to show that a correspondence β is LHC is the following one. Introduce a correspondence $\widehat{\beta}$; show that $\widehat{\beta}$ is LHC; show that* $\text{Cl}\,\widehat{\beta} = \beta$. *Then from the above Proposition 660, the desired result follows - see, for example, point 5 the proof of Proposition 674 below.*

13.3 Fixed point theorems

A thorough analysis of the many versions of fixed point theorems existing in the literature is outside the scope of this notes. Below, we present a useful relatively general version of fixed point theorems both in the case of functions and correspondences.

Theorem 662 ***(The Brouwer Fixed Point Theorem)***

For any $n \in \mathbb{N}\setminus\{0\}$, let S be a nonempty, compact, convex subset of $\mathbb{R}^n$. If $f : S \to S$ is a continuous function, then $\exists x \in S$ such that $f(x) = x$.

Proof. For a (not self-contained) proof, see Ok (2007), page 279. ■

Just to try to avoid having a Section without a proof, let's show the following extremely simple version of that theorem.

Proposition 663 *If $f : [0,1] \to [0,1]$ is a continuous function, then $\exists x \in [0,1]$ such that $f(x) = x$.*

Proof. If $f(0) = 0$ or $f(1) = 1$, the result is true. Then suppose otherwise, i.e., $f(0) \neq 0$ and $f(1) \neq 1$, i.e., since the domain of f is $[0,1]$, suppose that $f(0) > 0$ and $f(1) < 1$. Define

$$g : [0,1] \to \mathbb{R}, \qquad : x \mapsto x - f(x).$$

Clearly, g is continuous, $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$. Then, from the intermediate value for continuous functions, $\exists x \in [0,1]$ such that $g(x) = x - f(x) = 0$, i.e., $x = f(x)$, as desired. ■

Theorem 664 ***(Kakutani's Fixed Point Theorem)***

For any $n \in \mathbb{N}\setminus\{0\}$, let S be a nonempty, compact, convex subset of $\mathbb{R}^n$. If $\varphi : S \to\to S$ is a convex valued, closed correspondence, then $\exists x \in S$ such that $\varphi(x) \ni x$.

Proof. For a proof, see Ok (2007), page 331. ■

13.4 Application of the maximum theorem to the consumer problem

Definition 665 *(Mas Colell (1996), page 17) Commodities are goods and services available for purchases in the market.*

We assume the number of commodities is finite and equal to C. Commodities are indexed by superscript $c = 1, ..., C$.

Definition 666 *A commodity vector is an element of the commodity space $\mathbb{R}^C$.*

Definition 667 *(almost Mas Colell(1996), page 18) A consumption set is a subset of the commodity space $\mathbb{R}^C$. It is denoted by X. Its elements are the vector of commodities the individual can conceivably consume given the physical or institutional constraints imposed by the environment.*

Example 668 *See Mas colell pages 18, 19.*

Common assumptions on X are that it is convex,bounded below and unbounded. Unless otherwise stated, we make the following stronger

Assumption 1 $X = \mathbb{R}^C_+ := \left\{x \in \mathbb{R}^C : x \geq 0\right\}$.

Definition 669 $p \in \mathbb{R}^C$ *is the vector of commodity prices.*

Households' choices are limited also by an economic constraint: they cannot buy goods whose value is bigger than their wealth, i.e., it must be the case that $px \leq w$, where w is household's wealth.

Remark 670 w *can take different specifications. For example, we can have* $w = pe$*, where* $e \in \mathbb{R}^C$ *is the vector of goods owned by the household, i.e., her endowments.*

Assumption 2 All commodities are traded in markets at publicly observable prices, expressed in monetary unit terms.

Assumption 3 All commodities are assumed to be strictly goods (and not "bad"), i.e., $p \in \mathbb{R}^C_{++}$.

Assumption 4 Households behave as if they cannot influence prices.

Definition 671 *The budget set is*

$$\beta(p, w) := \left\{x \in \mathbb{R}^C_+ : px \leq w\right\}.$$

With some abuse of notation we define the budget correspondence as

$$\beta : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \to\to \mathbb{R}^C,\ \beta(p, w) = \left\{x \in \mathbb{R}^C_+ : px \leq w\right\}.$$

Definition 672 *The utility function is*

$$u : X \to \mathbb{R}, \qquad x \mapsto u(x)$$

Definition 673 *The Utility Maximization Problem* (UMP) *is*

$$\max_{x\in\mathbb{R}^C_+} \quad u(x) \qquad s.t. \quad px \leq w,\ or\ x \in \beta(p, w).$$

$\xi : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \to\to \mathbb{R}^C,\ \xi(p, w) = \arg\max(UMP)$ *is the demand correspondence.*

Theorem 674 ξ *is a non-empty valued, compact valued, closed and UHC correspondence and*

$$v : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \to \mathbb{R},\ v : (p, w) \mapsto \max(UMP),$$

i.e., the indirect utility function, is a continuous function.

Proof.

As an application of the (second version of) the Maximum Theorem, i.e., Theorem 658, we have to show that β is non-empty valued, compact valued, convex valued, closed and LHC.

1. β is non-empty valued.

$x = \left(\frac{w}{Cp^c}\right)_{c=1}^{C} \in \beta(p, w)$ (or, simpler, $0 \in \beta(p, w)$).

2. β is compact valued.

$\beta(p, w)$ is closed because is the intersection of the inverse image of closed sets via continuous functions.

$\beta(p, w)$ is bounded below by zero.

$\beta(p, w)$ is bounded above because for every c, $x^c \leq \frac{w - \sum_{c'\neq c} p^{c'} x^{c'}}{p^c} \leq \frac{w}{p^c}$, where the first inequality comes from the fact that $px \leq w$, and the second inequality from the fact that $p \in \mathbb{R}^C_{++}$ and $x \in \mathbb{R}^C_+$.

3. β is convex valued.

To see that, simply, observe that $(1 - \lambda) px' + \lambda px'' \leq (1 - \lambda) w + \lambda w = w$.

4. β is closed.

We want to show that for every sequence $\{(p_n, w_n)\}_n \subseteq \mathbb{R}^C_{++} \times \mathbb{R}_{++}$ such that

$(p_n, w_n) \to (p, w)$, $x_n \in \beta(p_n, w_n)$, $x_n \to x$,
it is the case that $x \in \beta(p, w)$.

Since $x_n \in \beta(p_n, w_n)$, we have that $p_n x_n \le w_n$ and $x_n \ge 0$. Taking limits of both sides of both inequalities, we get $px \le w$ and $x \ge 0$,i.e., $x \in \beta(p, w)$.

5. β is LHC.

We proceed as follows: a. *Int* β is LHC; b. *Cl Int* $\beta = \beta$. Then, from Proposition 660 the result follows.

a. Observe that $Int\ \beta(p, w) := \{x \in \mathbb{R}^C_+ : x \gg 0 \text{ and } px < w\}$ and that $Int\ \beta(p, w) \neq \varnothing$, since $x = \left(\frac{w}{2Cp^c}\right)_{c=1}^{C} \in Int\ \beta(p, w)$. We want to show that the following is true.

For every sequence $(p_n, w_n)_n \in \left(\mathbb{R}^C_{++} \times \mathbb{R}_{++}\right)^\infty$ such that $(p_n, w_n) \to (p, w)$ and for any $x \in Int\beta(p, w)$,

there exists a sequence $\{x_n\}_n \subseteq \mathbb{R}^C_+$ such that $\forall n$, $x_n \in Int\ \beta(p_n, w_n)$ and $x_n \to x$.

$p_n x - w_n \to px - w < 0$ (where the strict inequality follows from the fact that $x \in Int\ \beta(p, w)$. Then, $\exists N$ such that $n \ge N \Rightarrow p_n x - w_n < 0$.

For $n \le N$, choose an arbitrary $x_n \in Int\ \beta(p_n, w_n) \neq \varnothing$. Since $p_n x - w < 0$, for every $n > N$, there exists $\varepsilon_n > 0$ such that $z \in B(x, \varepsilon_n) \Rightarrow p_n z - w_n < 0$.

For any $n > N$, choose $x_n = x + \frac{1}{\sqrt{C}} \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\} \cdot \mathbf{1}$. Then,

$$d(x, x_n) = \left(C\left(\frac{1}{\sqrt{C}} \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\}\right)^2\right)^{\frac{1}{2}} = \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\} < \varepsilon_n,$$

i.e., $x_n \in B(x, \varepsilon_n)$ and therefore

$$p_n x_n - w_n < 0 \quad (1).$$

Since $x_n \gg x$, we also have

$$x_n \gg 0 \quad (2).$$

(1) and (2) imply that $x_n \in Int\ \beta(p_n, w_n)$. Moreover, since $x_n \gg x$, we have $0 \leqq \lim_{n\to+\infty}(x_n - x) = \lim_{n\to\infty} \frac{1}{\sqrt{C}} \cdot \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\} \cdot \mathbf{1} \leqq \lim_{n\to\infty} \frac{1}{n}\frac{1}{\sqrt{C}} \cdot \mathbf{1} = 0$, i.e., $\lim_{n\to+\infty} x_n = x$.[6]

b.

It follows from the fact that the budget correspondence is the intersection of the inverse images of half spaces via continuous functions.

2.

It follows from Proposition 675, part (4), and the Maximum Theorem.

■

Proposition 675 *For every* $(p, w) \in \mathbb{R}^C_{++} \times \mathbb{R}_{++}$,

(1) $\forall \alpha \in \mathbb{R}_{++}, \xi(\alpha p, \alpha w) = \xi(p, w)$;

(2) *if* u *is LNS,* $\forall x \in \mathbb{R}^C_+$, $x \in \xi(p, w) \Rightarrow px = w$;

(3) *if* u *is quasi-concave,* ξ *is convex valued;*

(4) *if* u *is strictly quasi-concave,* ξ *is single valued, i.e., it is a function.*

Proof.

(1)

It simply follows from the fact that $\forall \alpha \in \mathbb{R}_{++}, \beta(\alpha p, \alpha w) = \beta(p, w)$.

(2)

Suppose otherwise, then $\exists x' \in \mathbb{R}^C_+$ such that $x' \in \xi(p, w)$ and $px' < w$. Therefore,$\exists \varepsilon' > 0$ such that $B(x', \varepsilon') \subseteq \beta(p, w)$ (take $\varepsilon' = d(x', H(p, w))$). Then, from the fact that u is LNS, there exists x^* such that $x^* \in B(x', \varepsilon') \subseteq \beta(p, w)$ and $u(x^*) > u(x')$, i.e., $x' \notin \xi(p, w)$, a contradiction.

(3)

Assume there exist x', x'' such that $x', x'' \in \xi(p, w)$. We want to show that $\forall \lambda \in [0, 1]$, $x^\lambda := (1 - \lambda)x' + \lambda x'' \in \xi(p, w)$. Observe that $u(x') = u(x'') := u^*$. From the quasi-concavity of u, we

[6] Or simply

$$0 \le \lim_{n\to\infty} d(x, x_n) \le \lim_{n\to\infty} \frac{1}{n} = 0.$$

have $u(x^\lambda) \geq u^*$. We are therefore left with showing that $x^\lambda \in \beta(p,w)$, i.e., β is convex valued. To see that, simply, observe that $px^\lambda = (1-\lambda)px' + \lambda px'' \leq (1-\lambda)w + \lambda w = w$.

(4) Assume otherwise. Following exactly the same argument as above we have $x', x'' \in \xi(p,w)$, and $px^\lambda \leq w$. Since u is strictly quasi concave, we also have that $u(x^\lambda) > u(x') = u(x'') := u^*$, which contradicts the fact that $x', x'' \in \xi(p,w)$.

■

Proposition 676 *If u is a continuous LNS utility function, then the indirect utility function has the following properties.*

For every $(p,w) \in \mathbb{R}^C_{++} \times \mathbb{R}_{++}$,

(1) $\forall \alpha \in \mathbb{R}_{++}, v(\alpha p, \alpha w) = v(p,w)$;

(2) *Strictly increasing in w and for every c, non increasing in p^c;*

(3) *for every $\overline{v} \in \mathbb{R}$, $\{(p,w) : v(p,w) \leq \overline{v}\}$ is convex.*

(4) *continuous.*

Proof.

(1) It follows from Proposition 675 (2).

(2)

If w increases, say by Δw, then, from Proposition 675 (2), $px(p,w) < w + \Delta w$. Define $x(p,w) := x'$. Then,$\exists \varepsilon' > 0$ such that $B(x', \varepsilon') \subseteq \beta(p, w+\Delta w)$ (take $\varepsilon' = d(x', H(p, w+\Delta w))$). Then, from the fact that u is LNS, there exists x^* such that $x^* \in B(x', \varepsilon') \subseteq \beta(p, w+\Delta w)$ and $u(x^*) > u(x')$. The result follows observing that $v(p, w+\Delta w) \geq u(x^*)$.

Similar proof applies to the case of a decrease in p. Assume $\Delta p^{c'} < 0$. Define $\Delta := (\Delta^c)_{c=1}^C \in \mathbb{R}^C$ with $\Delta^c = 0$ iff $c \neq c'$ and $\Delta^{c'} = \Delta p^{c'}$. Then,

$px(p,w) = w \Rightarrow (p+\Delta)x(p,w) = px(p,w) + \overbrace{\Delta p^{c'} x^{c'}}^{(\leq 0)}(p,w) =$
$= w + \Delta p^{c'} x^{c'}(p,w) \leq w$. The remaining part of the proof is the same as in the case of an increase of w.

(3) Take $(p', w'), (p'', w'') \in \{(p,w) : v(p,w) \leq \underline{v}\} := S(\overline{v})$. We want to show that $\forall \lambda \in [0,1]$, $(p^\lambda, w^\lambda) := (1-\lambda)(p', w') + \lambda(p'', w'') \in S(\overline{v})$, i.e., $x \in \xi(p^\lambda, w^\lambda) \Rightarrow u(x) > \overline{v}$.

$x \in \xi(p^\lambda, w^\lambda) \Rightarrow p^\lambda x \leq w^\lambda \Leftrightarrow (1-\lambda)p' + \lambda p'' \leq (1-\lambda)w' + \lambda w''$.

Then, either $p'x \leq w'$ or $p''x \leq w''$. If $p'x \leq w'$, then $u(x) \leq v(p', w') \leq \overline{v}$. Similarly, if $p''x \leq w''$.

(4)

It was proved in Theorem 674.

■

Part III

Differential calculus in Euclidean spaces

Chapter 14

Partial derivatives and directional derivatives

14.1 Partial Derivatives

The[1] concept of partial derivative is not that different from the concept of "standard" derivative of a function from $\mathbb{R}$ to $\mathbb{R}$, in fact we are going to see that partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}$ are just standard derivatives of a naturally associated function from $\mathbb{R}$ to $\mathbb{R}$.

Recall that for any $k \in \{1, ..., n\}$, $e_n^k = (0, .., 1, ..., 0)$ is the $k-th$ vector in the canonical basis of $\mathbb{R}^n$.

Definition 677 *Let a set $S \subseteq \mathbb{R}^n$, a point $x_0 = (x_{0k})_{k=1}^n \in Int\ S$ and a function $f : S \to \mathbb{R}$ be given. If the following limit exists and it is finite*

$$\lim_{h\to 0} \frac{f\left(x_0 + he_n^k\right) - f(x_0)}{h} = \lim_{x_k \to x_{0k}} \frac{f\left(x_0 + (x_k - x_{0k})\, e_n^k\right) - f(x_0)}{x_k - x_{0k}} \tag{14.1}$$

then it is called the partial derivative of f with respect to the $k-th$ coordinate computed in x_0 and it is denoted by any of the following symbols

$$D_{x_k} f(x_0), \qquad D_k f(x_0), \qquad \frac{\partial f}{\partial x_k}(x_0), \qquad \frac{\partial f(x)}{\partial x_k}_{|x=x_0}.$$

Remark 678 *As said above, partial derivatives are not really a new concept. We are just treating f as a function of one variable at the time, keeping the other variables fixed. In other words, for simplicity taking $S = \mathbb{R}^n$ and using the notation of the above definition, we can define*

$$g_k : \mathbb{R} \to \mathbb{R}, \qquad g_k(x_k) = f\left(x_0 + (x_k - x_{0k})\, e_n^k\right)$$

a function of only one variable, and, by definition of g_k,

$$\begin{aligned} g_k'(x_{0k}) &= \lim_{x_k \to x_{0k}} \tfrac{g_k(x_{0k} + x_k - x_{0k}) - g_k(x_{0k})}{x_k - x_{0k}} = \\ &= \lim_{h\to 0} \tfrac{g_k(x_{0k}+h) - g_k(x_{0k})}{h} = \lim_{h \to 0} \tfrac{f\left(x_0 + he^k\right) - f(x_0)}{h} = D_{x_k} f(x_0). \end{aligned} \tag{14.2}$$

Example 679 *Given $f : \mathbb{R}^3 \to \mathbb{R}$,*

$$f(x_1, x_2, x_3) = e^{xy} \cos x + \sin yz$$

we have

$$\begin{pmatrix} D_{x_1} f(x) \\ D_{x_2} f(x) \\ D_{x_2} f(x) \end{pmatrix} = \begin{pmatrix} -(\sin x)\, e^{xy} + y(\cos x)\, e^{xy} \\ z \cos yz + x(\cos x)\, e^{xy} \\ y \cos yz \end{pmatrix}$$

[1] In this Part, I follow closely Section 5.14 and chapters 12 and 13 in Apostol (1974).

Remark 680 *Loosely speaking, we can give the following geometrical interpretation of partial derivatives. Given $f : \mathbb{R}^2 \to \mathbb{R}$ admitting partial derivatives, $\frac{\partial f(x_0)}{\partial x_1}$ is the slope of the graph of the function obtained cutting the graph of f with a plane which is*

orthogonal to the $x_1 - x_2$ plane, and

going through the line parallel to the x_1 axis and passing through the point x_0, line to which we have given the same orientation as the x_1 axis.

`picture to be added.`

Definition 681 *Given an open subset S in $\mathbb{R}^n$ and a function $f : S \to \mathbb{R}$, if $\forall k \in \{1, ..., n\}$, the limit in* (14.1) *exists, we call the gradient of f in x_0 the following vector*

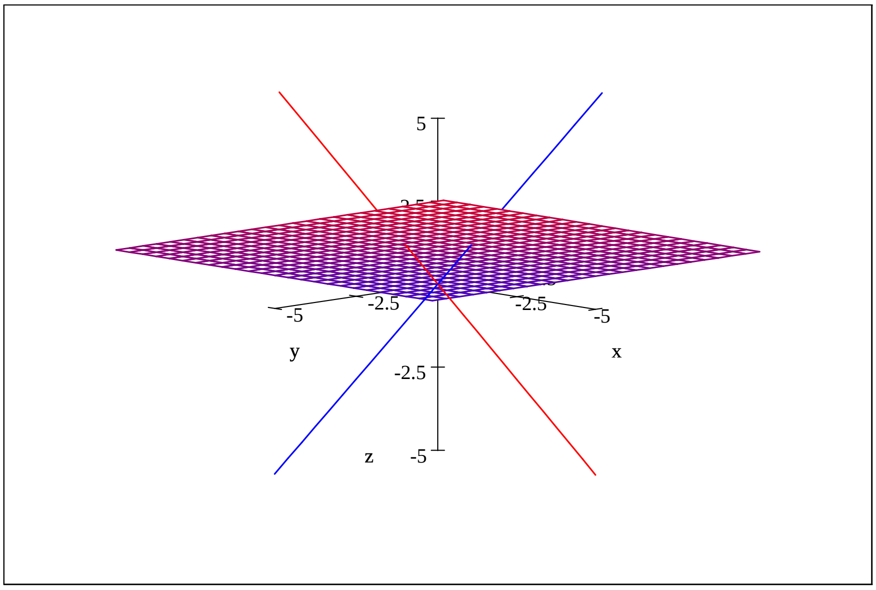
$$(D_k f(x_0))_{k=1}^n$$

and we denote it by

$$Df(x_0)$$

Remark 682 *The existence of the gradient for f in x_0 does not imply continuity of the function in x_0, as the following example shows.*

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x_1, x_2) = \begin{cases} x_1 + x_2 \quad if & \begin{array}{l} either\ x_1 = 0\ or\ x_2 = 0 \\ i.e.,\ (x_1, x_2) \in (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \end{array} \\ 1 & otherwise \end{cases}$$



$$D_1 f(0) = \lim_{x_1 \to 0} \frac{f(x_1, 0) - f(0,0)}{x_1 - 0} = \lim_{x_1 \to 0} \frac{x_1}{x_1 - 0} = 1$$

and similarly

$$D_2 f(0) = 1.$$

f is not continuous in 0: we want to show that $\exists \varepsilon > 0$ such that $\forall \delta > 0$ there exists $(x_1, x_2) \in \mathbb{R}^2$ such that $(x_1, x_2) \in B(0, \varepsilon)$ and $|f(x_1, x_2) - f(0,0)| \geq \varepsilon$. Take $\varepsilon = \frac{1}{2}$ and any $(x_1, x_2) \in B(0, \delta)$ such that $x_1 \neq 0$ and $x_2 \neq 0$. Then $|f(x_1, x_2) - f(0,0)| = 1 > \varepsilon$.

14.2 Directional Derivatives

A first generalization of the concept of partial derivative of a function is presented in Definition 684 below.

Definition 683 *Given*

$$f : S \subseteq \mathbb{R}^n \to \mathbb{R}^m, \qquad x \mapsto f(x),$$

$\forall i \in \{1, .., m,\}$, *the function*

$$f_i : S \subseteq \mathbb{R}^n \to \mathbb{R}^m, \qquad x \mapsto \ i-th \ component \ of \ f(x).$$

is called the $i-th$ component function of f.

Therefore,

$$\forall x \in S, \quad f(x) = (f_i(x))_{i=1}^m. \tag{14.3}$$

Definition 684 *Given $m, n \in \mathbb{N}\backslash\{0\}$, a set $S \subseteq \mathbb{R}^n$, $x_0 \in \ Int\ S$, $u \in \mathbb{R}^n$, $h \in \mathbb{R}$ such that $x_0 + hu \in S$, $f : S \to \mathbb{R}^m$, we call the directional derivative of f at x_0 in the direction u, denoted by the symbol*

$$f'(x_0; u),$$

the limit

$$\lim_{h \to 0} \frac{f(x_0 + hu) - f(x_0)}{h} \tag{14.4}$$

if it exists and it is finite.

Remark 685 *Assume that the limit in* (14.4) *exists and it is finite. Then, from* (14.3) *and using Proposition 493,*

$$f'(x_0; u) = \lim_{h \to 0} \frac{f(x_0 + hu) - f(x_0)}{h} = \left(\lim_{h \to 0} \frac{f_i(x_0 + hu) - f_i(x_0)}{h}\right)_{i=1}^m = (f_i'(x_0; u))_{i=1}^m.$$

If $u = e_n^j$, the $j-th$ element of the canonical basis in $\mathbb{R}^n$, we then have

$$f'\left(x_0; e_n^j\right) = \left(\lim_{h \to 0} \frac{f_i\left(x_0 + he_n^j\right) - f_i(x_0)}{h}\right)_{i=1}^m = \left(f_i'\left(x_0; e_n^j\right)\right)_{i=1}^m \overset{(*)}{=} \left(D_{x_j} f_i(x_0)\right)_{i=1}^m := D_{x_j} f(x_0) \tag{14.5}$$

where equality $(*)$ follows from (14.2).

We can then construct a matrix whose n columns are the above vectors, a matrix which involves all partial derivative of all component functions of f. That matrix is formally defined below.

Definition 686 *Assume that $f = (f_i)_{i=1}^m : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ admits all partial derivatives in x_0. The Jacobian matrix of f at x_0 is denoted by $Df(x_0)$ and is the following $m \times n$ matrix:*

$$\begin{bmatrix} D_{x_1} f_1(x_0) & ... & D_{x_j} f_1(x_0) & ... & D_{x_n} f_1(x_0) \\ ... & & ... & & ... \\ D_{x_1} f_i(x_0) & ... & D_{x_j} f_i(x_0) & ... & D_{x_n} f_i(x_0) \\ ... & & ... & & ... \\ D_{x_1} f_m(x_0) & ... & D_{x_j} f_m(x_0) & ... & D_{x_n} f_m(x_0) \end{bmatrix} = \begin{bmatrix} D_{x_1} f(x_0) & ... & D_{x_j} f(x_0) & ... & D_{x_n} f(x_0) \end{bmatrix} =$$

$$= \begin{bmatrix} f'\left(x_0; e_n^1\right) & ... & f'\left(x_0; e_n^j\right) & ... & f'\left(x_0; e_n^n\right) \end{bmatrix}.$$

Remark 687 *How to easily write the Jacobian matrix of a function.*

To compute the Jacobian of f is convenient to construct a table as follows.

1. In the first column, write the m vector component functions $f_1, ..., f_i, ..., f_m$ of f.

2. In the first row, write the subvectors $x_1, ..., x_j, ..., x_n$ of x.

3. For each i and j, write the partial Jacobian matrix $D_{x_j} f_i(x)$ in the entry at the intersection of the $i-th$ row and $j-th$ column.

We then obtain the following table,

$$\begin{array}{cc|ccc|}
 & & x_1 \quad ... & x_j \quad ... & x_n \\
f_1 & & D_{x_1} f_1(x) & D_{x_j} f_1(x) & D_{x_n} f_1(x) \\
... & & & & \\
f_i & & D_{x_1} f_i(x) & D_{x_j} f_i(x) & D_{x_n} f_i(x) \\
... & & & & \\
f_m & & D_{x_1} f_m(x) & D_{x_j} f_m(x) & D_{x_n} f_m(x)
\end{array}$$

where the Jacobian matrix is the part of the table between square brackets.

Example 688 *Given* $f : \mathbb{R}^4 \to \mathbb{R}^5$,

$$f(x,y,z,t) = \begin{pmatrix} \frac{xy}{x^2+1} \\ \frac{x+yz}{e^x} \\ \frac{xyzt}{e^t} \\ x+y+z+t \\ x^2+t^2 \end{pmatrix}$$

its Jacobian matrix is

$$\begin{bmatrix} \frac{y}{x^2+1} - 2x^2 \frac{y}{(x^2+1)^2} & \frac{x}{x^2+1} & 0 & 0 \\ \frac{1}{e^x} - \frac{1}{e^x}(x+yz) & \frac{z}{e^x} & \frac{y}{e^x} & 0 \\ ty\frac{z}{e^t} & tx\frac{z}{e^t} & tx\frac{y}{e^t} & xy\frac{z}{e^t} - txy\frac{z}{e^t} \\ 1 & 1 & 1 & 1 \\ 2x & 0 & 0 & 2t \end{bmatrix}_{5\times 4}$$

Remark 689 *From Remark 685,*

$$\forall u \in \mathbb{R}^n, \ f'(x_0;u) \ \textit{exists} \ \Rightarrow Df(x_0) \ \textit{exists} \tag{14.6}$$

On the other hand, the opposite implication does not hold true. Consider the example in Remark 682. There, we have seen that

$$D_x f(0) = D_y f(0) = 1$$

But if $u = (u_1, u_2)$ *with* $u_1 \neq 0$ *and* $u_2 \neq 0$, *we have*

$$\lim_{h\to 0} \frac{f(0+hu) - f(0)}{h} = \lim_{h\to 0} \frac{1-0}{h} = \infty$$

Remark 690 *Again loosely speaking, we can give the following geometrical interpretation of directional derivatives. Take* $f : \mathbb{R}^2 \to \mathbb{R}$ *admitting directional derivatives.* $f(x_0;u)$ *with* $\|u\| = 1$ *is the slope the graph of the function obtained cutting the graph of* f *with a plane which is*

orthogonal to the $x_1 - x_2$ *plane, and*

going through the line going through the points x_0 *and* $x_0 + u$, *line to which we have given the same orientation as* u.

`picture to be added.`

Example 691 *Take*

$$f : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto x \cdot x = \|x\|^2$$

Then, the existence of $f'(x_0;u)$ *can be checked computing the following limit.*

$$\lim_{h\to 0} \frac{f(x_0+hu)-f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+hu)(x_0+hu)-x_0x_0}{h} =$$

$$= \lim_{h\to 0} \frac{x_0x_0+hx_0u+hux_0+h^2uu-x_0x_0}{h} =$$

$$= \lim_{h\to 0} \frac{2hx_0u+h^2uu}{h} = \lim_{h\to 0} 2x_0u + huu = 2x_0u$$

Exercise 692 *Verify that*

$$f'(x_0; -u) = -f'(x_0; u).$$

Solution:

$$f'(x_0; -u) = \lim_{h \to 0} \frac{f(x_0 + h(-u)) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0 - hu) - f(x_0)}{h} =$$

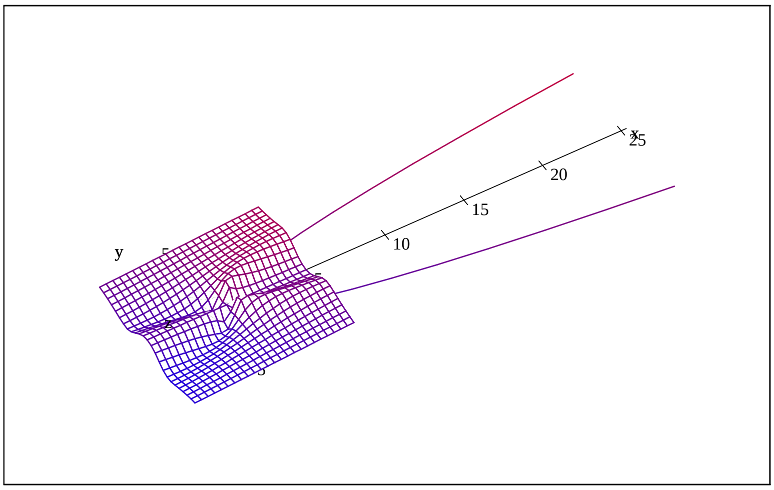
$$= -\lim_{k \to 0} \frac{f(x_0 + (-h)u) - f(x_0)}{-h} \overset{k:=-h}{=} -\lim_{k \to 0} \frac{f(x_0 + ku) - f(x_0)}{k} = -f'(x_0; u).$$

Remark 693 *It is **not** the case that*

$$\forall u \in \mathbb{R}^n,\ f'(x_0; u) \text{ exists } \Rightarrow f \text{ is continuous in } x_0 \tag{14.7}$$

as the following example shows. Consider

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^4} & if \quad x \neq 0 \\ 0 & if \quad x = 0, i.e., (x, y) \in \{0\} \times \mathbb{R} \end{cases}$$



Let's compute $f'(0; u)$. *If* $u_1 \neq 0$.

$$\lim_{h \to 0} \frac{f(0 + hu) - f(0)}{h} = \lim_{h \to 0} \frac{hu_1 \cdot h^2 u_2^2}{(h^2 u_1^2 + h^4 u_2^4) h} = \lim_{h \to 0} \frac{u_1 \cdot u_2^2}{u_1^2 + h^2 u_2^4} = \frac{u_2^2}{u_1}$$

If $u_1 = 0$,*we have*

$$\lim_{h \to 0} \frac{f(0 + hu) - f(0)}{h} = \lim_{h \to 0} \frac{f(0, hu_2) - f(0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

On the other hand, if $x = y^2$ *and* $x, y \neq 0$, *i.e., along the graph of the parabola* $x = y^2$ *except the origin, we have*

$$f(x, y) = f\left(y^2, y\right) = \frac{y^4}{y^4 + y^4} = \frac{1}{2}$$

while

$$f(0, 0) = 0.$$

Roughly speaking, the existence of partial derivatives in a given point in all directions implies "continuity along straight lines" through that point; it does not imply " continuity along all possible curves through that point", as in the case of the parabola in the picture above.

Remark 694 *We are now left with two problems:*

1. *Is there a definition of derivative whose existence implies continuity?*
2. *Is there any "easy" way to compute the directional derivative?*

Appendix `(to be corrected)`

There are other definitions of directional derivatives used in the literature.

Let the following objects be given: $m, n \in \mathbb{N} \setminus \{0\}$, a set $S \subseteq \mathbb{R}^n$, $x_0 \in$ *Int* S, $u \in \mathbb{R}^n$, $h \in \mathbb{R}$ such that $x_0 + hu \in S$, $f : S \to \mathbb{R}^m$,

Definition 695 *(our definition following* ***A****postol (1974)) We call the directional derivative of f at x_0 in the direction u according to Apostol, denoted by the symbol*

$$f'_A(x_0; u),$$

the limit

$$\lim_{h \to 0} \frac{f(x_0 + hu) - f(x_0)}{h} \tag{14.8}$$

if it exists and it is finite.

Definition 696 *(**G**irsanov (1972)) We call the directional derivative of f at x_0 in the direction u according to Girsanov, denoted by the symbol*

$$f'_G(x_0; u),$$

the limit

$$\lim_{\underline{h \to 0^+}} \frac{f(x_0 + hu) - f(x_0)}{h} \tag{14.9}$$

if it exists and it is finite.

Definition 697 *(**W**ikipedia) Take $u \in \mathbb{R}^n$ such that $\|u\| = 1$. We call the directional derivative of f at x_0 in the direction u according to Wikipedia, denoted by the symbol*

$$f'_W(x_0; u),$$

the limit

$$\lim_{\underline{h \to 0^+}} \frac{f(x_0 + hu) - f(x_0)}{h} \tag{14.10}$$

if it exists and it is finite.

Fact 1. For given $x_0 \in S, u \in \mathbb{R}^n$

$$A \Rightarrow G \Rightarrow W,$$

while the opposite implications do not hold true. In particular, to see way $A \not\Leftarrow G$, just take $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases},$$

and observe that while the right derivative in 0 is

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{1 - 1}{h} = 0,$$

while the left derivative is

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{0 - 1}{h} = +\infty.$$

Fact 2. For given $x_0 \in S$,

$$f'_W(x, u) \text{ exists} \Rightarrow f'_G(x, v) \text{ exists for any } v = \alpha u \text{ and } \alpha \in \mathbb{R}_{++}.$$

Proof.

$$f'_G(x, v) = \lim_{h \to 0^+} \tfrac{f(x_0 + hv) - f(x_0)}{h} = \alpha \lim_{h \to 0^+} \tfrac{f(x_0 + h\alpha u) - f(x_0)}{\alpha h} \overset{k = \alpha h > 0}{=}$$

$$= \alpha \lim_{h \to 0^+} \tfrac{f(x_0 + ku) - f(x_0)}{k} = \alpha f'_W(x, u).$$

Fact 3. Assume that $u \neq 0$ and $x_0 \in \mathbb{R}^n$. Then the following implications are true:

$$\forall u \in \mathbb{R}^n,\ f'_A(x,u) \text{ exists} \quad \Leftrightarrow \quad \forall u \in \mathbb{R}^n,\ f'_G(x,u) \text{ exists} \quad \Leftrightarrow \quad \forall u \in \mathbb{R}^n \text{ such that } \|u\| = 1,\ f'_W(x,u) \text{ exists.}$$

Proof.
From Fact 1, we are left with showing just two implications.
$G \Rightarrow A$.
We want to show that

$$\forall u \in \mathbb{R}^n, \lim_{h \to 0^+} \frac{f(x_0 + hu) - f(x_0)}{h} \in \mathbb{R} \Rightarrow \forall v \in \mathbb{R}^n, \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h} \in \mathbb{R}.$$

Therefore, it suffices to show that $l := \lim_{h \to 0^-} \frac{f(x_0+hv)-f(x_0)}{h} \in \mathbb{R}$.Take $u = -v$. Then,

$$l = \lim_{h \to 0^-} \frac{f(x_0 - hu) - f(x_0)}{h} = -\lim_{h \to 0^-} \frac{f(x_0 - hu) - f(x_0)}{-h} \overset{k=-h}{=} -\lim_{k \to 0^+} \frac{f(x_0 + ku) - f(x_0)}{k} \in \mathbb{R}.$$

$W \Rightarrow G$.
The proof of this implication is basically the proof of Fact 2.We want to show that

$$\forall u \in \mathbb{R}^n \text{ such that } \|u\| = 1, \lim_{h \to 0^+} \frac{f(x_0 + hu) - f(x_0)}{h} \in \mathbb{R} \Rightarrow \forall v \in \mathbb{R}^n \setminus \{0\}, l := \lim_{h \to 0^+} \frac{f(x_0 + hv) - f(x_0)}{h} \in \mathbb{R}.$$

In fact,

$$l := \lim_{h \to 0^+} \frac{f\left(x_0 + h\|v\| \frac{v}{\|v\|}\right) - f(x_0)}{h} \in \mathbb{R},$$

simply because $\left\|\frac{v}{\|v\|}\right\| = 1$.

Remark 698 *We can give the following geometrical interpretation of directional derivatives. First of all observe that from Proposition 701,*

$$f'(x_0; u) := \lim_{h \to 0} \frac{f(x_0 + hu) - f(x_0)}{h} = df_{x_0}(u).$$

If $f : \mathbb{R} \to \mathbb{R}$, we then have

$$f'(x_0; u) = f'(x_0) \cdot u.$$

Therefore, if $u = 1$, we have

$$f'(x_0; u) = f'(x_0),$$

and if $u > 0$, we have

$$sign\ f'(x_0; u) = sign\ f'(x_0).$$

Take now $f : \mathbb{R}^2 \to \mathbb{R}$ admitting directional derivatives. Then,

$$f'(x_0; u) = Df(x_0) \cdot u \quad with \quad \|u\| = 1$$

is the slope the graph of the function obtained cutting the graph of f with a plane which is orthogonal to the $x_1 - x_2$ plane, and along the line going through the points x_0 and $x_0 + u$, in the direction from x_0 to $x_0 + u$.

Chapter 15

Differentiability

15.1 Total Derivative and Differentiability

If $f : \mathbb{R} \to \mathbb{R}$, we say that f is differentiable in x_0, if the following limit exists and it is finite

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and we write

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

or, in equivalent manner,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - f'(x_0) \cdot h}{h} = 0$$

and

$$f(x_0 + h) - (f(x_0) + f'(x_0) \cdot h) = r(h)$$

where

$$\lim_{h \to 0} \frac{r(h)}{h} = 0,$$

or

$$f(x_0 + h) = f(x_0) + f'(x_0) \cdot h + r(h),$$

or using what said in Section 8.5, and more specifically using definition 8.8,

$$f(x_0 + h) = f(x_0) + l_{f'(x_0)}(h) + r(h)$$

where $l_{f'(x_0)} \in \mathcal{L}(\mathbb{R}, \mathbb{R})$

and $\lim_{h \to 0} \frac{r(h)}{h} = 0$

Definition 699 *Given a set $S \subseteq \mathbb{R}^n$, $x_0 \in Int\ S$, $f : S \to \mathbb{R}^m$, we say that f is differentiable at x_0 if*

there exists

a linear function $df_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$

such that for any $u \in \mathbb{R}^n$, $u \neq 0$, such that $x_0 + u \in S$,

$$\lim_{u \to 0} \frac{f(x_0 + u) - f(x_0) - df_{x_0}(u)}{\|u\|} = 0 \tag{15.1}$$

In that case, the linear function df_{x_0} is called the total derivative or the differential or simply the derivative of f at x_0.

Remark 700 *Obviously, given the condition of the previous Definition, we can say that f is differentiable at x_0 if there exists a linear function $df_{x_0}: \mathbb{R}^n \to \mathbb{R}^m$ such that $\forall u \in \mathbb{R}^n$ such that $x_0 + u \in S$*

$$f(x_0+u) = f(x_0) + df_{x_0}(u) + r(u), \qquad \textit{with} \qquad \lim_{u\to 0} \frac{r(u)}{\|u\|} = 0 \tag{15.2}$$

or

$$f(x_0+u) = f(x_0) + df_{x_0}(u) + \|u\| \cdot E_{x_0}(u), \qquad \textit{with} \qquad \lim_{u\to 0} E_{x_0}(u) = 0 \tag{15.3}$$

The above equations are called the first-order Taylor formula (of f at x_0 in the direction u). Condition (15.3) *is the most convenient one to use in many instances.*

Proposition 701 *Assume that $f : S \to \mathbb{R}^m$ is differentiable at x_0 , then*

$$\forall u \in \mathbb{R}^n, \qquad f'(x_0; u) = df_{x_0}(u).$$

Proof.

$$f'(x_0;u) := \lim_{h\to 0} \frac{f(x_0+hu) - f(x_0)}{h} \overset{(1)}{=}$$

$$= \lim_{h\to 0} \frac{f(x_0) + df_{x_0}(hu) + \|hu\| \cdot E_{x_0}(hu) - f(x_0)}{h} \overset{(2)}{=} \lim_{h\to 0} \frac{h df_{x_0}(u) + |h| \|u\| \cdot E_{x_0}(hu)}{h} \overset{(3)}{=}$$

$$= \lim_{h\to 0} df_{x_0}(u) + \lim_{h\to 0} sign\ (h) \cdot \|u\| \cdot E_{x_0}(hu) \overset{(4)}{=} df_{x_0}(u) + \|u\| \lim_{hu\to 0} sign\ (h) \cdot E_{x_0}(hu) \overset{(5)}{=} df_{x_0}(u),$$

where
(1) follows from (15.3) with hu in the place of u,
(2) from the fact that df_{x_0} is linear and therefore (Exercise) continuous, and from a property of a norm,
(3) from the fact that $\frac{|h|}{h} = sign\ (h)$, [1]
(4) from the fact that $h \to 0$ implies that $hu \to 0$,
(5) from the assumption that f is differentiable in x_0. ■

Remark 702 *The above Proposition implies that if the differential exists, then it is unique - from the fact that the limit is unique, if it exists.*

Proposition 703 *If $f : S \to \mathbb{R}^m$ is differentiable at x_0, then f is continuous at x_0.*

Proof. We have to prove that

$$\lim_{u\to 0} f(x_0+u) - f(x_0) = 0$$

i.e., from (15.3), it suffices to show that

$$\lim_{u\to 0} df_{x_0}(u) + \|u\| \cdot E_{x_0}(u) = df_{x_0}(0) + \lim_{u\to 0} \|u\| \cdot E_{x_0}(u) = 0$$

where the first equality follows from the fact that df_{x_0} is linear and therefore continuous, and the second equality from the fact again that df_{x_0} is linear, and therefore $df_x(0) = 0$, and from (15.2). ■

Remark 704 *The above Proposition is the answer to Question 1 in Remark 694. We still do not have a answer to Question 2 and another question naturally arises at this point:*
3. Is there an "easy" way of checking differentiability?

[1] *sign* is the function defined as follows:

$$sign : \mathbb{R} \to \{-1, 0+1\}, \quad x \mapsto \begin{cases} -1 & if \quad x<0 \\ 0 & if \quad x=0 \\ +1 & if \quad x>0. \end{cases}$$

15.2 Total Derivatives in terms of Partial Derivatives.

In Remark 706 below, we answer question 2 in Remark 694: Is there any "easy" way to compute the directional derivative?

Proposition 705 *Assume that* $f = (f_j)_{j=1}^m : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable in* x_0. *The matrix associated with* df_{x_0} *with respect to the canonical bases of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *is the Jacobian matrix* $Df(x_0)$, *i.e., using the notation of Section 8.5,*

$$[df_{x_0}] = Df(x_0),$$

i.e.,

$$\forall x \in \mathbb{R}^n, \qquad df_{x_0}(x) = Df(x_0) \cdot x. \tag{15.4}$$

Proof. From (8.6) in Section 8.5

$$[df_{x_0}] = \begin{bmatrix} df_{x_0}(e_n^1) & \dots & df_{x_0}(e_n^i) & \dots & df_{x_0}(e_n^n) \end{bmatrix}_{m\times n}$$

From Proposition 701,

$$\forall i \in \{1, ..., n\}, \; df_{x_0}(e^i) = f'(x_0; e^i),$$

and from (14.5)

$$f'(x_0; e^i) = (D_{x_i} f_j(x_0))_{j=1}^m.$$

Then

$$[df_{x_0}] = \begin{bmatrix} (D_{x_1} f_j(x_0))_{j=1}^m & \dots & (D_{x_i} f_j(x_0))_{j=1}^m & \dots & (D_{x_1} f_j(x_0))_{j=1}^m \end{bmatrix}_{m\times n},$$

as desired. ■

Remark 706 *From Proposition 701, part 1, and the above Proposition 705, we have that if* f *is differentiable in* x_0, *then* $\forall u \in \mathbb{R}^m$

$$\forall u \in \mathbb{R}^m, \qquad f'(x_0; u) = Df(x_0) \cdot u.$$

Remark 707 *From* (15.4), *we get*

$$\|df_{x_0}(x)\| = \|[Df(x_0)]_{mn} \; x\| \overset{(1)}{\leq} \sum_{j=1}^m |Df_j(x_0) \cdot x| \overset{(2)}{\leq} \sum_{j=1}^m \|Df_j(x_0)\| \cdot \|x\|$$

where (1) *follows from Remark 56,* (2) *from Cauchy-Schwarz inequality in* (53). *Therefore, defined* $\alpha := \sum_{j=1}^m \|Df_j(x_0)\|$, *we have that*

$$\|df_{x_0}(x)\| \leq \alpha \cdot \|x\|$$

and

$$\lim_{x \to 0} \|df_{x_0}(x)\| = 0$$

Remark 708 *We have seen that*

f *differentiable in* x_0	$\Rightarrow$	f *admits directional derivative in* x_0	$\underset{\not\Leftarrow}{\Rightarrow}$	$Df(x_0)$ *exists*
$\Downarrow$		(*not* $\Downarrow$)		(*not* $\Downarrow$)
f *continuous in* x_o		f *continuous in* x_o		f *continuous in* x_o

Therefore

$$f \text{ differentiable in } x_0 \quad \not\Leftarrow \quad Df(x_0) \text{ exists}$$

and

$$f \text{ differentiable in } x_0 \quad \not\Leftarrow \quad f \text{ admits directional derivative in } x_0$$

We still do not have an answer to question 3 in Remark 704: Is there an easy way of checking differentiability? We will provide an answer in Proposition 736.

Chapter 16

Some Theorems

We first introduce some needed definitions.

Definition 709 *Given an open $S \subseteq \mathbb{R}^n$,*

$$f : S \to \mathbb{R}^m, \qquad x := (x_j)_{j=1}^n \mapsto f(x) = (f_i(x))_{i=1}^m$$

$$I \subseteq \{1, ..., m\} \quad and \quad J \subseteq \{1, ..., n\},$$

the partial Jacobian of $(f_i)_{i \in I}$ with respect to $(x_j)_{j \in J}$ in $x_0 \in S$ is the following $(\#I) \times (\#J)$ submatrix of $Df(x_0)$

$$\left[\frac{\partial f_i(x_0)}{\partial x_j}\right]_{i \in I,\ j \in J},$$

and it is denoted by

$$D_{(x_j)_{j \in J}} (f_j)_{i \in I} (x_0)$$

Example 710 *Take:*
S an open subset of $\mathbb{R}^{n_1}$, with generic element $x' = (x_j)_{j=1}^{n_1}$,
T an open subset of $\mathbb{R}^{n_2}$, with generic element $x'' = (x_k)_{k=1}^{n_2}$ and

$$f : S \times T \to \mathbb{R}^m, \quad (x', x'') \mapsto f(x', x'')$$

Then, defined $n = n_1 + n_1$, we have

$$D_{x'} f(x_0) = \begin{bmatrix} D_{x_1} f_1(x_0) & ... & D_{x_{n_1}} f_1(x_0) \\ ... & & ... \\ D_{x_1} f_i(x_0) & ... & D_{x_{n_1}} f_i(x_0) \\ ... & & ... \\ D_{x_1} f_m(x_0) & ... & D_{x_{n_1}} f_m(x_0) \end{bmatrix}_{m \times n_1}$$

and, similarly,

$$f(x_0) := \begin{bmatrix} D_{x_{n_1+1}} f_1(x_0) & ... & D_{x_n} f_1(x_0) \\ ... & & ... \\ D_{x_{n_1+1}} f_i(x_0) & ... & D_{x_n} f_i(x_0) \\ ... & & ... \\ D_{x_{n_1+1}} f_m(x_0) & ... & D_{x_n} f_m(x_0) \end{bmatrix}_{m \times n_2}$$

and therefore

$$Df(x_0) := \begin{bmatrix} D_{x'} f(x_0) & D_{x''} f(x_0) \end{bmatrix}_{m \times n}$$

Definition 711 *Given an open set $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}$, assume that $\forall x \in S$, $Df(x) := \left(\frac{\partial f(x)}{\partial x_j}\right)_{j=1}^n$ exists. Then, $\forall j \in \{1, ..., n\}$, we define the $j-th$ partial derivative function as*

$$\frac{\partial f}{\partial x_j} : S \to \mathbb{R}, \qquad x \mapsto \frac{\partial f(x)}{\partial x_j}$$

Assuming that the above function has partial derivative with respect to x_k for $k \in \{1, ..., n\}$, we define it as the mixed second order partial derivative of f with respect to x_j and x_k and we write

$$\frac{\partial^2 f(x)}{\partial x_j \partial x_k} := \frac{\partial \frac{\partial f(x)}{\partial x_j}}{\partial x_k}$$

Definition 712 *Given $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, the Hessian matrix of f at x_0 is the $n \times n$ matrix*

$$D^2 f(x_0) := \left[\frac{\partial^2 f}{\partial x_j \partial x_k}(x_0)\right]_{j,k=1,...,n}$$

Remark 713 *$D^2 f(x_0)$ is the Jacobian matrix of the gradient function of f.*

Example 714 *Compute the Hessian function of $f : \mathbb{R}^3_{++} \to \mathbb{R}$,*

$$f(x, y, z) = \left(e^x \cos y + z^2 + x^2 \log y + \log x + \log z + 2t \log t\right)$$

We first compute the gradient:

$$\begin{pmatrix} 2x \ln y + (\cos y) e^x + \frac{1}{x} \\ -(\sin y) e^x + \frac{x^2}{y} \\ 2z + \frac{1}{z} \\ 2 \ln t + 2 \end{pmatrix}$$

and then the Hessian matrix

$$\begin{bmatrix} 2 \ln y + (\cos y) e^x - \frac{1}{x^2} & -(\sin y) e^x + 2\frac{x}{y} & 0 & 0 \\ -(\sin y) e^x + 2\frac{x}{y} & -(\cos y) e^x - \frac{x^2}{y^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{z^2} + 2 & 0 \\ 0 & 0 & 0 & \frac{2}{t} \end{bmatrix}$$

16.1 The chain rule

Proposition 715 *(Chain Rule) Given $S \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^m$,*

$$f : S \subseteq \mathbb{R}^n \to \mathbb{R}^m,$$

such that $\operatorname{Im} f \subseteq T$, *and*

$$g : T \subseteq \mathbb{R}^m \to \mathbb{R}^p,$$

assume that f and g are differentiable in x_0 and $y_0 = f(x_0)$, respectively. Then

$$h : S \subseteq \mathbb{R}^n \to \mathbb{R}^p, \qquad h(x) = (g \circ f)(x)$$

is differentiable in x_0 and

$$dh_{x_0} = dg_{f(x_0)} \circ df_{x_0}.$$

Proof. We want to show that there exists a linear function $dh_{x_0} : \mathbb{R}^n \to \mathbb{R}^p$ such that

$$h(x_0 + u) = h(x_0) + dh_{x_0}(u) + \|u\| \cdot E^*_{x_0}(u), \qquad \text{with} \qquad \lim_{u \to 0} E^*_{x_0}(u) = 0,$$

and $dh_{x_0} = dg_{f(x_0)} \circ df_{x_0}$.
Taking u sufficiently small (in order to have $x_0 + u \in S$), we have

$$h(x_0 + u) - h(x_0) = g[f(x_0 + u)] - g[f(x_0)] = g[f(x_0 + u)] - g(y_0)$$

and defined

$$v = f(x_0 + u) - y_0$$

we get

$$h(x_0 + u) - h(x_0) = g(y_0 + v) - g(y_0).$$

Since f is differentiable in x_0, we get

$$v = df_{x_0}(u) + \|u\| \cdot E_{x_0}(u), \qquad \text{with} \qquad \lim_{u\to 0} E_{x_0}(u) = 0. \tag{16.1}$$

Since g is differentiable in $y_0 = f(x_0)$, we get

$$g(y_0 + v) - g(y_0) = dg_{y_0}(v) + \|v\| \cdot E_{y_0}(v), \qquad \text{with} \qquad \lim_{v\to 0} E_{y_0}(v) = 0. \tag{16.2}$$

Inserting (16.1) in (16.2), we get

$$g(y_0+v)-g(y_0) = dg_{y_0}(df_{x_0}(u) + \|u\| \cdot E_{x_0}(u)) + \|v\| \cdot E_{y_0}(v) = dg_{y_0}(dg_{x_0}(u)) + \|u\| \cdot dg_{y_0}(E_{x_0}(u)) + \|v\| \cdot E_{y_0}(v)$$

Defined

$$E_{x_0}(u) := \begin{cases} 0 & \text{if} \quad u = 0 \\ df_{y_0}(E_{x_0}(u)) + \frac{\|v\|}{\|u\|} \cdot E_{y_0}(v) & \text{if} \quad u \neq 0 \end{cases},$$

we are left with showing that

$$\lim_{u\to 0} E_{x_0}(u) = 0.$$

Observe that

$$\lim_{u\to 0} df_{y_0}(E_{x_0}(u)) = 0$$

since linear functions are continuous and from (16.1). Moreover, since $\lim_{u\to 0} v = \lim_{u\to 0}(f(x_0+u) - y_0) = 0$, from (16.2), we get

$$\lim_{u\to 0} E_{y_0}(v) = 0.$$

Finally, we have to show that $\lim_{u\to 0} \frac{\|v\|}{\|u\|}$ is bounded. Now, from the definition of u and from (707), defined $\alpha := \sum_{j=1}^{m} \|Df_j(x_0)\|$,

$$\|v\| = \|df_{x_0}(u) + \|u\| \cdot E_{x_0}(u)\| \leq \|df_{x_0}(u)\| + \|u\| \, \|E_{x_0}(u)\| \leq (\alpha + \|E_{x_0}(u)\|) \cdot \|u\|$$

and

$$\lim_{u\to 0} \frac{\|v\|}{\|u\|} \leq \lim_{u\to 0} (\alpha + \|E_{x_0}(u)\|) = \alpha,$$

as desired. ■

Remark 716 *From Proposition 705 and Proposition 304, or simply* (8.10)*, we also have*

$$Dh(x_0)_{p\times n} = Dg(f(x_0))_{p\times m} \cdot Df(x_0)_{m\times n}.$$

Observe that $Dg(f(x_0))$ is obtained computing $Dg(y)$ and then substituting $f(x_0)$ in the place of y. We therefore also write $Dg(f(x_0)) = Dg(y)_{|y=f(x_0)}$

Exercise 717 *Compute dh_{x_0}, if $n = 1, p = 1$.*

Definition 718 *Given $f : S \subseteq \mathbb{R}^n \to \mathbb{R}^{m_1}$, $g : S \subseteq \mathbb{R}^n \to \mathbb{R}^{m_2}$,*

$$(f, g) : \mathbb{R}^n \to \mathbb{R}^{m_1+m_2}, \qquad (f, g)(x) = (f(x), g(x))$$

Remark 719 *Clearly,*

$$D(f, g)(x_0) = \begin{bmatrix} Df(x_0) \\ Dg(x_0) \end{bmatrix}$$

Example 720 *Given*

$$f : \mathbb{R} \to \mathbb{R}^2, \qquad : x \mapsto (\sin x, \cos x)$$

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \qquad : (y_1, y_2) \mapsto (y_1 + y_2, y_1 \cdot y_2)$$

$$h = g \circ f : \mathbb{R} \to \mathbb{R}^2, \qquad : x \mapsto (\sin x + \cos x, \sin x \cdot \cos x)$$

verify the conclusion of the Chain Rule Proposition.

$$Df(x) = \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix}$$

$$Dg(y) = \begin{bmatrix} 1 & 1 \\ y_2 & y_1 \end{bmatrix}$$

$$Dg(f(x)) = \begin{bmatrix} 1 & 1 \\ \cos x & \sin x \end{bmatrix}$$

$$Dg(f(x)) \cdot Df(x) = \begin{bmatrix} 1 & 1 \\ \cos x & \sin x \end{bmatrix} \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix} = \begin{bmatrix} \cos x - \sin x \\ \cos^2 x - \sin^2 x \end{bmatrix} = Dh(x)$$

Example 721 *Take*

$$g : \mathbb{R}^k \to \mathbb{R}^n, \qquad : t \mapsto g(t)$$

$$f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m, \qquad : (x,t) \mapsto f(x,t)$$

$$h : \mathbb{R}^k \to \mathbb{R}^m, \qquad : t \mapsto f(g(t), t)$$

Then

$$\widetilde{g} := (g, id_{\mathbb{R}^k}) : \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k, \qquad t \mapsto (g(t), t)$$

and

$$h = f \circ \widetilde{g} = f \circ (g, id_{\mathbb{R}^k},)$$

Therefore, assuming that f, g, h are differentiable,

$$[Dh(t_0)]_{m\times k} = [Df(g(t_0), t_0)]_{m\times(n+k)} \cdot \begin{bmatrix} Dg(t_0) \\ I \end{bmatrix}_{(n+k)\times k} =$$

$$= \left[[D_x f(g(t_0), t_0)]_{m\times n} \mid [D_t f(g(t_0), t_0)]_{m\times k} \right] \cdot \begin{bmatrix} [Dg(t_0)]_{n\times k} \\ I_{k\times k} \end{bmatrix} =$$

$$[D_x f(g(t_0), t_0)]_{m\times n} \cdot [Dg(t_0)]_{n\times k} + [D_t f(g(t_0), t_0)]_{m\times k}$$

In the case $k = n = m = 1$, the above expression

$$\frac{df(x = g(t), t)}{dt} = \frac{\partial f(g(t), t)}{\partial x} \frac{dg(t)}{dt} + \frac{\partial f(g(t), t)}{\partial t}$$

or

$$\frac{df(g(t), t)}{dt} = \frac{\partial f(x,t)}{\partial x}_{|x=g(t)} \cdot \frac{dg(t)}{dt} + \frac{\partial f(x,t)}{\partial t}_{|x=g(t)}$$

16.2 Mean value theorem

Proposition 722 *(Mean Value Theorem) Let S be an open subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}^m$ a differentiable function. Let $x, y \in S$ be such that the line segment joining them is contained in S, i.e.,*

$$L(x,y) := \{z \in \mathbb{R}^n : \exists \lambda \in [0,1] \text{ such that } z = (1-\lambda)x + \lambda y\} \subseteq S.$$

Then

$$\forall a \in \mathbb{R}^m, \qquad \exists\, z \in L(x,y) \qquad \text{such that} \qquad a \cdot [f(y) - f(x)] = a \cdot [Df(z) \cdot (y - x)]$$

Remark 723 *Under the assumptions of the above theorem, the following conclusion is **false:***

$$\exists\, z \in L(x,y) \text{ such that } f(y) - f(x) = Df(z) \cdot (y - x).$$

But if $f : S \to \mathbb{R}^{m=1}$, then setting $a \in \mathbb{R}^{m=1}$ equal to 1, we get that the above statement is indeed true.

Proof. of Proposition 722

Define $u = y - x$. Since S is open and $L(x,y) \subseteq S$, $\exists \delta > 0$ such that $\forall t \in (-\delta, 1+\delta)$ such that $x + tu = (1-t)x + ty \in S$. Taken $a \in \mathbb{R}^m$, define

$$F : (-\delta, 1+\delta) \to \mathbb{R}, \qquad : t \mapsto a \cdot f(x+tu) = \sum_{j+1}^{m} a_j \cdot f_j(x+tu)$$

Then

$$F'(t) = \sum_{j+1}^{m} a_j \cdot [Df_j(x+tu)]_{1\times n} \cdot u_{n\times 1} = a_{1\times m} \cdot [Df(x+tu)]_{m\times n} \cdot u_{n\times 1}$$

and F is continuous on $[0,1]$ and differentiable on $(0,1)$; then, we can apply "Calculus 1" Mean Value Theorem and conclude that

$$\exists \theta \in (0,1) \text{ such that } F(1) - F(0) = F'(\theta),$$

and by definition of F and u,

$$\exists \theta \in (0,1) \text{ such that } f(y) - f(x) = a \cdot Df(x+\theta u) \cdot (y-x)$$

which choosing $z = x + \theta u$ gives the desired result. ■

Remark 724 *Using the results we have seen on directional derivatives, the conclusion of the above theorem can be rewritten as follows.*

$$\exists\, z \in L(x,y) \text{ such that } f(y) - f(x) = f'(z, y-x)$$

As in the case of real functions of real variables, the Mean Value Theorem allows to give a simple relationship between sign of the derivative and monotonicity.

Definition 725 *A set $C \subseteq \mathbb{R}^n$ is convex if $\forall x_1,\, x_2 \in C$ and $\forall \lambda \in [0,1]$, $(1-\lambda)x_1 + \lambda x_2 \in C$.*

Proposition 726 *Let S be an open and convex subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}^m$ a differentiable function. If $\forall x \in S$, $df_x = 0$, then f is constant on S.*

Proof. Take arbitrary $x, y \in S$. Then since S is convex and f is differential, from the Mean Value Theorem, we have that

$$\forall a \in \mathbb{R}^m, \quad \exists\, z \in L(x,y) \quad \text{such that} \quad a\ [f(y) - f(x)] = a\,[Df(z) \cdot (y-x)] = 0.$$

Taken $a = f(y) - f(x)$, we get that

$$\|f(y) - f(x)\| = 0$$

and therefore

$$f(x) = f(y),$$

as desired. ■

Definition 727 *Given $x := (x_i)_{i=1}^n, y := (y_i)_{i=1}^n \in \mathbb{R}^n$,*

$$x \geq y \quad \textit{means} \quad \forall i \in \{1,...,n\}, \ \ x_i \geq y_i;$$

$$x > y \quad \textit{means} \quad x \geq y \ \wedge \ x \neq y;$$

$$x \gg y \quad \textit{means} \quad \forall i \in \{1,...,n\}, \ \ x_i > y_i.$$

Definition 728 *$f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is increasing if $\forall x, y \in S$, $x > y \Rightarrow f(x) \geq f(y)$.*
f is strictly increasing if $\forall x, y \in S$, $x > y \Rightarrow f(x) > f(y)$.

Proposition 729 *Take an open, convex subset S of $\mathbb{R}^n$, and $f : S \to \mathbb{R}$ differentiable.*
1. *If $\forall x \in S$, $Df(x) \geq 0$, then f is increasing;*
2. *If $\forall x \in S$, $Df(x) >> 0$, then f is strictly increasing.*

Proof. 1. Take $y \geq x$. Since S is convex, $L(x,y) \subseteq S$. Then from the Mean Value Theorem,

$$\exists\ z \in L(x,y) \text{ such that } f(y) - f(x) = Df(z) \cdot (y-x)$$

Since $y - x \geq 0$ and $Df(z) \geq 0$, the result follows.
2. Take $x > y$. Since S is convex, $L(x,y) \subseteq S$. Then from the Mean Value Theorem,

$$\exists\ z \in L(x,y) \text{ such that } f(y) - f(x) = Df(z) \cdot (y-x)$$

Since $y - x > 0$ and $Df(z) >> 0$, the result follows. ■

Exercise 730 *Is the following statement correct: "If $\forall x \in S$, $Df(x) > 0$, then f is strictly increasing" ?.*

Corollary 731 *Take an open, convex subset S of $\mathbb{R}^n$, and $f \in C^1(S,\mathbb{R})$. If $\exists x_0 \in S$ and $u \in \mathbb{R}^n \backslash \{0\}$ such that $f'(x_0,u) > 0$, then $\exists\ \bar{t} \in \mathbb{R}_{++}$ such that $\forall t \in [0,\bar{t})$,*

$$f(x_0 + tu) \geq f(x_0).$$

Proof. Since f is C^1 and $f'(x_0,u) = Df(x_0) \cdot u > 0$, $\exists r > 0$ such that

$$\forall x \in B(x_0,r), \quad f'(x,u) > 0.$$

Then $\forall t \in (-r,r)$, $\left\| x_0 + \frac{1}{\|u\|} tu - x_0 \right\| = t < r$, and therefore

$$f'\left(x_0 + \frac{t}{\|u\|} u, u\right) > 0$$

Then, from the Mean Value Theorem, $\forall t \in \left[0, \frac{r}{2}\right]$,

$$f(x_0 + tu) - f(x_0) = f'(x_0 + tu, u) \geq 0.$$

■

Definition 732 *Given a function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, $x_0 \in S$ is a point of local maximum for f if*

$$\exists \delta > 0 \text{ such that } \forall x \in B(x_0,\delta), \quad f(x_0) \geq f(x);$$

x_0 is a point of global maximum for f if

$$\forall x \in S, \quad f(x_0) \geq f(x).$$

$x_0 \in S$ is a point of strict local maximum for f if

$$\exists \delta > 0 \text{ such that } \forall x \in B(x_0,\delta), \quad f(x_0) > f(x);$$

x_0 is a point of strict global maximum for f if

$$\forall x \in S, \quad f(x_0) > f(x).$$

Local, global, strict minima are defined in obvious manner

Proposition 733 *If $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$ admits all partial derivatives in $x_0 \in Int\ S$ and x_0 is a point of local maximum or minimum, then $Df(x_0) = 0$.*

Proof. Since x_0 is a point of local maximum, $\exists \delta > 0$ such that $\forall x \in B(x_0,\delta)$, $f(x_0) \geq f(x)$. As in Remark 678, for any $k \in \{1,...,n\}$, define

$$g_k : \mathbb{R} \to \mathbb{R}, \qquad g(x_k) = f\left(x_0 + (x_k - x_{0k})\, e_n^k\right).$$

Then g_k has a local maximum point at x_{0k}. Then from Calculus 1,

$$g_k'(x_{0k}) = 0$$

Since, again from Remark 678, we have

$$D_k f(x_0) = g_k'(x_0).$$

the result follows. ■

16.3 A sufficient condition for differentiability

Definition 734 *$f = (f_i)_{i=1}^m : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable on $A \subseteq S$, or f is C^1 on S, or $f \in \mathcal{C}(A, \mathbb{R}^m)$ if $\forall i \in \{1, ..., m\}, j \in \{1, ..., n\}$,*

$$D_{x_j} f_i : A \to \mathbb{R}, \qquad x \mapsto D_{x_j} f_i (x) \qquad \text{is continuous.}$$

Definition 735 *$f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable on $A \subseteq S$, or f is C^2 on S, or $f \in \mathcal{C}^2(A, \mathbb{R}^m)$ if $\forall j, k \in \{1, ..., n\}$,*

$$\frac{\partial^2 f}{\partial x_j \partial x_k} : A \to \mathbb{R}, \qquad x \mapsto \frac{\partial^2 f}{\partial x_j \partial x_k}(x) \qquad \text{is continuous.}$$

Proposition 736 *If f is C^1 in an open neighborhood of x_0, then it is differentiable in x_0.*

Proof. See Apostol (1974), page 357 or Section "Existence of derivative", page 232, in Bartle (1964), where the significant case $f : \mathbb{R}^n \to \mathbb{R}^{m=1}$ is presented using Cauchy-Schwarz inequality. See also,Theorem 1, page 197, in Taylor and Mann (1984), for the case $f : \mathbb{R}^2 \to \mathbb{R}$. ■

The above result is the answer to Question 3 in Remark 704. To show that $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, it is enough to verify that all its partial derivatives, i.e., the entries of the Jacobian matrix, are continuous functions.

16.4 A sufficient condition for equality of mixed partial derivatives

Proposition 737 *If $f : S \subseteq \mathbb{R}^n \to \mathbb{R}^1$ is C^2 in an open neighborhood of x_0, then $\forall i$*

$$\frac{\partial \frac{\partial f}{\partial x_i}}{\partial x_k}(x_0) = \frac{\partial \frac{\partial f}{\partial x_k}}{\partial x_i}(x_0)$$

or

$$\frac{\partial^2 f}{\partial x_i \partial x_k}(x_0) = \frac{\partial^2 f}{\partial x_k \partial x_i}(x_0)$$

Proof. See Apostol (1974), Section 12.13, page 358. ■

16.5 Taylor's theorem for real valued functions

To get Taylor's theorem for functions $f : \mathbb{R}^m \to \mathbb{R}$, we introduce some notation in line with the definition of directional derivative:

$$f'(x, u) = \sum_{i=1}^{n} D_{x_i} f(x) \cdot u_i.$$

Definition 738 *Assume S is an open subset of $\mathbb{R}^m$ and the function $f : S \to \mathbb{R}$ admits partial derivatives at least up to order m, and $x \in S$, $u \in \mathbb{R}^m$. Then*

$$f''(x, u) := \sum_{i=1}^{n} \sum_{j=1}^{n} D_{i,j} f(x) \cdot u_i \cdot u_j,$$

$$f'''(x, u) := \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} D_{i,j,k} f(x) \cdot u_i \cdot u_j \cdot u_k$$

and similar definition applies to $f^{(m)}(x, u)$.

Proposition 739 *(Taylor's formula) Assume S is an open subset of $\mathbb{R}^m$ and the function $f : S \to \mathbb{R}$ admits partial derivatives at least up to order m, and $x \in S$, $u \in \mathbb{R}^m$. Assume also that all its partial derivative of order $< m$ are differentiable. If y and x are such that $L(y,x) \subseteq S$, then there exists $z \in L(y,x)$ such that*

$$f(y) = f(x) + \sum_{k=1}^{m-1} \frac{1}{k!} f^{(k)}(x, y-x) + \frac{1}{m!} f^{(m)}(z, y-z).$$

Proof. . Since S is open and $L(x,y) \subseteq S$, $\exists \delta > 0$ such that $\forall t \in (-\delta, 1+\delta)$ such that $x + t(y-x) \in S$. Define $g : (-\delta, 1+\delta) \to \mathbb{R}$

$$g(t) = f(x + t(y-x)).$$

From standard "Calculus 1" Taylor's theorem, we have that $\exists \theta \in (0,1)$ such that

$$f(y) - f(x) = g(1) - g(0) = \sum_{k=1}^{m-1} \frac{1}{k!} g^{(k)}(0) + \frac{1}{m!} g^{(m)}(\theta).$$

Then

$$g'(t) = Df(x + t(y-x)) \cdot (y-x) = \sum_{i=1}^{n} D_{x_i} f(x + t(y-x)) \cdot (y_i - x_i) = f'(x + t(y-x), y-x),$$

$$g''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{x_i,x_j} f(x + t(y-x)) \cdot (y_i - x_i) \cdot (y_j - x_j) = f''(x + t(y-x), y-x)$$

and similarly

$$g^{(m)}(t) = f'^{(m)}(x + t(y-x), y-x)$$

Then the desired result follow substituting 0 in the place of t where needed and choosing $z = x + \theta(y-x)$. ■

Chapter 17

Implicit function theorem

17.1 Some intuition

Below, we present an informal discussion of the Implicit Function Theorem. Assume that

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad (x,t) \mapsto f(x,t)$$

is at least C^1. The basic goal is to study the nonlinear equation

$$f(x,t) = 0,$$

where x can be interpreted as an endogenous variable and t as a parameter (or an exogenous variable). Assume that

$$\exists \left(x^0, t^0\right) \in \mathbb{R}^2 \text{ such that } f\left(x^0, t^0\right) = 0$$

and for some $\varepsilon > 0$

$$\exists \text{ a } C^1 \text{ function } g : \left(t^0 - \varepsilon, t^0 + \varepsilon\right) \to \mathbb{R}, t \mapsto g(t)$$

such that

$$f(g(t), t) = 0 \tag{17.1}$$

We can then say that g describes *the solution* to the equation

$$f(x,t) = 0,$$

in the unknown variable x and parameter t, in an open neighborhood of t^0. Therefore, using the Chain Rule - and in fact, Remark 721 - applied to both sides of (17.1), we get

$$\frac{\partial f(x,t)}{\partial x}_{|x=g(t)} \cdot \frac{dg(t)}{dt} + \frac{\partial f(x,t)}{\partial t}_{|x=g(t)} = 0$$

and

$$\text{assuming that } \frac{\partial f(x,t)}{\partial x}_{|x=g(t)} \neq 0$$

we have

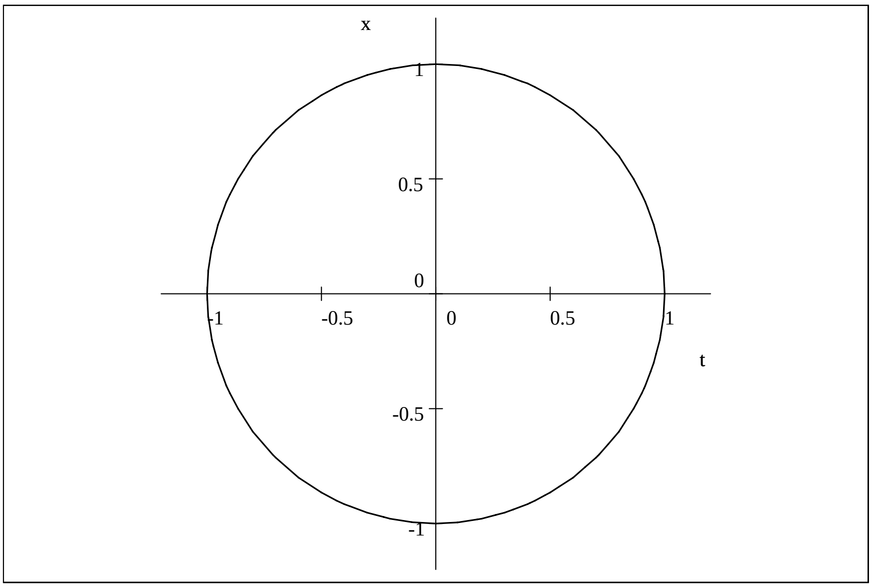
$$\frac{dg(t)}{dt} = -\frac{\frac{\partial f(x,t)}{\partial t}_{|x=g(t)}}{\frac{\partial f(x,t)}{\partial x}_{|x=g(t)}} \tag{17.2}$$

The above expression is the derivative of the function implicitly defined by (17.1) close to the value t^0. In other words, it is the slope of the level curve $f(x,t) = 0$ at the point $(t, g(t))$.

For example, taken

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x,t) \mapsto x^2 + t^2 - 1$$

$f(x,t) = 0$ describes the circle with center in the origin and radius equal to 1. Putting t on the horizontal axis and x on the vertical axis, we have the following picture.



Clearly

$$f((0,1)) = 0$$

As long as $t \in (-1, 1)$, $g(t) = \sqrt{1-t^2}$ is such that

$$f(g(t), t) = 0 \tag{17.3}$$

Observe that

$$\frac{d\left(\sqrt{1-t^2}\right)}{dt} = -\frac{t}{\sqrt{1-t^2}}$$

and

$$-\frac{\frac{\partial f(x,t)}{\partial t}_{|x=g(t)}}{\frac{\partial f(x,t)}{\partial x}_{|x=g(t)}} = -\frac{2t}{2x}_{|x=g(t)} = -\frac{t}{\sqrt{1-t^2}}$$

For example for $t = \frac{1}{\sqrt{2}}$, $g'(t) = -\frac{\frac{1}{\sqrt{2}}}{\sqrt{1-\frac{1}{2}}} = -1$.

Let's try to present a more detailed geometrical interpretation[1]. Consider the set $\left\{(x,t) \in \mathbb{R}^2 : f(x,t) = 0\right\}$ presented in the following picture.

`Insert picture a., page 80.`

In this case, does equation

$$f(x,t) = 0 \tag{17.4}$$

define x as a function of t? Certainly, the curve presented in the picture is not the graph of a function with x as dependent variable and t as an independent variable for all values of t in $\mathbb{R}$. In fact,

1. if $t \in (-\infty, t_1]$, there is only one value of x which satisfies equation (17.4);
2. if $t \in (t_1, t_2)$, there are two values of x for which $f(x,t) = 0$;
3. if $t \in (t_2, +\infty)$, there are no values satisfying the equation.

If we consider t belonging to an interval contained in (t_1, t_2), we have to to restrict the admissible range of variation of x in order to conclude that equation (17.4) defines x as a function of t in that interval. For example, we see that if the rectangle R is as indicated in the picture , the given equation defines x as a function of t, for well chosen domain and codomain - naturally associated with R. The graph of that function is indicated in the figure below.

`Insert picture b., page 80.`

[1] This discussion is taken from Sydsaeter (1981), page 80-81.

The size of R is limited by the fact that we need to define a *function* and therefore one and only one value has to be associated with t. Similar rectangles and associated solutions to the equation can be constructed for all other points on the curve, *except one:* (t_2, x_2). Irrespectively of how small we choose the rectangle around that point, there will be values of t close to t_2, say t', such that there are two values of x, say x' and x'', with the property that both (t', x') and (t', x'') satisfy the equation and lie inside the rectangle. Therefore, equation (17.4) does not define x as a function of t in an open neighborhood of the point (t_2, x_2). In fact, there the slope of the tangent to the curve is infinite. If you try to use expression (17.2) to compute the slope of the curve defined by $x^2 + t^2 = 1$ in the point $(1, 0)$, you get an expression with *zero* in the denominator.

On the basis of the above discussion, we see that it is crucial to require the condition

$$\frac{\partial f(x,t)}{\partial x}_{|x=g(t)} \neq 0$$

to insure the possibility of locally writing x as a solution (to (17.4)) function of t.

We can informally, summarize what we said as follow.

If f is C^1, $f(x_0, t_0) = 0$ and $\frac{\partial f(x,t)}{\partial x}_{|(x,t)=(x^0,t^0)} \neq 0$, then $f(x,t) = 0$ define x as a C^1 function g of t in an open neighborhood of t^0, and $g'(t) = -\frac{\frac{\partial f(x,t)}{\partial t}}{\frac{\partial f(x,t)}{\partial x}}_{|x=g(t)}$.

Next sections provide a formal statement and proof of the Implicit Function Theorem. Some work is needed.

17.2 Functions with full rank square Jacobian

Proposition 740 *Taken $a \in \mathbb{R}^n$, $r \in \mathbb{R}_{++}$, assume that*

1. $f := (f_i)_{i=1}^n : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on Cl $(B(a,r))$;

2. $\forall x \in B(a,r)$, $[Df(x)]_{n\times n}$ exists and $\det Df(x) \neq 0$;

3. $\forall x \in \mathcal{F}(B(a,r))$, $f(x) \neq f(a)$.

Then, $\exists \delta \in \mathbb{R}_{++}$ such that

$$f(B(a,r)) \supseteq B(f(a), \delta).$$

Proof. Define $B := B(a,r)$ and

$$g : \mathcal{F}(B) \to \mathbb{R}, \qquad x \mapsto \|f(x) - f(a)\|.$$

From Assumption 3, $\forall x \in \mathcal{F}(B)$, $g(x) > 0$. Moreover, since g is continuos and $\mathcal{F}(B)$ is compact, g attains a global minimum value $m > 0$ on $\mathcal{F}(B)$. Take $\delta = \frac{m}{2}$; to prove the desired result, it is enough to show that $T := B(f(a), \delta) \subseteq f(B)$, i.e., $\forall y \in T$, $y \in f(B)$. Define

$$h : \text{Cl } (B) \to \mathbb{R}, \qquad x \mapsto \|f(x) - y\|.$$

since h is continuos and Cl B is compact, h attains a global minimum in a point $c \in \overline{B}$. We now want to show that $c \in B$. Observe that, since $y \in T = B\left(f(a), \delta = \frac{m}{2}\right)$,

$$h(a) = \|f(a) - y\| < \frac{m}{2} \tag{17.5}$$

Therefore, since c is a global minimum point for h, it must be the case that $h(c) < \frac{m}{2}$. Now take $x \in \mathcal{F}(B)$; then

$$h(x) = \|f(x) - y\| = \|f(x) - f(a) - (y - f(a))\| \overset{(1)}{\geq} \|f(x) - f(a)\| - \|y - f(a)\| \overset{(2)}{\geq} g(x) - \frac{m}{2} \overset{(3)}{\geq} \frac{m}{2},$$

where (1) follows from Remark 54, (2) from (17.5) and (3) from the fact that g has minimum value equal to m. Therefore, $\forall x \in \mathcal{F}(B)$, $h(x) > h(a)$ and h does not attain its minimum on $\mathcal{F}(B)$. Then h and h^2 get their minimum at $c \in B$ [2]. Since

$$H(x) := h^2(x) = \|f(x) - y\|^2 = \sum_{i=1}^n (f_i(x) - y_i)^2$$

[2] $\forall x \in \overline{B}$, $h(x) \geq 0$ and $h(x) \geq h(c)$. Therefore, $h^2(x) \geq h^2(c)$.

from Proposition 733, $DH(c) = 0$, i.e.,

$$\forall k \in \{1, ..., n\}, \quad 2\sum_{i=1}^{n} D_{x_k} f_i(c) \cdot (f_i(c) - y_i) = 0$$

i.e.,

$$[Df(c)]_{n\times n} (f(c) - y)_{n\times 1} = 0.$$

Then, from assumption 2,

$$f(c) = y,$$

i.e., since $c \in B$, $y \in f(B)$, ad desired. ■

Proposition 741 *(1st sufficient condition for openness of a function)*
Let an open set $A \subseteq \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^n$ be given. If
1. f is continuous,
2. f is one-to-one,
3. $\forall x \in A$, $Df(x)$ exists and $\det Df(x) \neq 0$,

then f is open.

Proof. Taken $b \in f(A)$, there exists $a \in A$ such that $f(a) = b$. Since A is open, there exists $r \in \mathbb{R}_{++}$ such that $B(a, r) \subseteq A$. Moreover, since f is one-to-one and since $a \notin \mathcal{F}(B)$, $\forall x \in \mathcal{F}(B)$, $f(x) \neq f(a)$ .Then[3], for sufficiently small r, $\mathrm{Cl}(B(a, r)) \subseteq A$, and the assumptions of Proposition 740 are satisfied and there exists $\delta \in \mathbb{R}_{++}$ such that

$$f(A) \supseteq f(\mathrm{Cl}(B(a, r))) \supseteq B(f(a), \delta),$$

as desired. ■

Definition 742 *Given $f : S \to T$, and $A \subseteq S$, the function $f_{|A}$ is defined as follows*

$$f_{|A} : A \to f(A), \qquad f_{|A}(x) = f(x).$$

Proposition 743 *Let an open set $A \subseteq \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^n$ be given. If*
1. f is C^1,
2. $\exists a \in A$ such that $\det Df(a) \neq 0$,
then $\exists r \in \mathbb{R}_{++}$ such that f is one-to-one on $B(a, r)$, and, therefore, $f_{|B(a,r)}$ is invertible.

Proof. Consider $(\mathbb{R}^n)^n$ with generic element $z := (z^i)_{i=1}^n$, where $\forall i \in \{1, ..., n\}$, $z^i \in \mathbb{R}^n$, and define

$$h : \mathbb{R}^{n^2} \to \mathbb{R}, \qquad : (z^i)_{i=1}^n \mapsto \det \begin{bmatrix} Df_1(z^1) \\ ... \\ Df_i(z^i) \\ ... \\ Df_n(z^n) \end{bmatrix}.$$

Observe that h is continuous because f is C^1 and the determinant function is continuous in its entries. Moreover, from Assumption 2,

$$h(a, ..., a, ..., a) = \det Df(a) \neq 0.$$

Therefore, $\exists r \in \mathbb{R}_{++}$ such that

$$\forall (z^i)_{i=1}^n \in B((a, ..., a, ..., a), r'), \quad h\left((z^i)_{i=1}^n\right) \neq 0.$$

Observe that $B^* := B((a, ..., a, ..., a), r) \cap \beta \neq \varnothing$, where $\beta := \left\{(z^i)_{i=1}^n \in \mathbb{R}^{n^2} : \forall i \in \{1, ..., n\}, z^i = z^1\right\}$. Define

$$proj : (\mathbb{R}^n)^n \to \mathbb{R}^n, proj : (z^i)_{i=1}^n \mapsto z^1,$$

[3] Simply observe that $\forall r \in \mathbb{R}_{++}$, $\overline{B}\left(x, \frac{r}{2}\right) \subseteq B(x, r)$.

and observe that $proj\,(B^*) = B\,(a,r) \subseteq \mathbb{R}^n$ and $\forall i \in \{1,..,n\}, \forall z_i \in B\,(a,r)$, $\left(z^1,...,z^i...,z^n\right) \in B^*$ and therefore $h\left(z^1,...,z^i...,z^n\right) \neq 0$,or, summarizing,

$$\exists r \in \mathbb{R}_{++} \text{ such that } \forall i \in \{1,..,n\}, \forall z_i \in B\,(a,r), \quad h\left(z^1,...,z^i...,z^n\right) \neq 0$$

We now want to show that f is one-to-one on $B\,(a,r)$. Suppose otherwise, i.e., given $x,y \in B\,(a,r)$, $f\,(x) = f\,(y)$, but $x \neq y$. We can now apply the Mean Value Theorem (see Remark 723) to f^i for any $i \in w\,\{1,...,n\}$ on the segment $L\,(x,y) \subseteq B\,(a,r)$. Therefore $\forall i \in \{1,..,n\}$, $\exists z^i \in L\,(x,y)$ such that

$$\forall i \in \{1,..,n\}, \exists z^i \in L\,(x,y) \text{ such that } 0 = f_i\,(x) - f_i\,(y) = Df\left(z^i\right)(y-x)$$

i.e.,

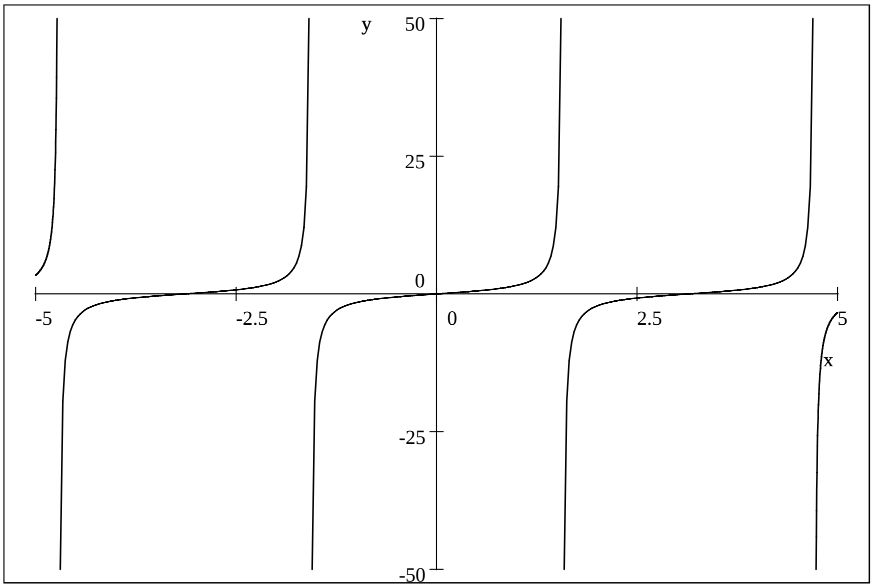
$$\begin{bmatrix} Df_1\left(z^1\right) \\ ... \\ Df_i\left(z^i\right) \\ ... \\ Df_n\left(z^n\right) \end{bmatrix} (y-x) = 0$$

Observe that $\forall i, z^i \in B\,(a,r)$ and therefore $\left(z^i\right)_{i=1}^n \in B\,((a,...,a,...,a), r'')$,and therefore

$$\det \begin{bmatrix} Df_1\left(z^1\right) \\ ... \\ Df_i\left(z^i\right) \\ ... \\ Df_n\left(z^n\right) \end{bmatrix} = h\left(\left(z^i\right)_{i=1}^n\right) \neq 0$$

and therefore $y = x$, a contradiction. ■

Remark 744 *The above result is not a global result, i.e., it is false that if f is C^1 and its Jacobian has full rank everywhere in the domain, then f is one to one. Just take the function* tan.



The next result gives a global property.

Proposition 745 *(2nd sufficient condition for openness of a function) Let an open set $A \subseteq \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^n$ be given. If*

1. f is C^1,

2. $\forall x \in A$, $\det Df\,(x) \neq 0$,

then f is an open function.

Proof. Take an open set $S \subseteq A$. From Proposition 743,$\forall x \in S$ there exists $r \in \mathbb{R}_{++}$ such that f is one-to-one on $B(x,r)$. Then, from Proposition 741, $f(B(x,r))$ is open in $\mathbb{R}^n$. We can then write $S = \cup_{x\in S} B(x,r)$ and

$$f(S) = f(\cup_{x\in S} B(x,r)) = \cup_{x\in S} f(B(x,r))$$

where the second equality follows from Proposition 614.2..f1, and then $f(S)$ is an open set. ■

17.3 The inverse function theorem

Proposition 741 shows that a C^1 function with full rank square Jacobian in a point a has a local inverse in an open neighborhood of a. The inverse function theorem give local differentiability properties of that local inverse function.

Lemma 746 *If g is the inverse function of $f : X \to Y$ and $A \subseteq X$, then $g_{|f(A)}$ is the inverse of $f_{|A}$, and*
if g is the inverse function of $f : X \to Y$ and $B \subseteq Y$, then $g_{|B}$ is the inverse of $f_{|g(B)}$.

Proof. Exercise. ■

Proposition 747 *Let an open set $S \subseteq \mathbb{R}^n$ and a function $f : S \to \mathbb{R}^n$ be given. If*
1. f is C^1, and
2. $\exists a \in S$, $\det Df(a) \neq 0$,
then there exist two open sets $X \subseteq S$ and $Y \subseteq f(S)$ and a unique function g such that
1. $a \in X$ and $f(a) \in Y$,
2. $Y = f(X)$,
3. f is one-to-one on X,
4. g is the inverse of f_X,
5. g is C^1.

Proof. Since f is C^1, $\exists r_1 \in \mathbb{R}_{++}$ such that $\forall x \in B(a,r_1)$, $\det Df(x) \neq 0$. Then, from Proposition 743, f is one-to-one on $B(a,r_1)$. Then take $r_2 \in (0,r_1)$, and define $B := B(a,r_2)$. Observe that Cl $(B)(a,r_2) \subseteq B(a,r_1)$ .Using the fact that f is one-to-one on $B(a,r_1)$ and therefore on $B(a,r_2)$, we get that Assumption 3 in Proposition 740 is satisfied - while the other two are trivially satisfied. Then, $\exists \delta \in \mathbb{R}_{++}$ such that

$$f(B(a,r_2)) \supseteq B(f(a),\delta) := Y.$$

Define also

$$X := f^{-1}(Y) \cap B, \tag{17.6}$$

an open set because Y and B are open sets and f is continuous. Since f is one-to-one and continuous on the compact set Cl (B) , from Proposition 622, there exists a unique continuous inverse function $\widehat{g} : f(\text{Cl }(B)) \to \text{Cl }(B)$ of $f_{|\text{Cl }(B)}$. From definition of Y,

$$Y \subseteq f(B) \subseteq f(\text{Cl }(B)). \tag{17.7}$$

From definition of X,

$$f(X) = Y \cap f(B) = Y.$$

Then, from Lemma 746,

$$g = \widehat{g}_Y \text{ is the inverse of } f_{|X}.$$

The above shows conclusions 1-4 of the Proposition. (About conclusion 1, observe that $a \in f^{-1}(B(f(a,\delta))) \cap B(a,r_2) = X$ and $f(a) \in f(X) = Y$.)
We are then left with proving condition 5.

Following what said in the proof of Proposition 743, we can define

$$h : \mathbb{R}^{n^2} \to \mathbb{R}, \qquad : \left(z^i\right)_{i=1}^n \mapsto \det \begin{bmatrix} Df_1\left(z^1\right) \\ ... \\ Df_i\left(z^i\right) \\ ... \\ Df_n\left(z^n\right) \end{bmatrix}.$$

and get that, from Assumption 2,

$$h\left(a, ..., a, ..., a\right) = \det Df\left(a\right) \neq 0,$$

and, see the proof of Proposition 743 for details,

$$\exists r \in \mathbb{R}_{++} \text{ such that } \forall i \in \{1, .., n\}, \forall z_i \in B\left(a, r\right), \quad h\left(z^1, ..., z^i ..., z^n\right) \neq 0, \tag{17.8}$$

and trivially also

$$\exists r \in \mathbb{R}_{++} \text{ such that } \forall z \in B\left(a, r\right), \quad h\left(z, ..., z ..., z\right) = \det Df\left(z\right) \neq 0. \tag{17.9}$$

Assuming, without loss of generality that we took $r_1 < r$, we have that

$$\text{Cl }\left(B\right) := \text{Cl }\left(B\right)\left(a, r_2\right) \subseteq B\left(a, r_1\right) \subseteq B\left(a, r\right).$$

Then $\forall z^1, ..., z^n \in \text{Cl }(B)$, $h\left(z^1, ..., z^i ..., z^n\right) \neq 0$. Writing $g = \left(g^j\right)_{j=1}^n$, we want to prove that $\forall i \in \{1, .., n\}$, g^i is C^1. We go through the following two steps: 1. $\forall y \in Y$, $\forall i, k \in \{1, .., n\}$, $D_{y_k} g^i\left(y\right)$ exists, and 2.it is continuous.

Step 1.

We want to show that the following limit exists and it is finite:

$$\lim_{h \to 0} \frac{g^i\left(y + he_n^k\right) - g^i\left(y\right)}{h}.$$

Define

$$\begin{array}{l} x = \left(x_i\right)_{i=1}^n = g\left(y\right) \subseteq X \subseteq \text{Cl }\left(B\right) \\ x' = \left(x_i'\right)_{i=1}^n = g\left(y + he_n^k\right) \subseteq X \subseteq \text{Cl }\left(B\right) \end{array} \tag{17.10}$$

Then

$$f\left(x'\right) - f\left(x\right) = \left(y + he_n^k\right) - y = he_n^k.$$

We can now apply the Mean Value Theorem to f^i for $i \in \{1, .., n\}$: $\exists z^i \in L\left(x, x'\right) \subseteq \text{Cl }(B)$, where the inclusion follows from the fact that $x, x' \in \text{Cl }(B)$ a convex set, such that

$$\forall i \in \{1, .., n\}, \quad \frac{f^i\left(x'\right) - f^i\left(x\right)}{h} = \frac{Df^i\left(z^i\right)\left(x' - x\right)}{h}$$

and therefore

$$\begin{bmatrix} Df_1\left(z^1\right) \\ ... \\ Df_i\left(z^i\right) \\ ... \\ Df_n\left(z^n\right) \end{bmatrix} \frac{1}{h}\left(x' - x\right) = e_n^k.$$

Define

$$A = \begin{bmatrix} Df_1\left(z^1\right) \\ ... \\ Df_i\left(z^i\right) \\ ... \\ Df_n\left(z^n\right) \end{bmatrix}$$

Then, from (17.8), the above system admits a unique solution, i.e., using (17.10),

$$\frac{g\left(y + he_n^k\right) - g\left(y\right)}{h} = \frac{1}{h}\left(x' - x\right)$$

and, using Cramer theorem (i.e., Theorem 356),

$$\frac{g\left(y+he_n^k\right)-g\left(y\right)}{h}=\frac{\varphi\left(z^1,...,z^n\right)}{h\left(z^1,...,z^n\right)}$$

where φ takes values which are determinants of a matrix involving entries of A. We are left with showing that

$$\lim_{h\to 0}\frac{\varphi\left(z^1,...,z^n\right)}{h\left(z^1,...,z^n\right)}$$

exists and it is finite, i.e., the limit of the numerator exists and its finite and the limit of the denominator exists is finite and nonzero.

Then, if $h\to 0$, $y+he_n^k\to y$, and, being g continuous, $x'\to x$ and, since $z^i\in L\left(x,x'\right)$, $z^i\to x$ for any i. Then, $h\left(z^1,...,z^n\right)\to h\left(x,...,x\right)\neq 0$,because, from 17.10, $x\in \text{Cl }(B)$ and from (17.9). Moreover, $\varphi\left(z^1,...,z^n\right)\to\varphi\left(x,...,x\right)$.

Step 2.

Since

$$\lim_{h\to 0}\frac{g^i\left(y+he_n^k\right)-g^i\left(y\right)}{h}=\frac{\varphi\left(x,...,x\right)}{h\left(x,...,x\right)}$$

and φ and h are continuous functions, the desired result follows. ■

17.4 The implicit function theorem

Theorem 748 *Given S,T open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively and a function*

$$f:S\times T\to\mathbb{R}^n,\quad :(x,t)\mapsto f\left(x,t\right),$$

assume that

1. *f is C^1,*

   there exists $(x_0,t_0)\in S\times T$ such that
2. *$f(x_0,t_0)=0$,*
3. *$D_xf\left(x_0,t_0\right)_{n\times n}$ is invertible.*

Then there exist $N(x_0)\subseteq S$ open neighborhood of x_0, $N(t_0)\subseteq T$ open neighborhood of t_0 and a unique function

$$g:N(t_0)\to N(x_0)$$

such that

1. *g is C^1,*
2. *$\{(x,t)\in N(x_0)\times N(t_0):f(x,t)=0\}=\{(x,t)\in N(x_0)\times N(t_0):x=g(t)\}:=graph\ g$.*[4]

Proof. See Apostol (1974), ■

Remark 749 *Conclusion 2. above can be rewritten as*

$$\forall t\in N\left(t_0\right),\quad f\left(g\left(t\right),t\right)=0 \tag{17.11}$$

Computing the Jacobian of both sides of (17.11)*, using Remark 721, we get*

$$\forall t\in N\left(t_0\right),\quad 0=\left[D_xf\left(g\left(t\right),t\right)\right]_{n\times n}\cdot\left[Dg\left(t\right)\right]_{n\times k}+\left[D_tf\left(g\left(t\right),t\right)\right]_{n\times k} \tag{17.12}$$

and using Assumption 3 of the Implicit Function Theorem, we get

$$\forall t\in N\left(t_0\right),\quad \left[Dg\left(t\right)\right]_{n\times k}=-\left[D_xf\left(g\left(t\right),t\right)\right]_{n\times n}^{-1}\cdot\left[D_tf\left(g\left(t\right),t\right)\right]_{n\times k}$$

Observe that (17.12) *can be rewritten as the following k systems of equations: $\forall i\in\{1,...,k\}$,*

$$\left[D_xf\left(g\left(t\right),t\right)\right]_{n\times n}\cdot\left[D_{t_i}g\left(t\right)\right]_{n\times 1}=-\left[D_{t_i}f\left(g\left(t\right),t\right)\right]_{n\times 1}$$

[4] Then $g\left(t_0\right)=x_0$.

Exercise 750 [5] *Discuss the application of the Implicit Function Theorem to* $f : \mathbb{R}^5 \to \mathbb{R}^2$

$$f(x_1, x_2, t_1, t_2, t_3) \mapsto \begin{pmatrix} 2e^{x_1} + x_2 t_1 - 4t_2 + 3 \\ x_2 \cos x_1 - 6x_1 + 2t_1 - t_3 \end{pmatrix}$$

at $\left(x^0, t^0\right) = (0, 1, 3, 2, 7)$.
Let's check that each assumption of the Theorem is verified.

1. $f(x_0, t_0) = 0$. *Obvious.*
2. f *is* C^1.

   We have to compute the Jacobian of the function and check that each entry is a continuous function.

	x_1	x_2	t_1	t_2	t_3
$2e^{x_1} + x_2 t_1 - 4t_2 + 3$	$2e^{x_1}$	t_1	x_2	-4	0
$x_2 \cos x_1 - 6x_1 + 2t_1 - t_3$	$-x_2 \sin x_1 - 6$	$\cos x_1$	2	0	-1

3. $\left[D_x f(x_0, t_0)\right]_{n \times n}$ *is invertible.*

$$\left[D_x f(x_0, t_0)\right] = \begin{bmatrix} 2e^{x_1} & t_1 \\ -x_2 \sin x_1 - 6 & \cos x_1 \end{bmatrix}_{|(0,1,3,2,7)} = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}$$

   whose determinant is 20.

Therefore, we can apply the Implicit Function Theorem and compute the Jacobian of $g : N(t_0) \subseteq \mathbb{R}^2 \to N(x_0) \subseteq \mathbb{R}^3$*:*

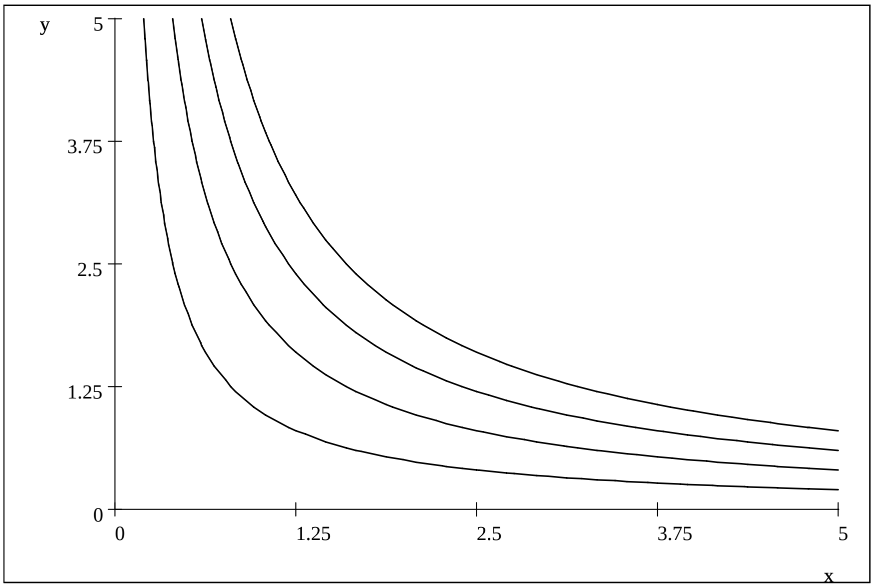
$$Dg(t) = -\begin{bmatrix} 2e^{x_1} & t_1 \\ -x_2 \sin x_1 - 6 & \cos x_1 \end{bmatrix}^{-1} \begin{bmatrix} x_2 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} =$$

$$= \frac{1}{6t_1 + 2(\cos x_1) e^{x_1} + t_1 x_2 \sin x_1} \begin{bmatrix} 2t_1 - x_2 \cos x_1 & 4 \cos x_1 & -t_1 \\ -6x_2 - 4e^{x_1} - x_2^2 \sin x_1 & 4x_2 \sin x_1 + 24 & 2e^{x_1} \end{bmatrix}$$

Exercise 751 *Given the utility function* $u : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$, $(x, y) \mapsto u(x, y)$ *satisfying the following properties*

i. u *is* C^2, *ii.* $\forall (x, y) \in \mathbb{R}^2_{++}$, $Du(x, y) >> 0$,*iii.* $\forall (x, y) \in \mathbb{R}^2_{++}$, $D_{xx}u(x, y) < 0, D_{yy}u(x, y) < 0, D_{xy}u(x, y) > 0$,

compute the Marginal Rate of Substitution in (x_0, y_0) *and say if the graph of each indifference curve is concave.*



[5] The example is taken from Rudin (1976), pages 227-228.

17.5 Some geometrical remarks on the gradient

In what follows we make some geometrical, not rigorous remarks on the meaning of the gradient, using the implicit function theorem. Consider an open subset X of $\mathbb{R}^2$, a C^1 function

$$f : X \to \mathbb{R}, \qquad : (x,y) \mapsto f(x,y)$$

where $a \in \mathbb{R}$. Assume that set

$$L(a) := \{(x,y) \in X : f(x,y) = a\}$$

is such that $\forall (x,y) \in X$, $\frac{\partial f(x,y)}{\partial y} \neq 0$ and $\frac{\partial f(x,y)}{\partial x} \neq 0$, then

1. $L(a)$ is the graph of a C^1 function from a subset of $\mathbb{R}$ to $\mathbb{R}$;
2. $(x^*, y^*) \in L(a) \Rightarrow$ the line going through the origin and the point $Df(x^*, y^*)$ is orthogonal to the line going through the origin and parallel to the tangent line to $L(a)$ at (x^*, y^*) ;or the line tangent to the curve $L(a)$ in (x^*, y^*) is orthogonal to the line to which the gradient belongs to.
3. $(x^*, y^*) \in L(a) \Rightarrow$ the directional derivative of f at (x^*, y^*) in the the direction u such that $\|u\| = 1$ is the largest one if $u = \frac{Df(x^*,y^*)}{\|Df(x^*,y^*)\|}$.

1. It follows from the Implicit Function Theorem.
2. The slope of the line going through the origin and the vector $Df(x^*, y^*)$ is

$$\frac{\frac{\partial f(x^*,y^*)}{\partial y}}{\frac{\partial f(x^*,y^*)}{\partial x}} \tag{17.13}$$

Again from the Implicit Function Theorem, the slope of the tangent line to $L(a)$ in (x^*, y^*) is

$$-\frac{\frac{\partial f(x^*,y^*)}{\partial x}}{\frac{\partial f(x^*,y^*)}{\partial y}} \tag{17.14}$$

The product between the expressions in (17.13) and (17.14) is equal to -1.
3. the directional derivative of f at (x^*, y^*) in the the direction u is

$$f'\left((x^*, y^*); u\right) = Df(x^*, y^*) \cdot u = \|Df(x^*, y^*)\| \cdot \|u\| \cdot \cos\theta$$

where θ is an angle in between the two vectors. Then the above quantity is the greatest possible iff $\cos\theta = 1$, i.e., u is colinear with $Df(x^*, y^*)$, i.e., $u = \frac{Df(x^*,y^*)}{\|Df(x^*,y^*)\|}$.

17.6 Extremum problems with equality constraints.

Given the open set $X \subseteq \mathbb{R}^n$, consider the C^1 functions

$$f : X \to \mathbb{R}, \qquad f : x \mapsto f(x),$$

$$g : X \to \mathbb{R}^m, \qquad g : x \mapsto g(x) := (g_j(x))_{j=1}^m$$

with $\underline{m \leq n.}$ Consider also the following "maximization problem":

$$(P) \qquad \max_{x \in X} \quad f(x) \qquad \text{subject to} \qquad g(x) = 0 \tag{17.15}$$

The set

$$C := \{x \in X : g(x) = 0\}$$

is called the constraint set associated with problem (17.15).

Definition 752 *The solution set to problem* (17.15) *is the set*

$$\{x^* \in C : \forall x \in C,\ f(x^*) \geq f(x)\},$$

and it is denoted by arg max (17.15)*.*

The function

$$\mathcal{L} : X \times \mathbb{R}^m \to \mathbb{R}, \quad \mathcal{L} : (x, \lambda) \mapsto f(x) + \lambda^T g(x)$$

is called Lagrange function associated with problem (17.15).

Theorem 753 *Given the open set $X \subseteq \mathbb{R}^n$ and the C^1 functions*

$$f : X \to \mathbb{R}, \qquad f : x \mapsto f(x), \quad g : X \to \mathbb{R}^m, \qquad g : x \mapsto g(x) := (g_j(x))_{j=1}^m,$$

assume that

1. f and g are C^1functions,

2. x_0 is a solution to problem (17.15),[6] *and*

3. $\operatorname{rank}[Dg(x_0)]_{m\times n} = m$*.*

Then, there exists $\lambda_0 \in \mathbb{R}^m$, such that, $D\mathcal{L}(x_0, \lambda_0) = 0$, i.e.,

$$\begin{cases} Df(x_0) + \lambda_0 Dg(x_0) = 0 \\ g(x_0) = 0 \end{cases} \tag{17.16}$$

Proof. Define $x' := (x_i)_{i=1}^m \in \mathbb{R}^m$ and $t = (x_{m+k})_{k=1}^{n-m} \in \mathbb{R}^{n-m}$ and therefore $x = (x', t)$. From Assumption 3, without loss of generality,

$$\det[D_{x'}g(x_0)]_{m\times m} \neq 0. \tag{17.17}$$

We want to show that there exists $\lambda_0 \in \mathbb{R}^m$ which is a solution to the system

$$[Df(x_0)]_{1\times n} + \lambda_{1\times m}[Dg(x_0)]_{m\times n} = 0. \tag{17.18}$$

We can rewrite (17.18) as follows

$$\left[\ D_{x'}f(x_0)_{1\times m} \quad | \quad D_t f(x_0)_{1\times(n-m)}\ \right] + \lambda_{1\times m}\left[\ D_{x'}g(x_0)_{m\times m} \quad | \quad D_t g(x_0)_{m\times(n-m)}\ \right] = 0$$

or

$$\begin{cases} \left[\ D_{x'}f(x_0)_{1\times m}\ \right] + \lambda_{1\times m}\left[\ D_{x'}g(x_0)_{m\times m}\ \right] = 0 \qquad (1) \\ \\ \left[\ D_t f(x_0)_{1\times(n-m)}\ \right] + \lambda_{1\times m}\left[\ D_t g(x_0)_{m\times(n-m)}\ \right] = 0 \qquad (2) \end{cases} \tag{17.19}$$

From (17.17), there exists a unique solution λ_0 to subsystem (1) in (17.19). If $n = m$, we are done. Assume now that $n > m$. We have now to verify that λ_0 is a solution to subsystem (2) in (17.19), as well. To get the desired result, we are going to use the Implicit Function Theorem. Summarizing, we hat that

$$1.\ g \text{ is } C^1, \qquad 2.\ g(x_0', t_0) = 0, \qquad 3. \quad \det[D_{x'}g(x_0', t_0)]_{m\times m} \neq 0,$$

i.e., all the assumption of the Implicit Function Theorem are verified. Then we can conclude that there exist $N(x_0) \subseteq \mathbb{R}^m$ open neighborhood of x_0', $N(t_0) \subseteq \mathbb{R}^{n-m}$ open neighborhood of t_0 and a unique function $\varphi : N(t_0) \to N(x_0)$ such that

$$1.\ \varphi \text{ is } C^1, \qquad 2.\ \varphi(t_0) = x_0'\ , \qquad 3. \quad \forall t \in N(t_0), \quad g(\varphi(t), t) = 0. \tag{17.20}$$

Define now

$$F : N(t_0) \subseteq \mathbb{R}^{n-m} \to \mathbb{R}, \qquad : t \mapsto f(\varphi(t), t),$$

and

$$G : N(t_0) \subseteq \mathbb{R}^{n-m} \to \mathbb{R}^m, \qquad : t \mapsto g(\varphi(t), t).$$

[6] The result does apply in the case in which x_0 is a local maximum for Problem (17.15). Obviously the result apply to the case of (local) minima, as well.

Then, from (17.20) and from Remark 749, we have that $\forall t \in N(t_0)$,

$$0 = [DG(t)]_{m\times(n-m)} = [D_{x'}g(\varphi(t), t)]_{m\times m} \cdot [D\varphi(t)]_{m\times(n-m)} + [D_t g(\varphi(t), t)]_{m\times(n-m)} . \quad (17.21)$$

Since[7], from (17.20), $\forall t \in N(t_0)$, $\quad g(\varphi(t), t) = 0$ and since

$$x_0 := (x'_0, t_0) \overset{(17.20)}{=} (\varphi(t_0), t_0) \quad (17.22)$$

is a solution to problem (17.15), we have that $f(x_0) = F(t_0) \geq F(t)$, i.e., briefly,

$$\forall t \in N(t_0), \qquad F(t_0) \geq F(t).$$

Then, from Proposition 733, $DF(t_0) = 0$. Then, from the definition of F and the Chain Rule, we have

$$[D_{x'}f(\varphi(t_0), t_0)]_{1\times m} \cdot [D\varphi(t_0)]_{m\times(n-m)} + [D_t f(\varphi(t_0), t_0)]_{1\times(n-m)} = 0. \quad (17.23)$$

Premultiplying (17.21) by λ, we get

$$\lambda_{1\times m} \cdot [D_{x'}g(\varphi(t), t)]_{m\times m} \cdot [D\varphi(t)]_{m\times(n-m)} + \lambda_{1\times m} \cdot [D_t g(\varphi(t), t)]_{m\times(n-m)} = 0. \quad (17.24)$$

Adding up (17.23) and (17.24) ,computed at $t = t_0$,we get

$$([D_{x'}f(\varphi(t_0), t_0)] + \lambda \cdot [D_{x'}g(\varphi(t_0), t_0)]) \cdot [D\varphi(t_0)] + [D_t f(\varphi(t_0), t_0)] + +\lambda \cdot [D_t g(\varphi(t_0), t_0)] = 0,$$

and from (17.22),

$$([D_{x'}f(x_0)] + \lambda \cdot [D_{x'}g(x_0)]) \cdot [D\varphi(t_0)] + [D_t f(x_0)] + \lambda \cdot [D_t g(x_0)] = 0. \quad (17.25)$$

Then, from the definition of λ_0 as the unique solution to (1) in (17.19) ,we have that $[D_{x'}f(x_0)] + \lambda_0 \cdot [D_{x'}g(x_0)] = 0$, and then from (17.25) computed at $\lambda = \lambda_0$, we have

$$[D_t f(x_0)] + \lambda_0 \cdot [D_t g(x_0)] = 0,$$

i.e., (2) in (17.19), the desired result. ■

[7] The only place where the proof has to be sligtgly changed to get the result for local maxima is here.

Part IV

Nonlinear programming

Chapter 18

Concavity

Consider[1] a set $X \subseteq \mathbb{R}^n$, a set $\Pi \subseteq \mathbb{R}^k$ and the functions $f : X \times \Pi \to \mathbb{R}$, $g : X \times \Pi \to \mathbb{R}^m$, $h : X \times \Pi \to \mathbb{R}^l$. The goal of this Chapter is to study the problem:

for given f, g, h and for given $\pi \in \Pi$,

$$\max_{x \in X} f(x, \pi) \quad s.t. \quad g(x, \pi) \geq 0 \quad and \quad h(x, \pi) = 0,$$

under suitable assumptions. The role of concavity (and differentiability) of the functions f,g and h is crucial.

In what follows, unless needed, we omit the depends on π.

18.1 Convex sets

Definition 754 *A set $C \subseteq \mathbb{R}^n$ is convex if $\forall x_1, x_2 \in C$ and $\forall \lambda \in [0, 1]$, $(1 - \lambda) x_1 + \lambda x_2 \in C$.*

Definition 755 *A set $C \subseteq \mathbb{R}^n$ is strictly convex if $\forall x_1, x_2 \in C$ such that $x_1 \neq x_2$, and $\forall \lambda \in (0, 1)$, $(1 - \lambda) x_1 + \lambda x_2 \in Int\ C$.*

Remark 756 *If C is strictly convex, then C is convex, but not vice-versa.*

Proposition 757 *The intersection of an arbitrary family of convex sets is convex.*

Proof. We want to show that given a family $\{C_i\}_{i \in I}$ of convex sets, if $x, y \in C := \cap_{i \in I} C_i$ then $(1 - \lambda) x + \lambda y \in C$. $x, y \in C$ implies that $x, y \in C_i$, $\forall i \in I$. Since C_i is convex, $\forall i \in I$, $\forall \lambda \in [0, 1]$, $(1 - \lambda) x + \lambda y \in C_i$, and $\forall \lambda \in [0, 1]$ $(1 - \lambda) x + \lambda y \in C$. ■

Exercise 758 $\forall n \in \mathbb{N}, \forall i \in \{1, ..., n\}$, *$I_i$ is an interval in $\mathbb{R}$, then*

$$\times_{i=1}^n I_i$$

is a convex set.

18.2 Different Kinds of Concave Functions

Maintained Assumptions in this Chapter. Unless otherwise stated,

$$X \text{ is an open and convex subset of } \mathbb{R}^n.$$

f is a function such that

$$f : X \to \mathbb{R}, \qquad : x \mapsto f(x).$$

[1] This part is based on Cass (1991).

For each type of concavity we study, we present

1. the definition in the case in which f is C^0 (i.e., continuous),

2. an attempt of a "partial characterization" of that definition in the case in which f is C^1 and C^2; by partial characterization, we mean a statement which is either sufficient or necessary for the concept presented in the case of continuous f;

3. the relationship between the different partial characterizations;

4. the relationship between the type of concavity and critical points and local or global extrema of f.

Finally, we study the relationship between different kinds of concavities.

The following pictures are taken from David Cass's Microeconomic Course I followed at the University of Pennsylvania (in 1985) and summarize points 1., 2. and 3. above.

Concave and Quasi-Concave Functions

Assume $X \subset \mathbb{R}^n$ is convex and open, $f: X \to \mathbb{R}$.

- f is **concave** if

f continuous

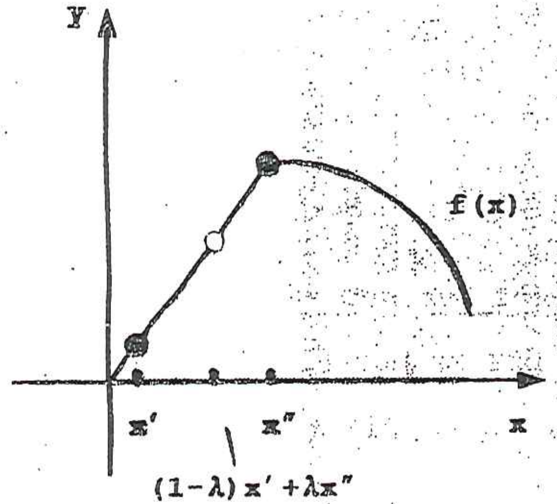
$\{(y,x) \in \mathbb{R}^{n+1} : x \in X \ \& \ y \leq f(x)\}$

is convex

$\Updownarrow$

$x', x'' \in X \ \& \ 0 \leq \lambda \leq 1 \Rightarrow$

$f((1-\lambda)x' + \lambda x'') \geq (1-\lambda)f(x') + \lambda f(x'')$



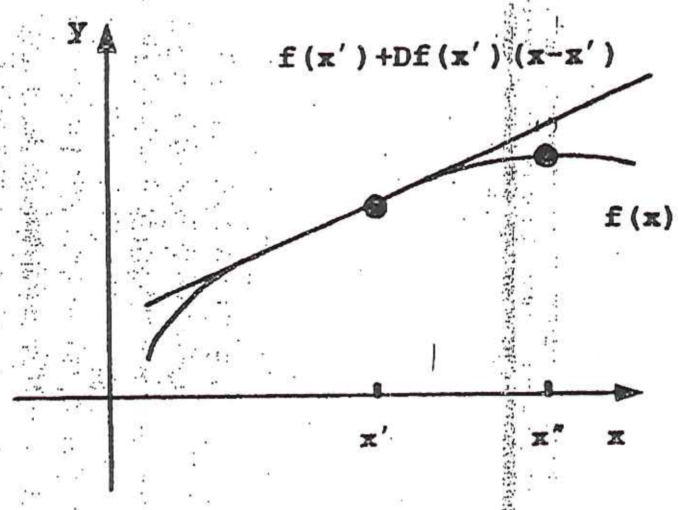
both "flats" and "kinks" are possible

$\Leftrightarrow$

f differentiable

$x', x'' \in X \Rightarrow$

$f(x'') \leq f(x') + Df(x')(x''-x')$



"flats" but not "kinks" are possible

$\Leftrightarrow$

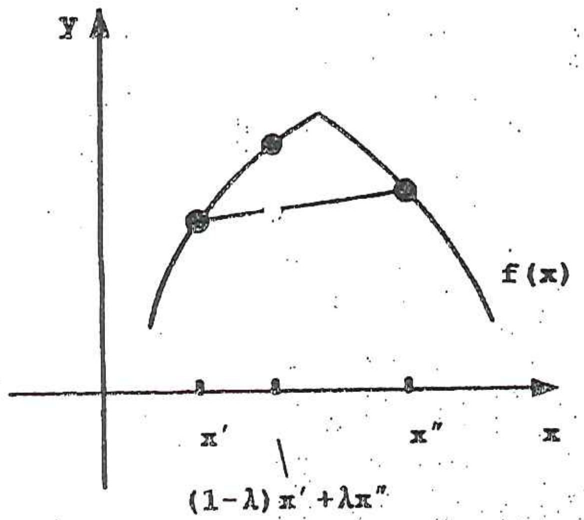
f twice differentiable

$x \in X \ \& \ \Delta x \in \mathbb{R}^n \Rightarrow$

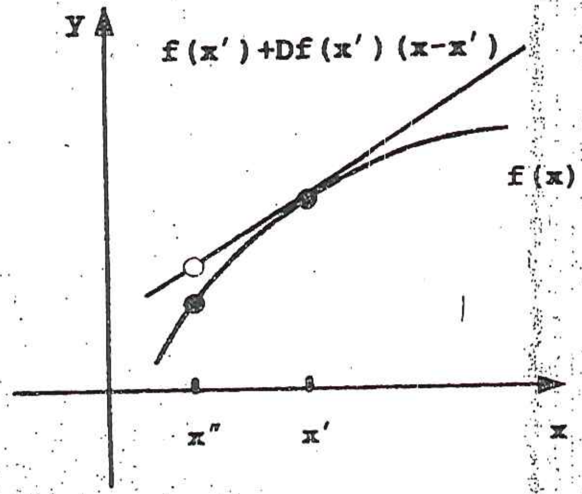
$\Delta x^T D^2 f(x) \Delta x \leq 0$

- f is **strictly concave** if it is concave and

f <u>continuous</u>	f <u>differentiable</u>	f <u>twice differentiable</u>
$x', x'' \in X,\ x' \neq x''\ \&\ 0 < \lambda < 1 \Rightarrow$ $f((1-\lambda)x' + \lambda x'') > (1-\lambda)f(x') + \lambda f(x'')$ $\Leftrightarrow$	$x', x'' \in X\ \&\ x' \neq x'' \Rightarrow$ $f(x'') < f(x') + Df(x')(x''-x')$ $\Leftarrow$	$x \in X,\ \Delta x \in \mathbb{R}^n\ \&\ \Delta x \neq 0 \Rightarrow$ $\Delta x^T D^2 f(x) \Delta x < 0$



"kinks" but not "flats" are possible



neither "flats" nor "kinks" are possible

Note: When X is open (as well convex), (i) if f is concave, then it is continuous, while (ii) it is meaningful to assume, for example, that f is differentiable (i.e., that f has a first-order differential at each point in X).

f is **quasi-concave** if

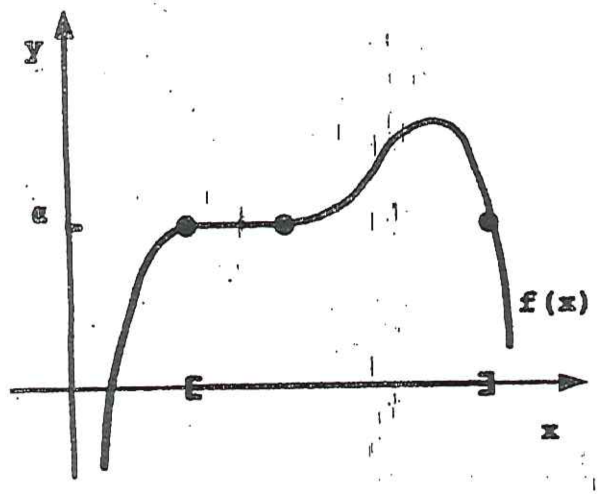
f continuous

$\{x \in X : f(x) \geq \alpha\}$ is convex for every $\alpha \in \mathbb{R}$

$\Updownarrow$

$x', x'' \in X \; \& \; 0 \leq \lambda \leq 1 \Rightarrow$

$f((1-\lambda)x' + \lambda x'') \geq \min\{f(x'), f(x'')\}$

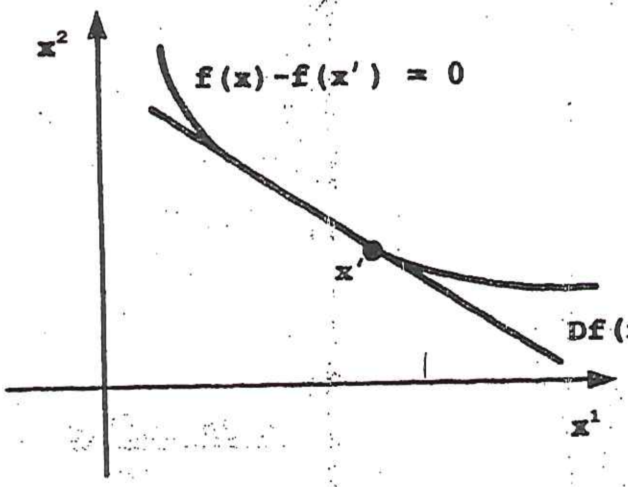


$\Leftrightarrow$

f differentiable

$x', x'' \in X \; \& \; f(x'') - f(x') \geq 0 \Rightarrow$

$Df(x')(x''-x) \geq 0$



$\Rightarrow$

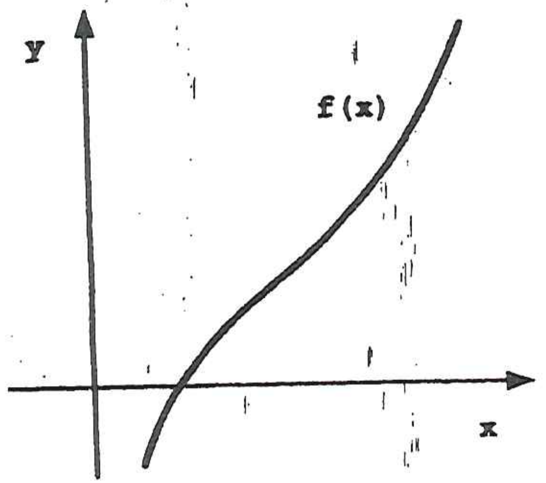
f twice differentiable

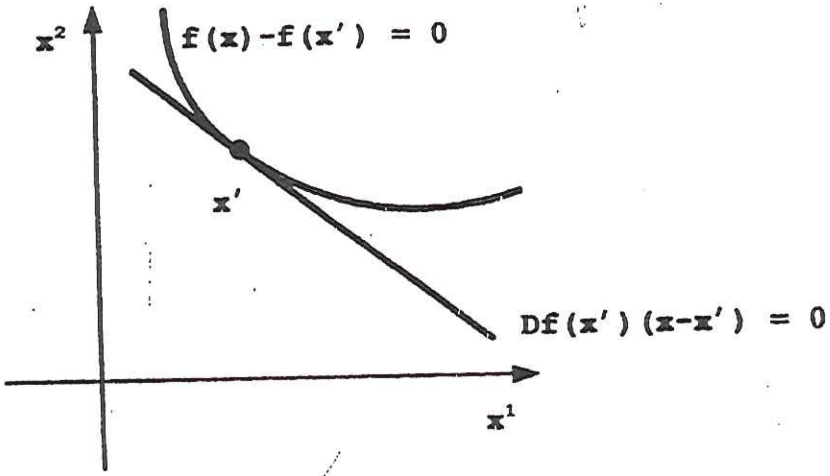
$x \in X, \Delta x \in \mathbb{R}^n, \; \& \; Df(x)\Delta x = 0 \Rightarrow$

$\Delta x^T D^2 f(x) \Delta x \leq 0$

f is (at worst) "single-peaked", and may have "flats" f is **strictly quasi-concave** if it is quasi-concave and

f <u>continuous</u>		f <u>differentiable</u>		f <u>twice differentiable</u>
$x', x'' \in X,\ x' \neq x''$ & $0 < \lambda < 1 \Rightarrow$		$x', x'' \in X,\ x' \neq x''$ & $f(x'') - f(x') \geq 0 \Rightarrow$		$x \in X,\ \Delta x \in \mathbb{R}^n,\ \Delta x \neq 0$ & $Df(x)\Delta x = 0 \Rightarrow$
$f((1-\lambda)x' + \lambda x'') > \min\{f(x'), f(x'')\}$	$\Leftarrow$	$Df(x')(x'' - x') > 0$	$\Leftarrow$	$\Delta x^T D^2 f(x) \Delta x < 0$





f has no "flats"

Note: There are alternative possible definitions of strict quasi-concavity when f is differentiable.
This is the definition most frequently adopted in economic applications.

18.2.1 Concave Functions.

Definition 759 *Consider a C^0 function f. f is concave iff $\forall\, x', x'' \in X$, $\forall\, \lambda \in [0,1]$,*

$$f((1-\lambda)x' + \lambda x'') \geq (1-\lambda) f(x') + \lambda f(x'').$$

Proposition 760 *Consider a C^0 function f.*
f is concave
$\Leftrightarrow$
$M = \{(x,y) \in X \times \mathbb{R} : y \leq f(x)\}$ is convex.

Proof.
[$\Rightarrow$]
Take $(x', y'), (x'', y'') \in M$. We want to show that

$$\forall\, \lambda \in [0,1], \; ((1-\lambda)x' + \lambda x'', (1-\lambda)y' + \lambda y'') \in M.$$

But, from the definition of M, we get that

$$(1-\lambda)y' + \lambda y'' \leq (1-\lambda)f(x') + \lambda f(x'') \leq f((1-\lambda)x' + \lambda x'').$$

[$\Leftarrow$]
From the definition of M, $\forall\, x', x'' \in X$, $(x', f(x')) \in M$ and $(x'', f(x'')) \in M$.
Since M is convex,

$$((1-\lambda)x' + \lambda x'',\; (1-\lambda) f(x') + \lambda f(x'')) \in M$$

and from the definition of M,

$$(1-\lambda) f(x') + \lambda f(x'') \leq f(\lambda x' + (1-\lambda) x'')$$

as desired.
■

Proposition 761 *(Some properties of concave functions).*
1. If $f, g : X \to \mathbb{R}$ are concave functions and $a, b \in \mathbb{R}_+$, then the function $af + bg : X \to \mathbb{R}$, $af + bg : x \mapsto af(x) + bg(x)$ is a concave function.
2. If $f : X \to \mathbb{R}$ is a concave function and $F : A \to \mathbb{R}$, with $A \supseteq Im\, f$, is nondecreasing and concave, then $F \circ f$ is a concave function.

Proof.
1. This result follows by a direct application of the definition.
2. Let $x', x'' \in X$ and $\lambda \in [0,1]$. Then

$$(F \circ f)((1-\lambda)x' + \lambda x'') \overset{(1)}{\geq} F((1-\lambda) f(x') + \lambda f(x'')) \overset{(2)}{\geq} (1-\lambda) \cdot (F \circ f)(x') + \lambda \cdot (F \circ f)(x''),$$

where (1) comes from the fact that f is concave and F is non decreasing, and
(2) comes from the fact that F is concave.
■

Remark 762 *(from Sydsæter (1981)). With the notation of part 2 of the above Proposition, the assumption that F is concave cannot be dropped, as the following example shows. Take $f, F : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $f(x) = \sqrt{x}$ and $F(y) = y^3$. Then f is concave and F is strictly increasing, but $F \circ f(x) = x^{\frac{3}{2}}$ and its second derivative is $\frac{3}{4}x^{-\frac{1}{2}} > 0$. Then, from Calculus I, we know that $F \circ f$ is strictly convex and therefore it is not concave.*

Of course, the monotonicity assumption cannot be dispensed either. Consider $f(x) = -x^2$ and $F(y) = -y$. Then, $(F \circ f)(x) = x^2$, which is not concave.

Proposition 763 *Consider a differentiable function f.*
f is concave
$\Leftrightarrow$
$\forall x', x'' \in X$, $f(x'') - f(x') \leq Df(x')(x'' - x')$.

Proof.
[⇒]
From the definition of concavity, we have that for $\lambda \in (0,1)$,

$$(1-\lambda) f(x') + \lambda f(x'') \le f(x' + \lambda (x'' - x')) \qquad \Rightarrow$$

$$\lambda (f(x'') - f(x')) \le f(x' + \lambda (x'' - x')) - f(x') \quad \Rightarrow$$

$$f(x'') - f(x') \le \frac{f\left(x'+\lambda\left(x''-x'\right)\right)-f(x')}{\lambda}.$$

Taking limits of both sides of the lasts inequality for $\lambda \to 0$, we get the desired result.
[⇐]
Consider $x', x'' \in X$ and $\lambda \in (0,1)$. For $\lambda \in \{0,1\}$, the desired result is clearly true. Since X is convex, $x^\lambda := (1-\lambda) x' + \lambda x'' \in X$. By assumption,

$$f(x'') - f(x^\lambda) \le Df(x^\lambda)(x'' - x^\lambda) \text{ and}$$

$$f(x') - f(x^\lambda) \le Df(x^\lambda)(x' - x^\lambda)$$

Multiplying the first expression by λ, the second one by $(1-\lambda)$ and summing up, we get

$$\lambda(f(x'') - f(x^\lambda)) + (1-\lambda)(f(x') - f(x^\lambda)) \le Df(x^\lambda)(\lambda(x'' - x^\lambda) + (1-\lambda)(x' - x^\lambda))$$

Since

$$\lambda(x'' - x^\lambda) + (1-\lambda)(x' - x^\lambda) = x^\lambda - x^\lambda = 0,$$

we get

$$\lambda f(x'') + (1-\lambda) f(x') \le f(x^\lambda),$$

i.e., the desired result.
■

Definition 764 *Given a symmetric matrix $A_{n\times n}$, A is negative semidefinite if $\forall x \in \mathbb{R}^n$, $xAx \le 0$. A is negative definite if $\forall x \in \mathbb{R}^n \setminus \{0\}$, $xAx < 0$.*

Proposition 765 *Consider a C^2 function f.*
f is concave
⇔
$\forall x \in X$, $D^2 f(x)$ is negative semidefinite.

Proof.
[⇒]
We want to show that $\forall u \in \mathbb{R}^n$, $\forall x_0 \in X$, it is the case that $u^T D^2 f(x_0) u \le 0$. Since X is open, $\forall x_0 \in X$ $\exists a \in \mathbb{R}_{++}$ such that $|h| < a \Rightarrow (x_0 + hu) \in X$. Taken $I := (-a, a) \subseteq \mathbb{R}$, define

$$g : I \to \mathbb{R}, \quad g : h \mapsto f(x_0 + hu) - f(x_0) - Df(x_0)hu.$$

Observe that

$$g'(h) = D_x f(x_0 + hu) \cdot u + Df(x_0) \cdot u$$

and

$$g''(h) = u \cdot D^2 f(x_0 + hu) \cdot u$$

Since f is a concave function, from Proposition 763, we have that $\forall\, h \in I$, $g(h) \le 0$. Since $g(0) = 0$, $h = 0$ is a maximum point. Then, $g'(0) = 0$ and

$$g''(0) \le 0 \quad (1).$$

Moreover, $\forall h \in I$, $g'(h) = Df(x_0 + hu)u - Df(x_0)u$ and $g''(h) = u^T D^2 f(x_0 + hu)u$. Then,

$$g''(0) = u \cdot D^2 f(x_0) \cdot u \quad (2).$$

(1) and (2) give the desired result.
[⇐]
Consider $x, x^0 \in X$. From Taylor's Theorem (see Proposition 739), we get

$$f(x) = f(x^0) + Df(x^0)(x - x^0) + \frac{1}{2}\left(x - x^0\right)^T D^2 f(\overline{x})(x - x^0)$$

where $\overline{x} = x^0 + \theta(x - x^0)$, for some $\theta \in (0, 1)$. Since, by assumption, $\left(x - x^0\right)^T D^2 f(\overline{x})(x - x^0) \leq 0$, we have that

$$f(x) - f(x^0) \leq Df(x^0)(x - x^0),$$

the desired result.
■

Some Properties.

Proposition 766 *Consider a concave function f. If x_0 is a local maximum point, then it is a global maximum point.*

Proof.
By definition of local maximum point, we know that $\exists \delta > 0$ such that $\forall x \in B(x_0, \delta)$, $f(x_0) \geq f(x)$. Take $y \in X$; we want to show that $f\left(x^0\right) \geq f(y)$.
Since X is convex,

$$\forall \lambda \in [0, 1], \ (1 - \lambda) x^0 + \lambda y \in X.$$

Take $\lambda^0 > 0$ and sufficiently small to have $\left(1 - \lambda^0\right) x^0 + \lambda^0 y \in B(x^0, \delta)$. To find such λ^0, just solve the inequality $\left\|\left(1 - \lambda^0\right) x^0 + \lambda^0 y - x^0\right\| = \left\|\lambda^0 \left(y - x^0\right)\right\| = \left|\lambda^0\right| \left\|\left(y - x^0\right)\right\| < \delta$, where, without loss of generality, $y \neq x^0$.
Then,

$$f\left(x^0\right) \geq f\left(\left(1 - \lambda^0\right) x^0 + \lambda^0 y\right) \overset{f \text{ concave}}{\geq} \left(1 - \lambda^0\right) f(x^0) + \lambda^0 f(y),$$

or $\lambda^0 f(x^0) \geq \lambda^0 f(y)$. Dividing both sides of the inequality by $\lambda^0 > 0$, we get $f(x^0) \geq f(y)$.
■

Proposition 767 *Consider a differentiable and concave function f. If $Df(x^0) = 0$, then x^0 is a global maximum point.*

Proof.
From Proposition 763, if $Df(x^0) = 0$, we get that $\forall x \in X$, $f\left(x^0\right) \geq f(x)$, the desired result.
■

18.2.2 Strictly Concave Functions.

Definition 768 *Consider a C^0 function f. f is strictly concave iff $\forall x', x'' \in X$ such that $x' \neq x''$, $\forall \lambda \in (0, 1)$,*

$$f((1 - \lambda)x' + \lambda x'') > (1 - \lambda) f(x') + \lambda f(x'').$$

Proposition 769 *Consider a C^1 function f.*
f is strictly concave
$\Leftrightarrow \forall x', x'' \in X$ such that $x' \neq x''$,

$$f(x'') - f(x') < Df(x')(x'' - x').$$

Proof.
[⇒]
Since strict concavity implies concavity, it is the case that

$$\forall x', x'' \in X, \ f(x'') - f(x') \leq Df(x')(x'' - x'). \tag{18.1}$$

By contradiction, suppose f is not strictly concave. Then, from 18.1, we have that

$$\exists\, x', x'' \in X,\ x' \neq x'' \text{ such that } f(x'') = f(x') + Df(x')(x'' - x'). \tag{18.2}$$

From the definition of strict concavity and 18.2, for $\lambda \in (0,1)$,

$$f((1-\lambda)x' + \lambda x'') > (1-\lambda) f(x') + \lambda f(x') + \lambda Df(x')(x'' - x')$$

or

$$f((1-\lambda)x' + \lambda x'') > f(x') + \lambda Df(x')(x'' - x'). \tag{18.3}$$

Applying 18.1 to the points $x(\lambda) := (1-\lambda)x' + \lambda x''$ and x', we get that for $\lambda \in (0,1)$,

$$f((1-\lambda)x' + \lambda x'') \leq f(x') + Df(x')((1-\lambda)x' + \lambda x'' - x')$$

or

$$f((1-\lambda)x' + \lambda x'') \leq f(x') + \lambda Df(x')(x'' - x'). \tag{18.4}$$

And 18.4 contradicts 18.3.
[$\Leftarrow$] The proof is very similar to that one in Proposition 760.
■

Proposition 770 *Consider a C^2 function f. If*

$$\forall x \in X, \quad D^2 f(x) \text{ is negative definite,}$$

then f is strictly concave.

Proof.
The proof is similar to that of Proposition 765.
■

Remark 771 *In the above Proposition, the opposite implication does not hold. The standard counterexample is $f : \mathbb{R} \to \mathbb{R}, f : x \mapsto -x^4$.*

Some Properties.

Proposition 772 *Consider a strictly concave, C^0 function f. If x^0 is a local maximum point, then it is a strict global maximum point, i.e., the unique global maximum point.*

Proof.
First, we show that a. it is a global maximum point, and then b. the desired result.
a. It follows from the fact that strict concavity is stronger than concavity and from Proposition 766.
b. Suppose otherwise, i.e., $\exists x', x^0 \in X$ such that $x' \neq x^0$ and both of them are global maximum points. Then, $\forall \lambda \in (0,1)$, $(1-\lambda)x' + \lambda x^0 \in X$, since X is convex, and

$$f\left((1-\lambda)x' + \lambda x^0\right) > (1-\lambda) f(x') + \lambda f\left(x^0\right) = f(x') = f\left(x^0\right),$$

a contradiction.
■

Proposition 773 *Consider a strictly concave, differentiable function f. If $Df\left(x^0\right) = 0$, then x^0 is a strict global maximum point.*

Proof.
Take an arbitrary $x \in X$ such that $x \neq x^0$. Then from Proposition 769, we have that $f(x) < f(x^0) + Df(x^0)(x - x^0) = f\left(x^0\right)$, the desired result.
■

18.2.3 Quasi-Concave Functions.

Definitions.

Definition 774 *Consider a C^0 function f. f is quasi-concave iff $\forall x', x'' \in X$, $\forall \lambda \in [0,1]$,*

$$f((1-\lambda)x' + \lambda x'') \geq \min\{f(x'), f(x'')\}.$$

Proposition 775 *If $f : X \to \mathbb{R}$ is a quasi-concave function and $F : \mathbb{R} \to \mathbb{R}$ is non decreasing, then $F \circ f$ is a quasi-concave function.*

Proof.
Without loss of generality, assume

$$f(x'') \geq f(x') \quad (1).$$

Then, since f is quasi-concave, we have

$$f((1-\lambda)x' + \lambda x'') \geq f(x') \quad (2).$$

Then,

$$F(f((1-\lambda)x' + \lambda x'')) \overset{(a)}{\geq} F(f(x')) \overset{(b)}{=} \min\{F(f(x')), F(f(x''))\},$$

where (a) comes from (2) and the fact that F is nondecreasing, and
(b) comes from (1) and the fact that F is nondecreasing.
■

Proposition 776 *Consider a C^0 function f. f is quasi-concave $\Leftrightarrow$ $\forall \alpha \in \mathbb{R}$, $B(\alpha) := \{x \in X : f(x) \geq \alpha\}$ is convex.*

Proof.
[$\Rightarrow$] [Strategy: write what you want to show].
We want to show that $\forall \alpha \in \mathbb{R}$ and $\forall \lambda \in [0,1]$, we have that

$$\langle x', x'' \in B(a)\rangle \Rightarrow \langle (1-\lambda)x' + \lambda x'' \in B(\alpha)\rangle,$$

i.e.,

$$\langle f(x') \geq \alpha \text{ and } f(x'') \geq \alpha\rangle \Rightarrow \langle f((1-\lambda)x' + \lambda x'') \geq \alpha\rangle.$$

But by Assumption,

$$f((1-\lambda)x' + \lambda x'') \geq \min\{f(x'), f(x'')\} \overset{\text{def } x', x''}{\geq} \alpha.$$

[$\Leftarrow$]
Consider arbitrary $x', x'' \in X$. Define $\alpha := \min\{f(x'), f(x'')\}$. Then $x', x'' \in B(\alpha)$. By assumption, $\forall \lambda \in [0,1]$, $(1-\lambda)x' + \lambda x'' \in B(\alpha)$, i.e.,

$$f((1-\lambda)x' + \lambda x'') \geq \alpha := \min\{f(x'), f(x'')\}.$$

■

Proposition 777 *Consider a differentiable function f. f is quasi-concave $\Leftrightarrow$ $\forall x', x'' \in X$,*

$$f(x'') - f(x') \geq 0 \quad \Rightarrow \quad Df(x')(x'' - x') \geq 0.$$

Proof.
[$\Rightarrow$] [Strategy: Use the definition of directional derivative.]
Take x', x'' such that $f(x'') \geq f(x')$. By assumption,

$$f((1-\lambda)x' + \lambda x'') \geq \min\{f(x'), f(x'')\} = f(x')$$

and

$$f((1-\lambda)x'+\lambda x'')-f(x')\geq 0.$$

Dividing both sides of the above inequality by $\lambda>0$, and taking limits for $\lambda\to 0^+$, we get

$$\lim_{\lambda\to 0^+}\frac{f(x'+\lambda(x''-x'))-f(x')}{\lambda}=Df(x')(x''-x')\geq 0.$$

[⇐]
Without loss of generality, take

$$f(x')=\min\{f(x'),f(x'')\}\quad (1).$$

Define

$$\varphi:[0,1]\to\mathbb{R},\ \varphi:\lambda\mapsto f((1-\lambda)x'+\lambda x'').$$

We want to show that

$$\forall\lambda\in[0,1],\ \varphi(\lambda)\geq\varphi(0).$$

Suppose otherwise, i.e., $\exists\,\lambda^*\in[0,1]$ such that $\varphi(\lambda^*)<\varphi(0)$. Observe that in fact it cannot be $\lambda^*\in\{0,1\}$: if $\lambda^*=0$, we would have $\varphi(0)<\varphi(0)$, and if $\lambda^*=1$, we would have $\varphi(1)<\varphi(0)$, i.e., $f(x'')<f(x')$, contradicting (1). Then, we have that

$$\exists\lambda^*\in(0,1)\ \text{such that}\ \varphi(\lambda^*)<\varphi(0)\quad (2).$$

Observe that from (1), we also have that

$$\varphi(1)\geq\varphi(0)\quad (3).$$

Therefore, see Lemma 778, $\exists\,\lambda^{**}>\lambda^*$ such that

$$\varphi'(\lambda^{**})>0\quad (4),\ \text{and}$$
$$\varphi(\lambda^{**})<\varphi(0)\quad (5).$$

From (4), and using the definition of φ', and the Chain Rule,[2] we get

$$0<\varphi'(\lambda^{**})=[Df((1-\lambda^{**})x'+\lambda^{**}x'')](x''-x')\quad (6).$$

Define $x^{**}:=(1-\lambda^{**})x'+\lambda^{**}x''$. From (5), and the assumption, we get that

$$f(x^{**})<f(x').$$

Therefore, by assumption,

$$0\leq Df(x^{**})(x'-x^{**})=Df(x^{**})(-\lambda^{**})(x''-x'),$$

i.e.,

$$[Df(x^{**})]^T(x''-x')\leq 0\quad (7).$$

But (7) contradicts (6).
■

Lemma 778 *Consider a function $g:[a,b]\to\mathbb{R}$ with the following properties:*
1. g is differentiable on (a,b);
2. there exists $c\in(a,b)$ such that $g(b)\geq g(a)>g(c)$.
Then, $\exists\,t\in(c,b)$ such that $g'(t)>0$ and $g(t)<g(a)$.

[2] Defined $v:[0,1]\to X\subseteq\mathbb{R}^n,\lambda\mapsto(1-\lambda)x'+\lambda x''$, we have that $\varphi=f\circ v$. Therefore, $\varphi'(\lambda^*)=Df(v(\lambda^*))\cdot Dv(\lambda^*)$.

Proof.

Without loss of generality and to simplify notation, assume $g(a) = 0$. Define $A := \{x \in [c, b] : g(x) = 0\}$.

Observe that $A = [c, b] \cap g^{-1}(0)$ is closed; and it is non empty, because g is continuous and by assumption $g(c) < 0$ and $g(b) \geq 0$.

Therefore, A is compact, and we can define $\xi := \min A$.

Claim. $x \in [c, \xi) \Rightarrow g(x) < 0$.

Suppose not, i.e., $\exists y \in (c, \xi)$ such that $g(y) \geq 0$. If $g(y) = 0$, ξ could not be $\min A$. If $g(y) > 0$, since $g(c) < 0$ and g is continuous, there exists $x' \in (c, y) \subseteq (c, \xi)$, again contradicting the definition of ξ. End of the proof of the Claim.

Finally, applying Lagrange Theorem to g on $[c, \xi]$, we have that $\exists t \in (c, \xi)$ such that $g'(t) = \frac{g(\xi) - g(c)}{\xi - c}$. Since $g(\xi) = 0$ and $g(c) < 0$, we have that $g'(t) < 0$. From the above Claim, the desired result then follows.

■

Proposition 779 *Consider a C^2 function f. If f is quasi-concave then*

$$\forall x \in X, \forall \Delta \in \mathbb{R}^n \text{ such that } Df(x) \cdot \Delta = 0, \quad \Delta^T D^2 f(x) \Delta \leq 0.$$

Proof.

`for another proof- see Laura’ s file`

Suppose otherwise, i.e., $\exists\, x \in X$, and $\exists \Delta \in \mathbb{R}^\kappa$ such that $Df(x) \cdot \Delta = 0$ and $\Delta^T D^2 f(x^0)\Delta > 0$.

Since the function $h : X \to R$, $h : x \mapsto \Delta^T D^2 f(x) \Delta$ is continuous and X is open, $\forall\, \lambda \in [0, 1]$, $\exists \varepsilon > 0$ such that if $\| x - x^0 \| < \varepsilon$, then

$$\Delta \cdot D^2 f\left(\lambda x + (1 - \lambda) x^0\right) \cdot \Delta > 0 \qquad (1).$$

Define $\overline{x} := x^0 + \mu \frac{\Delta}{\|\Delta\|}$, with $0 < \mu < \varepsilon$. Then,

$$\|\overline{x} - x^0\| = \|\mu \frac{\Delta}{\|\Delta\|}\| = \mu < \varepsilon$$

and $\overline{x}$ satisfies (1). Observe that

$$\Delta = \frac{\| \Delta \|}{\mu}(\overline{x} - x^0).$$

Then, we can rewrite (1) as

$$\left(\overline{x} - x^0\right)^T D^2 f\left(\lambda \overline{x} + (1 - \lambda) x^0\right)\left(\overline{x} - x^0\right) > 0$$

From Taylor Theorem, $\exists\, \lambda \in (0, 1)$ such that

$$f(\overline{x}) = f(x^0) + (\overline{x} - x^0)^T Df\left(x^0\right) + \frac{1}{2}(\overline{x} - x^0)^T D^2 f(\lambda \overline{x} + (1 - \lambda) x^0)(\overline{x} - x^0).$$

Since $Df(x^0)\Delta = 0$ and from (1), we have

$$f(\overline{x}) > f(x^0) \qquad (2).$$

Letting $\widetilde{x} = x^0 + \mu(-\Delta / \| \Delta \|)$, using the same procedure as above, we can conclude that

$$f(\widetilde{x}) > f(x^0)\ (3).$$

But, since $x^0 = \frac{1}{2}(\overline{x} + \widetilde{x})$, (2) and (3) contradict the Definition of quasi-concavity.

■

Remark 780 *In the above Proposition, the opposite implication does not hold. Consider $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto x^4$.From Proposition 776, that function is clearly not quasi-concave. Take $\alpha > 0$. Then $B(\alpha) = \left\{x \in \mathbb{R} : x^4 \geq \alpha\right\} = (-\infty, -\sqrt{\alpha}) \cup (\sqrt{\alpha}, +\infty)$ which is not convex.*

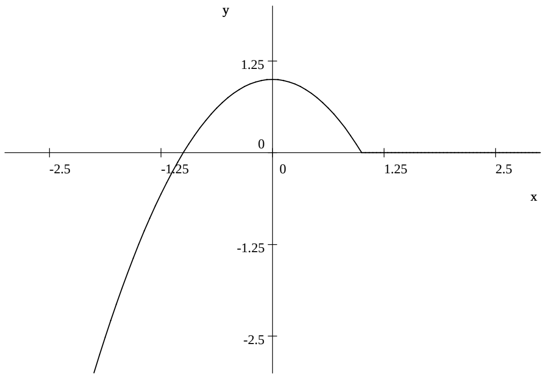
On the other hand observe the following. $f'(x) = 4x^3$ and $4x^3\Delta = 0$ if either $x = 0$ or $\Delta = 0$. In both cases $\Delta 12x^2\Delta = 0$. (This is example is taken from Avriel M. and others (1988), page 91).

Some Properties.

Remark 781 *Consider a quasi concave function f. It is NOT the case that*

if x_0 is a local maximum point , then it is a global maximum point. To see that, consider the following function.

$$f : \mathbb{R} \to \mathbb{R}, \quad f : x \mapsto \begin{cases} -x^2 + 1 & if \quad x < 1 \\ 0 & if \quad x \geq 1 \end{cases}$$



Proposition 782 *Consider a C^0 quasi-concave function f. If x_0 is a strict local maximum point, then it is a strict global maximum point.*

Proof.

By assumption, $\exists \delta > 0$ such that if $x \in B(x_0, \delta) \cap X$ and $x_0 \neq x$, then $f(x_0) > f(x)$.
Suppose the conclusion of the Proposition is false; then $\exists x' \in X$ such that $f(x') \geq f(x_0)$.
Since f is quasi-concave,

$$\forall \lambda \in [0, 1], \quad f((1 - \lambda) x_0 + \lambda x') \geq f(x_0). \quad (1)$$

For sufficiently small λ, $(1 - \lambda) x_0 + \lambda x' \in B(x_0, \delta)$ and (1) above holds, contradicting the fact that x_0 is the *strict* local maximum point.

■

Proposition 783 *Consider $f : (a, b) \to \mathbb{R}$. f monotone $\Rightarrow$ f quasi-concave.*

Proof.

Without loss of generality, take $x'' \geq x'$.

Case 1. f is increasing. Then $f(x'') \geq f(x')$. If $\lambda \in [0, 1]$, then $(1 - \lambda) x' + \lambda x'' = x' + \lambda (x'' - x') \geq x'$ and therefore $f((1 - \lambda) x' + \lambda x'') \geq f(x')$.

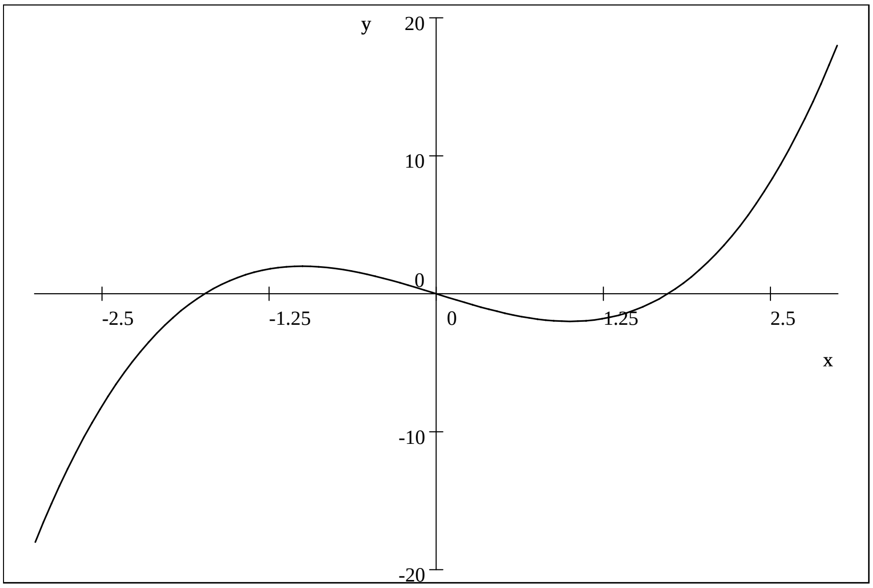
Case 2. f is decreasing. Then $f(x'') \leq f(x')$. If $\lambda \in [0, 1]$, then $(1 - \lambda) x' + \lambda x'' = (1 - \lambda) x' - (1 - \lambda) x'' + x'' = x'' - (1 - \lambda)(x'' - x') \leq x''$and therefore $f((1 - \lambda) x' + \lambda x'') \geq f(x'')$.

■

Remark 784 *The following statement is false: If f_1 and f_2 are quasi-concave and $a, b \in \mathbb{R}_+$, then $af_1 + bf_2$ is quasi-concave.*

It is enough to consider $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^3 + x$, and $f_2(x) = -4x$. Since $f_1' > 0$, then f_1 and, of course, f_2 are monotone and then, from Proposition 783, they are quasi-concave. On the other hand, $g(x) = f_1(x) + f_2(x) = x^3 - x$ has a strict local maximum in $x = -1$which is not a strict global maximum, and therefore, from Proposition 782, g is not quasi-concave.

$x^3 - 3x$



Remark 785 *Consider a differentiable quasi-concave function f. It is NOT the case that if $Df(x^0) = 0$, then x^0 is a global maximum point.*
Just consider $f : \mathbb{R} \to \mathbb{R}, f : x \mapsto x^3$ and $x_0 = 0$, and use Proposition 783.

18.2.4 Strictly Quasi-concave Functions.

Definitions.

Definition 786 *Consider a C^0 function f. f is strictly quasi-concave iff $\forall\ x', x'' \in X$, such that $x' \neq x''$, and $\forall\ \lambda \in (0,1)$, we have that*

$$f((1-\lambda)x' + \lambda x'') > \min\{f(x'), f(x'')\}.$$

Proposition 787 *Consider a C^0 function f. f is strictly quasi-concave $\Rightarrow \forall \alpha \in \mathbb{R}$, $B(\alpha) := \{x \in X : f(x) \geq \alpha\}$ is strictly convex.*

Proof.
Taken an arbitrary α and $x', x'' \in B(\alpha)$, with $x' \neq x''$, we want to show that $\forall \lambda \in (0,1)$, we have that

$$x^\lambda := (1-\lambda)x' + \lambda x'' \in Int\, B(\alpha)$$

Since f is strictly quasi-concave,

$$f(x^\lambda) > \min\{f(x'), f(x'')\} \geq \alpha$$

Since f is C^0, there exists $\delta > 0$ such that $\forall x \in B(x^\lambda, \delta)$

$$f(x) > \alpha$$

i.e., $B(x^\lambda, \delta) \subseteq B(\alpha)$, as desired. Of course, we are using the fact that $\{x \in X : f(x) > \alpha\} \subseteq B(\alpha)$.
■

Remark 788 *Observe that in Proposition 787, the opposite implication does not hold true: just consider $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto 1$.*
Observe that $\forall \alpha \leq 1$, $B(\alpha) = \mathbb{R}$, and $\forall \alpha > 1$, $B(\alpha) = \varnothing$. On the other hand, f is not strictly quasi-concave.

Definition 789 *Consider a differentiable function f. f is differentiable-strictly-quasi-concave iff $\forall\ x', x'' \in X$, such that $x' \neq x''$, we have that*

$$f(x'') - f(x') \geq 0 \ \ \Rightarrow \ \ Df(x')(x'' - x') > 0.$$

Proposition 790 *Consider a differentiable function f.*
If f is differentiable-strictly-quasi-concave, then f is strictly quasi-concave.

Proof.
The proof is analogous to the case of quasi concave functions.
■

Remark 791 *Given a differentiable function, it is not the case that strict-quasi-concavity implies differentiable-strict-quasi-concavity.*
$f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto x^3$ a. is differentiable and strictly quasi concave and b. it is not differentiable-strictly-quasi-concave.
a. f is strictly increasing and therefore strictly quasi concave - see Fact below.
b. Take $x' = 0$ and $x'' = 1$. Then $f(1) = 1 > 0 = f(0)$. But $Df(x')(x'' - x') = 0 \cdot 1 = 0 \not> 0$.

Remark 792 *If we restrict the class of differentiable functions to whose with non-zero gradients everywhere in the domain, then differentiable-strict-quasi-concavity and strict-quasi-concavity are equivalent (see Balasko (1988), Math. 7.2.).*

Fact. Consider $f : (a, b) \to \mathbb{R}$. f strictly monotone $\Rightarrow$ f strictly quasi concave.

Proof.
By assumption, $x' \neq x''$, say $x' < x''$ implies that $f(x') < f(x'')$ (or $f(x') > f(x'')$). If $\lambda \in (0, 1)$, then $(1 - \lambda) x' + \lambda x'' > x'$ and therefore $f((1 - \lambda) x' + \lambda x'') > \min \{f(x'), f(x'')\}$.
■

Proposition 793 *Consider a C^2 function f. If*

$$\forall x \in X, \forall \Delta \in \mathbb{R}^n \setminus \{0\}, \text{we have that } \left\langle Df(x)\,\Delta = 0 \Rightarrow \Delta^T D^2 f(x) \Delta < 0 \right\rangle,$$

then f is differentiable-strictly-quasi-concave.

Proof.

Suppose otherwise, i.e., there exist $x', x'' \in X$ such that

$$x' \neq x'', \ f(x'') \geq f(x') \quad \text{and} \quad Df(x')(x'' - x') \leq 0.$$

Since X is an open set, $\exists a \in \mathbb{R}_{++}$ such the following function is well defined:

$$g : [-a, 1] \to \mathbb{R}, g : h \mapsto f((1 - h) x' + h x'').$$

Since g is continuous, there exists $h_m \in [0, 1]$ which is a global minimum. We now proceed as follows. Step 1. $h_m \notin \{0, 1\}$. Step 2. h_m is a strict local maximum point, a contradiction.
Preliminary observe that

$$g'(h) = Df(x' + h(x'' - x')) \cdot (x'' - x')$$

and

$$g''(h) = (x'' - x')^T \cdot D^2 f(x' + h(x'' - x')) \cdot (x'' - x').$$

Step 1. If $Df(x')(x'' - x') = 0$, then, by assumption,

$$(x'' - x')^T \cdot D^2 f(x' + h(x'' - x')) \cdot (x'' - x') < 0.$$

Therefore, *zero* is a strict local maximum (see, for example, Theorem 13.10, page 378, in Apostol (1974)). Therefore, there exists $h^* \in \mathbb{R}$ such that $g(h^*) = f(x' + h^*(x'' - x')) < f(x') = g(0)$.

If

$$g'(0) = Df(x')(x'' - x') < 0,$$

then there exists $h^{**} \in \mathbb{R}$ such that

$$g(h^{**}) = f(x' + h^*(x'' - x')) < f(x') = g(0).$$

Moreover, $g(1) = f(x'') \geq f(x')$. In conclusion, neither *zero* nor *one* can be global minimum points for g on $[0,1]$.

Step 2. Since the global minimum point $h_m \in (0,1)$, we have that

$$0 = g'(h_m) = Df(x' + h_m(x'' - x'))(x'' - x').$$

Then, by assumption,

$$g''(0) = (x'' - x')^T \cdot D^2 f(x' + h_m(x'' - x')) \cdot (x'' - x') < 0,$$

but then h_m is a strict local maximum point, a contradiction.

■

Remark 794 *Differentiable-strict-quasi-concavity does not imply the condition presented in Proposition 793. $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto -x^4$ is differentiable-strictly-quasi-concave (in next section we will show that strict-concavity implies differentiable-strict-quasi-concavity). On the other hand, take $x^* = 0$. Then $Df(x^*) = 0$. Therefore, for any $\Delta \in \mathbb{R}^n \setminus \{0\}$, we have $Df(x^*)\Delta = 0$, but $\Delta^T D^2 f(x^*)\Delta = 0 \not< 0$.*

Some Properties.

Proposition 795 *Consider a differentiable-strictly-quasi-concave function f.*
x^* is a strict global maximum point $\Leftrightarrow$ $Df(x^*) = 0$.

Proof.

[$\Rightarrow$] Obvious.

[$\Leftarrow$] From the contropositive of the definition of differentiable-strictly-quasi-concave function, we have:

$\forall\, x^*, x'' \in X$, such that $x^* \neq x''$, it is the case that $Df(x^*)(x'' - x^*) \leq 0 \Rightarrow f(x'') - f(x^*) < 0$ or $f(x^*) > f(x'')$. Since $Df(x^*) = 0$, then the desired result follows.

■

Remark 796 *Obviously, we also have that if f is differentaible-strictly-quasi-concave, it is the case that:*
x^* local maximum point $\Rightarrow$ x^* is a strict maximum point.

Remark 797 *The above implication is true also for continuous strictly quasi concave functions. (Suppose otherwise, i.e., $\exists\, x' \in X$ such that $f(x') \geq f(x^*)$. Since f is strictly quasi-concave, $\forall \lambda \in (0,1)$, $f((1-\lambda)x^* + \lambda x') > f(x^*)$, which for sufficiently small λ contradicts the fact that x^* is a local maximum point.*

Is there a definition of ?−concavity weaker than concavity and such that:
If f is a ?−concave function, then
x^* is a global maximum point iff $Df(x^*) = 0$.
The answer is given in the next section.

18.2.5 Pseudo-concave Functions.

Definition 798 *Consider a differentiable function f. f is pseudo-concave iff*

$$\forall x', x'' \in X, \;\; f(x'') > f(x') \Rightarrow Df(x')(x'' - x') > 0,$$

or

$$\forall x', x'' \in X, \;\; Df(x')(x'' - x') \leq 0 \Rightarrow f(x'') \leq f(x').$$

Proposition 799 *If f is a pseudo-concave function, then*

$$x^* \text{ is a global maximum point} \Leftrightarrow Df(x^*) = 0.$$

Proof.
[$\Rightarrow$] Obvious.
[$\Leftarrow$] $Df(x^*) = 0 \Rightarrow \forall x \in X,\ \ Df(x^*)(x - x^*) \leq 0 \Rightarrow f(x) \leq f(x^*)$.
■

Remark 800 *Observe that the following "definition of pseudo-concavity" will not be useful:*

$$\forall x', x'' \in X,\ \ Df(x')(x'' - x') \geq 0 \Rightarrow f(x'') \leq f(x') \tag{18.5}$$

For such a definition the above Proposition would still apply, but it is not weaker than concavity. Simply consider the function $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto -x^2$. That function is concave, but it does not satisfy condition (18.5). Take $x' = -2$ and $x'' = -1$. Then, $f'(x')(x'' - x') = 4(-1 - (-2)) = 4 > 0$, but $f(x'') = -1 > f(x') = -4$.

We summarize some of the results of this subsection in the following tables.

Class of function	**Fundamental properties**		
	C $\Rightarrow$ G max	L max $\Rightarrow$ G max	Uniqueness of G. max
Strictly concave	Yes	Yes	Yes
Concave	Yes	Yes	No
Diff.ble-str.-q.-conc.	Yes	Yes	Yes
Pseudoconcave	Yes	Yes	No
Quasiconcave	No	No	No

where C stands for property of being a critical point, and L and G stand for local and global, respectively. Observe that the first, the second and the last row of the second column apply to the case of C^0 and not necessarily differentiable functions.

18.3 Relationships among Different Kinds of Concavity

The relationships among different definitions of concavity in the case of *differentiable functions* are summarized in the following table.

				strict concavity		
				$\Downarrow$	$\searrow$	
linearity	$\Rightarrow$	affinity	$\Rightarrow$	concavity		
				$\Downarrow$		
				pseudo-concavity	$\Leftarrow$	differentiable-strict-quasi-concavity
				$\Downarrow$		
				quasi-concavity		

All the implications which are not implied by those explicitly written do not hold true.

In what follows, we prove the truth of each implication described in the table and we explain why the other implications do no hold.

Recall that
1. $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function iff $\forall x', x'' \in \mathbb{R}^n$, $\forall a, b \in \mathbb{R}$ $f(ax' + bx'') = af(x') + bf(x'')$;
2. $g : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function iff there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ and $c \in \mathbb{R}^m$ such that $\forall x \in \mathbb{R}^n$, $g(x) = f(x) + c$.

$\boxed{SC \Rightarrow C}$
Obvious ("$a > b \Rightarrow a \geq b$").

$\boxed{C \Rightarrow PC}$

From the assumption and from Proposition 763, we have that $f(x'') - f(x') \leq Df(x')(x'' - x')$. Then $f(x'') - f(x') > 0 \Rightarrow Df(x')(x'' - x') > 0$.

$\boxed{PC \ \Rightarrow QC}$

Suppose otherwise, i.e., $\exists\, x', x'' \in X$ and $\exists \lambda^* [0,1]$ such that

$$f\left((1-\lambda^*) x' + \lambda^* x''\right) < \min\left\{f(x'), f(x'')\right\}.$$

Define $x(\lambda) := (1-\lambda)x' + \lambda x''$. Consider the segment $L(x', x'')$ joining x' to x''. Take $\overline{\lambda} \in \arg\min_\lambda f(x(\lambda))$ $s.t.$ $\lambda \in [0,1]$. $\overline{\lambda}$ is well defined from the Extreme Value Theorem. Observe that $\overline{\lambda} \neq 0, 1$, because $f(x(\lambda^*)) < \min\left\{f(x(0)) = f(x'), f(x(1)) = f(x'')\right\}$.

Therefore, $\forall \lambda \in [0,1]$ and $\forall \mu \in (0,1)$,

$f\left(x\left(\overline{\lambda}\right)\right) \leq f\left((1-\mu)x\left(\overline{\lambda}\right) + \mu x(\lambda)\right)$.

$$(1-\mu)x\left(\overline{\lambda}\right) + \mu x(\lambda)$$

$$\downarrow$$

$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \uparrow & \uparrow & & \uparrow & \uparrow \\ x' & x(\lambda) & & x\left(\overline{\lambda}\right) & x'' \end{array}$$

Then,

$$\forall \lambda \in [0,1], \quad 0 \leq \lim_{\mu \to 0^+} \frac{f\left((1-\mu)x\left(\overline{\lambda}\right) + \mu x(\lambda)\right) - f\left(x\left(\overline{\lambda}\right)\right)}{\mu} = Df\left(x\left(\overline{\lambda}\right)\right)\left(x(\lambda) - x\left(\overline{\lambda}\right)\right).$$

Taking $\lambda = 0, 1$ in the above expression, we get:

$$Df\left(x\left(\overline{\lambda}\right)\right)\left(x' - x\left(\overline{\lambda}\right)\right) \geq 0 \quad (1)$$

and

$$Df\left(x\left(\overline{\lambda}\right)\right)\left(x'' - x\left(\overline{\lambda}\right)\right) \geq 0 \quad (2).$$

Since

$$x' - x\left(\overline{\lambda}\right) = x' - \left(1 - \overline{\lambda}\right)x' - \overline{\lambda}x'' = -\overline{\lambda}\left(x'' - x'\right) \quad (3),$$

and

$$x'' - x\left(\overline{\lambda}\right) = x'' - \left(1 - \overline{\lambda}\right)x' - \overline{\lambda}x'' = \left(1 - \overline{\lambda}\right)\left(x'' - x'\right) \quad (4),$$

substituting (3) in (1), and (4) in (2), we get

$$\overset{(-)}{-\overline{\lambda}} \cdot \left[Df\left(x\left(\overline{\lambda}\right)\right) \cdot \left(x'' - x'\right)\right] \geq 0,$$

and

$$\overset{(+)}{\left(1 - \overline{\lambda}\right)} \cdot \left[Df\left(x\left(\overline{\lambda}\right)\right) \cdot \left(x'' - x'\right)\right] \geq 0.$$

Therefore,

$$0 = Df\left(x\left(\overline{\lambda}\right)\right) \cdot \left(x'' - x'\right) = Df\left(x\left(\overline{\lambda}\right)\right) \cdot \left(1 - \overline{\lambda}\right) \cdot \left(x'' - x'\right) \overset{(4)}{=}$$

$$= Df\left(x\left(\overline{\lambda}\right)\right) \cdot \left(x'' - x\left(\overline{\lambda}\right)\right).$$

Then, by pseudo-concavity,

$$f\left(x''\right) \leq f\left(x\left(\overline{\lambda}\right)\right) \quad (5).$$

By assumption,

$$f\left(x\left(\lambda^*\right)\right) < f\left(x''\right) \quad (6).$$

(5) and (6) contradict the definition of $\overline{\lambda}$.

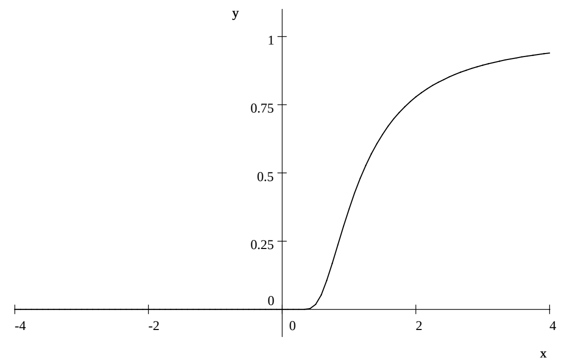
$\boxed{DSQC \Rightarrow PC}$
Obvious.

$\boxed{SC \Rightarrow DSQC}$
Obvious.

$\boxed{L \Rightarrow C}$
Obvious.

$\boxed{C \nRightarrow SC}$
$f : \mathbb{R} \to \mathbb{R}, \; f : x \mapsto x.$

$\boxed{QC \nRightarrow PC}$
$$f : \mathbb{R} \to \mathbb{R}, \; f : x \mapsto \begin{cases} 0 & if \quad x \leq 0 \\ e^{-\frac{1}{x^2}} & if \quad x > 0 \end{cases}$$



f is clearly nondecreasing and therefore, from Lemma 783, quasi-concave.
f is not pseudo-concave: $0 < f(-1) > f(1) = 0$, but
$f'(1)(-1-1) = 0 \cdot (-2) = 0$.

$\boxed{PC \nRightarrow C}$, $\boxed{DSQC \nRightarrow C}$ and $\boxed{DSQC \nRightarrow SC}$
Take $f : (1, +\infty) \to \mathbb{R}, \; f : x \mapsto x^3$.

Take $x' < x''$. Then $f(x'') > f(x')$. Moreover, $\overset{(>0)}{Df}(x')\overset{(>0)}{(x'' - x')} > 0$. Therefore, f is $DSQC$ and therefore PC. Since $f''(x) > 0$, f is strictly convex and therefore it is not concave and, a fortiori, it is not strictly concave.

$\boxed{PC \nRightarrow DSQC}$, $\boxed{C \nRightarrow DSQC}$
Consider $f : \mathbb{R} \to \mathbb{R}, \; f : x \mapsto 1$. f is clearly concave and PC, as well ($\forall x', x'' \in \mathbb{R}, Df(x')(x'' - x') \geq 0$). Moreover, any point in $\mathbb{R}$ is a critical point, but it is not the unique global maximum point. Therefore, from Proposition 795, f is not differentiable - strictly - quasi - concave.

$\boxed{QC \nRightarrow DSQC}$
If so, we would have $QC \Rightarrow DSQC \Rightarrow PC$, contradicting the fact that $QC \nRightarrow PC$.

$\boxed{C \nRightarrow L}$ and $\boxed{SC \nRightarrow L}$
$f : \mathbb{R} \to \mathbb{R}, \; f : x \mapsto -x^2$.

18.3.1 Hessians and Concavity.

In this subsection, we study the relation between submatrices of a matrix involving the Hessian matrix of a C^2 function and the concavity of that function.

Definition 801 *Consider a matrix $A_{n\times n}$. Let $1 \leq k \leq n$.*

A $k-th$ order principal submatrix (minor) of A is the (determinant of the) square submatrix of A obtained deleting $(n-k)$ rows and $(n-k)$ columns in the same position. Denote these matrices by $\widetilde{D}_k^i$.

The $k-th$ order leading principal submatrix (minor) of A is the (determinant of the) square submatrix of A obtained deleting the last $(n-k)$ rows and the last $(n-k)$ columns. Denote these matrices by D_k.

Example 802 *Consider*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then

$$\widetilde{D}_1^1 = a_{11},\ \widetilde{D}_1^2 = a_{22},\ \widetilde{D}_1^3 = a_{33}, \quad D_1 = \widetilde{D}_1^1 = a_{11};$$

$$\widetilde{D}_2^1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ \widetilde{D}_2^2 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix},\ \widetilde{D}_2^3 = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix},$$

$$D_2 = \widetilde{D}_2^1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix};$$

$$D_3 = \widetilde{D}_3^1 = A.$$

Definition 803 *Consider a C^2 function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$. The bordered Hessian of f is the following matrix*

$$B_f(x) = \begin{bmatrix} 0 & Df(x) \\ [Df(x)]^T & D^2 f(x) \end{bmatrix}.$$

Theorem 804 *(Simon, (1985), Theorem 1.9.c, page 79 and Sydsaeter (1981), Theorem 5.17, page 259). Consider a C^2 function $f : X \to \mathbb{R}$.*

1. If $\forall x \in X, \forall k \in \{1, ..., n\}$,
$sign\left(k - \text{leading principal minor of } D^2 f(x)\right) = sign\ (-1)^k$,
then f is strictly concave.

2. $\forall x \in X, \forall k \in \{1, ..., n\}$,
$sign\left(\text{non zero } k - \text{ principal minor of } D^2 f(x)\right) = sign\ (-1)^k$,
iff f is concave.

3. If $n \geq 2$ and $\forall x \in X, \forall k \in \{3, ..., n+1\}$,
$sign\left(k - \text{leading principal minor of } Bf(x)\right) = sign\ (-1)^{k-1}$,
then f is pseudo concave and, therefore, quasi-concave.

4. If f is quasi-concave, then $\forall x \in X, \forall k \in \{2, ..., n+1\}$,
$sign\left(\text{non zero } k - \text{leading principal minors of } Bf(x)\right) = sign\ (-1)^{k-1}$

Remark 805 *It can be proved that Conditions in part 1 and 2 of the above Theorem are sufficient for $D^2 f(x)$ being negative definite and equivalent to $D^2 f(x)$ being negative semidefinite, respectively.*

Remark 806 *(From Sydsaetter (1981), page 239) It is tempting to conjecture that a function f is concave iff*

$$\forall x \in X, \forall k \in \{1, ..., n\},\ sign\left(\text{non zero } k - \textbf{\textit{leading}} \text{ principal minor of } D^2 f(x)\right) = sign\ (-1)^k, \tag{18.6}$$

That conjecture is false. Consider

$$f:\mathbb{R}^3\to\mathbb{R},\qquad f:(x_1,x_2,x_3)\mapsto -x_2^2+x_3^2.$$

Then $Df(x)=(0,-2x_2,2x_3)$ *and*

$$D^2f(x)=\begin{bmatrix}0&0&0\\0&-2&0\\0&0&2\end{bmatrix}$$

All the leading principal minors of the above matrix are zero, and therefore Condition 18.6 is satisfied, but f *is not a concave function. Take* $x'=(0,0,0)$ *and* $x''(0,0,1)$*. Then*

$$\forall\lambda\in(0,1),\ \ f((1-\lambda)x'+\lambda x'')=\lambda^2<(1-\lambda)f(x')+\lambda f(x'')=\lambda$$

Example 807 *Consider* $f:\mathbb{R}^2_{++}\to\mathbb{R}$, $f:(x,y)\mapsto x^\alpha y^\beta$, *with* $\alpha,\beta\in\mathbb{R}_{++}$. *Observe that* $\forall(x,y)\in\mathbb{R}^2_{++}$, $f(x,y)>0$. *Verify that*

1. *if* $\alpha+\beta<1$, *then* f *is strictly concave;*
2. $\forall\alpha,\beta\in\mathbb{R}_{++}$, f *is quasi-concave;*
3. $\alpha+\beta\le1$ *if and only if* f *is concave.*

1.
$D_xf(x,y)=\alpha x^{\alpha-1}y^\beta=\frac{\alpha}{x}f(x,y)$;
$D_yf(x,y)=\beta x^{\alpha}y^{\beta-1}=\frac{\beta}{y}f(x,y)$;
$D^2_{x,x}f(x,y)=\alpha(\alpha-1)x^{\alpha-2}y^\beta=\frac{\alpha(\alpha-1)}{x^2}f(x,y)$;
$D^2_{y,y}f(x,y)=\beta(\beta-1)x^{\alpha}y^{\beta-2}=\frac{\beta(\beta-1)}{y^2}f(x,y)$;
$D^2_{x,.y}f(x,y)=\alpha\beta x^{\alpha-1}y^{\beta-1}=\frac{\alpha}{x}\frac{\beta}{y}f(x,y)$.
$D^2f(x,y)=f(x,y)\begin{bmatrix}\frac{\alpha(\alpha-1)}{x^2}&\frac{\alpha}{x}\frac{\beta}{y}\\\frac{\alpha}{x}\frac{\beta}{y}&\frac{\beta(\beta-1)}{y^2}\end{bmatrix}$.
a. $\frac{\alpha(\alpha-1)}{x^2}<0\Leftrightarrow\alpha\in(0,1)$.
b. $\frac{\alpha(\alpha-1)\beta(\beta-1)-\alpha^2\beta^2}{x^2y^2}=\frac{1}{x^2y^2}\left(\alpha\beta(\alpha\beta-\alpha-\beta+1)-\alpha^2\beta^2\right)=$
$=\frac{1}{x^2y^2}\alpha\beta(1-\alpha-\beta)>0\overset{\alpha,\beta>0}{\Leftrightarrow}\alpha+\beta<1$.
In conclusion, if $\alpha,\beta\in(0,1)$ *and* $\alpha+\beta<1$, *then* f *is strictly concave.*
2.
Observe that

$$f(x,y)=g(h(x,y))$$

where

$$h:\mathbb{R}^2_{++}\to\mathbb{R},\quad(x,y)\mapsto\alpha\ln x+\beta\ln y$$

$$g:\mathbb{R}\to\mathbb{R},\quad z\mapsto e^z$$

Since h *is strictly concave (why?) and therefore quasi-concave and* g *is strictly increasing, the desired result follows from Proposition 775.*
3.
Obvious from above results.

Chapter 19

Maximization Problems

Let the following objects be given:

1. an open convex set $X \subseteq \mathbb{R}^n$, $n \in \mathbb{N}\setminus\{0\}$;
2. $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$, $h : X \to \mathbb{R}^k$, $m, k \in \mathbb{N}\setminus\{0\}$, with f, g, h at least differentiable.

The goal of this Chapter is to study the problem.

$$\begin{array}{lll} \max_{x \in X} & f(x) & \\ s.t. & g(x) \geq 0 & (1) \\ & h(x) = 0 & (2) \end{array} \tag{19.1}$$

f is called **objective function**; x **choice variable** vector; (1) and (2) in (19.1) **constraints**; g and h **constraint functions**;

$$C := \{x \in X : g(x) \geq 0 \quad \text{and} \quad h(x) = 0\}$$

is the constraint set.

To solve the problem (19.1) means to describe the following set

$$\{x^* \in C : \forall x \in C,\ f(x^*) \geq f(x)\}$$

which is called solution set to problem (19.1) and it is also denoted by arg max (19.1). We will proceed as follows.

1. We will analyze in detail the problem with inequality constraints, i.e.,

$$\begin{array}{lll} \max_{x \in X} & f(x) & \\ s.t. & g(x) \geq 0 & (1) \end{array}$$

2. We will analyze in detail the problem with equality constraints, i.e.,

$$\begin{array}{lll} \max_{x \in X} & f(x) & \\ s.t. & h(x) = 0 & (2) \end{array}$$

3. We will describe how to solve the problem with both equality and inequality constraints, i.e.,

$$\begin{array}{lll} \max_{x \in X} & f(x) & \\ s.t. & g(x) \geq 0 & (1) \\ & h(x) = 0 & (2) \end{array}$$

19.1 The case of inequality constraints: Kuhn-Tucker theorems

Consider the open and convex set $X \subseteq \mathbb{R}^n$ and the differentiable functions $f : X \to \mathbb{R}$, $g := \left(g^j\right)_{j=1}^m : X \to \mathbb{R}^m$. The problem we want to study is

$$\max_{x \in X} f(x) \qquad s.t. \qquad g(x) \geq 0. \tag{19.2}$$

Definition 808 *The Kuhn-Tucker system (or conditions) associated with problem 19.2 is*

$$\begin{cases} Df(x) + \lambda Dg(x) & = & 0 & (1) \\ \lambda & \geq & 0 & (2) \\ g(x) & \geq & 0 & (3) \\ \lambda g(x) & = & 0 & (4) \end{cases} \tag{19.3}$$

Equations (1) *are called first order conditions; equations* (2) , (3) *and* (4) *are called complementary slackness conditions.*

Remark 809 $(x, \lambda) \in X \times \mathbb{R}^m$ *is a solution to Kuhn-Tucker system iff it is a solution to any of the following systems:*

1.
$$\begin{cases} \frac{\partial f(x)}{\partial x_i} + \sum_{j=1}^m \lambda_j \frac{\partial g_j(x)}{\partial x_i} & = & 0 & for\ i = 1, ..., n & (1) \\ \lambda_j & \geq & 0 & for\ j = 1, ..., m & (2) \\ g_j(x) & \geq & 0 & for\ j = 1, ..., m & (3) \\ \lambda_j g_j(x) & = & 0 & for\ j = 1, ..., m & (4) \end{cases}$$

2.
$$\begin{cases} Df(x) + \lambda Dg(x) & = & 0 & & (1) \\ \min\{\lambda_j, g_j(x)\} & = & 0 & for\ j = 1, ..., m & (2) \end{cases}$$

Moreover, $(x, \lambda) \in X \times \mathbb{R}^m$ is a solution to Kuhn-Tucker system iff it is a solution to

$$Df(x) + \lambda Dg(x) \quad = \quad 0 \qquad (1)$$

and for each $j = 1, ..., m$, to **one** of the following conditions

$$\begin{array}{lllll} \text{either} & (& \lambda_j > 0 & \text{and} & g_j(x) = 0 &) \\ \text{or} & (& \lambda_j = 0 & & g_j(x) > 0 &) \\ \text{or} & (& \lambda_j = 0 & & g_j(x) = 0 &) \end{array}$$

Definition 810 *Given* $x^* \in X$, *we say that* j *is a binding constraint at* x^* *if* $g_j(x^*) = 0$. *Let*

$$J^*(x^*) := \{j \in \{1, ..., m\} : g_j(x^*) = 0\}$$

and

$$g^* := (g_j)_{j \in J^*(x^*)}, \quad \widehat{g} := (g_j)_{j \notin J^*(x^*)}$$

Definition 811 $x^* \in \mathbb{R}^n$ *satisfies the constraint qualifications associated with problem 19.2 if it is a solution to*

$$\max_{x \in \mathbb{R}^n} Df(x^*)x \qquad s.t. \qquad Dg^*(x^*)(x - x^*) \quad \geq \quad 0 \tag{19.4}$$

The above problem is obtained from 19.2

1. replacing g with g^*;
2. linearizing f and g^* around x^*, i.e., substituting f and g^* with $f(x^*) + Df(x^*)(x - x^*)$ and $g(x^*) + Dg(x^*)(x - x^*)$, respectively;
3. dropping redundant terms, i.e., the term $f(x^*)$ in the objective function, and the term $g^*(x^*) = 0$ in the constraint.

Theorem 812 *Suppose* x^* *is a solution to problem 19.2 and to problem 19.4, then there exists* $\lambda^* \in \mathbb{R}^m$ *such that* (x^*, λ^*) *satisfies Kuhn-Tucker conditions.*

The proof of the above theorem requires the following lemma, whose proof can be found for example in Chapter 2 in Mangasarian (1994).

Lemma 813 *(Farkas) Given a matrix* $A_{m \times n}$ *and a vector* $a \in \mathbb{R}^n$,
either 1. there exists $\lambda \in \mathbb{R}^m_+$ *such that* $a = \lambda A$,
or 2. there exists $y \in \mathbb{R}^n$ *such that* $Ay \geq 0$ *and* $ay < 0$,
but not both.

Proof. of Theorem 812

(main steps: 1. use the fact x^* is a solution to problem 19.4; 2. apply Farkas Lemma; 3. choose $\lambda^* = (\lambda \text{ from Farkas }, 0))$.

Since x^* is a solution to problem 19.4, for any $x \in \mathbb{R}^n$ such that $Dg^*(x^*)(x - x^*) \geq 0$ it is the case that $Df(x^*)x^* \geq Df(x^*)x$ or

$$Dg^*(x^*)(x - x^*) \geq 0 \Rightarrow [-Df(x^*)](x - x^*) \geq 0. \tag{19.5}$$

Applying Farkas Lemma identifying

$$a \quad \text{with} \quad -Df(x^*)$$

and

$$A \quad \text{with} \quad Dg^*(x^*)$$

we have that either

1. there exists $\lambda \in \mathbb{R}^m_+$ such that

$$-Df(x^*) = \lambda Dg^*(x^*) \tag{19.6}$$

or 2. there exists $y \in \mathbb{R}^n$ such that

$$Dg^*(x^*)y \geq 0 \quad \text{and} \quad -Df(x^*)y < 0 \tag{19.7}$$

but not both 1 and 2.

Choose $x = y + x^*$ and therefore you have $y = x - x^*$. Then, 19.7 contradicts 19.5. Therefore, 1. above holds.

Now, choose $\lambda^* := (\lambda, 0) \in \mathbb{R}^{m^*} \times \mathbb{R}^{m-m^*}$, we have that

$$Df(x^*) + \lambda^* Dg(x^*) = Df(x^*) + (\lambda, 0)\begin{pmatrix} Dg^*(x^*) \\ D\widehat{g}(x^*) \end{pmatrix} = Df(x^*) + \lambda Dg(x^*) = 0$$

where the last equality follows from 19.6;

$\lambda^* \geq 0$ by Farkas Lemma;

$g(x^*) \geq 0$ from the assumption that x^* solves problem 19.2;

$\lambda^* g(x^*) = (\lambda, 0)\begin{pmatrix} g^*(x^*) \\ \widehat{g}(x^*) \end{pmatrix} = \lambda g^*(x) = 0$, where the last equality follows from the definition of g^*. ■

Theorem 814 *If x^* is a solution to problem* (19.2) *and*
either for $j = 1, .., m$, g_j is pseudo-concave and $\exists x^{++} \in X$ such that $g(x^{++}) \gg 0$,
or rank $Dg^*(x^*) = m^* := \#J^*(x^*)$,
then x^* solves problem (19.4).

Proof. We prove the conclusion of the theorem under the first set of conditions.

Main steps: 1. suppose otherwise: $\exists\ \widetilde{x}$ ... ; 2. use the two assumptions; 3. move from x^* in the direction $x^\theta := (1-\theta)\widetilde{x} + \theta x^{++}$.

Suppose that the conclusion of the theorem is false. Then there exists $\widetilde{x} \in \mathbb{R}^n$ such that

$$Dg^*(x^*)(\widetilde{x} - x^*) \geq 0 \quad \text{and} \quad Df(x^*)(\widetilde{x} - x^*) > 0 \tag{19.8}$$

Moreover, from the definition of g^* and x^{++}, we have that

$$g^*\left(x^{++}\right) >> 0 = g^*(x^*)$$

Since for $j = 1, .., m$, g_j is pseudo-concave we have that

$$Dg^*(x^*)\left(x^{++} - x^*\right) \gg 0 \tag{19.9}$$

Define

$$x^\theta := (1-\theta)\widetilde{x} + \theta x^{++}$$

with $\theta \in (0,1)$. Observe that

$$x^{\theta} - x^{*} = (1-\theta)\,\widetilde{x} + \theta x^{++} - (1-\theta)\,x^{*} - \theta x^{*} = (1-\theta)\left(\widetilde{x} - x^{*}\right) + \theta\left(x^{++} - x^{*}\right)$$

Therefore,

$$Dg^{*}\left(x^{*}\right)\left(x^{\theta} - x^{*}\right) = (1-\theta)\,Dg^{*}\left(x^{*}\right)\left(\widetilde{x} - x^{*}\right) + \theta Dg^{*}\left(x^{*}\right)\left(x^{++} - x^{*}\right) \gg 0 \tag{19.10}$$

where the last equality come from 19.8 and 19.9.
Moreover,

$$Df\left(x^{*}\right)\left(x^{\theta} - x^{*}\right) = (1-\theta)\,Df\left(x^{*}\right)\left(\widetilde{x} - x^{*}\right) + \theta Df\left(x^{*}\right)\left(x^{++} - x^{*}\right) \gg 0 \tag{19.11}$$

where the last equality come from 19.8 and a choice of θ sufficiently small.[1]
Observe that from Remark 706, 19.10 and 19.11 we have that

$$\left(g^{*}\right)'\left(x^{*}, x^{\theta}\right) \gg 0$$

and

$$f'\left(x^{*}, x^{\theta}\right) > 0$$

Therefore, using the fact that X is open, and that $\widehat{g}\left(x^{*}\right) \gg 0$, there exists γ such that

$$\begin{array}{l} x^{*} + \gamma\left(x^{\theta} - x^{*}\right) \in X \\ g^{*}\left(x^{*} + \gamma\left(x^{\theta} - x^{*}\right)\right) \gg g^{*}\left(x^{*}\right) = 0 \\ f^{*}\left(x^{*} + \gamma\left(x^{\theta} - x^{*}\right)\right) > f\left(x^{*}\right) \\ \widehat{g}\left(x^{*} + \gamma\left(x^{\theta} - x^{*}\right)\right) \gg 0 \end{array} \tag{19.12}$$

But then 19.12 contradicts the fact that x^{*} solves problem (19.2). ■
From Theorems 812 and 814, we then get the following corollary.

Theorem 815 *Suppose x^{*} is a solution to problem 19.2, and one of the following constraint qualifications hold:*
a. for $j = 1, ..., m$, g_j is pseudo-concave and there exists $x^{++} \in X$ such that $g\left(x^{++}\right) \gg 0$
b. $\operatorname{rank} Dg^{*}\left(x^{*}\right) = \#J^{*}$,
Then there exists $\lambda^{*} \in \mathbb{R}^{m}$ such that $\left(x^{*}, \lambda^{*}\right)$ solves the system 19.3.

Theorem 816 *If f is pseudo-concave, and for $j = 1, ..., m$, g_j is quasi-concave, and $\left(x^{*}, \lambda^{*}\right)$ solves the system 19.3, then x^{*} solves problem 19.2.*

Proof. Main steps: 1. suppose otherwise and use the fact that f is pseudo-concave; 2. for $j \in J^{*}\left(x^{*}\right)$, use the quasi-concavity of g_j; 3. for $j \in J^{*}\left(x^{*}\right)$, use (second part of) kuhn-Tucker conditions; 4. Observe that 2. and 3. above contradict the first part of Kuhn-Tucker conditions.)
Suppose otherwise, i.e., there exists $\widehat{x} \in X$ such that

$$g\left(\widehat{x}\right) \geq 0 \quad \text{and} \quad f\left(\widehat{x}\right) > f\left(x^{*}\right) \tag{19.13}$$

From 19.13 and the fact that f pseudo-concave, we get

$$Df\left(x^{*}\right)\left(\widehat{x} - x^{*}\right) > 0 \tag{19.14}$$

From 19.13, the fact that $g^{*}\left(x^{*}\right) = 0$ and that g_j is quasi-concave, we get that

$$\text{for } \ j \in J^{*}\left(x^{*}\right), \quad Dg^{j}\left(x^{*}\right)\left(\widehat{x} - x^{*}\right) \geq 0$$

---

[1] Assume that $\theta \in (0,1)$, $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$. We want to show that there exist $\theta^{*} \in (0,1)$ such that

$$(1-\theta)\,\alpha + \theta\beta > 0$$

i.e.,

$$\alpha > \theta\,(\alpha - \beta)$$

If $(\alpha - \beta) = 0$, the claim is true.
If $(\alpha - \beta) > 0$, any $\theta < \frac{\alpha}{\alpha - \beta}$ will work (observe that $\frac{\alpha}{\alpha - \beta} > 0$).
If $(\alpha - \beta) < 0$, the claim is clearly true because $0 < \alpha$ and $\theta\,(\alpha - \beta) < 0$.

and since $\lambda^* \geq 0$,

$$\text{for} \quad j \in J^*(x^*), \quad \lambda_j^* Dg^j(x^*)(\widehat{x} - x^*) \geq 0 \tag{19.15}$$

For $j \in \widehat{J}(x^*)$, from Kuhn-Tucker conditions, we have that $g_j(x^*) > 0$ and $\lambda_j^* = 0$, and therefore

$$\text{for} \quad j \in \widehat{J}(x^*), \quad \lambda_j^* Dg^j(x^*)(\widehat{x} - x^*) = 0 \tag{19.16}$$

But then from 19.14, 19.15 and 19.16, we have

$$Df(x^*)(\widehat{x} - x^*) + \lambda^* Dg(x^*)(\widehat{x} - x^*) > 0$$

contradicting Kuhn-Tucker conditions. ■

We can summarize the above results as follows. Call (M) the problem

$$\max_{x \in X} f(x) \qquad s.t. \qquad g(x) \quad \geq \quad 0 \tag{19.17}$$

and define

$$M := \arg\max(M) \tag{19.18}$$

$$S := \{x \in X : \exists \lambda \in \mathbb{R}^m \text{ such that } (x, \lambda) \text{ is a solution to Kuhn-Tucker system } (19.3)\} \tag{19.19}$$

1. Assume that one of the following conditions hold:

   (a) for $j = 1, ..., m$, g_j is pseudo-concave and there exists $x^{++} \in X$ such that $g(x^{++}) \gg 0$

   (b) rank $Dg^*(x^*) = \#J^*$.
   Then
   $$x^* \in M \Rightarrow x^* \in S$$

2. Assume that both the following conditions hold:

   (a) f is pseudo-concave, and

   (b) for $j = 1, ..., m$, g_j is quasi-concave.
   Then
   $$x^* \in S \Rightarrow x^* \in M.$$

19.1.1 On uniqueness of the solution

The following proposition is a useful tool to show uniqueness.

Proposition 817 *The solution to problem*

$$\max_{x \in X} f(x) \quad s.t. \ g(x) \geq 0 \qquad (P)$$

either does not exist or it is unique if one of the following conditions holds

1. *f is strictly quasi-concave, and*
   for $j \in \{1, ..., m\}$, g_j is quasi-concave;

2. *f is quasi-concave and locally non-satiated (i.e., $\forall x \in X, \forall \varepsilon > 0$, there exists $x' \in B(x, \varepsilon)$ such that $f(x') > f(x)$), and*
   for $j \in \{1, ..., m\}$, g_j is strictly quasi-concave.

Proof. 1.

Since $g_j$is quasi concave $V^j := \{x \in X : g_j(x) \geq 0\}$ is convex. Since the intersection of convex sets is convex $V = \cap_{j=1}^m V^j$ is convex.

Suppose that both x' and x'' are solutions to problem (P) and $x' \neq x''$. Then for any $\lambda \in (0, 1)$,

$$(1 - \lambda) x' + \lambda x'' \in V \tag{19.20}$$

because V is convex, and

$$f((1-\lambda)x' + \lambda x'') > \min\{f(x'), f(x'')\} = f(x') = f(x'') \tag{19.21}$$

because f is strictly-quasi-concave.
But (19.20) and (19.21) contradict the fact that x' and x'' are solutions to problem (P).
2.
Observe that V is strictly convex because each V^j is strictly convex. Suppose that both x' and x'' are solutions to problem (P) and $x' \neq x''$. Then for any $\lambda \in (0,1)$,

$$x(\lambda) := (1-\lambda)x' + \lambda x'' \in Int\ V$$

i.e., $\exists \varepsilon > 0$ such that $B(x(\lambda), \varepsilon) \subseteq V$. Since f is locally non-satiated, there exists $x' \in B(x(\lambda), \varepsilon) \subseteq V$ such that

$$f(\widehat{x}) > f(x(\lambda)) \tag{19.22}$$

Since f is quasi-concave,

$$f(x(\lambda)) \geq f(x') = f(x'') \tag{19.23}$$

(19.22) and (19.23) contradict the fact that x' and x'' are solutions to problem (P). ■

Remark 818 *1. If f is strictly increasing (i.e., $\forall x', x'' \in X$ such that $x' > x''$, we have that $f(x') > f(x'')$) or strictly decreasing, then f is locally non-satiated.*
2. If f is affine and not constant, then f is quasi-concave and Locally NonSatiated.
Proof of 2.
$f : \mathbb{R}^n \to \mathbb{R}$ affine and not constant means that there exists $a \in \mathbb{R}$ and $b \in \mathbb{R}\setminus\{0\}$ such that $f : x \mapsto a + b^T x$. Take an arbitrary $\overline{x}$ and $\varepsilon > 0$. For $i \in \{1, ..., n\}$, define $\alpha_i := \frac{\varepsilon}{k} \cdot (sign\ b_i)$ and $\widetilde{x} := \overline{x} + (\alpha_i)_{i=1}^n$, with $k \neq 0$ and which will be computed below. Then
$f(\widetilde{x}) = a + b\overline{x} + \sum_{i=1}^n \frac{\varepsilon}{k}|b_i| > f(\overline{x})$;
$\|\widetilde{x} - \overline{x}\| = \left\|\frac{\varepsilon}{k} \cdot ((sign\ b_i) \cdot b_i)_{i=1}^n\right\| = \frac{\varepsilon}{k} \cdot \left\|\sqrt{\sum_{i=1}^n (b_i)^2}\right\| = \frac{\varepsilon}{k} \cdot \|b\| < \varepsilon$ *if* $k > \frac{1}{\|b\|}$.

Remark 819 *In part 2 of the statement of the Proposition f **has** to be both quasi-concave and Locally NonSatiated.*
a. Example of f quasi-concave (and g_j strictly-quasi-concave) with more than one solution:

$$\max_{x \in \mathbb{R}} 1 \quad s.t. \quad x + 1 \geq 0 \quad 1 - x \geq 0$$

The set of solution is $[-1, +1]$
a. Example of f Locally NonSatiated (and g_j strictly-quasi-concave) with more than one solution:

$$\max_{x \in \mathbb{R}} x^2 \quad s.t. \quad x + 1 \geq 0 \quad 1 - x \geq 0$$

The set of solutions is $\{-1, +1\}$.

19.2 The Case of Equality Constraints: Lagrange Theorem.

Consider the C^1 functions

$$f : X \to \mathbb{R}, \qquad f : x \mapsto f(x),$$

$$g : X \to \mathbb{R}^m, \qquad g : x \mapsto g(x) := (g_j(x))_{j=1}^m$$

with $\underline{m \leq n.}$ Consider also the following “”maximization problem:

$$(P) \qquad \max_{x \in X} \quad f(x) \qquad s.t. \qquad g(x) = 0 \tag{19.24}$$

$$\mathcal{L} : X \times \mathbb{R}^m \to \mathbb{R}, \quad \mathcal{L} : (x, \lambda) \mapsto f(x) + \lambda g(x)$$

is called Lagrange function associated with problem (17.15).
We recall below the statement of Theorem 753.

Theorem 820 *(Necessary Conditions)*
Assume that rank $[Dg(x^*)] = m$.
Under the above condition, we have that
x^* *is a local maximum for* (P)
$\Rightarrow$
there exists $\lambda^* \in \mathbb{R}^m$, *such that*

$$\begin{cases} Df(x^*) + \lambda^* Dg(x^*) = 0 \\ g(x^*) = 0 \end{cases} \tag{19.25}$$

Remark 821 *The full rank condition in the above Theorem cannot be dispensed. The following example shows a case in which* x^* *is a solution to maximization problem (19.24),* $Dg(x^*)$ *does not have full rank and there exists no* λ^* *satisfying Condition 19.25. Consider*

$$\max_{(x,y)\in\mathbb{R}^2} x \quad s.t. \quad \begin{array}{l} x^3 - y = 0 \\ x^3 + y = 0 \end{array}$$

The constraint set is $\{(0,0)\}$ and therefore the solution is just $(x^*, y^*) = (0,0)$. The Jacobian matrix of the constraint function is

$$\begin{bmatrix} 3x^2 & -1 \\ 3x^2 & 1 \end{bmatrix}_{|(x^*,y^*)} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

which does have full rank.

$$(0,0) = Df(x^*, y^*) + (\lambda_1, \lambda_2) Dg(x^*, y^*) =$$
$$= (1,0) + (\lambda_1, \lambda_2) \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = (1, -\lambda_1 + \lambda_2),$$

from which it follows that there exists no λ^* solving the above system.

Theorem 822 *(Sufficient Conditions)*
Assume that
1. f *is pseudo-concave,*
2. for $j = 1, ..., m$, g_j *is quasi concave.*
Under the above conditions, we have what follows.
[there exist $(x^*, \lambda^*) \in X \times \mathbb{R}^m$ *such that*
3. $\lambda^* \geq 0$,
4. $Df(x^*) + \lambda^* Dg(x^*) = 0$, *and*
5. $g(x^*) = 0$ *]*
$\Rightarrow$
x^* *solves* (P).

Proof.
Suppose otherwise, i.e., there exists $\widehat{x} \in X$ such that

$$for\ j = 1, ..., m, \quad g_j(\widehat{x}) = g_j(x^*) = 0 \quad (1), \text{ and}$$

$$f(\widehat{x}) > f(x^*) \quad (2).$$

Quasi-concavity of g_j and (1) imply that

$$Dg^j(x^*)(\widehat{x} - x^*) \geq 0 \quad (3).$$

Pseudo concavity of f and (2) imply that

$$Df(x^*)(\widehat{x} - x^*) > 0 \quad (4).$$

But then

$$0 \overset{Assumption}{=} [Df(x^*) + \lambda Dg(x^*)](\widehat{x} - x^*) = Df(x^*) \overset{(>0)}{(\widehat{x} - x^*)} + \overset{(\geq 0)}{\lambda} Dg(x^*) \overset{(\geq 0)}{(\widehat{x} - x^*)} > 0,$$

a contradiction.
■

19.3 The Case of Both Equality and Inequality Constraints.

Consider

the open and convex set $X \subseteq \mathbb{R}^n$ and the differentiable functions $f : X \to \mathbb{R},\ g : X \to \mathbb{R}^m,\ h : X \to \mathbb{R}^l$.

Consider the problem

$$\max_{x \in X} f(x) \qquad s.t. \qquad \begin{array}{rcl} g(x) & \geq & 0 \\ h(x) & = & 0. \end{array} \tag{19.26}$$

Observe that

$$h(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow \text{ for } k = 1, ..., l, \ \ h^k(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow \text{ for } k = 1, ..., l, \ \ h^{k1}(x) := h^k(x) \geq 0 \text{ and } h^{k2}(x) := -h^k(x) \geq 0.$$

Defined $h^{\cdot 1}(x) := \left(h^{k1}(x)\right)_{k=1}^{l}$ and $h^{\cdot 2}(x) := \left(h^{k2}(x)\right)_{k=1}^{l}$, problem 19.26 with associated multipliers can be rewritten as

$$\max_{x \in X} f(x) \qquad s.t. \qquad \begin{array}{rcll} g(x) & \geq & 0 & \lambda \\ h^{\cdot 1}(x) & \geq & 0 & \mu_1 \\ h^{\cdot 2}(x) & \geq & 0 & \mu_2 \end{array} \tag{19.27}$$

The Lagrangian function of the above problem is

$$\mathcal{L}(x; \lambda, \mu_1, \mu_2) = f(x) + \lambda^T g(x) + (\mu_1 - \mu_2)^T h(x) =$$

$$= f(x) + \sum_{j=1}^{m} \lambda_j g_j(x) + \sum_{k=1}^{l} \left(\mu_1^k - \mu_2^k\right)^T h(x),$$

and the Kuhn-Tucker Conditions are:

$$\begin{array}{ll} Df(x) + \lambda^T Dg(x) + (\mu_1 - \mu_2)^T Dh(x) = 0 & \\ g_j(x) \geq 0,\ \lambda_j \geq 0,\ \lambda_j g_j(x) = 0, & \text{for } j = 1, ..., m, \\ h^k(x) = 0,\ \left(\mu_1^k - \mu_2^k\right) := \mu \gtreqless 0, & \text{for } \ k = 1, ..., l. \end{array} \tag{19.28}$$

Theorem 823 *Assume that f, g and h are C^2 functions and that*

either rank $\begin{bmatrix} Dg^*(x^*) \\ Dh(x^*) \end{bmatrix} = m^* + l,$

or for $j = 1, ..., m,\ -g_j$ is pseudoconcave, and $\forall k,\ h_k$ and $-h_k$ are pseudoconcave

Under the above conditions,

if x^* solves 19.26, then $\exists\, (x^*, \lambda^*, \mu^*) \in X \times \mathbb{R}^m \times \mathbb{R}^l$ which satisfies the associated Kuhn-Tucker conditions.

Proof. The above conditions are called "Weak reverse convex constraint qualification" (Mangasarian (1969)) or "Reverse constraint qualification" (Bazaraa and Shetty (1976)). The needed result is presented and proved in

Mangasarian[2],- see 4, page 172 and Theorem 6, page 173, and Bazaraa and Shetty (1976) - see 7 page 148, and theorems 6.2.3, page 148 and Theorem 6.2.4, page 150.

See also El-Hodiri (1991), Theorem 1, page 48 and Simon (1985), Theorem 4.4. (iii), page 104.

■

■

[2] What Mangasarian calls a linear function is what we call an affine function.

Theorem 824 *Assume that*
f is pseudo-concave, and
for $j = 1, ..., m$, g_j is quasi-concave, and for $k = 1, ..., l$, h^k is quasi-concave and $-h^k$ is quasi-concave.
Under the above conditions,
if $(x^*, \lambda^*, \mu^*) \in X \times \mathbb{R}^m \times \mathbb{R}^l$ satisfies the Kuhn-Tucker conditions associated with 19.26, then x^* solves 19.26.

Proof.
This follows from Theorems proved in the case of inequality constraints.
■
Similarly, to what we have done in previous sections, we can summarize what said above as follows.
Call (M_2) the problem

$$\max_{x \in X} f(x) \qquad s.t. \qquad \begin{array}{rcl} g(x) & \geq & 0 \\ h(x) & = & 0. \end{array} \tag{19.29}$$

and define

$$M_2 := \arg\max (M_2)$$

$$S_2 := \{x \in X : \exists \lambda \in \mathbb{R}^m \text{ such that } (x, \lambda) \text{ is a solution to Kuhn-Tucker system } (19.28)\}$$

1. Assume that one of the following conditions hold:
   (a) $rank \begin{bmatrix} Dg^*(x^*) \\ Dh(x^*) \end{bmatrix} = m^* + l$,or
   (b) for $j = 1, ..., m$, g_j is linear, and $h(x)$ is affine.
   Then
   $$M_2 \subseteq S_2$$
2. Assume that both the following conditions hold:
   (a) f is pseudo-concave, and
   (b) for $j = 1, ..., m$, g_j is quasi-concave, and for $k = 1, ..., l$, h^k is quasi-concave and $-h^k$ is quasi-concave.
   Then
   $$M_2 \supseteq S_2$$

19.4 Main Steps to Solve a (Nice) Maximization Problem

We have studied the problem

$$\max_{x \in X} f(x) \qquad s.t. \qquad g(x) \geq 0 \qquad (M)$$

which we call a maximization problem in the "canonical form", i.e., a maximization problem with constraints in the form of "$\geq$", and we have defined

$$M := \arg\max (M)$$

$$C := \{x \in X : g(x) \geq 0\}$$

$$S := \{x \in X : \exists \lambda \in \mathbb{R}^m \text{ such that } (x, \lambda) \text{ satisfies Kuhn-Tucker Conditions } (19.3)\}$$

Recall that X is an open, convex subset of $\mathbb{R}^n$, $f : X \to \mathbb{R}$, $\forall j \in \{1, ..., m\}$, $g_j : X \to \mathbb{R}$ and $g := (g_j)_{j=1}^m : X \to \mathbb{R}^m$.

In many cases, we have to study the following problem

$$\max_x f(x) \quad s.t. \quad g(x) \geq 0, \tag{M'}$$

in which the set X is not specified.

We list the main steps to try to solve (M').

1. Canonical form.

Write the problem in the (in fact, our definition of) canonical form. Sometimes the problem contains a parameter $\pi \in \Pi$ an open subset of $\mathbb{R}^k$. Then we should write: for given $\pi \in \Pi$

$$\max_x f(x, \pi) \quad s.t. \quad g(x, \pi) \geq 0.$$

2. The set X and the functions f and g.

a. Define the functions $\widetilde{f}$, $\widetilde{g}$ naturally arising from the problem with domain equal to their definition set, where the definition set of a function φ is the largest set $\mathcal{D}_\varphi$ which can be the domain of that function.

b. Determine X. A possible choice for X is the intersection of the "definition set" of each function, , i.e.,

$$X = \mathcal{D}_f \cap \mathcal{D}_{g_1} \cap ... \cap \mathcal{D}_{g_m}$$

c. Check if X is open and convex.

d. To apply the analysis described in the previous sections, show, if possible, that f and g are of class C^2 or at least C^1.

3. Existence.

Try to apply the Extreme Value Theorem. If f is at least C^1, then f is continuous and therefore we have to check if the constraint set C is non-empty and compact. Recall that a set S in $\mathbb{R}^n$ is compact if and only if S is closed (in $\mathbb{R}^n$) and bounded.

Boundedness has to be shown "brute force", i.e., using the specific form of the maximization problem.

If $X = \mathbb{R}^n$, then $C := \{x \in X : g(x) \geq 0\}$ is closed, because of the following well-known argument:

$C = \cap_{j=1}^m g_j^{-1}([0, +\infty))$; since g_j is C^2 (or at least C^1) and therefore continuous, and $[0, +\infty)$ closed, $g_j^{-1}([0, +\infty))$ is closed in $X = \mathbb{R}^n$; then C is closed because intersection of closed sets.

A problem may arise if X is an open proper subset of $\mathbb{R}^n$. In that case the above argument shows that C is a closed set in $X \neq \mathbb{R}^n$ and therefore it is not necessarily closed in $\mathbb{R}^n$. A possible way out is the following one.

Verify that while the definition set of f is X, the definition set of g is $\mathbb{R}^n$. If $\mathcal{D}_{\widetilde{g}} = \mathbb{R}^n$, then from the above argument

$$\widetilde{C} := \{x \in \mathbb{R}^n : \widetilde{g}(x) \geq 0\}$$

is closed (in $\mathbb{R}^n$). Observe that

$$C = \widetilde{C} \cap X.$$

Then, we are left with showing that $\widetilde{C} \subseteq X$ and therefore $\widetilde{C} \cap X = \widetilde{C}$ and then

$$C = \widetilde{C}$$

If $\widetilde{C}$ is compact, C is compact as well.[3]

4. Number of solutions.

See subsection 19.1.1. In fact, summarizing what said there, we know that the solution to (M), if any, is unique if

1. f is strictly-quasi-concave, and for $j \in \{1, ..., m\}$, g_j is quasi-concave; **or**
2. for $j \in \{1, ..., m\}$, g_j is strictly-quasi-concave and

[3] Observe that the above argument does not apply to che case in which

$$C = \left\{x \in \mathbb{R}^2_{++} : w - x_1 - x_2 \geq 0\right\}:$$

in that case,

$$\widetilde{C} = \left\{x \in \mathbb{R}^2 : w - x_1 - x_2 \geq 0\right\} \nsubseteq \mathbb{R}^2_{++}.$$

either a. f is quasi-concave and locally non-satiated,
or b. f is affine and non-costant,
or c. f is quasi-concave and strictly monotone,
or d. f is quasi-concave and $\forall x \in X$, $Df(x) >> 0$,
or e. f is quasi-concave and $\forall x \in X$, $Df(x) << 0$.

5. Necessity of K-T conditions.

Check if the conditions which insure that $M \subseteq S$ hold, i.e.,
either a. for $j = 1, ..., m$, g_j is pseudo-concave and there exists $x^{++} \in X$ such that $g(x^{++}) \gg 0$,
or b. rank $Dg^*(x^*) = \# J^*$.

If those conditions holds, each property we show it holds for elements of S does hold *a fortiori* for elements of M.

6. Sufficiency of K-T conditions.

Check if the conditions which insure that $M \supseteq S$ hold, i.e., that
f is pseudo-concave and for $j = 1, ..., m$, g_j is quasi-concave.

If those conditions holds, each property we show it does *not* hold for elements of S does not hold *a fortiori* for elements of M.

7. K-T conditions.

Write the Lagrangian function and then the Kuhn-Tucker conditions.

8. Solve the K-T conditions.

Try to solve the system of Kuhn-Tucker conditions in the unknown variables (x, λ). To do that;
either, analyze all cases,
or, try to get a "good conjecture" and check if the conjecture is correct.

Example 825 *Discuss the problem*

$$\max_{(x_1,x_2)} \frac{1}{2}\log(1+x_1) + \frac{1}{3}\log(1+x_2) \qquad s.t. \qquad \begin{array}{rcl} x_1 & \geq & 0 \\ x_2 & \geq & 0 \\ x_1 + x_2 & \leq & w \end{array}$$

with $w > 0$.

1. Canonical form.

For given $w \in \mathbb{R}_{++}$,

$$\max_{(x_1,x_2)} \frac{1}{2}\log(1+x_1) + \frac{1}{3}\log(1+x_2) \qquad s.t. \qquad \begin{array}{rcl} x_1 & \geq & 0 \\ x_2 & \geq & 0 \\ w - x_1 - x_2 & \geq & 0 \end{array} \tag{19.30}$$

2. The set X and the functions f and g.

a.

$$\begin{array}{ll} \widetilde{f} : (-1,+\infty)^2 \to \mathbb{R}, & (x_1,x_2) \mapsto \frac{1}{2}\log(1+x_1) + \frac{1}{3}\log(1+x_2) \\ \widetilde{g}_1 : \mathbb{R}^2 \to \mathbb{R} & (x_1,x_2) \mapsto x_1 \\ \widetilde{g}_2 : \mathbb{R}^2 \to \mathbb{R} & (x_1,x_2) \mapsto x_2 \\ \widetilde{g}_3 : \mathbb{R}^2 \to \mathbb{R} & (x_1,x_2) \mapsto w - x_1 - x_2 \end{array}$$

b.

$$X = (-1,+\infty)^2$$

and therefore f and g are just $\widetilde{f}$ and $\widetilde{g}$ restricted to X.

c. X is open and convex because Cartesian product of open intervals which are open, convex sets.

d. Let's try to compute the Hessian matrices of f, g_1, g_2, g_3. Gradients are

$$\begin{array}{rcl} Df(x_1,x_2), & = & \left(\frac{1}{2(x_1+1)}, \frac{1}{3(x_2+1)}\right) \\ D\widetilde{g}_1(x_1,x_2) & = & (1,0) \\ D\widetilde{g}_2(x_1,x_2) & = & (0,1) \\ D\widetilde{g}_3(x_1,x_2) & = & (-1,-1) \end{array}$$

Hessian matrices are

$$\begin{array}{rcl} D^2 f(x_1,x_2), & = & \begin{bmatrix} -\frac{1}{2(x_1+1)^2} & 0 \\ 0 & -\frac{1}{3(x_2+1)^2} \end{bmatrix} \\ D^2 \widetilde{g}_1(x_1,x_2) & = & 0 \\ D^2 \widetilde{g}_2(x_1,x_2) & = & 0 \\ D^2 \widetilde{g}_3(x_1,x_2) & = & 0 \end{array}$$

In fact, g_1, g_2, g_3 *are affine functions. In conclusion,* f *and* g_1, g_2, g_3 *are* C^2. *In fact,* g_1 *and* g_2 *are linear and* g_3 *is affine.*

2. Existence.

C *is clearly bounded:* $\forall x \in C$,

$$(0,0) \le (x_1,x_2) \le (w,w)$$

In fact, the first two constraint simply say that $(x_1,x_2) \ge (0,0)$. *Moreover, from the third constraint* $x_1 \le w - x_2 \le w$, *simply because* $x_2 \ge 0$; *similar argument can be used to show that* $x_2 \le w$.

To show closedness, use the strategy proposed above.

$$\widetilde{C} := \{x \in \mathbb{R}^n : g(x) \ge 0\}$$

is obviously closed. Since $\widetilde{C} \subseteq \mathbb{R}^2_+$, *because of the first two constraints,* $\widetilde{C} \subseteq X := (-1,+\infty)^2$ *and therefore* $C = \widetilde{C} \cap X = \widetilde{C}$ *is closed.*

We can then conclude that C *is compact and therefore* $\arg\max(19.30) \neq \varnothing$.

4. Number of solutions.

From the analysis of the Hessian and using Theorem 804, parts 1 ad 2, we have that f *is strictly concave:*

$$-\frac{1}{2(x_1+1)^2} < 0$$

$$\det \begin{bmatrix} -\frac{1}{2(x_1+1)^2} & 0 \\ 0 & -\frac{1}{3(x_2+1)^2} \end{bmatrix} = \frac{1}{2(x_1+1)^2} \cdot \frac{1}{3(x_2+1)^2} > 0$$

Moreover g_1, g_2, g_3 *are affine and therefore concave. From Proposition 817, part 1, the solution is unique.*

5. Necessity of K-T conditions.

Since each g_j *is affine and therefore pseudo-concave, we are left with showing that there exists* $x^{++} \in X$ *such that* $g(x^{++}) >> 0$. *Just take* $(x_1^{++}, x_2^{++}) = \frac{w}{4}(1,1)$:

$$\begin{array}{rcl} \frac{w}{4} & > & 0 \\ \frac{w}{4} & > & 0 \\ w - \frac{w}{4} - \frac{w}{4} = \frac{w}{2} & > & 0 \end{array}$$

Therefore

$$M \subseteq S$$

6. Sufficiency of K-T conditions.

f *is strictly concave and therefore pseudo-concave, and each* g_j *is linear and therefore quasi-concave. Therefore*

$$M \supseteq S$$

7. K-T conditions.

$$\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2,\mu;w) = \frac{1}{2}\log(1+x_1) + \frac{1}{3}\log(1+x_2) + \lambda_1 x_1 + \lambda_2 x_2 + \mu(w - x_1 - x_2)$$

$$\begin{cases} \frac{1}{2(x_1+1)} + \lambda_1 - \mu & = 0 \\ \frac{1}{3(x_2+1)} + \lambda_2 - \mu & = 0 \\ \min\{x_1,\lambda_1\} & = 0 \\ \min\{x_2,\lambda_2\} & = 0 \\ \min\{w - x_1 - x_2, \mu\} & = 0 \end{cases}$$

8. Solve the K-T conditions.

Conjecture: $x_1 > 0$ *and therefore* $\lambda_1 = 0$*;* $x_2 > 0$ *and therefore* $\lambda_2 = 0$*;* $w - x_1 - x_2 = 0$*.The Kuhn-Tucker system becomes:*

$$\begin{cases} \frac{1}{2(x_1+1)} - \mu & = & 0 \\ \frac{1}{3(x_2+1)} - \mu & = & 0 \\ w - x_1 - x_2 & = & 0 \\ \mu & \geq & 0 \\ x_1 > 0, x_2 > 0 \\ \lambda_1 = 0, \lambda_2 = 0 \end{cases}$$

Then,

$$\begin{cases} \frac{1}{2(x_1+1)} & = & \mu \\ \frac{1}{3(x_2+1)} & = & \mu \\ w - x_1 - x_2 & = & 0 \\ \mu & > & 0 \\ x_1 > 0, x_2 > 0 \\ \lambda_1 = 0, \lambda_2 = 0 \end{cases}$$

$$\begin{cases} x_1 & = & \frac{1}{2\mu} - 1 \\ x_2 & = & \frac{1}{3\mu} - 1 \\ w - \left(\frac{1}{2\mu} - 1\right) - \left(\frac{1}{3\mu} - 1\right) & = & 0 \\ \mu & > & 0 \\ x_1 > 0, x_2 > 0 \\ \lambda_1 = 0, \lambda_2 = 0 \end{cases}$$

$0 = w - \left(\frac{1}{2\mu} - 1\right) - \left(\frac{1}{3\mu} - 1\right) = w - \frac{5}{6\mu} + 2$*; and* $\mu = \frac{5}{6(w+2)} > 0$*. Then* $x_1 = \frac{1}{2\mu} - 1 = \frac{6(w+2)}{2 \cdot 5} - 1 = \frac{3w+6-5}{5} = \frac{3w+1}{5}$ *and* $x_2 = \frac{1}{3\mu} - 1 = \frac{6(w+2)}{3 \cdot 5} - 1 = \frac{2w+4-5}{5} = \frac{2w-1}{5}$.

Summarizing

$$\begin{cases} x_1 & = & \frac{3w+1}{5} > 0 \\ x_2 & = & \frac{2w-1}{5} > 0 \\ \mu & = & \frac{5}{6(w+2)} > 0 \\ \lambda_1 = 0, \lambda_2 = 0 \end{cases}$$

Observe that while $x_1 > 0$ *for any value of* w*,* $x_2 > 0$*iff* $w > \frac{1}{2}$*. Therefore, for* $w \in \left(0, \frac{1}{2}\right]$*, the above one is not a solution, and we have to come up with another conjecture;*

$x_1 = w$ *and therefore* $\lambda_1 = 0$*;* $x_2 = 0$ *and* $\lambda_2 \geq 0$*;* $w - x_1 - x_2 = 0$ *and* $\mu \geq 0$*.The Kuhn-Tucker conditions become*

$$\begin{cases} \frac{1}{2(w+1)} - \mu & = & 0 \\ \frac{1}{3} + \lambda_2 - \mu & = & 0 \\ \lambda_1 & = & 0 \\ x_2 & = & 0 \\ \lambda_2 & \geq & 0 \\ x_1 & = & w \\ \mu & \geq & 0 \end{cases}$$

and

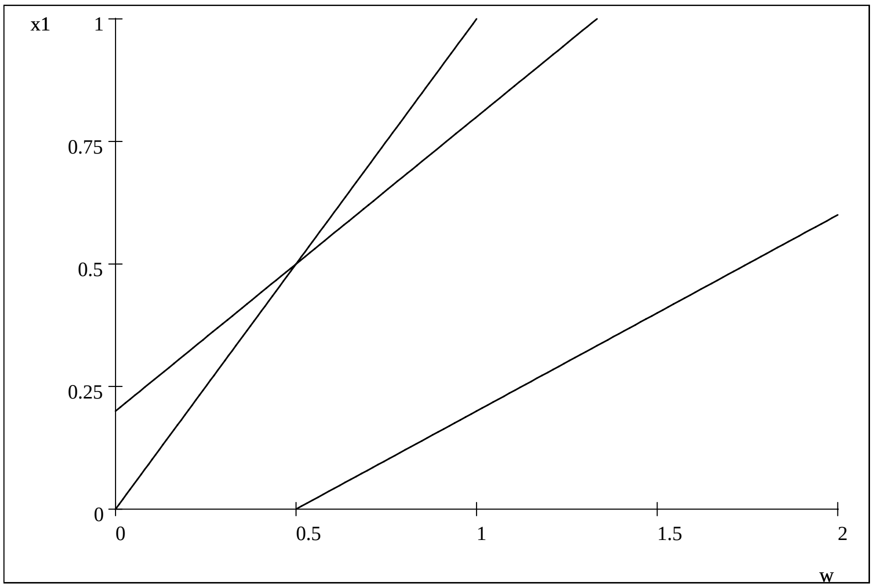
$$\begin{cases} \mu & = & \frac{1}{2(w+1)} > 0 \\ \lambda_2 & = & \frac{1}{2(w+1)} - \frac{1}{3} = \frac{3-2w-2}{6(w+1)} = \frac{1-2w}{6(w+1)} \\ \lambda_1 & = & 0 \\ x_2 & = & 0 \\ \lambda_2 & \geq & 0 \\ x_1 & = & w \end{cases}$$

$\lambda_2 = \frac{1-2w}{6(w+1)} = 0$ *if* $w = \frac{1}{2}$*,and .*$\lambda_2 = \frac{1-2w}{6(w+1)} > 0$ *if* $w \in \left(0, \frac{1}{2}\right)$

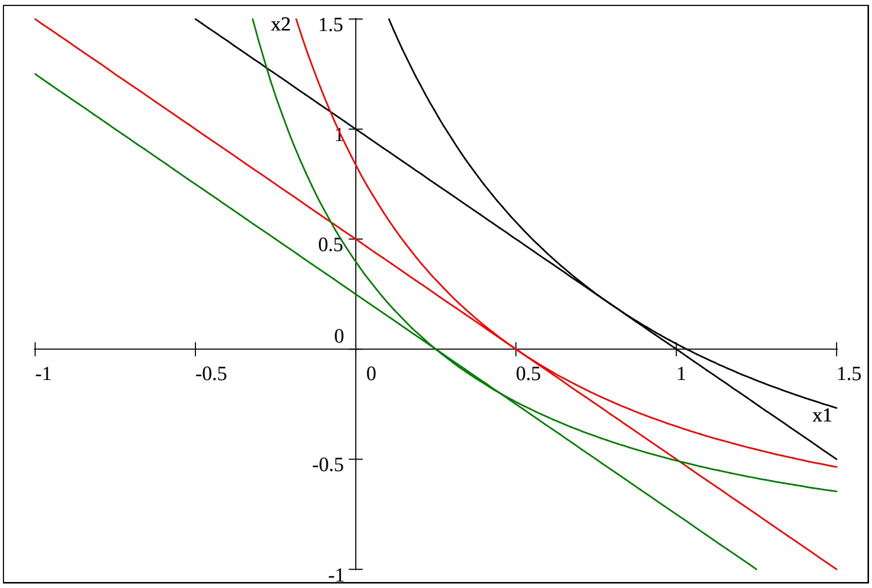
Summarizing, the unique solution x^* *to the maximization problem is*

if $w \in \left(0, \frac{1}{2}\right)$,	*then*	$x_1^* = w$,	$\lambda_1^* = 0$	*and*	$x_2^* = 0$,	$\lambda_2^* > 0$
if $w = \frac{1}{2}$,	*then*	$x_1^* = w$,	$\lambda_1^* = 0$		$x_2^* = 0$,	$\lambda_2^* = 0$
if $w \in \left(\frac{1}{2}, +\infty\right)$,	*then*	$x_1^* = \frac{3w+1}{5} > 0$	$\lambda_1^* = 0$	*and*	$x_2^* = \frac{2w-1}{5} > 0$	$\lambda_2^* = 0$

The graph of x_1^* as a function of w is presented below `(please, complete the picture)`



The graph below shows constraint sets for $w = \frac{1}{4}, \frac{1}{2}, 1$ and some significant level curve of the objective function.



Observe that in the example, we get that if $\lambda_2^* = 0$, the associated constraint $x_2 \geq 0$ is not significant. See Subsection 19.6.2, for a discussion of that statement.

Of course, several problems may arise in applying the above procedure. Below, we describe some commonly encountered problems and some possible (partial) solutions.

19.4.1 Some problems and some solutions

1. The set X.

X is not open.

Rewrite the problem in terms of an open set X' and some added constraints. A standard example is the following one.

$$\max_{x \in \mathbb{R}^n_+} f(x) \quad s.t. \quad g(x) \geq 0$$

which can be rewritten as

$$\max_{x\in\mathbb{R}^n} f(x) \quad s.t. \quad \begin{array}{l} g(x)\geq 0 \\ x\geq 0 \end{array}$$

2. Existence.

a. The constraint set is not compact. If the constraint set is not compact, it is sometimes possible to find another maximization problem such that

i. its constraint set is compact and nonempty, and

ii. whose solution set is contained in the solution set of the problem we are analyzing.

A way to try to achieve both i. and ii. above is to "restrict the constraint set (to make it compact) without eliminating the solution of the original problem". Sometimes, a problem with the above properties is the following one.

$$\max_{x\in X} f(x) \qquad s.t. \qquad \begin{array}{lcl} g(x) & \geq & 0 \\ f(x)-f(\widehat{x}) & \geq & 0 \end{array} \qquad (P1)$$

where $\widehat{x}$ is an element of X such that $g(\widehat{x})\geq 0$.

Observe that $(P-$ cons$)$ is an example of $(P1)$.

In fact, while, condition 1. above., i.e., the compactness of the constraint set of $(P1)$ depends upon the specific characteristics of X, f and g, condition ii. above is satisfied by problem $(P1)$, as shown in detail below.

Define

$$M := \arg\max(P) \quad M^1 := \arg\max(P1)$$

and V and V^1 the constraint sets of Problems (P) and $(P1)$, respectively. Observe that

$$V^1 \subseteq V \tag{19.31}$$

If V^1 is compact, then $M^1\neq\emptyset$ and the only thing left to show is that $M^1\subseteq M$, which is always insured as proved below.

Proposition 826 $M^1\subseteq M$.

Proof. If $M^1=\emptyset$, we are done.

Suppose that $M^1\neq\emptyset$, and that the conclusion of the Proposition is false, i.e., there exists $x^1\in M^1$ such that

a. $x^1\in M^1$, and b. $x^1\notin M$, or

a. $\forall x\in X$ such that $g(x)\geq 0$ and $f(x)\geq f(\widehat{x})$, we have $f\left(x^1\right)\geq f(x)$;

and

b. either i. $x^1\notin V$,

or ii. $\exists\widetilde{x}\in X$ such that

$$g(\widetilde{x})\geq 0 \tag{19.32}$$

and

$$f(\widetilde{x})>f\left(x^1\right) \tag{19.33}$$

Let's show that i. and ii. cannot hold.

i.

It cannot hold simply because $V^1\subseteq V$, from 19.31.

ii.

Since $x^1\in V^1$,

$$f\left(x^1\right)\geq f(\widehat{x}) \tag{19.34}$$

From (19.33) and (19.34), it follows that

$$f(\widetilde{x})>f(\widehat{x}) \tag{19.35}$$

But (19.32), (19.35) and (19.33) contradict the definition of x^1, i.e., a. above ■

b. Existence without the Extreme Value Theorem If you are not able to show existence, but

i. sufficient conditions to apply Kuhn-Tucker conditions hold, and
ii. you are able to find a solution to the Kuhn-Tucker conditions,
then a solution exists.

19.5 The Implicit Function Theorem and Comparative Statics Analysis

The Implicit Function Theorem can be used to study how solutions ($x \in X \subseteq \mathbb{R}^n$) to maximizations problems and, if needed, associated Lagrange or Kuhn-Tucker multipliers ($\lambda \in \mathbb{R}^m$) change when parameters ($\pi \in \Pi \subseteq \mathbb{R}^k$) change. That analysis can be done if the solutions to the maximization problem (and the multipliers) are solution to a system of equation of the form

$$F_1(x, \pi) = 0$$

with (# choice variables) = (# dimension of the codomain of F_1), or

$$F_2(\xi, \pi) = 0$$

where $\xi := (x, \lambda)$, and (# choice variables *and* multipliers) = (# dimension of the codomain of F_2),

To apply the Implicit Function Theorem, it must be the case that the following conditions do hold.

1. (# choice variables x) = (# dimension of the codomain of F_1), or
   (# choice variables *and* multipliers) = (# dimension of the codomain of F_2).
2. F_i has to be at least C^1.That condition is insured if the above systems are obtained from maximization problems characterized by functions f, g which are at least C^2: usually the above systems contain some form of first order conditions, which are written using first derivatives of f and g.
3. $F_1(x^*, \pi_0) = 0$ or $F_2(\xi^*, \pi_0) = 0$. The existence of a solution to the system is usually the result of the strategy to describe how to solve a maximization form - see above Section 19.4.
4. $\det [D_x F_1(x^*, \pi_0)]_{n \times n} \neq 0$ or $\det [D_\xi F_2(\xi^*, \pi_0)]_{(n+m)\times(n+m)} \neq 0$. That condition has to be verified directly on the problem.

If the above conditions are verified, the Implicit Function Theorem allow to conclude what follows (in reference to F_2).

There exist an open neighborhood $N(\xi^*) \subseteq X$ of ξ^*, an open neighborhood $N(\pi_0) \subseteq \Pi$ of π_0 and a unique C^1 function $g : N(\pi_0) \subseteq \Pi \subseteq \mathbb{R}^p \to N(\xi^*) \subseteq X \subseteq \mathbb{R}^n$ such that $\forall \pi \in N(\pi_0),\ F(g(\pi), \pi) = 0$ and

$$Dg(\pi) \quad = - \left[D_\xi F(\xi, \pi)_{|\xi = g(\pi)} \right]^{-1} \cdot \left[D_\pi F(\xi, \pi)_{|\xi = g(\pi)} \right]$$

Therefore, using the above expression, we may be able to say if the increase in any value of any parameter implies an increase in the value of any choice variable (or multiplier).

Three significant cases of application of the above procedure are presented below. We are going to consider C^2 functions defined on open subsets of Euclidean spaces.

19.5.1 Maximization problem without constraint

Assume that the problem to study is

$$\max_{x \in X} f(x, \pi)$$

and that
1. f is concave;

2. There exists a solution x^* to the above problem associated with π_0.
Then, from Proposition 767, we know that x^* is a solution to

$$Df(x, \pi_0) = 0$$

Therefore, we can try to apply the Implicit Function Theorem to

$$F_1(x, \pi) = Df(x, \pi_0)$$

An example of application of the strategy illustrated above is presented in Section 20.3.

19.5.2 Maximization problem with equality constraints

Consider a maximization problem

$$\max_{x \in X} \quad f(x, \pi) \qquad s.t. \qquad g(x, \pi) = 0,$$

Assume that necessary and sufficient conditions to apply Lagrange Theorem hold and that there exists a vector (x^*, λ^*) which is a solution (not necessarily unique) associated with the parameter π_0. Therefore, we can try to apply the Implicit Function Theorem to

$$F_2(\xi, \pi) = \begin{pmatrix} Df(x^*, \pi_0) + \lambda^* Dg(x^*, \pi_0) \\ g(x^*, \pi_0). \end{pmatrix} \tag{19.36}$$

19.5.3 Maximization problem with Inequality Constraints

Consider the following maximization problems with inequality constraints. For given $\pi \in \Pi$,

$$\max_{x \in X} f(x, \pi) \quad s.t. \quad g(x, \pi) \geq 0 \tag{19.37}$$

Moreover, assume that the set of solutions of that problem is nonempty and characterized by the set of solutions of the associated Kuhn-Tucker system, i.e., using the notation of Subsection 19.1,

$$M = S \neq \varnothing.$$

We have seen that we can write Kuhn-Tucker conditions in one of the two following ways, beside some other ones,

$$\begin{cases} Df(x, \pi) + \lambda Dg(x, \pi) & = & 0 & (1) \\ \lambda & \geq & 0 & (2) \\ g(x, \pi) & \geq & 0 & (3) \\ \lambda g(x, \pi) & = & 0 & (4) \end{cases} \tag{19.38}$$

or

$$\begin{cases} Df(x, \pi) + \lambda Dg(x, \pi) & = & 0 & & (1) \\ \min\{\lambda_j, g_j(x, \pi)\} & = & 0 & \text{for } j \in \{1, ..., m\} & (2) \end{cases} \tag{19.39}$$

The Implicit Function Theorem *cannot* be applied to either system (19.38) or system (19.39): system (19.38) contains *inequalities*; system (19.39) involves functions which are *not differentiable.* We present below conditions under which the Implicit Function Theorem can be anyway applied to allow to make comparative statics analysis. Take a solution (x^*, λ^*, π_0) to the above system(s). Assume that

$$\text{for each } j, \text{ either } \lambda_j^* > 0 \text{ or } g_j(x^*, \pi_0) > 0.$$

In other words, there is no j such that $\lambda_j = g_j(x^*, \pi_0) = 0$. Consider a partition $J^*, \widehat{J}$ of $\{1, .., m\}$, and the resulting Kuhn-Tucker conditions.

$$\begin{cases} Df(x^*, \pi_0) + \lambda^* Dg(x^*, \pi_0) = 0 & \\ \lambda_j^* > 0 & \text{for } j \in J^* \\ g_j(x^*, \pi_0) = 0 & \text{for } j \in J^* \\ \lambda_j^* = 0 & \text{for } j \in \widehat{J} \\ g_j(x^*, \pi_0) > 0 & \text{for } j \in \widehat{J} \end{cases} \tag{19.40}$$

Define

$$\begin{aligned} g^*(x^*, \pi_0) &:= (g_j(x^*, \pi_0))_{j \in J^*} \\ \widehat{g}(x^*, \pi_0) &:= (g_j(x^*, \pi_0))_{j \in \widehat{J}} \\ \lambda^{**} &:= (\lambda_j^*)_{j \in J^*} \\ \widehat{\lambda}^* &:= (\lambda_j^*)_{j \in \widehat{J}} \end{aligned}$$

Write the system of equations obtained from system (19.40) eliminating strict inequality constraints and substituting in the zero variables:

$$\begin{cases} Df(x^*, \pi_0) + \lambda^{**} Dg^*(x^*, \pi_0) & = & 0 \\ g^*(x^*, \pi_0) & = & 0 \end{cases} \tag{19.41}$$

Observe that the number of equations is equal to the number of "remaining" unknowns and they are

$$n + \# J^*$$

i.e., Condition 1 presented at the beginning of the present Section 19.5 is satisfied. Assume that the needed rank condition does hold and we therefore can apply the Implicit Function Theorem to

$$F_2(\xi, \pi) = \begin{pmatrix} Df(x^*, \pi_0) + \lambda^{**} Dg^*(x^*, \pi_0) \\ g^*(x^*, \pi_0) \end{pmatrix} = 0$$

Then. we can conclude that there exists a unique C^1 function φ defined in an open neighborhood N_1 of π_0 such that

$$\forall \pi \in N_1, \qquad \varphi(\pi) := (x^*(\pi), \lambda^{**}(\pi))$$

is a solution to system (19.41) at π.

Therefore, by definition of φ,

$$\begin{cases} Df(x^*(\pi), \pi) + \lambda^{**}(\pi)^T Dg^*(x^*(\pi), \pi) & = & 0 \\ g^*(x^*(\pi), \pi) & = & 0 \end{cases} \tag{19.42}$$

Since φ is continuous and $\lambda^{**}(\pi_0) > 0$ and $\widehat{g}(x^*(\pi_0), \pi_0) > 0$, there exist an open neighborhood $N_2 \subseteq N_1$ of π_0 such that $\forall \pi \in N_2$

$$\begin{cases} \lambda^{**}(\pi) & > & 0 \\ \widehat{g}(x(\pi), \pi) & > & 0 \end{cases} \tag{19.43}$$

Take also $\forall \pi \in N_2$

$$\left\{ \widehat{\lambda}^*(\pi) \quad = \quad 0 \right. \tag{19.44}$$

Then, systems (19.42), (19.43) and (19.44) say that $\forall \pi \in N_2$, $\left(x(\pi), \lambda^*(\pi), \widehat{\lambda}(\pi)\right)$ satisfy Kuhn-Tucker conditions for problem (19.37) and therefore, since $C = M$, they are solutions to the maximization problem.

The above conclusion *does not hold true* if Kuhn-Tucker conditions are of the following form

$$\begin{cases} Df(x, \pi) + \lambda^T Dg(x, \pi) = 0 & \\ \lambda_j = 0, \quad g_j(x, \pi) = 0 & \text{for } j \in J' \\ \lambda_j > 0, \quad g_j(x, \pi) = 0 & \text{for } j \in J'' \\ \lambda_j = 0, \quad g_j(x, \pi) > 0 & \text{for } j \in \widehat{J} \end{cases} \tag{19.45}$$

where $J' \neq \emptyset$, J'' and $\widehat{J}$ is a partition of J.

In that case, applying the same procedure described above, i.e., eliminating strict inequality constraints and substituting in the zero variables, leads to the following systems in the unknowns $x \in \mathbb{R}^n$ and $(\lambda_j)_{j \in J''} \in \mathbb{R}^{\# J''}$:

$$\begin{cases} Df(x, \pi) + (\lambda_j)_{j \in J''} D(g_j)_{j \in J''}(x, \pi) = 0 & \\ g_j(x, \pi) = 0 & \text{for } j \in J' \\ g_j(x, \pi) = 0 & \text{for } j \in J'' \end{cases}$$

and therefore the number of equation is $n + \#J'' + \#J' > n + \#J''$,simply because we are considering the case $J' \neq \emptyset$. Therefore the crucial condition

$$(\#\text{ choice variables } \textit{and} \text{ multipliers}) = (\#\text{ dimension of the codomain of } F_2)$$

is violated.

Even if the Implicit Function Theorem could be applied to the equations contained in (19.45), in an open neighborhood of π_0 we could have

$$\lambda_j(\pi) < 0 \text{ and/or } \quad g_j(x(\pi), \pi) < 0 \quad \text{for } j \in J'$$

Then $\varphi(\pi)$ would be solutions to a set of equations and inequalities which are not Kuhn-Tucker conditions of the maximization problem under analysis, and therefore $x(\pi)$ would not be a solution to the that maximization problem.

An example of application of the strategy illustrated above is presented in Section 20.1.

19.6 The Envelope Theorem and the meaning of multipliers

19.6.1 The Envelope Theorem

Consider the problem (M) : for given $\pi \in \Pi$,

$$\max_{x \in X} f(x, \pi) \qquad s.t. \quad g(x, \pi) = 0$$

Assume that for every π, the above problem admits a unique solution characterized by Lagrange conditions and that the Implicit function theorem can be applied. Then, there exists an open set $\mathcal{O} \subseteq \Pi$ such that

$$\begin{array}{ll} x : \mathcal{O} \to X, \; x : \pi \mapsto \arg\max(P) \;, & \\ v : \mathcal{O} \to \mathbb{R}, \; v : \pi \mapsto \max(P) & \text{and} \\ \lambda : \mathcal{O} \to \mathbb{R}^m, \; \pi \mapsto \text{ unique Lagrange multiplier vector} & \end{array}$$

are differentiable functions.

Theorem 827 *For any $\pi^* \in \mathcal{O}$ and for any pair of associated $(x^*, \lambda^*) := (x(\pi^*), \lambda(\pi^*))$, we have*

$$D_\pi v(\pi^*) = D_\pi \mathcal{L}(x^*, \lambda^*, \pi^*)$$

i.e.,

$$D_\pi v(\pi^*) = D_\pi f(x^*, \pi^*) + \lambda^* D_\pi g(x^*, \pi^*)$$

Remark 828 *Observe that the above analysis applies also to the case of inequality constraints, as long as the set of binding constraints does not change.*

Proof. of Theorem 827 By definition of $v(.)$ and $x(.)$, we have that

$$\forall \pi \in \mathcal{O}, \;\; v(\pi) = f(x(\pi), \pi). \quad (1)$$

Consider an arbitrary value π^* and the unique associate solution $x^* = x(\pi^*)$ of problem (P). Differentiating both sides of (1) with respect to π and computing at π^*, we get

$$[D_\pi v(\pi^*)]_{1\times k} = \left[D_x f(x, \pi)_{|(x^*, \pi^*)}\right]_{1\times n} \cdot \left[D_\pi x(\pi)_{|\pi=\pi^*}\right]_{n\times k} + \left[D_\pi f(x, \pi)_{|(x^*, \pi^*)}\right]_{1\times k} \quad (2)$$

From Lagrange conditions

$$D_x f(x, \pi)_{|(x^*, \pi^*)} = -\lambda^* D_x g(x, \pi)_{|(x^*, \pi^*)} \quad (3),$$

where λ^* is the unique value of the Lagrange multiplier. Moreover

$$\forall \pi \in \mathcal{O}, \;\; g(x(\pi), \pi) = 0. \quad (4)$$

Differentiating both sides of (4) with respect to π and computing at π^*, we get

$$\left[D_x g(x,\pi)_{|(x^*,\pi^*)}\right]_{m\times n}\cdot\left[[D_\pi x(\pi)]_{|\pi=\pi^*}\right]_{n\times k}+\left[D_\pi g(x,\pi)_{|(x^*,\pi^*)}\right]_{m\times k}=0\quad(5).$$

Finally,

$$[D_\pi v(\pi^*)]_{1\times k}\overset{(2),(3)}{=}-\lambda^* D_x g(x,\pi)_{|(x^*,\pi^*)}D_\pi x(\pi)_{|\pi=\pi^*}+D_\pi f(x,\pi)_{|(x^*,\pi^*)}\overset{(5)}{=}$$

$$=D_\pi f(x,\pi)_{|(x^*,\pi^*)}+\lambda^* D_\pi g(x,\pi)_{|(x^*,\pi^*)}$$

■

19.6.2 On the meaning of the multipliers

The main goal of this subsection is to try to formalize the following statements.

1. The fact that $\lambda_j=0$ indicates that the associated constraint $g_j(x)\geq 0$ is not significant - see Proposition 829 below.

2. The fact that $\lambda_j>0$ indicates that a way to increase the value of the objective function is to violate the associated constraint $g_j(x)\geq 0$ - see Proposition 830 below.

For simplicity, consider the case $m=1$. Let (CP) be the problem

$$\max_{x\in X} f(x)\quad s.t.\quad g(x)\geq 0$$

and (UP) the problem

$$\max_{x\in X} f(x)$$

with f strictly quasi-concave, g is quasi-concave and solutions to both problem exist . Define $x^*:=\arg\max(CP)$ with associated multiplier λ^*, and $x^{**}:=\arg\max(UP)$.

Proposition 829 *If $\lambda^*=0$, then $x^*=\arg\max(UP)\Leftrightarrow x^*=\arg\max(CP)$.*

Proof. By the assumptions of this section, the solution to (CP) exists, is unique, it is equal to x^* and there exists λ^* such that

$$\begin{array}{ll}Df(x^*)+\lambda^* Dg(x^*) & =0\\ \min\{g(x^*),\lambda^*\} & =0.\end{array}$$

Moreover, the solution to (UP) exists, is unique and it is the solution to

$$Df(x)\quad=0.$$

Since $\lambda^*=0$, the desired result follows. ■

Take $\varepsilon>0$ and $k\in(-\varepsilon,+\infty)$. Let (CPk) be the problem

$$\max_{x\in X} f(x)\quad s.t.\quad g(x)\geq k$$

Let

$$\widehat{x}:(-\varepsilon,+\infty)\to X,\quad k\mapsto\arg\max(CPk)$$

$$\widehat{v}:(-\varepsilon,+\infty)\to\mathbb{R},\quad k\mapsto\max(CPk):=f(\widehat{x}(k))$$

Let $\widehat{\lambda}(k)$ be such that $\left(\widehat{x}(k),\widehat{\lambda}(k)\right)$ is the solution to the associated Kuhn-Tucker conditions. Observe that

$$x^*=\widehat{x}(0),\quad\lambda^*=\widehat{\lambda}(0)\tag{19.46}$$

Proposition 830 *If $\lambda^*>0$, then $\widehat{v}'(0)<0$.*

Proof. From the envelope theorem,

$$\forall k \in (-\varepsilon, +\infty), \widehat{v}'(k) = \frac{\partial \left(f(x) + \lambda\left(g(x) - k\right)\right)}{\partial k}_{|\widehat{x}(k), \widehat{\lambda}(k)} = -\widehat{\lambda}(k)$$

and from (19.46)

$$\widehat{v}'(0) = -\widehat{\lambda}(0) = -\lambda^* < 0.$$

■

Remark 831 *Consider the following problem. For given $a \in \mathbb{R}$,*

$$\max_{x \in X} f(x) \quad s.t. \quad g(x) - a \geq 0 \tag{19.47}$$

Assume that the above problem is "well-behaved" and that $x(a) = \arg\max$ (19.47), $v(a) = f(x(a))$ and $(x(a), \lambda(a))$ is the solution of the associated Kuhn-Tucker conditions. Then, applying the Envelope Theorem we have

$$v'(a) = \lambda(a)$$

Chapter 20

Applications to Economics

20.1 The Walrasian Consumer Problem

The utility function of a household is

$$u : \mathbb{R}^C_{++} \to \mathbb{R} \qquad : x \mapsto u(x).$$

Assumption. u is a $\mathcal{C}^2$ function; u is differentiably strictly increasing, i.e., $\forall x \in \mathbb{R}^C_{++},\ Du(x) \gg 0$; u is differentiably strictly quasi-concave, i.e., $\forall x \in \mathbb{R}^C_{++},\ \ \Delta x \neq 0$ and $Du(x)\Delta x = 0 \Rightarrow \Delta x^T D^2 u(x)\Delta x < 0$; for any $\underline{u} \in \mathbb{R}$, $\left\{x \in \mathbb{R}^C_{++} : u(x) \geq \underline{u}\right\}$ is closed in $\mathbb{R}^C$.

The maximization problem for household h is

$$(P1) \qquad \max_{x \in \mathbb{R}^C_{++}} \quad u(x) \quad s.t. \quad px - w \leq 0.$$

The budget set of the above problem is clearly not compact. But, in the Appendix, we show that the solution of $(P1)$ are the same as the solutions of $(P2)$ and $(P3)$ below. Observe that the constraint set of $(P3)$ is compact.

$$(P2) \qquad \max_{x \in \mathbb{R}^C_{++}} \quad u(x) \quad s.t. \quad px - w = 0;$$

$$(P3) \qquad \max_{x \in \mathbb{R}^C_{++}} \quad u(x) \quad s.t. \quad \begin{array}{c} px - w \leq 0; \\ u(x) \geq u(e^*), \end{array}$$

where $e^* \in \left\{x \in \mathbb{R}^C_{++} : px \leq w\right\}$.

Theorem 832 *Under the Assumptions (smooth 1-5), $\xi_h(p, w_h)$ is a $\mathcal{C}^1$ function.*

Proof.
Observe that, from it can be easily shown that, ξ is a function.
We want to show that $(P2)$ satisfies necessary and sufficient conditions to Lagrange Theorem, and then apply the Implicit Function Theorem to the First Order Conditions of that problem.
The necessary condition is satisfied because $D_x[px - w] = p \neq 0$;
Define also
$\mu : \mathbb{R}^C_{++} \times \mathbb{R}^{C-1}_{++} \to \mathbb{R}^C_{++}$,
$\mu : (p, w) \mapsto$ Lagrange multiplier for $(P2)$.
The sufficient conditions are satisfied because: from Assumptions (smooth 4), u is differentiably strictly quasi-concave; the constraint is linear; the Lagrange multiplier μ is strictly positive -see below.

The Lagrangian function for problem $(P2)$ and the associated First Order Conditions are described below.

$$\mathcal{L}(x, \mu, p, w) = u(x) + \mu \cdot (-px + w)$$

$$\begin{array}{lllll} (FOC) & (1) & Du(x) - \mu p & = & 0 \\ & (2) & -px + w & = & 0 \end{array}$$

Define

$$F : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \times \mathbb{R}^C_{++} \times \mathbb{R}_{++} \to \mathbb{R}^C \times \mathbb{R},$$

$$F : (x, \mu, p, w) \mapsto \begin{pmatrix} Du(x) - \mu p \\ -px + w \end{pmatrix}.$$

As an application of the Implicit Function Theorem, it is enough to show that $D_{(x,\mu)} F(x, \mu, p, w)$ has full row rank $(C+1)$.
Suppose $D_{(x,\mu)} F$ does not have full rank; then there would exist
$\Delta x \in \mathbb{R}^C$ and $\Delta\mu \in \mathbb{R}$ such that $\Delta := (\Delta x, \Delta\mu) \neq 0$ and $D_{(x,\mu)} F \cdot (\Delta x, \Delta\mu) = 0$, or

$$\begin{bmatrix} D^2u(x) & -p^T \\ -p & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\mu \end{bmatrix} = 0,$$

or

$$\begin{array}{lll} (a) & D^2u(x)\Delta x - p^T\Delta\mu & = 0 \\ (b) & -p\Delta x & = 0 \end{array}$$

.

The idea of the proof is to contradict Assumption u3.

Claim 1. $\Delta x \neq 0$.
By assumption it must be $\Delta \neq 0$ and therefore, if $\Delta x = 0$, $\Delta\mu \neq 0$. Since $p \in \mathbb{R}^C_{++}$, $p^T\Delta\mu \neq 0$. Moreover, if $\Delta x = 0$, from (a), we would have $p^T\Delta\mu = 0$, a contradiction. .

Claim 2. $Du \cdot \Delta x = 0$.
From $(FOC1)$, we have $Du \cdot \Delta x - \mu_h p \cdot \Delta x = 0$; using (b) the desired result follows .

Claim 3. $\Delta x^T D^2 u \cdot \Delta x = 0$.
Premultiplying (a) by Δx^T, we get $\Delta x^T D^2u(x)\Delta x - \Delta x^T p^T \Delta\mu = 0$. Using (b), the result follows.

Claims 1, 2 and 3 contradict Assumption u3.

■

The above result gives also a way of computing $D_{(p,w)} x(p, w)$, as an application of the Implicit Function Theorem .

Since

	x	μ	p	w
$Du(x) - \mu p$	D^2u	$-p^T$	$-\mu I_C$	0
$-px + w$	$-p$	0	$-x$	1

$$\left[D_{(p,w)}(x,\mu)(p,w)\right]_{(C+1)\times(C+1)} = \begin{bmatrix} D_p x & D_w x \\ D_p\mu & D_p\mu \end{bmatrix} =$$

$$= -\begin{bmatrix} D^2u & -p^T \\ -p & 0 \end{bmatrix}^{-1}_{(C+1)\times(C+1)} \begin{bmatrix} -\mu I_C & 0 \\ -x & 1 \end{bmatrix}_{(C+1)\times(C+1)}$$

To compute the inverse of the above matrix, we can use the following fact about the inverse of partitioned matrix (see for example, Goldberger, (1963), page 26)

Let A be an $n \times n$ nonsingular matrix partitioned as

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

where $E_{n_1\times n_1}$, $F_{n_1\times n_2}$, $G_{n_2\times n_1}$, $H_{n_2\times n_2}$ and $n_1 + n_2 = n$. Suppose that E and $D := H - GE^{-1}F$ are non singular. Then

$$A^{-1} = \begin{bmatrix} E^{-1}\left(I + FD^{-1}GE^{-1}\right) & -E^{-1}FD^{-1} \\ -D^{-1}GE^{-1} & D^{-1} \end{bmatrix}.$$

If we assume that D^2u is negative definite and therefore invertible, we have

$$\begin{bmatrix} D^2u & -p^T \\ -p & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \left(D^2\right)^{-1}\left(I + \delta^{-1}p^T p\left(D^2\right)^{-1}\right) & \delta^{-1}\left(D^2\right)^{-1}p^T \\ \delta^{-1}p\left(D^2\right)^{-1} & \delta^{-1} \end{bmatrix}$$

where $\delta = -p\left(D^2\right)^{-1}p^T \in \mathbb{R}_{++}$.

And

$$[D_p x(p,w)]_{C\times C} = -\left[\begin{array}{cc} (D^2)^{-1}\left(I+\delta^{-1}p^Tp\,(D^2)^{-1}\right) & \delta^{-1}(D^2)^{-1}p^T \end{array}\right]\left[\begin{array}{c} \mu I_C \\ x \end{array}\right] =$$
$$= -\mu\,(D^2)^{-1}\left(I+\delta^{-1}p^Tp\,(D^2)^{-1}\right) - \delta^{-1}(D^2)^{-1}p^Tx = -\delta^{-1}(D^2)^{-1}\left[\mu\left(\delta I + p^Tp\,(D^2)^{-1}\right)+p^Tx\right]$$

$$[D_w x(p,w)]_{C\times 1} = -\left[\begin{array}{cc} (D^2)^{-1}\left(I+\delta^{-1}p^Tp\,(D^2)^{-1}\right) & \delta^{-1}(D^2)^{-1}p^T \end{array}\right]\left[\begin{array}{c} 0 \\ -1 \end{array}\right] = \delta^{-1}(D^2)^{-1}p^T$$

$$[D_p\mu(p,w)]_{1\times C} = -\left[\begin{array}{cc} \delta^{-1}p\,(D^2)^{-1} & \delta^{-1} \end{array}\right]\left[\begin{array}{c} \mu I_C \\ x \end{array}\right] = -\delta^{-1}\left(\mu p\,(D^2)^{-1}+x\right).$$

$$[D_w\mu(p,w)]_{1\times 1} = -\left[\begin{array}{cc} \delta^{-1}p\,(D^2)^{-1} & \delta^{-1} \end{array}\right]\left[\begin{array}{c} 0 \\ -1 \end{array}\right] = \delta^{-1}.$$

.

.

As a simple application of the Envelope Theorem, we also have that, defined the indirect utility function as

$$v : \mathbb{R}^{C+1}_{++} \to \mathbb{R},\ v : (p,w) \mapsto u(x(p,w)),$$

we have that

$$D_{(p,w)}v(p,w) = \lambda\left[\begin{array}{cc} -x^T & 1 \end{array}\right].$$

20.2 Production

Definition 833 *A production vector (or input-output or netput vector) is a vector $y := (y^c)_{c=1}^C \in \mathbb{R}^C$ which describes the net outputs of C commodities from a production process. Positive numbers denote outputs, negative numbers denote inputs, zero numbers denote commodities neither used nor produced.*

Observe that, given the above definition, py is the profit of the firm.

Definition 834 *The set of all feasible production vectors is called the production set $Y \subseteq \mathbb{R}^C$. If $y \in Y$, then y can be obtained as a result of the production process; if $y \notin Y$,that is not the case.*

Definition 835 *The Profit Maximization Problem (PMP) is*

$$\max_y py \quad s.t. \quad y \in Y.$$

It is convenient to describe the production set Y using a function $F : \mathbb{R}^C \to \mathbb{R}$ called *the transformation function.* That is done as follows:
$Y = \{y \in \mathbb{R}^C : F(y) \geq 0\}$.
We list below a smooth version of the assumptions made on Y, using the transformation function.
Some assumption on $F(.)$.
(1) $\exists y \in \mathbb{R}^C$ such that $F(y) \geq 0$.
(2) F is C^2.
(3) (No Free Lunch) If $y \geq 0$, then $F(y) < 0$.
(4) (Possibility of Inaction) $F(0) = 0$.
(5) (F is differentiably strictly decreasing) $\forall y \in \mathbb{R}^C$, $DF(y) \ll 0$
(6) (Irreversibility) If $y \neq 0$ and $F(y) \geq 0$, then $F(-y) < 0$.
(7) (F is differentiably strictly concave) $\forall \Delta \in \mathbb{R}^C \setminus \{0\}$, $\Delta^T D^2F(y)\Delta < 0$.

Definition 836 *Consider a function $F(.)$ satisfying the above properties and a strictly positive real number N. The Smooth Profit Maximization Problem (SPMP) is*

$$\max_y py \quad s.t. \quad F(y) \geq 0 \text{ and } \|y\| \leq N. \tag{20.1}$$

Remark 837 *For any solution to the above problem it must be the case that* $F(y) = 0$. *Suppose there exists a solution* y' *to (SPMP) such that* $F(y') > 0$. *Since* F *is continuous, in fact* C^2, *there exists* $\varepsilon > 0$ *such that* $z \in B(y', \varepsilon) \Rightarrow F(z) > 0$. *Take* $z' = y' + \frac{\varepsilon \cdot \mathbf{1}}{C}$. *Then,* $d(y', z') := \left(\sum_{c=1}^{C}\left(\frac{\varepsilon}{C}\right)^2\right)^{\frac{1}{2}} = \left(C\left(\frac{\varepsilon}{C}\right)^2\right)^{\frac{1}{2}} = \left(\frac{\varepsilon^2}{C}\right)^{\frac{1}{2}} = \frac{\varepsilon}{\sqrt{C}} < \varepsilon$. *Therefore* $z' \in B(y', \varepsilon)$ *and*

$$F(z') > 0 \quad (1).$$

But,

$$pz' = py' + p\frac{\varepsilon \cdot \mathbf{1}}{C} > py' \quad (2).$$

(1) *and* (2) *contradict the fact that* y' *solves (SPMP).*

Proposition 838 *If a solution with* $\|y\| < N$ *to* $(SPMP)$ *exists . Then* $y : \mathbb{R}^C_{++} \to \mathbb{R}^C$, $p \mapsto \arg\max(20.1)$ *is a well defined* C^1*function.*

Proof.

Let's first show that $y(p)$ is single valued.

Suppose there exist $y, y' \in y(p)$ with $y \neq y'$. Consider $y^\lambda := (1-\lambda)y + \lambda y'$. Since $F(.)$ is strictly concave, it follows that $F(y^\lambda) > (1-\lambda)F(y) + \lambda F(y') \geq 0$, where the last inequality comes from the fact that $y, y' \in y(p)$. But then $F(y^\lambda) > 0$. Then following the same argument as in Remark 837, there exists $\varepsilon > 0$ such that $z' = y^\lambda + \frac{\varepsilon \cdot \mathbf{1}}{C}$ and $F(z') > 0$. But $pz' > py^\lambda = (1-\lambda)py + \lambda py' = py$, contradicting the fact that $y \in y(p)$.

Let's now show that y is C^1

From Remark 837 and from the assumption that $\|y\| < N$, (SPMP) can be rewritten as $\max_y py \; s.t. \; F(y) = 0$. We can then try to apply Lagrange Theorem.

Necessary conditions: $DF(y) \ll 0$;

sufficient conditions: py is linear and therefore pseudo-concave; $F(.)$ is differentiably strictly concave and therefore quasi-concave; the Lagrange multiplier λ is strictly positive -see below.

Therefore, the solutions to $(SPMP)$ are characterized by the following First Order Conditions, i.e., the derivative of the Lagrangian function with respect to y and λ equated to zero:

$$\mathcal{L}(y, p) = py + \lambda F(y). \qquad \begin{array}{cc} y & p \\ p + \lambda DF(y) = 0 & F(y) = 0 \end{array}$$

.

Observe that $\lambda = -\frac{p1}{D_{y^1}F(y)} > 0$.

As usual to show differentiability of the choice function we take derivatives of the First Order Conditions.

$$\begin{array}{lll} & y & \lambda \\ p + \lambda DF(y) = 0 & D^2F(y) & [DF(y)]^T \\ F(y) = 0 & DF(y) & 0 \end{array}$$

We want to show that the above matrix has full rank. By contradiction, assume that there exists $\Delta := (\Delta y, \Delta\lambda) \in \mathbb{R}^C \times \mathbb{R}$, $\Delta \neq 0$ such that

$$\begin{bmatrix} D^2F(y) & [DF(y)]^T \\ DF(y) & 0 \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta\lambda \end{bmatrix} = 0,$$

i.e.,

$$D^2F(y) \cdot \Delta y + [DF(y)]^T \cdot \Delta\lambda = 0 \quad (a),$$

$$DF(y) \cdot \Delta y = 0 \qquad (b).$$

Premultiplying (a) by Δy^T, we get $\Delta y^T \cdot D^2F(y) \cdot \Delta y + \Delta y^T \cdot [DF(y)]^T \cdot \Delta\lambda = 0$. From (b), it follows that $\Delta y^T \cdot D^2F(y) \cdot \Delta y = 0$, contradicting the differentiably strict concavity of $F(.)$.

(3)

From the Envelope Theorem, we know that if $(\overline{y}, \overline{\lambda})$ is the unique pair of solution-multiplier associated with $\overline{p}$, we have that

$D_p\pi(p)_{|\overline{p}} = D_p(py)_{|(\overline{p},\overline{y})} + \overline{\lambda} D_p F(y)_{|(\overline{p},\overline{y})}$.

Since $D_p(py)_{|(\overline{p},\overline{y})} = \overline{y}$, $D_p F(y) = 0$ and, by definition of $\overline{y}$, $\overline{y} = y(\overline{p})$, we get $D_p\pi(p)_{|\overline{p}} = y(\overline{p})$, as desired.

(4)

From (3) , we have that $D_p y(p) = D_p^2 \pi(p)$. Since $\pi(.)$ is convex -see Proposition **??** (2)- the result follows.

(5)

From Proposition **??** (4) and the fact that $y(p)$ is single valued, we know that $\forall \alpha \in \mathbb{R}_{++}$, $y(p) - y(\alpha p) = 0$. Taking derivatives with respect to α, we have $D_p y(p)_{|(\alpha p)} \cdot p = 0$. For $\alpha = 1$, the desired result follows.

■

20.3 The demand for insurance

Consider an individual whose wealth is

$W - d$	with probability	π, and
W	with probability	$1 - \pi$,

where $W > 0$ and $d > 0$.

Let the function

$$u : A \to \mathbb{R}, \; u : c \to u(c)$$

be the individual's Bernoulli function.

Assumption 1. $\forall c \in \mathbb{R}$, $u'(c) > 0$ and $u''(c) < 0$.

Assumption 2. u is bounded above.

An insurance company offers a contract with following features: the potentially insured individual pays a premium p in each state and receives d if the accident occurs. The (potentially insured) individual can buy a quantity $a \in \mathbb{R}$ of the contract. In the case, she pays a premium $(a \cdot p)$ in each state and receives a reimbursement $(a \cdot d)$ if the accident occurs. Therefore, if the individual buys a quantity a of the contract, she get a wealth described as follows

$$\begin{array}{lll} W_1 := W - d - ap + ad & \text{with probability} & \pi, \text{ and} \\ W_2 := W - ap & \text{with probability} & 1 - \pi. \end{array} \tag{20.2}$$

Remark 839 *It is reasonable to assume that* $p \in (0, d)$.

Define

$$U : \mathbb{R} \to \mathbb{R}, \;\; U : a \mapsto \pi u(W - d - ap + ad) + (1 - \pi) u(W - ap) .$$

Then the individual solves the following problem. For given, $W \in \mathbb{R}_{++}, d \in \mathbb{R}_{++}, p \in (0, d)$, $\pi \in (0, 1)$

$$\max_{a \in \mathbb{R}} \quad U(a) \quad (M) \tag{20.3}$$

To show existence of a solution, we introduce the problem presented below. For given $W \in \mathbb{R}_{++}, d \in \mathbb{R}_{++}, p \in (0, d)$, $\pi \in (0, 1)$

$$\max_{a \in \mathbb{R}} U(a) \quad s.t. \quad U(a) \geq U(0) \quad (M')$$

Defined$A^* := \arg\max(M)$ and $A' := \arg\max(M')$, the existence of solution to (M) , follows from the Proposition below.

Proposition 840 *1.* $A' \subset A^*$. *2.* $A' \neq \emptyset$.

Proof.
Exercise
■
To show that the solution is unique, observe that

$$U'(a) = \pi u'(W - d + a(d-p))(d-p) + (1-\pi)u'(W - ap)(-p) \tag{20.4}$$

and therefore

$$U''(a) = \overset{(+)}{\pi} u''(W - d \overset{(-)}{+} a(d-p))\overset{(+)}{(d-p)^2} + \overset{(+)}{(1-\pi)}u''\overset{(-)}{(W-ap)}\overset{(+)}{p^2} < 0.$$

Summarizing, the unique solution of problem (M) is the unique solution of the equation:

$$U'(a) = 0.$$

Definition 841 $a^* : \mathbb{R}_{++} \times (0,1) \times (0,d) \times \mathbb{R}_{++} \to \mathbb{R}$,
$a^* : (d, \pi, p, W) \mapsto \arg\max(M)$.
$U^* : \Theta \to \mathbb{R}$,
$U^* : \theta \mapsto \pi u(W - d + a^*(\theta)(d-p)) + (1-\pi)u(W - a^*(\theta)p)$

Proposition 842 *The signs of the derivatives of a^* and U^* with respect to θ are presented in the following table*[1]*:*

		d	π	p	W
a^*		> 0 *if* $a^* \in [0,1]$	> 0	$\gtreqless 0$	≤ 0 *if* $a^* \leq 1$
U^*		≤ 0 *if* $a^* \in [0,1]$	≤ 0 *if* $a^* \in [0,1]$	≤ 0 *if* $a^* \geq 0$	> 0

Proof. Exercise. ■

[1] Conditions on $a^*(\theta)$ contained in the table can be expressed in terms of exogenous variables.

Part V

Problem Sets

Chapter 21

Exercises

21.1 Linear Algebra

1.

Show that the set of pair of real numbers is **not** a vector space with respect to the following operations:

(i). $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (ka, b)$;
(ii) $(a, b) + (c, d) = (a + c, b)$ and $k(a, b) = (ka, kb)$.

2.

Show that W is *not* a vector subspace of $\mathbb{R}^3$ on $\mathbb{R}$ if
(i) $W = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$;
(ii) $W = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$,
(iii) $W = \mathbb{Q}^3$.

3.

Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Show the W is a vector subspace of Vif
(i) $W = \{f \in V : f(1) = 0\}$;
(ii) $W = \{f \in V : f(1) = f(2)\}$.

4.

Show that
(i).

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$$

is a vector subspace of $\mathbb{R}^3$;
(ii).

$$S = \{(1, -1, 0), (0, 1, -1)\}$$

is a basis for V.

5.

Show the following fact.
Proposition. Let a matrix $A \in \mathbb{M}(n, n)$, with $n \in \mathbb{N}\setminus\{0\}$ be given. The set

$$\mathcal{C}_A := \{B \in \mathbb{M}(n, n) : BA = AB\}$$

is a vector subspace of $\mathbb{M}(n, n)$ (with respect to the field $\mathbb{R}$).

6.

Let U and V be vector subspaces of a vector space W. Show that

$$U + V := \{w \in W : \exists u \in U \text{ and } v \in V \text{ such that } w = u + v\}$$

is a vector subspace of W.

7.
Show that the following set of vectors is linearly independent:

$$\{(1,1,1,),(0,1,1),(0,0,1)\}.$$

8. Using the definition, find the change-of-basis matrix from

$$S=\{u_1=(1,2),\ u_2=(3,5)\}$$

to

$$E=\{e_1=(1,0),\ e_2=(0,1)\}$$

and from E to S. Check the conclusion of Proposition 205, i.e., that one matrix is the inverse of the other one.

9. Find the determinant of

$$C=\begin{bmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{bmatrix}$$

10. Say for which values of $k\in\mathbb{R}$ the following matrix has rank a. 4, b. 3:

$$A:=\begin{bmatrix} k+1 & 1 & -k & 2 \\ -k & 1 & 2-k & k \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

11.
Diagonalize

$$A=\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}.$$

12. Let

$$A=\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}.$$

Find: (a) all eigenvalues of A and corresponding eigenspaces, (b) an invertible matrix P such that $D=P^{-1}AP$ is diagonal.

13.
Show that similarity between matrices is an equivalence relation.

14.
Let A be a square symmetric matrix with real entries. Show that eigenvalues are real numbers. Show that if $\lambda_i\neq\lambda_j$ for $i\neq j$ then corresponding eigenvectors are orthogonal, i.e., their scalar product is zero.

15.
Show that

$$V=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1-x_2=0\right\}$$

is a vector subspace of $\mathbb{R}^3$ and find a basis for V.

16.

Given

$$l:\mathbb{R}^4\to\mathbb{R}^4,\quad l(x_1,x_2,x_3,x_4)=\begin{pmatrix} x_1, & x_1+x_2, & x_1+x_2+x_3, & x_1+x_2+x_3+x_4\end{pmatrix}$$

show it is linear, compute the associated matrix with respect to canonical bases, and compute $\ker l$ and Iml.

17.
Complete the text below.
Proposition. Assume that $l\in L(V,U)$ and $\ker l=\{0\}$. Then,

$$\forall u\in \operatorname{Im} l,\quad \text{there exists a unique } v\in V \text{ such that } l(v)=u.$$

Proof.
Since ...., by definition, there exists $v\in V$ such that

$$l(v)=u. \tag{21.1}$$

Take $v'\in V$ such that $l(v')=u$. We want to show that

$$\text{........................} \tag{21.2}$$

Observe that

$$l(v)-l(v')\overset{(a)}{=}\text{........................} \tag{21.3}$$

where (a) follows from ....
Moreover,

$$l(v)-l(v')\overset{(b)}{=}\text{........................}, \tag{21.4}$$

where (b) follows from ....
Therefore,

$$l(v-v')=0,$$

and, by definition of $\ker l$,

$$\text{.........................} \tag{21.5}$$

Since, ..., from (21.5), it follows that

$$v-v'=0.$$

18.
Let the following sets be given:

$$V=\left\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 : x_1-x_2+x_3-x_4=0\right\}$$

and

$$W=\left\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 : x_1+x_2+x_3+x_4=0\right\}$$

If possible, find a basis of $V\cap W$.

19.
Say if the following statement is true or false.
Let V and U be vector spaces on $\mathbb{R}$, W a vector subspace of U and $l\in\mathcal{L}(V,U)$. Then $l^{-1}(W)$ is a vector subspace of V.

20.
Let the following full rank matrices

$$A=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\qquad B=\begin{bmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}$$

be given. Say for which values of $k \in \mathbb{R}$, the following linear system has solutions.

$$\begin{bmatrix} 1 & a_{11} & a_{12} & 0 & 0 & 0 \\ 2 & a_{21} & a_{22} & 0 & 0 & 0 \\ 3 & 5 & 6 & b_{11} & b_{12} & 0 \\ 4 & 7 & 8 & b_{21} & b_{22} & 0 \\ 1 & a_{11} & a_{12} & 0 & 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} k \\ 1 \\ 2 \\ 3 \\ k \end{bmatrix}$$

21.
Consider the following Proposition contained in Section 8.1 in the class Notes:
Proposition .$\forall v \in V$,

$$[l]_{\mathbf{v}}^{\mathbf{u}} \cdot [v]_{\mathbf{v}} = [l(v)]_{\mathbf{u}} \tag{21.6}$$

Verify the above equality in the case in which
a.

$$l : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

b. the basis $\mathbf{v}$ of the domain of l is

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

c. the basis $\mathbf{u}$ of the codomain of l is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\},$$

d.

$$v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

22.
Complete the following proof.
Proposition. Let
$n, m \in \mathbb{N} \setminus \{0\}$ such that $m > n$, and
a vector subspace L of $\mathbb{R}^m$ such that $\dim L = n$
be given. Then, there exists $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that $\text{Im } l = L$.
Proof. Let $\{v^i\}_{i=1}^n$ be a basis of $L \subseteq \mathbb{R}^m$. Take $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\forall i \in \{1, ..., n\}, \qquad l_2(e_n^i) = v^i,$$

where e_n^i is the i–th element in the canonical basis in $\mathbb{R}^n$. Such function does exists and, in fact, it is unique as a consequence of a Proposition in the Class Notes that we copy below:
....
Then, from the Dimension theorem

$$\dim \text{Im} l = \text{...}$$

Moreover,

$$L = \text{..........} \{v^i\}_{i=1}^n \subseteq \text{...............}$$

Summarizing,

$$L \subseteq \text{Im } l, \ \dim L = n \text{ and } \dim \text{Im} \, l \leq n,$$

and therefore

$$\dim \text{Im} l = n.$$

Finally, from Proposition ....in the class Notes since $L \subseteq \text{Im } l$, $\dim L = n$ and $\dim \text{Im} l = n$, we have that $\text{Im} \, l = L$, as desired.
Proposition .... in the class Notes says what follows:

....

23.
Say for which value of the parameter $a \in \mathbb{R}$ the following system has one, infinite or no solutions

$$\begin{cases} ax_1 & + & x_2 & = & 1 \\ x_1 & + & x_2 & = & a \\ 2x_1 & + & x_2 & = & 3a \\ 3x_1 & + & 2x_2 & = & a \end{cases}$$

24.
Say for which values of k,the system below admits one, none or infinite solutions.

$$A(k) \cdot x = b(k)$$

where $k \in \mathbb{R}$, and

$$A(k) \equiv \begin{bmatrix} 1 & 0 \\ 1-k & 2-k \\ 1 & k \\ 1 & k-1 \end{bmatrix}, \qquad b(k) \equiv \begin{bmatrix} k-1 \\ k \\ 1 \\ 0 \end{bmatrix}.$$

25.
Let $\mathcal{V} = \{v^1, v^2, ..., v^n\}$ be a set of vectors in $\mathbb{R}^n$ such that for any $i, j \in \{1, ..., n\}$,

$$v^i \cdot v^j = \begin{cases} 0 & if \quad i \neq j \\ \\ 1 & if \quad i = j. \end{cases} \tag{21.7}$$

Show that $\mathcal{V}$ is a basis of $\mathbb{R}^n$.
26.
Given a vector space V on a field F, sets $A, B \subseteq V$ and $k \in F$, we define

$$A + B := \{v \in V : \text{ there exist } a \in A \text{ and } b \in B \text{ such that } v = a + b\},$$

$$kA := \{v \in V : \text{ there exist } a \in A \text{ such that } v = ka\}.$$

Given a vector space V on a field F, a linear function $T \in \mathcal{L}(V, V)$ and W vector subspace of V, W is said to be T-invariant if

$$T(W) \subseteq W.$$

Let W be both S-invariant and T-invariant and let $k \in F$. Show that
a. W is $S + T$-invariant;
b. W is $S \circ T$-invariant;
c. W is kT-invariant.
27.
Show that the set of all 2×2 symmetric real matrices is a vector subspace of $\mathbb{M}(2, 2)$ and compute its dimension.
28.
Let V be a vector space on a field F and W a vector subspace of V. Show that
a. $W + W = W$, and
b. for any $\alpha \in F \setminus \{0\}$, $\alpha W = W$.
29.
Let $\mathcal{P}_n(\mathbb{R})$ be the set polynomials of degree smaller or equal than $n \in \mathbb{N}_+$ on the set of real numbers , i.e.,

$$\mathcal{P}_n(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \text{ such that } \exists a_0, a_1, ..., a_n \in \mathbb{R} \text{ such that for any } t \in \mathbb{R}, f(t) = \sum_{i=0}^{n} a_i t^i \right\}.$$

Show that $\mathcal{P}_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^{n+1}$.

21.2 Some topology in metric spaces

21.2.1 Basic topology in metric spaces

1.
Do Exercise 435: Let d be a metric on a non-empty set X. Show that

$$d' : X \times X \to \mathbb{R}, d(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

is a metric on X.

2.
Let X be the set of continuous real valued functions with domain $[0,1] \subseteq \mathbb{R}$ and

$$d(f,g) = \int_0^1 f(x) - g(x)\, dx,$$

where the integral is the Riemann Integral (that one you learned in Calculus 1). Show that (X,d) is not a metric space.

3.
Do Exercise 452 for $n = 2$: $\forall n \in \mathbb{N}, \forall i \in \{1,...,n\}, \forall a_i, b_i \in \mathbb{R}$ with $a_i < b_i$,

$$\times_{i=1}^{n} (a_i, b_i)$$

is $(\mathbb{R}^n, d_2)$ open.

4.
Show the second equality in Remark 460:

$$\cap_{n=1}^{+\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

5.
Say if the following set is $(\mathbb{R}, d_2)$ open or closed:

$$S := \left\{ x \in \mathbb{R} : \exists\ n \in \mathbb{N} \setminus \{0\} \text{ such that } x = (-1)^n\ \frac{1}{n} \right\}$$

6.
Say if the following set is $(\mathbb{R}, d_2)$ open or closed:

$$A := \cup_{n=1}^{+\infty} \left(\frac{1}{n}, 10 - \frac{1}{n}\right).$$

7.
Do Exercise 470: show that $\mathcal{F}(S) = \mathcal{F}\left(S^C\right)$.

8.
Do Exercise 471: show that $\mathcal{F}(S)$ is a closed set.

9.
Let the metric space $(\mathbb{R}, d_2)$ be given. Find $Int\ S, \text{Cl}\ (S), \mathcal{F}(S), D(S), Is(S)$ and say if S is open or closed for $S = \mathbb{Q}$, $S = (0,1)$ and $S = \left\{x \in \mathbb{R} : \exists n \in \mathbb{N}_+ \text{ such that } x = \frac{1}{n}\right\}$.

10.
Show that the following statements are **false**:
a. $\text{Cl}\ (Int\ S) = S$,
b. $Int\ \text{Cl}\ (S) = S$.

11.
Given $S \subseteq \mathbb{R}$, say if the following statements are true or false.
a. S is an open bounded interval $\Rightarrow S$ is an open set;
b. S is an open set $\Rightarrow S$ is an open bounded interval;
c. $x \in \mathcal{F}(S) \Rightarrow x \in D(S)$;
d. $x \in D(S) \Rightarrow x \in \mathcal{F}(S)$.

12.
Using the definition of convergent sequences, show that the following sequences do converge:
a. $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ such that $\forall n \in \mathbb{N}$, $x_n = 1$;
b. $(x_n)_{n\in\mathbb{N}\setminus\{0\}} \in \mathbb{R}^\infty$ such that $\forall n \in \mathbb{N}\setminus\{0\}$, $x_n = \frac{1}{n}$.

13.
Using Proposition 497, show that $[0,1]$ is $(\mathbb{R}, d_2)$ closed.

14.
Show the following result: A subset of a discrete space, i.e., a metric space with the discrete metric, is compact if and only if it is finite.

15.
Say if the following statement is true: An open set is not compact.

16.
Using the definition of compactness, show the following statement: Any open ball in $\left(\mathbb{R}^2, d_2\right)$ is not compact.

17.
Show that $f(A \cup B) = f(A) \cup f(B)$.

18.
Show that $f(A \cap B) \neq f(A) \cap (B)$.

19.
Using the characterization of continuous functions in terms of open sets, show that for any metric space (X, d) the constant function is continuous.

20.
a. Say if the following sets are $(\mathbb{R}^n, d_2)$ compact:
i.

$$\mathbb{R}^n_+,$$

ii.

$$\forall x \in \mathbb{R}^n \text{ and} \forall r \in \mathbb{R}_{++},, \; Cl\ B(x,r).$$

b. Say if the following set is $(\mathbb{R}, d_2)$ compact:

$$\left\{x \in \mathbb{R} : \exists n \in \mathbb{N}\setminus\{0\} \text{ such that } x = \frac{1}{n}\right\}.$$

21.
Given the continuous functions

$$g : \mathbb{R}^n \to \mathbb{R}^m$$

show that the following set is closed

$$\{x \in \mathbb{R}^n : g(x) \geq 0\}$$

22.
Assume that $f:\mathbb{R}^m \to \mathbb{R}^n$ is continuous. Say if

$$X = \{x \in \mathbb{R}^m : f(x) = 0\}$$

is (a) closed, (b) is compact.

23.
Using the characterization of continuous functions in terms of open sets, show that the following function is not continuous

$$f:\mathbb{R}\to\mathbb{R}, \quad f(x) = \begin{cases} x & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$$

24.
Using the Extreme Value Theorem, say if the following maximization problems have solutions (with $\| \ \|$ being the Euclidean norm).

$$\max_{x\in\mathbb{R}^n} \sum_{i=1}^n x_i \quad s.t. \quad \|x\| \leq 1$$

$$\max_{x\in\mathbb{R}^n} \sum_{i=1}^n x_i \quad s.t. \quad \|x\| < 1$$

$$\max_{x\in\mathbb{R}^n} \sum_{i=1}^n x_i \quad s.t. \quad \|x\| \geq 1$$

25.
Let $(E, \|\|_E)$, $(F, \|\|_F)$ be normed vector spaces. A function $f : (E, \|\|_E)$, $(F, \|\|_F)$ is bounded if

$$\exists M \in \mathbb{R}_{++} \text{ such that } \forall x \in E \quad \|f(x)\|_F \leq M.$$

Show that given a linear function $l : E \to F$,

$$l \text{ is bounded} \Leftrightarrow l = 0.$$

26.
$f : (X, d) \to \mathbb{R}$ is upper semicontinuos at $x_0 \in X$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d(x - x_0) < \delta \Rightarrow f(x) < f(x_0) + \varepsilon.$$

f is upper semicontinuous if is upper semicontinuos at any $x_0 \in X$.
Show that the following statements are equivalent:
a. f is upper semicontinuos;
b. for any $\alpha \in \mathbb{R}$, $\{x \in X : f(x) < \alpha\}$ is (X, d) open;
c. for any $\alpha \in \mathbb{R}$, $\{x \in X : f(x) \geq \alpha\}$ is (X, d) closed.

27.
Let A be a subset of $(\mathbb{R}^n, d)$ where d is the Euclidean distance. Show that if A is $(\mathbb{R}^n, d)$ open, then for any $x \in X$, $\{x\} + A$ is $(\mathbb{R}^n, d)$ open.
Hint: use the fact that for any $x, y \in \mathbb{R}^n$, $d(x, y) = \|x - y\|$ and therefore, for any $a \in \mathbb{R}^n$, $d(a + x, a + y) = \|a + x - a - y\| = \|x - y\| = d(x, y)$.

28.
Let (X, d) be a metric space. Show that if K_1 and K_2 are compact subsets of X, then $K_1 + K_2$ is compact.

29.
Given two metric spaces (E, d_1) and (F, d_2), a function $f : E \to F$ is an isometry with respect to d_1 and d_2 if $\forall x_1, x_2 \in E$,

$$d_2(f(x_1), f(x_2)) = d_1(x_1, x_2).$$

Show that if $f : E \to F$ is an isometry then
a.f is one-to-one;
b. $\hat{f} : E \to f(E)$ is invertible;
c. f is continuous.

21.2.2 Correspondences

To solve the following exercises on correspondences, we need some preliminary definitions.[1]

A set $C \subseteq \mathbb{R}^n$ is convex if $\forall x_1,\, x_2 \in C$ and $\forall \lambda \in [0,1]$, $(1-\lambda)\, x_1 + \lambda x_2 \in C$.

A set $C \subseteq \mathbb{R}^n$ is strictly convex if $\forall x_1,\, x_2 \in C$ and $\forall \lambda \in (0,1)$, $(1-\lambda)\, x_1 + \lambda x_2 \in Int\ C$.

Consider an open and convex set $X \subseteq \mathbb{R}^n$ and a continuous function $f : X \to \mathbb{R}$, f is quasi-concave iff $\forall\ x', x'' \in X,\ \forall\ \lambda \in [0,1]$,

$$f((1-\lambda)x' + \lambda x'') \geq \min \{f(x'), f(x'')\}.$$

f is strictly quasi-concave

Definition 843 *iff* $\forall\ x', x'' \in X$, *such that* $x' \neq x''$, *and* $\forall\ \lambda \in (0,1)$, *we have that*

$$f((1-\lambda)x' + \lambda x'') > \min \{f(x'), f(x'')\}.$$

We define the budget correspondence as

Definition 844

$$\beta : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \to\to \mathbb{R}^C,\ \beta(p,w) = \left\{x \in \mathbb{R}^C_{+} : px \leq w\right\}.$$

The Utility Maximization Problem (UMP) is

Definition 845

$$\max_{x \in \mathbb{R}^C_+} \quad u(x) \qquad s.t. \quad px \leq w,\ \ or\ x \in \beta(p,w)$$

$\xi : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \to\to \mathbb{R}^C,\ \xi(p,w) = \arg\max (UMP)$ *is the demand correspondence.*

The Profit Maximization Problem (PMP) is

$$\max_y py \quad s.t. \quad y \in Y.$$

Definition 846 *The supply correspondence is*

$$y : \mathbb{R}^C_{++} \to\to \mathbb{R}^C,\ y(p) = \arg\max(PMP).$$

We can now solve some exercises. (`the numbering has to be changed`)

1.
Show that ξ is non-empty valued.

2.
Show that for every $(p,w) \in \mathbb{R}^C_{++} \times \mathbb{R}_{++}$,
(a) if u is quasiconcave, ξ is convex valued;
(b) if u is strictly quasiconcave, ξ is single valued, i.e., it is a function.

3.
Show that β is closed.

4.
If a solution to (PMP) exists, show the following properties hold.
(a) If Y is convex, $y(.)$ is convex valued;
(b) If Y is strictly convex (i.e., $\forall \lambda \in (0,1)$, $y^\lambda := (1-\lambda)\, y' + \lambda y'' \in Int\ Y$), $y(.)$ is single valued.

5
Consider $\phi_1, \phi_2 : [0,2] \to\to \mathbb{R}$,

$$\phi_1(x) = \begin{cases} \left[-1 + 0.25 \cdot x, x^2 - 1\right] & if \quad x \in [0,1) \\ [-1, 1] & if \quad x = 1 \\ \left[-1 + 0.25 \cdot x, x^2 - 1\right] & if \quad x \in (1,2] \end{cases},$$

and

$$\phi_2(x) = \begin{cases} \left[-1 + 0.25 \cdot x, x^2 - 1\right] & if \quad x \in [0,1) \\ [-0.75, -0.25] & if \quad x = 1 \\ \left[-1 + 0.25 \cdot x, x^2 - 1\right] & if \quad x \in (1,2] \end{cases}.$$

[1] The definition of quasi-concavity and strict quasi-concavity will be studied in detail in Chapter .

Say if ϕ_1 and ϕ_2 are LHC, UHC, closed, convex valued, compact valued.
6.
Consider $\phi : \mathbb{R}_+ \rightarrow\rightarrow \mathbb{R}$,

$$\phi(x) = \begin{cases} \left\{\sin \frac{1}{x}\right\} & if \quad x > 0 \\ [-1, 1] & if \quad x = 0 \end{cases},$$

Say if ϕ is LHC, UHC, closed.
7.
Consider $\phi : [0, 1] \rightarrow\rightarrow [-1, 1]$

$$\phi(x) = \begin{cases} [0, 1] & if \quad x \in \mathbb{Q} \cap [0, 1] \\ [-1, 0] & if \quad x \in [0, 1] \backslash \mathbb{Q} \end{cases}.$$

Say if ϕ is LHC, UHC, closed.
8.
Consider $\phi_1, \phi_2 : [0, 3] \rightarrow\rightarrow \mathbb{R}$,

$$\phi_1(x) = \left[x^2 - 2, x^2\right],$$

and

$$\phi_2(x) = \left[x^2 - 3, x^2 - 1\right],$$

$$\phi_3(x) := (\phi_1 \cap \phi_2)(x) := \phi_1(x) \cap \phi_2(x).$$

Say if ϕ_1, ϕ_2 and ϕ_3 are LHC, UHC, closed.

21.3 Differential Calculus in Euclidean Spaces

1 .
Using the definition, compute the partial derivative of the following function in an arbitrary point (x_0, y_0) :

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = 2x^2 - xy + y^2.$$

2 .
If possible, compute partial derivatives of the following functions.
a. $f(x, y) = x \cdot \arctan \frac{y}{x}$;
b. $f(x, y) = x^y$;
c. $f(x, y) = (\sin(x + y))^{\sqrt{x+y}}$ in $(0, 3)$

3,
Given the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & if \quad x + y \neq 0 \\ \\ 0 & \quad \text{otherwise} \end{cases},$$

show that it admits both partial derivatives in $(0, 0)$ and it is not continuous in $(0, 0)$.

4 .
Using the definition, compute the directional derivative $f'((1, 1); (\alpha_1, \alpha_2))$ with $\alpha_1, \alpha_2 \neq 0$ for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$f(x, y) = \frac{x + y}{x^2 + y^2 + 1}.$$

5 .
Using the definition, show that the following function is differentiable: $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$f(x, y) = x^2 - y^2 + xy \tag{21.8}$$

Comment: this exercise requires some tricky computations. Do not spend too much time on it. Do this exercise after having studied Proposition 736.

6 .
Using the definition, show that the following functions are differentiable.
a. $l \in L(\mathbb{R}^n, \mathbb{R}^m)$;
b. the projection function $f : \mathbb{R}^n \to \mathbb{R}$, $f : \left(x^i\right)_{i=1}^n \mapsto x^1$.
7.
Show the following result which was used in the proof of Proposition 701. A linear function $l : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.

8 .
Compute the Jacobian matrix of $f : \mathbb{R}^2 \to \mathbb{R}^3$,

$$f(x, y) = (\sin x \cos y, \qquad \sin x \sin y, \qquad \cos x \cos y)$$

9 .
Given differentiable functions $g, h : \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}\setminus\{0\}$, compute the Jacobian matrix of $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = \left(g(x) \cdot h(z), \quad \frac{g(h(x))}{y}, \quad e^{x \cdot g(h(x))}\right)$

10 .
Compute total derivative and directional derivative at x_0 in the direction u.
a.

$$f : \mathbb{R}^3_{++} \to \mathbb{R}, \qquad f(x_1, x_2, x_3) = \frac{1}{3}\log x_1 + \frac{1}{6}\log x_2 + \frac{1}{2}\log x_3$$

$x_0 = (1, 1, 2)$, $u = \frac{1}{\sqrt{3}}(1, 1, 1)$;
b.

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - x_3^2 - 2x_1x_2 - 6x_2x_3$$

$x_0 = (1, 0, -1)$, $u = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$;
c.

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x_1, x_2) = x_1 \cdot e^{x_1 x_2}$$

$x_0 = (0, 0)$, $u = (2, 3)$.

11 .
Given

$$f(x, y, z) = \left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}},$$

show that if $(x, y, z) \neq 0$, then

$$\frac{\partial^2 f(x, y, z)}{\partial x^2} + \frac{\partial^2 f(x, y, z)}{\partial y^2} + \frac{\partial^2 f(x, y, z)}{\partial z^2} = 0$$

12 .
Given the C^2 functions $g, h : \mathbb{R} \to \mathbb{R}_{++}$, compute the Jacobian matrix of

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \qquad f(x, y, z) = \left(\frac{g(x)}{h(z)}, \qquad g(h(x)) + xy, \qquad \ln(g(x) + h(x))\right)$$

13 .
Given the functions

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(x, y) = \begin{pmatrix} e^x + y \\ e^y + x \end{pmatrix}$$

$$g:\mathbb{R}\to\mathbb{R},\quad x\mapsto g(x)$$
$$h:\mathbb{R}\to\mathbb{R}^2,\qquad h(x)=f(x,g(x))$$

Assume that g is C^2. a. compute the differential of h in 0; b. check the conclusion of the Chain Rule.

14 .
Let the following differentiable functions be given.

$$f:\mathbb{R}^3\to\mathbb{R}\qquad (x_1,x_2,x_3)\mapsto f(x_1,x_2,x_3)$$
$$g:\mathbb{R}^3\to\mathbb{R}\qquad (x_1,x_2,x_3)\mapsto g(x_1,x_2,x_3)$$
$$a:\mathbb{R}^3\to\mathbb{R}^3,\qquad (x_1,x_2,x_3)\mapsto\begin{pmatrix} f(x_1,x_2,x_3)\\ g(x_1,x_2,x_3)\\ x_1\end{pmatrix}$$
$$b:\mathbb{R}^3\to\mathbb{R}^2,\qquad (y_1,y_2,y_3)\mapsto\begin{pmatrix} g(y_1,y_2,y_3)\\ f(y_1,y_2,y_3)\end{pmatrix}$$

Compute the directional derivative of the function $b\circ a$ in the point $(0,0,0)$ in the direction $(1,1,1)$.

15 .
Using the theorems of Chapter 16, show that the function in (21.8) is differentiable.

16 .
Given
$$f:\mathbb{R}^3\to\mathbb{R}^1,\qquad (x,y,z)\mapsto z+x+y^3+2x^2y^2+3xyz+z^3-9,$$
say if you can apply the Implicit Function Theorem to the function in $(x_0,y_0,z_0)=(1,1,1)$ and, if possible, compute $\frac{\partial x}{\partial z}$ and $\frac{\partial y}{\partial z}$ in $(1,1,1)$.

17 .
Using the notation of the statement of the Implicit Function Theorem presented in the Class Notes, say if that Theorem can be applied to the cases described below; if it can be applied, compute the Jacobian of g. (Assume that a solution to the system $f(x,t)=0$ does exist).

a. $f:\mathbb{R}^4\to\mathbb{R}^2$,
$$f(x_1,x_2,t_1,t_2)\mapsto\begin{pmatrix} x_1^2-x_2^2+2t_1+3t_2\\ x_1x_2+t_1-t_2\end{pmatrix}$$

b. $f:\mathbb{R}^4\to\mathbb{R}^2$,
$$f(x_1,x_2,t_1,t_2)\mapsto\begin{pmatrix} 2x_1x_2+t_1+t_2^2\\ x_1^2+x_2^2+t_1^2-2t_1t_2+t_2^2\end{pmatrix}$$

c. $f:\mathbb{R}^4\to\mathbb{R}^2$,
$$f(x_1,x_2,t_1,t_2)\mapsto\begin{pmatrix} t_1^2-t_2^2+2x_1+3x_2\\ t_1t_2+x_1-x_2\end{pmatrix}$$

18.
Say under which conditions, if $z^3-xz-y=0$, then
$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{3z^2+x}{(3z^2-x)^3}$$

19. Do Exercise 751: Let the utility function $u:\mathbb{R}^2_{++}\to\mathbb{R}_{++}$, $(x,y)\mapsto u(x,y)$ be given. Assume that it satisfies the following properties i. u is C^2, ii. $\forall(x,y)\in\mathbb{R}^2_{++}$, $Du(x,y)>>0$,iii.

$\forall (x,y) \in \mathbb{R}^2_{++}$, $D_{xx}u(x,y) < 0, D_{yy}u(x,y) < 0, D_{xy}u(x,y) > 0$. Compute the Marginal Rate of Substitution in (x_0, y_0) and say if the graph of each indifference curve is concave.

20.

Let the function $f, g : \mathbb{R}^n \to \mathbb{R}$ be given and assume that

for every $x_0 \in \mathbb{R}^n$ and every $u \in \mathbb{R}^n$, the directional derivatives $f'(x_0; u)$ and $g'(x_0; u)$ do exist.

Define $h : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x) \cdot g(x)$. If possible, compute $h'(x_0; u)$ for every $x_0 \in \mathbb{R}^n$ and every $u \in \mathbb{R}^n$.

21.

A function $f : \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto f(x)$ is homogenous of degree $n \in \mathbb{N}_+$ if

$$\text{for every } x = (x_1, x_2) \in \mathbb{R}^2 \text{ and every } a \in \mathbb{R}_+,\ f(ax_1, ax_2) = a^n f(x_1, x_2).$$

Show that if f is homogenous of degree n and $f \in C^1(\mathbb{R}^2, \mathbb{R})$, then

$$\text{for every } x = (x_1, x_2) \in \mathbb{R}^2, \qquad x_1 \cdot D_{x_1} f(x_1, x_2) + x_2 \cdot D_{x_2} f(x_1, x_2) = n f(x_1, x_2).$$

21.4 Nonlinear Programming

1. [2] Determine, if possible, the nonnegative parameter values for which the following functions $f : X \to \mathbb{R}$, $f : (x_i)_{i=1}^n := x \mapsto f(x)$ are concave, pseudo-concave, quasi-concave, strictly concave.

(a) $X = \mathbb{R}_{++}$, $f(x) = \alpha x^\beta$;

(b) $X = \mathbb{R}^n_{++}$, $n \geq 2$, $f(x) = \sum_{i=1}^n \alpha_i (x_i)^{\beta_i}$ (for pseudo-concavity and quasi-concavity consider only the case $n = 2$).

(c) $X = \mathbb{R}$, $f(x) = \min\{\alpha, \beta x - \gamma\}$.

2.

a. Discuss the following problem. For given $\pi \in (0,1)$, $a \in (0, +\infty)$,

$$\max_{(x_1,x_2)} \pi \cdot u(x) + (1-\pi) u(y) \quad \text{s.t.} \quad \begin{array}{l} y \leq a - \frac{1}{2}x \\ y \leq 2a - 2x \\ x \geq 0 \\ y \geq 0 \end{array}$$

where $u : \mathbb{R} \to \mathbb{R}$ is a C^2 function such that $\forall z \in \mathbb{R}$, $u'(z) > 0$ and $u''(z) < 0$.

b. Say if there exist values of (π, a) such that $(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{2}{3}a, \frac{2}{3}a, \lambda_1, 0, 0, 0\right)$, with $\lambda_1 > 0$, is a solution to Kuhn-Tucker conditions, where for $j \in \{1,2,3,4\}$, λ_j the multiplier associated with constraint j.

c. "Assuming" that the first, third and fourth constraint hold with a strict inequality, and the multiplier associated with the second constraint is strictly positive, describe in detail how to compute the effect of a change of a or π on a solution of the problem.

3.

a. Discuss the following problem. For given $\pi \in (0,1)$, $w_1, w_2 \in \mathbb{R}_{++}$,

$$\max_{(x,y,m) \in \mathbb{R}^2_{++} \times \mathbb{R}} \ \pi \log x + (1-\pi) \log y \qquad s.t \quad \begin{array}{l} w_1 - m - x \geq 0 \\ w_2 + m - y \geq 0 \end{array}$$

b. Compute the effect of a change of w_1 on the component x^* of the solution.

c. Compute the effect of a change of π on the objective function computed at the solution of the problem.

4.

[2] Exercise 1 is taken from David Cass' problem sets for his Microeconomics course at the University of Pennsylvania.

a. Discuss the following problem.

$$\min_{(x,y)\in\mathbb{R}^2} \quad x^2+y^2-4x-6y \qquad s.t. \qquad \begin{array}{l} x+y\leq 6 \\ y\leq 2 \\ x\geq 0 \\ y\geq 0 \end{array}$$

Let (x^*,y^*) be a solution to the the problem.
b. Can it be $x^*=0$?
c. Can it be $(x^*,y^*)=(2,2)$?.

5.
Characterize the solutions to the following problems.
(a) (consumption-investment)

$$\begin{array}{ll} \max_{(c_1,c_2,k)\in\mathbb{R}^3} & u(c_1)+\delta u(c_2) \\ & s.t. \\ & c_1+k\leq e \\ & c_2\leq f(k) \\ & c_1,c_2,k\geq 0, \end{array}$$

where $u:\mathbb{R}\to\mathbb{R},\ u'>0, u''<0;\ f:\mathbb{R}_+\to\mathbb{R},\ f'>0, f''<0$ and such that $f(0)=0;\ \delta\in(0,1), e\in\mathbb{R}_{++}$. After having written Kuhn Tucker conditions, consider just the case in which $c_1,c_2,k>0$.
(b) (labor-leisure)

$$\begin{array}{ll} \max_{(x,l)\in\mathbb{R}^2} & u(x,l) \\ & s.t. \\ & px+wl\leq w\bar{l} \\ & l\leq \bar{l} \\ & x,l\geq 0, \end{array}$$

where $u:\mathbb{R}^2$ is C^2, $\forall(x,l)\ \ Du(x,l)\gg 0, u$ is differentiably strictly quasi-concave, i.e.,$\forall(x,l)$, if $\Delta\neq 0$ and $Du(x,l)\cdot\Delta=0$, then $\Delta^T D^2u\ \Delta<0$; $p>0,\ w>0$ and $\bar{l}>0$.
Describe solutions for which $x>0$ and $0<l<\bar{l}$,

6.
(a) Consider the model described in Exercise 6. (a). What would be the effect on consumption (c_1,c_2) of an increase in initial endowment e?
What would be the effect on (the value of the objective function computed at the solution of the problem) of an increase in initial endowment e?
Assume that $f(k)=ak^\alpha$, with $a\in\mathbb{R}_{++}$ and $\alpha\in(0,1)$. What would be the effect on consumption (c_1,c_2) of an increase in a?
(b) Consider the model described in Exercise 6. (b).What would be the effect on leisure l of an increase in the wage rate w? in the price level p?
What would be the effect on (the value of the objective function computed at the solution of the problem) of an increase in the wage rate w? in the price level p?.
7.
Show that if $f:\mathbb{R}^2\to\mathbb{R},\ x\mapsto f(x)$ is homogenous of degree 1, then

$$f \text{ is concave} \Leftrightarrow \text{ for any } x,y\in\mathbb{R}^2,\ f(x+y)\geq f(x)+f(y).$$

Chapter 22

Solutions

22.1 Linear Algebra

1.

We want to prove that the following sets are **not** vector spaces. Thus, it is enough to find a counter-example violating one of the conditions defining vector spaces.

(i) The definition violates the so-called M2 distributive assumption since for $k = 1$ and $a = b = 1$,

$$(1+1)\cdot(1,1) = 2\cdot(1,1) = (2,1) : \text{while} : 1\cdot(1,1) + 1\cdot(1,1) = (2,2)$$

(ii) The definition violates the so-called A4 commutative property since for $a = b = c = 1$ and $d = 0$,

$$(1,1) + (1,0) = (2,1) \neq (1,0) + (1,1) = (2,0)$$

2.

(i) Take any $w := (x,y,z) \in W$ with $z > 0$, and $\alpha \in \mathbb{R}$ with $\alpha < 0$; then $\alpha w = (\alpha x, \alpha y, \alpha z)$ with $\alpha z < 0$ and therefore $w \notin W$.

(ii) Take any nonzero $w \in W$ and define $\alpha = 2/||w||$. Observe that $||\alpha w|| = 2 > 1$ and therefore $\alpha w \notin W$.

(iii) Multiplication of any nonzero element of $\mathbb{Q}^3$ by $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ will give an element of $\mathbb{R}^3\setminus\mathbb{Q}^3$ instead of $\mathbb{Q}^3$.

3.

We use Proposition 138. Therefore, we have to check that

a. $0 \in W$; b. $\forall u, v \in W, \forall \alpha, \beta \in F,\ \alpha u + \beta v \in W$.

Define simply by 0 the function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R},\ f(x) = 0$.

(i) a. Since $0(1) = 0$, $0 \in W$.

b.

$$\alpha u(1) + \beta v(1) = \alpha \cdot 0 + \beta \cdot 0 = 0,$$

where the first equality follows from the assumption that $u, v \in W$. Then, indeed, $\alpha u + \beta v \in W$.

(ii) a. Since $0(1) = 0 = 0(2)$, we have that $0 \in W$.

b.

$$\alpha u(1) + \beta v(1) = \alpha u(2) + \beta v(2),$$

where the equality follows from the assumption that $u, v \in W$ and therefore $u(1) = u(2)$ and $v(1) = v(2)$.

4.

Again we use Proposition 138 and we have to check that

a. $0 \in W$; b. $\forall u, v \in W, \forall \alpha, \beta \in F,\ \alpha u + \beta v \in W$.

a. $(0,0,0) \in W$ simply because $0 + 0 + 0 = 0$.

b. Given $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in V$, i.e., such that

$$u_1 + u_2 + u_3 = 0 \quad \text{and} \quad v_1 + v_2 + v_3 = 0, \tag{22.1}$$

we have

$$\alpha u+\beta v=(\alpha u_1+\beta v_1,\alpha u_2+\beta v_2,\alpha u_3+\beta v_3)\,.$$

Then,

$$\alpha u_1+\beta v_1+\alpha u_2+\beta v_2+\alpha u_3+\beta v_3=\alpha\left(u_1+u_2+u_3\right)+\beta\left(v_1+v_2+v_3\right)\overset{(22.1)}{=}0.$$

(ii) We have to check that a. S is linearly independent and b. *span* $S=V$.
a.

$$\alpha\left(1,-1,0\right)+\beta\left(0,1,-1\right)=\begin{pmatrix}\alpha & -\alpha+\beta & -\beta\end{pmatrix}=0$$

implies that $\alpha=\beta=0$.
b. Taken $(x_1,x_2,x_3)\in V$, we want to find $\alpha,\beta\in\mathbb{R}$ such that $(x_1,x_2,x_3)=\alpha\left(1,-1,0\right)+\beta\left(0,1,-1\right)=\begin{pmatrix}\alpha & -\alpha+\beta & -\beta\end{pmatrix}$, i.e., we want to find $\alpha,\beta\in\mathbb{R}$ such that

$$\begin{cases} x_1 & = & \alpha \\ x_2 & = & -\alpha+\beta \\ x_3 & = & -\beta \\ x_1+x_2+x_3 & = & 0 \end{cases}$$

$$\begin{cases} -x_2-x_3 & = & \alpha \\ x_2 & = & -\alpha+\beta \\ x_3 & = & -\beta \end{cases}$$

Then, $\alpha=-x_2-x_3$, $\beta=-x_3$ is the (unique) solution to the above system.

5.
1. $0\in\mathbb{M}\left(n,n\right):A0=0A=0$.
2. $\forall\alpha,\beta\in\mathbb{R}$ and $\forall B,B'\in\mathcal{C}_A$,

$$\left(\alpha B+\beta B'\right)A=\alpha BA+\beta B'A=\alpha AB+\beta AB'=A\alpha B+A\beta B'=A\left(\alpha B+\beta B'\right).$$

6.
i. $0\in U+V$, because $0\in U$ and $0\in V$.
ii. Take $\alpha,\beta\in F$ and $w^1,w^2\in U+V$. Then there exists $u^1,u^2\in U$ and $v^1,v^2\in V$ such that $w^1=u^1+v^1$ and $w^2=u^2+v^2$. Therefore,

$$\alpha w^1+\beta w^2=\alpha\left(u^1+v^1\right)+\beta\left(u^2+v^2\right)=\left(\alpha u^1+\beta u^2\right)+\left(\alpha v^1+\beta v^2\right)\in U+V,$$

because U and V are vector spaces and therefore $\alpha u^1+\beta v^1\in U$ and $\alpha u^2+\beta v^2\in V$.
7.
We want to show that if $\sum_{i=1}^3\beta_i v_i=0$, then $\beta_i=0$ for all i. Note that $\sum_{i=1}^3\beta_i v_i=(\beta_1,\beta_1+\beta_2,\beta_1+\beta_2+\beta_3)=0$, which implies the desired result.
8.
We want to apply Definition 203: Consider a vector space V of dimension n and two bases $\mathbf{v}=\left\{v^i\right\}_{i=1}^n$ and $\mathbf{u}=\left\{u^k\right\}_{k=1}^n$ of V. Then,

$$P=\begin{bmatrix}\left[u^1\right]_{\mathbf{v}} & \ldots & \left[u^k\right]_{\mathbf{v}} & \ldots & \left[u^n\right]_{\mathbf{v}}\end{bmatrix}\in\mathbb{M}\left(n,n\right),$$

is called the change-of-basis matrix from the basis $\mathbf{v}$ to the basis $\mathbf{u}$. Then in our case, the change-of-basis matrix from S to E is

$$P=\begin{bmatrix}\left[e^1\right]_S & \left[e^2\right]_S\end{bmatrix}\in\mathbb{M}\left(2,2\right).$$

Moreover,using also Proposition 205, the change-of-basis matrix from S to E is

$$Q=\begin{bmatrix}\left[v^1\right]_E & \left[v^2\right]_E\end{bmatrix}=P^{-1}.$$

Computation of $\left[e^1\right]_S$. We want to find α and β such that $e_1=\alpha u_1+\beta u_2$, i.e., $(1,0)=\alpha\left(1,2\right)+\beta\left(3,5\right)=\begin{pmatrix}\alpha+3\beta & 2\alpha+5\beta\end{pmatrix}$, i.e.,

$$\begin{cases} \alpha+3\beta & = & 1 \\ \\ 2\alpha+5\beta & = & 0 \end{cases}$$

whose solution is $\alpha = -5, \beta = 2$.

Computation of $\left[e^2\right]_S$. We want to find α and β such that $e_2 = \alpha u_1 + \beta u_2$, i.e., $(0,1) = \alpha(1,2) + \beta(3,5) = \begin{pmatrix} \alpha + 3\beta & 2\alpha + 5\beta \end{pmatrix}$, i.e.,

$$\begin{cases} \alpha + 3\beta & = & 0 \\ \\ 2\alpha + 5\beta & = & 1 \end{cases}$$

$$\begin{cases} \alpha + 3\beta = 0 \\ 2\alpha + 5\beta = 1 \end{cases}$$

whose solution is $\alpha = 3, \beta = -1$. Therefore,

$$P = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

Since E is the canonical basis, we have

$$S = \{u_1 = (1,2),\ u_2 = (3,5)\}$$

$$Q = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}.$$

Finally

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

as desired.

9.

Easiest way: use row and column operations to change C to a triangular matrix.

$$\det C = \det \begin{bmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{bmatrix} = \left[R^1 \leftrightarrow R^3\right] = -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 2 & 1 & 1 & -2 & 1 \\ 6 & 2 & 1 & 0 & 5 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 2 & 1 & 1 & -2 & 1 \\ 6 & 2 & 1 & 0 & 5 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -2R^1 + R^2 \to R^2 \\ -6R^1 + R^3 \to R^3 \\ -3R^1 + R^4 \to R^4 \\ R^1 + R^5 \to R^5 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & -4 & -11 & 12 & -13 \\ 0 & -3 & -4 & 9 & -10 \\ 0 & 0 & -1 & 2 & 5 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & -4 & -11 & 12 & -13 \\ 0 & -3 & -4 & 9 & -10 \\ 0 & 0 & -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 4R^2 + R^3 \to R^3 \\ 3R^2 + R^4 \to R^4 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 5 & 3 & 5 \\ 0 & 0 & -1 & 2 & 5 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 5 & 3 & 5 \\ 0 & 0 & -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} -5R^3 + R^4 \to R^4 \\ R^3 + R^5 \to R^5 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & -17 & -30 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & -17 & -30 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix} = \left[\frac{6}{17}R^4 + R^5 \to R^5\right] = -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & -17 & -30 \\ 0 & 0 & 0 & 0 & \frac{24}{17} \end{bmatrix}$$

$$= -\det\begin{bmatrix}1 & 1 & 2 & -2 & 3\\ 0 & -1 & -3 & 2 & -5\\ 0 & 0 & 1 & 4 & 7\\ 0 & 0 & 0 & -17 & -30\\ 0 & 0 & 0 & 0 & \frac{24}{17}\end{bmatrix} = -(1)(-1)(1)(-17)(\frac{24}{17}) \Longrightarrow \det C = -24$$

10.

Observe that it cannot be 4 as $rank(A) \leq \min\{\#rows, \#collums\}$. It's easy to check that $rank(A) = 3$ by using elementary operations on rows and columns of A :

$-2R^3 + R^1 \to R^1$, $C^4 + C^1 \to C^1$, $C^4 + C^3 \to C^3$, $C^1 + C^3 \to C^3$, $-C^2 + C^3 \to C^3$,

to get

$$\begin{bmatrix}k-1 & 1 & 0 & 0\\ 0 & 0 & 1 & k\\ 0 & 0 & 0 & -1\end{bmatrix}$$

which has the last three columns independent for any k.

11.

First we find the eigenvalues of the matrix:

$$A = \begin{bmatrix}4 & 2\\ 3 & -1\end{bmatrix} \Longrightarrow \det[tI - A] = \begin{bmatrix}t-4 & -2\\ -3 & t+1\end{bmatrix} = t^2 - 3t - 10$$

The solutions for $\det[tI - A] = 0$ and, therefore, the eigenvalues of the matrix are $\lambda_1 = 5$ and $\lambda_2 = -2$. They are different and thus by Proposition 262, the matrix P composed of the corresponding eigenvectors $v_1 = (2,1)'$ and $v_2 = (-1,3)'$ (for example) does the trick. The answer is

$$P = \begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix}5 & 0\\ 0 & -2\end{bmatrix}.$$

12.

For the eigenvalues, we look at the solutions for $\det[tI - A] = 0$

$$A = \begin{bmatrix}2 & 2\\ 1 & 3\end{bmatrix} \Longrightarrow \det[tI - A] = \begin{bmatrix}t-2 & -2\\ -1 & t-3\end{bmatrix} = t^2 - 5t + 4$$

So that the eigenvalues are 1 and 4. For the eigenspaces, we substitute each eigenvalue in the $tI - A$ matrix.

For $\lambda = 1$

$$I - A = \begin{bmatrix}-1 & -2\\ -1 & -2\end{bmatrix},$$

and then $\{(x,y) \in \mathbb{R}^2 : 2x = -y\}$ is the eigenspace for $\lambda = 1$.

For $\lambda = 4$

$$4I - A = \begin{bmatrix}2 & -2\\ -1 & 1\end{bmatrix},$$

and$\{(x,y) \in \mathbb{R}^2 : x = y\}$ is the eigenspace for $\lambda = 4$.

Finally, for the P matrix, we consider one eigenvector belonging to each of the eigenspaces, for example, $(2,-1)$ for $\lambda = 1$ and $(1,1)$ for $\lambda = 4$. Then, by Proposition 241, we can construct P as

$$P = \begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}$$

13.

Recall that: A matrix $B \in \mathbb{M}(n,n)$ is similar to a matrix $A \in \mathbb{M}(n,n)$ if there exists an invertible matrix $P \in \mathbb{M}(n,n)$ such that

$$B = P^{-1}AP.$$

An equivalence relation is a relation which is reflexive, symmetric and transitive.
a. reflexive. $B = I^{-1}BI$.
b. symmetric. $B = P^{-1}AP \Rightarrow A = PBP^{-1}$.
c. transitive. $B = P^{-1}AP$ and $C = Q^{-1}BQ \Rightarrow C = Q^{-1}BQ = Q^{-1}\left(P^{-1}AP\right)Q = (PQ)^{-1}A(PQ)$.

14.

1. Let A be a square matrix symmetric. Let $\lambda \in \mathbb{C}$ be an eigenvalue, $\bar{\lambda}$ its conjugate, and X a corresponding eigenvector - thus $X \neq 0$. * denotes the conjugate of a matrix or a vector according to the context. We have that

$$\bar{\lambda}X^*X = (\lambda X)^*.X = (AX)^*X = X^*A^*X$$

Or, A being symmetric $A^* = A$. Thus,

$$\bar{\lambda}X^*X = X^*AX = X^*(\lambda X) = \lambda X^*X$$

Since $X^*X \neq 0$, it must be that $\bar{\lambda} = \lambda$. Then $\lambda \in \mathbb{R}$.

2. Let now X_i and X_j be two eigenvectors associated with the two distinct eigenvectors λ_i and λ_j.
Then

$$\lambda_i X_i^* X_j \underbrace{=}_{\lambda_i \in R} (\lambda_i X_i)^* X_j = (AX_i)^* X_j \underbrace{=}_{A \text{ is symmetric}} X_i^* A X_j = \lambda_j X_i^* X_j$$

Since $\lambda_i \neq \lambda_j$, it must be that $X_i^* X_j = 0$, i.e., X_i and X_j are orthogonal.

15.
Defined

$$l : \mathbb{R}^3 \to \mathbb{R}, \quad (x_1, x_2, x_3) \mapsto x_1 - x_2,$$

it is easy to check that l is linear and $V = \ker l$,a vector space. Moreover,

$$[l] = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}.$$

Therefore, $\dim \ker l = 3 - \text{rank }[l] = 3 - 1 = 2$. $u_1 = (1,1,0)$ and $u_2 = (0,0,1)$ are independent elements of V. Therefore, from Remark 184, u_1, u_2 are a basis for V.

16.
Linearity is easy. By definition, l is linear if $\forall u, v \in \mathbb{R}^4$ and $\forall \alpha, \beta \in \mathbb{R}, l(\alpha u + \beta v) = \alpha l(u) + \beta l(v)$. Then,

$$\alpha l(u) + \beta l(v) =$$
$$=\alpha(u_1,u_1+u_2,u_1+u_2+u_3,u_1+u_2+u_3+u_4)++\beta(v_1,v_1+v_2,v_1+v_2+v_3,v_1+v_2+v_3+v_4) =$$
$$=(\alpha u_1+\beta v_1,\alpha u_1+\alpha u_2+\beta v_1+\beta v_2,\alpha u_1+\alpha u_2+\alpha u_3+\beta v_1+\beta v_2+\beta v_3,\alpha u_1+\alpha u_2+\alpha u_3+\alpha u_4+\beta v_1+\beta v_2+\beta v_3+\beta v_4) =$$
$$= l(\alpha u + \beta v)$$

and then l is linear.

$$[l] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Therefore, $\dim \ker l = 4 - \text{rank }[l] = 4 - 4 = 0$. Moreover, $\dim \operatorname{Im} l = 4$ and the column vectors of $[l]$ are a basis of $\operatorname{Im} l$.

17.
Proposition. Assume that $l \in L(V, U)$ and $\ker l = \{0\}$. Then,

$$\forall u \in Iml, \quad \text{there exists a unique } v \in V \text{ such that } l(v) = u.$$

Proof.
Since $u \in \mathtt{Im}l$, by definition, there exists $v \in V$ such that

$$l(v) = u. \tag{22.2}$$

Take $v' \in V$ such that $l(v') = u$. We want to show that

$$v = v'. \tag{22.3}$$

Observe that

$$l(v) - l(v') \overset{(a)}{=} u - u = 0, \tag{22.4}$$

where (a) follows from (22.2) and (22.3).
Moreover,

$$l(v) - l(v') \overset{(b)}{=} l(v - v'), \tag{22.5}$$

where (b) follows from the assumption that $l \in \mathcal{L}(V, U)$.
Therefore,

$$l(v - v') = 0,$$

and, by definition of $\ker l$,

$$v - v' \in \ker l. \tag{22.6}$$

Since, by assumption, $\ker l = \{0\}$, from (22.6), it follows that

$$v - v' = 0.$$

18.
Both V and W are ker of linear function; therefore V, W and $V \cap W$ are vector subspaces of $\mathbb{R}^4$. Moreover

$$V \cap W = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \left\{ \begin{array}{l} x_1 - x_2 + x_3 - x_4 = 0 \\ \\ x_1 + x_2 + x_3 + x_4 = 0 \end{array} \right. \right\}$$

$$\text{rank} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 2$$

Therefore, $\dim \ker l = \dim V \cap W = 4 - 2 = 2$.
Let's compute a basis of $V \cap W$:

$$\left\{ \begin{array}{l} x_1 - x_2 = -x_3 + x_4 \\ \\ x_1 + x_2 = -x_3 - x_4 \end{array} \right.$$

After taking sum and subtraction we get following expression

$$\left\{ \begin{array}{l} x_1 = -x_3 \\ \\ x_2 = -x_4 \end{array} \right.$$

A basis consists two linearly independent vectors. For example,

$$\{(-1, 0, 1, 0), (0, -1, 0, 0)\}.$$

19.
By proposition 138, we have to show that
1. $0 \in l^{-1}(W)$,
2. $\forall \alpha, \beta \in \mathbb{R}$ and $v^1, v^2 \in l^{-1}(W)$ we have that $\alpha v^1 + \beta v^2 \in l^{-1}(W)$.
1.

$$l(0) \overset{l \in \mathcal{L}(V,U)}{=} 0 \overset{W \text{ vector space}}{\in} W$$

,

2.Since $v^1, v^2 \in l^{-1}(W)$,

$$l\left(v^1\right), l\left(v^2\right) \in W. \tag{22.7}$$

Then

$$l\left(\alpha v^1+\beta v^2\right) \overset{l\in\mathcal{L}(V,U)}{=} \alpha l\left(v^1\right)+\beta l\left(v^2\right) \overset{(a)}{\in} W$$

where (a) follows from (22.7) and the fact that W is a vector space.

20.
Observe that

$$\det\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 5 & 6 & b_{11} & b_{12} & 0 \\ 7 & 8 & b_{21} & b_{22} & 0 \\ a_{11} & a_{12} & 0 & 0 & k \end{bmatrix} = \det A \cdot \det B \cdot k.$$

Then, if $k \neq 0$,then the rank of both matrix of coefficients and augmented matrix is 5 and the set of solution to the system is an affine subspace of $\mathbb{R}^6$ of dimension 1. If $k=0$,then the system is

$$\begin{bmatrix} 1 & a_{11} & a_{12} & 0 & 0 & 0 \\ 2 & a_{21} & a_{22} & 0 & 0 & 0 \\ 3 & 5 & 6 & b_{11} & b_{12} & 0 \\ 4 & 7 & 8 & b_{21} & b_{22} & 0 \\ 1 & a_{11} & a_{12} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \end{bmatrix},$$

which is equivalent to the system

$$\begin{bmatrix} 1 & a_{11} & a_{12} & 0 & 0 & 0 \\ 2 & a_{21} & a_{22} & 0 & 0 & 0 \\ 3 & 5 & 6 & b_{11} & b_{12} & 0 \\ 4 & 7 & 8 & b_{21} & b_{22} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix},$$

whose set of solution is an affine subspace of $\mathbb{R}^6$ of dimension 2.

21.

$$[l]_{\mathbf{v}}^{\mathbf{u}} := \left[\left[l\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_{\mathbf{u}}, \left[l\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_{\mathbf{u}}\right] = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathbf{u}}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathbf{u}}\right] = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix},$$

$$[v]_{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$[l(v)]_{\mathbf{u}} = \left[\begin{pmatrix} 7 \\ -1 \end{pmatrix}\right]_{\mathbf{u}} = \begin{bmatrix} -9 \\ 8 \end{bmatrix}$$

$$[l]_{\mathbf{v}}^{\mathbf{u}} \cdot [v]_{\mathbf{v}} = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \end{bmatrix}$$

22.
Let
$n, m \in \mathbb{N}\backslash\{0\}$ such that $m>n$, and
a vector subspace L of $\mathbb{R}^m$ such that $\dim L=n$
be given. Then, there exists $l \in \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right)$ such that $\operatorname{Im} l=L$.
Proof. Let $\left\{v^i\right\}_{i=1}^n$ be a basis of $L \subseteq \mathbb{R}^m$. Take $l \in \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right)$ such that

$$\forall i \in\{1, \ldots, n\}, l_2\left(e_n^i\right)=v^i,$$

where e_n^i is the i–th element in the canonical basis in $\mathbb{R}^n$. Such function does exists and, in fact, it is unique as a consequence of a Proposition in the Class Notes that we copy below:

Let V and U be finite dimensional vectors spaces such that $S = \{v^1, ..., v^n\}$ is a basis of V and $\{u^1, ..., u^n\}$ is a set of arbitrary vectors in U. Then there exists a unique linear function $l : V \to U$ such that $\forall i \in \{1, ..., n\}$, $l(v^i) = u^i$ - see Proposition 273, page 82.

Then, from the Dimension theorem

$$\dim \text{Im} l = n - \dim \ker l \leq n.$$

Moreover, $L = \text{span}\, \{v^i\}_{i=1}^n \subseteq \text{Im} l$. Summarizing,

$$L \subseteq \texttt{Im}\ l\ ,\ \dim L = n \text{ and } \dim \texttt{Im} l \leq n,$$

and therefore

$$\dim \text{Im} l = n.$$

Finally, from Proposition 179 in the class Notes since $L \subseteq \texttt{Im}\ l$, $\dim L = n$ and $\dim \text{Im} l = n$, we have that $\texttt{Im} l = L$, as desired.

Proposition 179 in the class Notes says what follows: Proposition. Let W be a subspace of an $n-$dimensional vector space V. Then, 1. $\dim W \leq n$; 2. If $\dim W = n$, then $W = V$.

23.

$$[A \mid b] = \begin{bmatrix} a & 1 & 1 \\ 1 & 1 & a \\ 2 & 1 & 3a \\ 3 & 2 & a \end{bmatrix} \Longrightarrow [R^1 \leftrightarrow R^2] \Longrightarrow \begin{bmatrix} 1 & 1 & a \\ a & 1 & 1 \\ 2 & 1 & 3a \\ 3 & 2 & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & a \\ a & 1 & 1 \\ 2 & 1 & 3a \\ 3 & 2 & a \end{bmatrix} \Longrightarrow \begin{matrix} -aR^1 + R^2 \to R^2 \\ -2R^1 + R^3 \to R^3 \\ -3R^1 + R^4 \to R^4 \end{matrix} \Longrightarrow \begin{bmatrix} 1 & 1 & a \\ 0 & 1-a & 1-a^2 \\ 0 & -1 & a \\ 0 & -1 & -2a \end{bmatrix} := [A'(a) \mid b'(a)]$$

Since

$$\det \begin{bmatrix} 1 & 1 & a \\ 0 & -1 & a \\ 0 & -1 & -2a \end{bmatrix} = 3a,$$

We have that if $a \neq 0$, then $\text{rank}\ A \leq 2 < 3 = \text{rank}\ [A'(a) \mid b'(a)]$, and the system has no solutions. If $a = 0$, $[A'(a) \mid b'(a)]$ becomes

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

whose rank is 3 and again the system has no solutions.

24.

$$[A(k)\,|b(k)] \equiv \begin{bmatrix} 1 & 0 & k-1 \\ 1-k & 2-k & k \\ 1 & k & 1 \\ 1 & k-1 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 0 & k-1 \\ 1 & k & 1 \\ 1 & k-1 & 0 \end{bmatrix} = 2 - 2k$$

If $k \neq 1$, the system has no solutions. If $k = 1$,

$$[A(1)\,|b(1)] \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Then, if $k = 1$, there exists a unique solution.
25.
The following Proposition is contained in the class Notes.

Proposition 847 *Let V be a vector space of dimension n.*

1. *If $S = \{u^1, ..., u^n\} \subseteq V$ is a linearly independent set, then it is a basis of V;*
2. *If* $\operatorname{span}(u^1, ..., u^n) = V$, *then $\{u^1, ..., u^n\}$ is a basis of V.*

From that Proposition, it suffices to show that $\mathcal{V}$ is linearly independent, i.e., given $(\alpha_i)_{i=1}^n \in \mathbb{R}^n$,

if

$$\sum_{i=1}^n \alpha_i v^i = 0 \tag{22.8}$$

then

$$\boxed{(\alpha_i)_{i=1}^n = 0.}$$

Now, for any $j \in \{1, ..., n\}$, we have

$$0 \overset{(1)}{=} \left(\sum_{i=1}^n \alpha_i v^i\right) v^j \overset{(2)}{=} \sum_{i=1}^n \alpha_i v^i v^j \overset{(3)}{=} \alpha_j,$$

where (1) follows from (22.8),
(2) follows from properties of the scalar product;
(3) follows from (21.7).
26.
a. Let $w \in W$. By assumption, $S(w) \in W$ and $T(w) \in W$ and since W is a subspace, $S(w) + T(w) \in W$[1]. Therefore, $(S + T)(w) = S(w) + T(w) \in W$, as desired.
b. Let $w \in W$. By assumption, $T(w) \in W$. Then $(S \circ T)(w) = S(T(w)) \in W$ since W is S-invariant.
c. Let $w \in W$. By assumption and recalling Proposition 138 in Villanacci(20 September 2012), $(kT)(w) = kT(w) \in W$.
27.
The set of 2×2 symmetric real matrices is

$$\mathcal{S} = \{A \in \mathbb{M}(2,2) : A = A^T\}.$$

We want to show that
i. $0 \in \mathcal{S}$ and
ii. for any $\alpha, \beta \in \mathbb{R}$, for any $A, B \in \mathcal{S}$, $\alpha A + \beta B \in \mathcal{S}$.
i. $0 = 0^T-$
2.$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T = \alpha A + \beta B$.
We want to show that

$$\mathcal{B} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

is a basis of $\mathcal{S}$ and therefore $\dim \mathcal{S} = 3$. $\mathcal{B}$ is clearly linearly independent. Moreover,

[1] see Proposition 138 in Villanacci(20 September, 2012)

$$\mathcal{S} = \left\{ A \in \mathbb{M}(2,2) : \exists a,b,c \in \mathbb{R} \text{ such that } A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right\},$$

and

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

i.e., span $(\mathcal{B})) = \mathcal{S}$, as desired.

28.

a. $[\supseteq]$ Taken $w \in W$,we want to find $w^1, w^2 \in W$ such that $w = w^1 + w^2$.take $w^1 = w$ and $w^2 = 0 \in W$.

$[\subseteq]$ Take $w^1, w^2 \in W$. Then $w^1 + w^2 \in W$ by definition ofvector space.

b. $[\supseteq]$ Let $w \in W$. Then $\frac{1}{\alpha} w \in W$ and $\alpha \left(\frac{1}{\alpha} w \right) w \in \alpha W$.

$[\subseteq]$ It follows from the definition of vector space.

29.

The isomorphism is $\varphi : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^{n+1}$

$$f(t) = \sum_{i=0}^{n} a_i t^i \mapsto (a_i)_{i=0}^{n}.$$

Indeed , φ is linear

...

and defined $\psi : \mathbb{R}^{n+1} \to \mathcal{P}_n(\mathbb{R})$,

$$(a_i)_{i=0}^{n} \to f(t) = \sum_{i=0}^{n} a_i t^i,$$

we have that $\varphi \circ \psi = id_{|\mathcal{P}_n(\mathbb{R})}$ and $\psi \circ \varphi = id_{|\mathbb{R}^{n+1}}$

....

22.2 Some topology in metric spaces

22.2.1 Basic topology in metric spaces

1.

To prove that d' is a metric, we have to check the properties listed in Definition 421.

a. $d'(x,y) \geq 0$, $d'(x,y) = 0 \Leftrightarrow x = y$

By definition of $d'(x,y)$, it is always going to be positive as $d(x,y) \geq 0$. Furthermore, $d'(x,y) = 0 \iff d(x,y) = 0 \iff x = y$.

b. $d'(x,y) = d'(y,x)$

Applying the definition

$$d'(x,y) = d'(y,x) \iff \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)}$$

but $d(x,y) = d(y,x)$ so we have

$$\frac{d(x,y)}{1+d(x,y)} = \frac{d(x,y)}{1+d(x,y)}$$

c. $d'(x,z) \leq d'(x,y) + d'(y,z)$

Applying the definition

$$d'(x,z) \leq d'(x,y) + d'(y,z) \iff \frac{d(x,z)}{1+d(x,z)} \leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$

Multiplying both sides by $[1+d(x,z)][1+d(x,y)][1+d(y,z)]$

$$d(x,z)[1+d(x,y)][1+d(y,z)] \leq d(x,y)[1+d(x,z)][1+d(y,z)] + d(y,z)[1+d(x,z)][1+d(x,y)]$$

Simplifying we obtain

$$\boxed{d(x,z) \leq d(x,y) + d(y,z)} + [[1+d(x,z)][1+d(x,y)][1+d(y,z)] + 2[1+d(x,y)][1+d(y,z)]]$$

which concludes the proof.

2.

It is enough to show that one of the properties defining a metric does not hold.

It can be $d(f,g) = 0$ and $f \neq g$. Take

$$f(x) = 0, \forall x \in [0,1],$$

and

$$g(x) = -2x + 1$$

Then,

$$\int_0^1 (-2x+1)\, dx = 0.$$

It can be $d(f,g) < 0$.Consider the null function and the function that take value 1 for all x in $[0;1]$. Then $d(0,1) = -\int_0^1 1\, dx$. by linearity of the Riemann integral, which is equal to -1. Then, $d(0,1) < 0$.

3.

Define $S = (a_1,b_1) \times (a_2,b_2)$ and take $x^0 := \left(x_1^0, x_2^0\right) \in S$. Then, for $i \in \{1,2\}$, there exist $\varepsilon_i > 0$ such that $x_i^0 \in B\left(x_i^0, \varepsilon_i\right) \subseteq (a_i,b_i)$. Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then, for $i \in \{1,2\}$, $x_i^0 \in B\left(x_i^0, \varepsilon\right) \subseteq (a_i,b_i)$ and, defined $B =.B\left(x_1^0,\varepsilon\right) \times B\left(x_2^0,\varepsilon\right)$, we have that $x^0 \in B \subseteq S$. It then suffices to show that $B\left(x^0,\varepsilon\right) \subseteq B$. Observe that

$$x \in B\left(x^0, \varepsilon\right) \Leftrightarrow d\left(x, x^0\right) < \varepsilon,$$

$$d\left(\left(x_1^0, 0\right), \left(x_1, 0\right)\right) = \sqrt{\left(x_1^0 - x_1\right)^2} = \left|x_1^0 - x_1\right|,$$

and

$$d\left(\left(x_1^0, 0\right), \left(x_1, 0\right)\right) = \sqrt{\left(x_1^0 - x_1\right)} \leq \sqrt{\left(x_1^0 - x_1\right)^2 + \left(x_1^0 - x_1\right)^2} = d\left(x, x^0\right).$$

4.
Show the second equality in Remark 460:

$$\cap_{n=1}^{+\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

5.

$$S = \left\{-1, +\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, ...\right\}$$

The set is not open: it suffices to find $x \in S$ and such that $x \notin Int\ S$; take for example -1. We want to show that it false that

$$\exists \varepsilon > 0 \text{ such that } (-1-\varepsilon, -1+\varepsilon) \subseteq S.$$

In fact, $\forall \varepsilon > 0$, $-1 - \frac{\varepsilon}{2} \in (-1-\varepsilon, -1+\varepsilon)$, but $-1 - \frac{\varepsilon}{2} \notin S$. The set is not closed. It suffices to show that $\mathcal{F}(S)$ is not contained in S, in fact that $0 \notin S$ (obvious) and $0 \in \mathcal{F}(S)$. We want to show that $\forall \varepsilon > 0$, $B(0, \varepsilon) \cap S \neq \varnothing$.In fact, $(-1)^n \frac{1}{n} \in B(0, \varepsilon)$ if n is even and $(-1)^n \frac{1}{n} = \frac{1}{n} < \varepsilon$. It is then enough to take n even and $n > \frac{1}{\varepsilon}$.

6.

$$A = (0, 10)$$

The set is $(\mathbb{R}, d_2)$ open, as a union of infinite collection of open sets. The set is not closed, because A^c is not open. 10 or 0 do not belongs to $Int(A^C)$

7.
The solution immediately follow from Definition of boundary of a set: Let a metric space (X, d) and a set $S \subseteq X$ be given. x is an boundary point of S if
any open ball centered in x intersects both S and its complement in X, i.e., $\forall r \in \mathbb{R}_{++}, \quad B(x, r) \cap S \neq \varnothing \qquad \wedge \qquad B(x, r) \cap S^C \neq \varnothing$.
As you can see nothing changes in definition above if you replace the set with its complement.
8.

$$\begin{aligned} x \in (\mathcal{F}(S))^C \quad & \Leftrightarrow x \notin (\mathcal{F}(S)) \\ & \Leftrightarrow \neg\left(\forall r \in \mathbb{R}_{++}, B(x, r) \cap S \neq \varnothing \ \wedge \ B(x, r) \cap S^C \neq \varnothing\right) \\ & \Leftrightarrow \exists r \in \mathbb{R}_{++} \text{ such that } B(x, r) \cap S = \varnothing \ \vee \ B(x, r) \cap S^C \neq \varnothing \\ & \Leftrightarrow \exists r \in \mathbb{R}_{++} \text{ such that } B(x, r) \subseteq S^C \ \vee \ B(x, r) \subseteq S \\ & \Leftrightarrow x \in Int\ S^C \ \vee x \in Int\ S \\ & \overset{(1)}{\Leftrightarrow} \exists r_x^* \in \mathbb{R}_{++} \text{ such that either a. } B(x, r_x^*) \subseteq Int\ S^C \text{ or b. } B(x, r_x^*) \subseteq Int\ S. \end{aligned} \tag{22.9}$$

where (1) follows from the fact that the Interior of a set is an open set.
If case a. in (22.9) holds true, then, using Lemma 554, $B(x, r_x^*) \subseteq (\mathcal{F}(S))^C$ and similarly for case b., as desired.

9.

	Int S	$\mathrm{Cl}(S)$	$\mathcal{F}(S)$	$D(S)$	$Is(S)$	open or closed
$S=\mathbb{Q}$	$\varnothing$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\varnothing$	neither open nor closed
$S=(0,1)$	$(0,1)$	$[0,1]$	$\{0,1\}$	$[0,1]$	$\varnothing$	open
$S=\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}_+}$	$\varnothing$	$S\cup\{0\}$	$S\cup\{0\}$	$\{0\}$	S	neither open nor closed

10.

a.

Take $S=\mathbb{N}$. Then, *Int* $S=\varnothing$, Cl $(\varnothing)=\varnothing$, and Cl $(Int\ S)=\varnothing\neq\mathbb{N}=S$.

b.

Take $S=\mathbb{N}$. Then, Cl $(S)=\mathbb{N}$, *Int* $\mathbb{N}=\varnothing$, and *Int* Cl $(S)=\varnothing\neq\mathbb{N}=S$.

11.

a.

True. If S is an open bounded interval, then $\exists a,b\in\mathbb{R}$, $a<b$ such that $S=(a,b)$. Take $x\in S$ and $\delta=\min\{|x-a|,|x-b|\}$. Then $I(x,\delta)\subseteq(a,b)$.

b.

False. $(0,1)\cup(2,3)$ is an open set, but it is not an open interval.

c.

False. Take $S:=\{0,1\}$. $0\in\mathcal{F}(S)$, but $0\notin D(S)$

d. .

False. Take $S(0,1)$. $\frac{1}{2}\in D(S)\,0$, but $\frac{1}{2}\notin\mathcal{F}(S)$.

12.

Recall that: A sequence $(x_n)_{n\in\mathbb{N}}\in X^\infty$ is said to be (X,d) convergent to $x_0\in X$ (or convergent with respect to the metric space (X,d)) if $\forall\varepsilon>0,\exists n_0\in\mathbb{N}$ such that $\forall n>n_0,\quad d(x_n,x_0)<\varepsilon$.

a.

$(x_n)_{n\in N}\in\mathbb{R}^\infty$ such that $\forall n\in N,: x_n=1$

Let $\epsilon>0$ then by definition of $(x_n)_{n\in N}$, $\forall n>0$, $d(x_n,1)=0<\epsilon$. So that

$$\lim_{n\to\infty}x_n=1$$

b.

$(x_n)_{n\in N}\in\mathbb{R}^\infty$ such that $\forall n\in N,: x_n=\frac{1}{n}$

Let $\epsilon>0$. Because N is unbounded, $\exists n_0\in N$, such that $n_0>\frac{1}{\epsilon}$. Then $\forall n>n_0$, $d(x_n,0)=\frac{1}{n}<\frac{1}{n_0}<\epsilon$. Then, by definition of a limit, we proved that

$$\lim_{n\to\infty}x_n=0$$

13.

Take $(x_n)_{n\in\mathbb{N}}\in[0,1]^\infty$ such that $x_n\to x_0$; we want to show that $x_0\in[0,1]$. Suppose otherwise, i.e., $x_0\notin[0,1]$.

Case 1. $x_0<0$. By definition of convergence, chosen $\varepsilon=-\frac{x_0}{2}>0$, there exists $n_\varepsilon\in\mathbb{N}$ such that $\forall n>n_\varepsilon$, $d(x_n,x_0)<\varepsilon$, i.e., $|x_n-x_0|<\varepsilon=-\frac{x_0}{2}$, i.e., $x_0+\frac{x_0}{2}<x_n<x_0-\frac{x_0}{2}=\frac{x_0}{2}<0$. Summarizing, $\forall n>n_\varepsilon$, $x_n\notin[0,1]$, contradicting the assumption that $(x_n)_{n\in\mathbb{N}}\in[0,1]^\infty$.

Case 2. $x_0>1$. Similar to case 1.

14.

This is Example 7.15, page 150, Morris (2007):

1. In fact, we have the following result: Let (X,d) be a metric space and $A=\{x_1,...,x_n\}$ any finite subset of X.Then A is compact, as shown below.

Let O_i, $i\in I$ be any family of open sets such that $A\subseteq\cup_{i\in I}O_i$. Then for each $x_j\in A$, there exists O_{i_j} such that $x_j\in O_{i_j}$. Then $A\subseteq O_{i_1}\cup O_{i_2}\cup...\cup O_{i_n}$. Therefore A is compact.

2. Conversely, let A be compact. Then the family of singleton sets $O_x = \{x\}$, $x \in A$ is such that each O_x is open and $A \subseteq \cup_{x\in A} O_x$. Since A is compact, there exists $O_{x_1}, O_{x_2}, ..., O_{x_n}$ such that $A \subseteq O_{x_1} \cup O_{x_2} \cup ... \cup O_{x_n}$, that is, $A \subseteq \{x_1, ..., x_n\}$. Hence, A is finite.

15.
In general it is false. For example in a discrete metric space: see previous exercise.

16.
Take an open ball $B(x,r)$. Consider $\mathcal{S} = \left\{B\left(x, r\left(1-\frac{1}{n}\right)\right)\right\}_{n\in\mathbb{N}\setminus\{0,1\}}$. Observe that S is an open cover of $B(x,r)$; in fact $\cup_{n\in\mathbb{N}\setminus\{0,1\}} B\left(x, r\left(1-\frac{1}{n}\right)\right) = B(x,r)$,as shown below.

[$\subseteq$] $x' \in \cup_{n\in\mathbb{N}\setminus\{0,1\}} B\left(x, r\left(1-\frac{1}{n}\right)\right) \Leftrightarrow \exists n_{x'} \mathbb{N}\setminus\{0,1\}$ such $x \in B\left(x, r\left(1-\frac{1}{n_{x'}}\right)\right) \subseteq B(x,r)$.

[$\supseteq$] Take $x' \in B(x,r)$. Then, $d(x,x') < r$. Take n such that $d(x',x) < r\left(1-\frac{1}{n_{x'}}\right)$, i.e., $n > \frac{r}{r-d(x',x)}$ (and $n>1$), then $x' \in B\left(x, r\left(1-\frac{1}{n}\right)\right)$.

Consider an arbitrary subcover of $\mathcal{S}$, i.e.,

$$\mathcal{S}' = \left\{B\left(x, r\left(1-\frac{1}{n}\right)\right)\right\}_{n\in\mathcal{N}}$$

with $\#\mathcal{N} = N \in \mathbb{N}$. Define $n^* = \min\{n \in \mathcal{N}\}$. Then $\cup_{n\in\mathcal{N}} B\left(x, r\left(1-\frac{1}{n}\right)\right) = B\left(x, r\left(1-\frac{1}{n^*}\right)\right)$, and if $d(x',x) \in \left(r\left(1-\frac{1}{n^*}\right), r\right)$, then $x' \in B(x,r)$ and $x' \notin \cup_{n\in\mathcal{N}} B\left(x, r\left(1-\frac{1}{n}\right)\right)$.

17.
1st proof.
We have to show that $f(A\cup B) \subseteq f(A) \cup f(B)$ and $f(A) \cup f(B) \subseteq f(A\cup B)$.

To prove the first inclusion, take $y \in f(A\cup B)$; then $\exists x \in A\cup B$ such that $f(x) = y$. Then either $x \in A$ or $x \in B$ that implies $f(x) = y \in A$ or $f(x) = y \in B$. In both case $y \in f(A) \cup f(B)$

We now show the opposite e inclusion. Let $y \in f(A) \cup f(B)$, then $y \in f(A)$ or $y \in f(B)$, but $y \in f(A)$ implies that $\exists x \in A$ such that $f(x) = y$. The same implication for $y \in f(B)$. As results, $y = f(x)$ in either case with $x \in A \cup B$ i.e. $y \in f(A\cup B)$.

2nd proof.

$$\begin{aligned}
&y \in f(A\cup B) \Leftrightarrow \\
&\Leftrightarrow \exists x \in A\cup B \text{ such that } f(x) = y \\
&\Leftrightarrow (\exists x \in A \text{ such that } f(x) = y) \vee (\exists x \in B \text{ such that } f(x) = y) \\
&\Leftrightarrow (y \in f(A)) \vee (y \in f(B)) \\
&\Leftrightarrow y \in f(A) \cup f(B)
\end{aligned}$$

18.:
First proof. Take $f = \sin$, $A = [-2\pi, 0]$, $B = [0, 2\pi]$.
Second proof. Consider

$$f : \{0,1\} \to \mathbb{R}, \qquad x \mapsto 1$$

Then take $A = \{0\}$ and $B = \{1\}$. Then $A \cap B = \emptyset$, so $f(A\cap B) = \emptyset$. But as $f(A) = f(B) = \{1\}$, we have that $f(A) \cap f(B) = \{1\} \neq \emptyset$.

19.
Take $c \in \mathbb{R}$ and define the following function

$$f : X \to Y, \quad f(x) = c.$$

It suffices to show that the preimage of every open subset of the domain is open in the codomain. The inverse image of any open set K is either X (if $c \in K$) or $\varnothing$ (if $c \notin K$), which are both open sets.

20.

a.
i. $\mathbb{R}^n_+$ is not bounded, then by Proposition 516 it is not compact.
ii. $Cl\ B(x,r)$ is compact.
From Proposition 516, it suffices to show that the set is closed and bounded.
$Cl\ B(x,r)$ is closed from Proposition 473.
$Cl\ B(x,r)$ is bounded because $Cl\ B(x,r) = \{y \in \mathbb{R}^n :\ d(x,y) \le r\} \subseteq B(x, 2r)$.
Let's show in detail the equality.
i. $Cl\ B(x,r) \subseteq C$.

The function $d_x : \mathbb{R}^n \to \mathbb{R}$, $d_x(y) = d(x,y) := \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{1}{2}}$ is continuous. Therefore, $C = d_x^{-1}([0,r])$ is closed. Since $B(x,r) \subseteq C$, by definition of closure, the desired result follows.

ii. $Cl\ B(x,r) \supseteq C$.

From Corollary 557, it suffices to show that $Ad\ B(x,r) \supseteq C$. If $d(y,x) < r$, we are done. Suppose that $d(y,x) = r$.We want to show that for every $\varepsilon > 0$, we have that $B(x,r) \cap B(y,\varepsilon) \neq \varnothing$.If $\varepsilon > r$, then $x \in B(x,r) \cap B(y,\varepsilon)$. Now take, $\varepsilon \le r$. It is enough to take a point "very close to y inside $B(x,r)$"". For example, we can verify that $z \in B(x,r) \cap B(y,\varepsilon)$, where $z = x + \left(1 - \frac{\varepsilon}{2r}\right)(y - x)\big)$.Indeed,

$$d(x,z) = \left(1 - \frac{\varepsilon}{2r})d(y,x)\right) = (1 - \frac{\varepsilon}{2r})r = r - \frac{\varepsilon}{2} < r,$$

and

$$d(y,z) = \frac{\varepsilon}{2r} d(y,x) = \frac{\varepsilon}{2r} r = \frac{\varepsilon}{2} < \varepsilon.$$

c.
See solution to Exercise 5, where it was shown that S is not closed and therefore using Proposition 516, we can conclude S is not compact.

21.
Observe that given for any $j \in \{1, ..., m\}$, the continuous functions $g_j : \mathbb{R}^n \to \mathbb{R}$ and $g = (g_j)_{j=1}^m$, we can define

$$C := \{x \in \mathbb{R}^n : g(x) \ge 0\}.$$

Then C is closed, because of the following argument:

$C = \cap_{j=1}^m g_j^{-1}([0, +\infty))$; since g_j is continuous, and $[0, +\infty)$ is closed, then $g_j^{-1}([0, +\infty))$ is closed in $\mathbb{R}^n$; then C is closed because intersection of closed sets.

22.
The set is closed, because $X = f^{-1}(\{0\})$.
The set is not compact: take f as the constant function.

23.
Let$V = B_Y(1,\varepsilon)$, be an open ball around the value 1 of the codomain, with $\varepsilon < 1$. $f^{-1}(V) = \{0\} \cup B_X(1,\varepsilon)$ is the union of an open set and a closed set, so is neither open nor closed.

24.
To apply the Extreme Value Theorem, we first have to check if the function to be maximized is continuous. Clearly, the function $\sum_{i=1}^n x_i$ is continuous as is the sum of affine functions. Therefore, to check for the existence of solutions for the problems we only have to check for the compactness of the restrictions.

The first set is closed, because it is the inverse image of the closed set $[0,1]$ via the continuous function $\|\|$. The first set is bounded as well by definition. Therefore the set is compact and the function is continuous, we can apply Extreme Value theorem. The second set is not closed, therefore it is not compact and Extreme Value theorem can not be applied. The third set is unbounded, and therefore it is not compact and the Extreme Value theorem can not be applied.

25.
[$\Leftarrow$]
Obvious.
[$\Rightarrow$]
We want to show that $l \neq 0 \Rightarrow l$ is not bounded, i.e., $\forall M \in \mathbb{R}_{++}, \exists x \in E$ such that $\|l(x)\|_F > M$.

Since $l \neq 0, \exists y \in E \backslash \{0\}$ such that $l(y) \neq 0$. Define $x = \frac{2M}{\|l(x)\|_F} y$. Then

$$\|l\,(y)\|_F = \left\| l\left(\frac{2M}{\|l\,(x)\|_F} x\right)\right\|_F = \frac{2M}{\|l\,(x)\|_F} \|l\,(x)\|_F = 2M > M,$$

as desired.

26.

$a \Rightarrow b$.

Take an arbitrary $\alpha \in \mathbb{R}$; if $\{x \in X : f\,(x) < \alpha\} = f^{-1}\,((-\infty, \alpha)) = \varnothing$, we are done. Otherwise, take $x_0 \in f^{-1}\,((-\infty, \alpha))$, Then, $\alpha - f\,(x_0) := \varepsilon > 0$ and by definition of upper semicontinuity , we have

$$\exists \delta > 0 \text{ such that } d\,(x - x_0) < \delta \Rightarrow f\,(x) < f\,(x_0) + \varepsilon = f\,(x_0) + \alpha - f\,(x_0) = \alpha,$$

i.e., $B\,(x_0, \delta) \subseteq f^{-1}\,((-\infty, \alpha))$, i.e., the desired result.

$b \Leftrightarrow c$.

$\{x \in X : f\,(x) < \alpha\} = X \setminus \{x \in X : f\,(x) \geq \alpha\}$.

$b \Rightarrow a$.

We want to show that

$$\forall x_0 \in X,\ \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } B\,(x_0, \delta) \subseteq f^{-1}\,((-\infty, f\,(x_0) + \varepsilon)).$$

But by assumption $f^{-1}\,((-\infty, f\,(x_0) + \varepsilon))$ is an open set and contains x_0 and therefore the desired result follows.

27.

Take $y \in \{x\} + A$. Then there exists $a \in A$ such that $y = x + a$ and since A is open there exists $\varepsilon > 0$ such that

$$a \in B\,(a, \varepsilon) \subseteq A. \tag{22.10}$$

We want to show that

i. $\{x\} + B\,(a, \varepsilon)$ is an open ball centered at $y = x + a$, i.e., $\{x\} + B\,(a, \varepsilon) = B\,(x + a, \varepsilon)$, and

ii. $B\,(x + a, \varepsilon) \subseteq \{x\} + A$.

i.

[$\subseteq$] $y \in \{x\} + B\,(a, \varepsilon) \Leftrightarrow \exists z \in X$ such that $d\,(z, a) < \varepsilon$ and $y = x + z \Rightarrow d\,(y, x + a) = d\,(x + z, x + a) \overset{Hint}{=} d\,(z, a) < \varepsilon \Rightarrow y \in B\,(x + a, \varepsilon)$.

[$\supseteq$] $y \in B\,(x + a, \varepsilon) \Leftrightarrow d\,(y, x + a) < \varepsilon$. Now since $y = x + (y - x)$ and $d\,(y - x, a) = \|y - x - a\| = \|y - (x + a)\| = d\,(y, x + a) < \varepsilon$, we get the desired conclusion.

ii.

$y \in B\,(x + a, \varepsilon) \Leftrightarrow \|y - (x + a)\| < \varepsilon$. Since $y = x + (y - x)$ and $\|(y - x) - a\| < \varepsilon$, i.e., $y - x \in B\,(a, \varepsilon) \overset{(22.10)}{\subseteq} A$, we get the desired result.

28.

By assumption, for any $i \in \{1, 2\}$ and for any $\{x_i^n\}_n \subseteq K_i$, there exists $x_i \in K_i$ such that, up to a subsequence $x_i^n \to x_i$.Take $\{y^n\} \subseteq K_1 + K_2 = K$. Then $\forall n$, $y^n = x_1^n + x_2^n$ with $x_i^n \in K_i$, $i = 1, 2$.Thus taking converging subsequences of $(x_i^n)_n$, $i \in \{1, 2\}$, we get $y^n \to x_1 + x_2 \in K$ as desired.

29.

a. We want to show that $\forall x_1, x_2 \in E$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Indeed

$$0 = d(f(x_1), f(x_2)) = d(x_1, x_2) \Rightarrow x_1 = x_2.$$

b. It follows from a.

c. We want to show that $\forall x_0 \in E, \forall \varepsilon > 0, \exists \delta > 0$ such that

$$d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \varepsilon.$$

Take $\delta = \varepsilon$. Then,

$$d_1(x, x_0) < \varepsilon \Rightarrow d_2(f(x), f(x_0)) < \varepsilon.$$

22.2.2 Correspondences

1.

Since u is a continuous function, from the Extreme Value Theorem , we are left with showing that for every (p,w) , $\beta(p,w)$ is non empty and compact,i.e., β is non empty valued and compact valued.

$x = \left(\frac{w}{Cp^c}\right)_{c=1}^{C} \in \beta(p,w)$.

$\beta(p,w)$ is closed because is the intersection of the inverse image of two closed sets via continuous functions.

$\beta(p,w)$ is bounded below by zero.

$\beta(p,w)$ is bounded above because for every c, $x^c \le \frac{w-\sum_{c'\neq c} p^{c'}x^{c'}}{p^c} \le \frac{w}{p^c}$, where the first inequality comes from the fact that $px \le w$, and the second inequality from the fact that $p \in \mathbb{R}^C_{++}$ and $x \in \mathbb{R}^C_{+}$.

2.

(a)Consider $x', x'' \in \xi(p,w)$. We want to show that $\forall \lambda \in [0,1]$, $x^\lambda := (1-\lambda)x' + \lambda x'' \in \xi(p,w)$. Observe that $u(x') = u(x'') := u^*$. From the quasiconcavity of u, we have $u\left(x^\lambda\right) \ge u^*$. We are therefore left with showing that $x^\lambda \in \beta(p,w)$, i.e., β is convex valued. To see that, simply, observe that $px^\lambda = (1-\lambda)px' + \lambda px'' \le (1-\lambda)w + \lambda w = w$.

(b) Assume otherwise. Following exactly the same argument as above we have $x', x'' \in \xi(p,w)$, and $px^\lambda \le w$. Since u is strictly quasi concave, we also have that $u\left(x^\lambda\right) > u(x') = u(x'') := u^*$, which contradicts the fact that $x', x'' \in \xi(p,w)$.

3.

We want to show that for every (p,w) the following is true. For every sequence $\{(p_n,w_n)\}_n \subseteq \mathbb{R}^C_{++} \times \mathbb{R}_{++}$ such that

$(p_n,w_n) \to (p,w)$, $x_n \in \beta(p_n,w_n)$, $x_n \to x$,

it is the case that $x \in \beta(p,w)$.

Since $x_n \in \beta(p_n,w_n)$, we have that $p_n x_n \le w_n$. Taking limits of both sides, we get $px \le w$,i.e., $x \in \beta(p,w)$.

4.

(a) We want to show that $\forall y', y'' \in y(p)$, $\forall \lambda \in [0,1]$, it is the case that $y^\lambda := (1-\lambda)y' + \lambda y'' \in y(p)$, i.e., $y^\lambda \in Y$ and $\forall y \in Y$, $py^\lambda \ge py$.

$y^\lambda \in Y$ simply because Y is convex.

$py^\lambda := (1-\lambda)py' + \lambda py'' \overset{y',y'' \in y(p)}{\ge} (1-\lambda)py + \lambda py = py.$

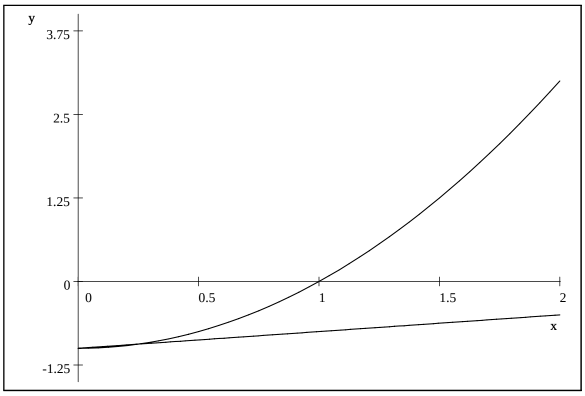
(b)Suppose not; then $\exists y', y'' \in Y$ such that $y' \neq y''$ and such that

$$\forall y \in Y,\ py' = py'' > py \quad (1)\,.$$

Since Y is strictly convex,$\forall \lambda \in (0,1)$, $y^\lambda := (1-\lambda)y' + \lambda y'' \in Int\ Y$. Then, $\exists \varepsilon > 0$ such that $B\left(y^\lambda, \varepsilon\right) \subseteq Y$. Consider $y^* := y^\lambda + \frac{\varepsilon}{2C}\mathbf{1}$, where $\mathbf{1} := (1,...,1) \in \mathbb{R}^C$. $d\left(y^*, y^\lambda\right) = \sqrt{\sum_{c=1}^{C}\left(\frac{\varepsilon}{2C}\right)^2} = \frac{\varepsilon}{2\sqrt{C}}$. Then, $y^* \in B\left(y^\lambda, \varepsilon\right) \subseteq Y$ and, since $p \gg 0$, we have that $py^* > py^\lambda = py' = py''$, contradicting (1) .

5.

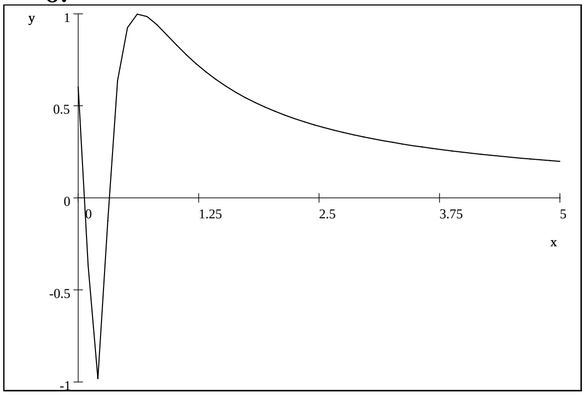
This exercise is taken from Beavis and Dobbs (1990), pages 74-78.



For every $x \in [0,2]$, both $\phi_1(x)$ and $\phi_2(x)$ are closed, bounded intervals and therefore convex and compact sets. Clearly ϕ_1 is closed and ϕ_2 is not closed.

$\phi_1$and ϕ_2 are clearly UHC and LHC for $x \neq 1$. Using the definitions, it is easy to see that for $x = 1$, ϕ_1 is UHC, and not LHC and ϕ_2 is LHC and not UHC.

6.



For every $x > 0$, ϕ is a continuous function. Therefore, for those values of x, ϕ is both UHC and LHC.

ϕ is UHC in 0. For every neighborhood of $[-1,1]$ and for any neighborhood of $\{0\}$ in $\mathbb{R}^+$, $\phi(x) \subseteq [-1,1]$.

ϕ is not LHC in 0. Take the open set $V = \left(\frac{1}{2}, \frac{3}{2}\right)$;we want to show that $\forall \varepsilon > 0\ \exists z^* \in (0,\varepsilon)$ such that $\phi(z^*) \notin \left(\frac{1}{2}, \frac{3}{2}\right)$. Take $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and $z^* = \frac{1}{n\pi}$. Then $0 < z^* < \varepsilon$ and $\sin z^* = \sin n\pi = 0 \notin \left(\frac{1}{2}, \frac{3}{2}\right)$.

Since ϕ is UHC and closed valued, from Proposition 16 is closed.

7.

ϕ is not closed. Take $x_n = \frac{\sqrt{2}}{2n} \in [0,1]$ for every $n \in \mathbb{N}$. Observe that $x_n \to 0$. For every $n \in \mathbb{N}$, $y_n = -1 \in \phi(x_n)$ and $y_n \to -1$. But $-1 \in \phi(0) = [0,1]$.

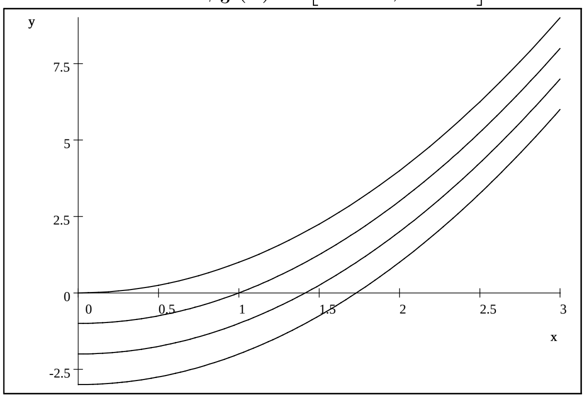
ϕ is not UHC. Take $x = 0$ and a neighborhood $V = \left(-\frac{1}{2}, \frac{3}{2}\right)$ of $\phi(0) = [0,1]$. Then $\forall \varepsilon > 0$, $\exists x^* \in (0,\varepsilon) \setminus \mathbb{Q}$. Therefore, $\phi(x^*) = [-1,0] \nsubseteq V$.

ϕ is not LHC. Take $x = 0$ and the open set $V = \left(\frac{1}{2}, \frac{3}{2}\right)$.Then $\phi(0) \cap \left(\frac{1}{2}, \frac{3}{2}\right) = [0,1] \cap \left(\frac{1}{2}, \frac{3}{2}\right) = \left(\frac{1}{2}, 1\right] \neq \varnothing$. But, as above, $\forall \varepsilon > 0$, $\exists x^* \in (0,\varepsilon) \setminus \mathbb{Q}$. . Then $\phi(x^*) \cap V = [-1,0] \cap \left(\frac{1}{2}, \frac{3}{2}\right) = \varnothing$.

8.

(This exercise is taken from Klein, E. (1973), *Mathematical Methods in Theoretical Economics, Academic Press*, New York, NY, page 119).

Observe that $\phi_3(x) = \left[x^2 - 2, x^2 - 1\right]$.



$\phi_1([0,3]) = \left\{(x,y) \in \mathbb{R}^2 : x \geq 0, \times \leq 3, y \geq x^2 - 2, y \leq x^2\right\}$. $\phi_1([0,3])$ is defined in terms of weak inequalities and continuous functions and it is closed and therefore ϕ_1 is closed. Similar argument applies to ϕ_2 and ϕ_3.

Since $[-10,10]$ is a compact set such that $\phi_1([0,3]) \subseteq [-10,10]$, from Proposition 17, ϕ_1 is UHC. Similar argument applies to ϕ_2 and ϕ_3.

ϕ_1 is LHC. Take an arbitrary $\overline{x} \in [0,3]$ and a open set V with non-empty intersection with $\phi_1(\overline{x}) = \left[\overline{x}^2 - 2, \overline{x}^2\right]$. To fix ideas, take $V = \left[\varepsilon, \overline{x}^2 + \varepsilon\right]$, with $\varepsilon \in \left(0, \overline{x}^2\right)$. Then, take $U = \left(\sqrt{\varepsilon}, \sqrt{\overline{x}^2 + \varepsilon}\right)$. Then for every $x \in U$, $\left\{x^2\right\} \subseteq V \cap \phi_1(x)$.

Similar argument applies to ϕ_2 and ϕ_3.

22.3 Differential Calculus in Euclidean Spaces

1 .

The partial derivative of f with respect to the first coordinate at the point (x_0, y_0), is - if it exists and is finite -

$$\lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

$$\begin{aligned} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} &= \frac{2x^2 - xy_0 + y_0^2 - (2x_0^2 - x_0y_0 + y_0^2)}{x - x_0} \\ &= \frac{2x^2 - 2x_0^2 - (xy_0 - x_0y_0)}{x - x_0} \\ &= 2(x + x_0) - y_0 \end{aligned}$$

Then

$$D_1 f(x_0, y_0) = 4x_0 - y_0$$

The partial derivative of f with respect to the second coordinate at the point (x_0, y_0), is - if it exists and is finite -

$$\lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

$$\begin{aligned} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} &= \frac{2x_0^2 - x_0y + y^2 - (2x_0^2 - x_0y_0 + y_0^2)}{y - y_0} \\ &= \frac{-x_0(y - y_0) + y^2 - y_0^2}{y - y_0} \\ &= -x_0 + (y + y_0) \end{aligned}$$

$$D_2 f(x_0, y_0) = -x_0 + 2y_0.$$

2 .

a.

The domain of f is $\mathbb{R}\backslash\{0\} \times \mathbb{R}$. As arctan is differentiable over the whole domain, we may compute the partial derivative over the whole domain of f at the point (x, y) - we omit from now on the superscript $_0$

$$\begin{aligned} D_1 f(x, y) &= \arctan \frac{y}{x} + x(-\frac{y}{x^2}) \frac{1}{1 + (\frac{y}{x})^2} \\ &= \arctan \frac{y}{x} - y \frac{1}{1 + (\frac{y}{x})^2} \\ D_2 f(x, y) &= x \frac{1}{x} \frac{1}{1 + (\frac{y}{x})^2} = \frac{1}{1 + (\frac{y}{x})^2} \end{aligned}$$

b.

The function is defined on $\mathbb{R}_{++} \times \mathbb{R}$. and

$$\forall (x, y) \in \mathbb{R}_{++} \times \mathbb{R}, \quad f(x, y) = e^{y \ln x}$$

Thus as exp and ln are differentiable over their whole respective domain, we may compute the partial derivatives :

$$\begin{aligned} D_1 f(x, y) &= \frac{y}{x} e^{y \ln x} = y x^{y-1} \\ D_2 f(x, y) &= \ln(x) e^{y ln x} = \ln(x) x^y \end{aligned}$$

c.

$f(x,y) = (\sin(x+y))^{\sqrt{x+y}} = e^{\sqrt{x+y}\ln[\sin(x+y)]}$ in $(0,3)$.
We check that $\sin(0+3) > 0$ so that the point belongs to the domain of the function. Both partial derivatives in (x,y) have the same expression since f is symmetric with respect to x and y.

$$D_1 f(x,y) = D_2 f(x,y) = \left[\frac{1}{2\sqrt{x+y}}\ln[\sin(x+y)] + \sqrt{x+y}\cot(x+y)\right](\sin(x+y))^{\sqrt{x+y}},$$

and

$$D_1 f(0,3) = D_2 f(0,3) = \left[\frac{1}{2\sqrt{3}}\ln[\sin(3)] + \sqrt{3}\tan(3)\right](\sin(3))^{\sqrt{3}}$$

3.

a.

$$D_x f(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0;$$

b.

$$D_y f(0,0) = \lim_{k\to 0}\frac{f(0,k) - f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0;$$

c.

Consider the sequences $(x_n)_{n\in N}$ and $(y_n)_{n\in N}$ such that $\forall n \in N, : x_n = y_n = \frac{1}{n}$. We have that $\lim_{n\to 0}(x_n, y_n) = 0_{\mathbb{R}^2}$, but

$$f(x_n, y_n) = \frac{\frac{1}{n^2}}{\frac{1}{n^2} + \frac{1}{n^2}}.$$

Then

$$\lim_{n\to 0} f(x_n, y_n) = \frac{1}{2} \neq f(0,0) = 0$$

Thus, f is not continuous in $(0,0)$.

4 .

$$f'((1,1);(\alpha_1,\alpha_2)) = \lim_{h\to 0}\frac{f(1+h\alpha_1, 1+h\alpha_2) - f(1,1)}{h} =$$

$$\lim_{h\to 0}\frac{1}{h}\left[\frac{2+h(\alpha_1+\alpha_2)}{(1+h\alpha_1)^2 + (1+h\alpha_2)^2 + 1} - \frac{2}{3}\right] = \cdots$$

$$= -\frac{\alpha_1+\alpha_2}{9}$$

5 .

1st answer.

We will show the existence of a linear function $T_{(x_0,y_0)}(x,y) = a(x-x_0) + b(y-y_0)$ such that the definition of differential is satisfied. After substituting, we want to show that

$$\lim_{(x,y)\to(x_0,y_0)}\frac{|x^2 - y^2 + xy - x_0^2 + y_0^2 - x_0y_0 - a(x-x_0) - b(y-y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

Manipulate the numerator of the above to get $NUM = |(x-x_0)(x+x_0-a+y) - (y-y_0)(y+y_0+b-x_0)|$. Now the ratio R whose limit we are interested to obtain satisfies

$$0 \le R \le \frac{|x-x_0||x+x_0-a+y|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} + \frac{|y-y_0||y+y_0+b-x_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

$$\le |x-x_0||x+x_0-a+y| + |y-y_0||y+y_0+b-x_0|.$$

For $a = 2x_0 + y_0$ and $b = x_0 - 2y_0$ we get the limit of R equal zero as required.

2nd answer.

As a polynomial function, we know that f is $C^1(\mathbb{R}^2)$ so we can "guess" that the derivative at the point (x,y) is the following linear application:

$$\begin{aligned} df_{(x,y)} &: \mathbb{R}^2 \to \mathbb{R} \\ & u \mapsto (2x+y, -2y+x).\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{aligned}$$

In order to comply with the definition, we have to prove that

$$\lim_{u \to 0} \frac{f\left((x,y)+(u_1,u_2)\right) - f\left((x,y)\right) - df_{(x,y)}(u)}{||u||} = 0$$

$$\begin{aligned} f\left((x,y)+(u_1,u_2)\right) - f\left((x,y)\right) - df_{(x,y)}(u) &= (x+u_1)^2 - (y+u_2)^2 + (x+u_1)(y+u_2) \\ &\quad - x^2 + y^2 - xy - (2x+y, -2y+x).\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= 2xu_1 + u_1^2 - 2yu_2 - u_2^2 + xu_2 + yu_1 + u_1u_2 \\ &\quad - (2x+y)u_1 - (-2y+x)u_2 \\ &= u_1^2 - u_2^2 + u_1u_2 \\ &= O(||u||^2) \end{aligned}$$

Or, we know that $\lim_{u \to 0} \frac{O(||u||^2)}{||u||} = 0$. Then we have proven that the candidate was indeed the derivative of f at point (x,y).

6 .
a) given $x_0 \in \mathbb{R}^n$ we need to find $T_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$ linear and E_{x_0} with $\lim_{v \to 0} E_{x_0}(v) = 0$. Take $T_{x_0} = l$ and $E_{x_0} \equiv 0$. Then, the desired result follows.
b) projection is linear, so by a) is differentiable.
7.
From the definition of continuity, we want to show that $\forall x_0 \in \mathbb{R}^n$, $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\|x - x_0\| < \delta \Rightarrow \|l(x) - l(x_0)\| < \varepsilon$. Defined $[l] = A$, we have that

$$\begin{aligned} &\|l(x) - l(x_0)\| = \|A \cdot x - x_0\| = \\ &= \left\|R^1(A) \cdot (x - x_0), ..., R^m(A) \cdot (x - x_0)\right\| \overset{(1)}{\leq} \textstyle\sum_{i=1}^m \left|R^i(A) \cdot (x - x_0)\right| \overset{(2)}{\leq} \\ &\leq \textstyle\sum_{i=1}^m \left\|R^i(A)\right\| \cdot \|x - x_0\| \leq m \cdot \left(\max_{i \in \{1,...,m\}} \left\{\left\|R^i(A)\right\|\right\}\right) \cdot \|x - x_0\|, \end{aligned} \tag{22.11}$$

where (1) follows from Remark 56 and (2) from Proposition 53.4, i.e., Cauchy-Schwarz inequality.Take

$$\delta = \frac{\varepsilon}{m \cdot \left(\max_{i \in \{1,...,m\}} \left\{R^i(A)\right\}\right)}.$$

Then we have that $\|x - x_0\| < \delta$ implies that $\|x - x_0\| \cdot m \cdot \left(\max_{i \in \{1,...,m\}} \left\{R^i(A)\right\}\right) < \varepsilon$, and from (22.11), $\|l(x) - l(x_0)\| < \varepsilon$, as desired.

8 .

$$Df(x,y) = \begin{bmatrix} \cos x \cos y & -\sin x \sin y \\ \cos x \sin y & \sin x \cos y \\ -\sin x \cos y & -\cos x \sin y \end{bmatrix}.$$

9 .

$$Df(x,y,z) = \begin{bmatrix} g'(x)h(z) & 0 & g(x)h'(z) \\ g'(h(x))h'(x)/y & -g(h(x))/y^2 & 0 \\ \exp(xg(h(x))((g(h(x)) + g'(h(x))h'(x)x)) & 0 & 0 \end{bmatrix}.$$

10 .

a.

f is differentiable over its domain since $x \mapsto \log x$ is differentiable over $\mathbb{R}_{++}$. Then from Proposition 705, we know that

$$[df_x] = Df(x) = \begin{bmatrix} \frac{1}{3x_1}, & \frac{1}{6x_2}, & \frac{1}{2x_3} \end{bmatrix}$$

Then

$$[df_{x_0}] = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{4} \end{bmatrix}$$

By application of Remark 706, we have that

$$f'(x_0, u) = Df(x_0).u = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{4} \end{bmatrix} . \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{\sqrt{3}}{4}$$

b.

As a polynomial expression, f is differentiable over its domain.

$$[df_x] = Df(x) = \begin{bmatrix} 2x_1 - 2x_2, & 4x_2 - 2x_1 - 6x_3, & -2x_3 - 6x_2 \end{bmatrix}$$

Then

$$[df_{x_0}] = \begin{bmatrix} 2 & 4 & 2 \end{bmatrix}$$

and

$$f'(x_0, u) = Df(x_0).u = \begin{bmatrix} 2 & 4 & 2 \end{bmatrix} . \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0$$

c.

$$[df_x] = Df(x) = \begin{bmatrix} e^{x_1 x_2} + x_2 x_1 e^{x_1 x_2}, & x_1^2 e^{x_1 x_2} \end{bmatrix}$$

Then

$$[df_{x_0}] = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$f'(x_0, u) = Df(x_0).u = \begin{bmatrix} 1, & 0 \end{bmatrix} . \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2$$

11 .

Then since f is in $C^\infty(\mathbb{R} \backslash \{0\})$, we know that f admits partial derivative *functions* and that these functions admit themselves partial derivatives in (x, y, z). Since, the function is symmetric in its arguments, it is enough to compute explicitly $\frac{\partial^2 f(x,y,z)}{\partial x^2}$.

$$\frac{\partial f}{\partial x}(x, y, z) = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

Then

$$\frac{\partial^2 f}{\partial x^2}(x, y, z) = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} + 3x^2(x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

Then $\forall (x, y, z) \in \mathbb{R}^3 \backslash \{0\}$,

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x, y, z) + \frac{\partial^2 f}{\partial y^2}(x, y, z) + \frac{\partial^2 f}{\partial z^2}(x, y, z) &= -3(x^2 + y^2 + z^2)^{-\frac{3}{2}} + (3x^2 + 3y^2 + 3z^2)(x^2 + y^2 + z^2)^{-\frac{5}{2}} \\ &= -3(x^2 + y^2 + z^2)^{-\frac{3}{2}} + 3(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ &= 0 \end{aligned}$$

12 .

$g, h \in C^2(\mathbb{R}, \mathbb{R}_{++})^2$

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(x, y, z) = \left(\frac{g(x)}{h(z)}, \quad g(h(x)) + xy, \quad \ln(g(x) + h(x)) \right)$$

Since $Im(g) \subseteq \mathbb{R}_{++}$, $(x, y, z) \mapsto \frac{g(x)}{h(z)}$ is differentiable as the ratio of differentiable functions. And since $Im(g+h) \subseteq \mathbb{R}_{++}$ and that ln is differentiable over $\mathbb{R}_{++}$, $x \mapsto \ln(g(x)+h(x))$ is differentiable by proposition 619. Then

$$Df(x,y,z) = \begin{bmatrix} \frac{g'(x)}{h(z)} & 0 & -h'(z)\frac{g(x)}{h(z)^2} \\ h'(x)g'(h(x)) + y & x & 0 \\ \frac{g'(x)+h'(x)}{g(x)+h(x)} & 0 & 0 \end{bmatrix}$$

13 .

a.

Since

$$h(x) = \begin{pmatrix} e^x + g(x) \\ e^{g(x)} + x \end{pmatrix},$$

then

$$[dh_x] = Dh(x) = \begin{pmatrix} e^x + g'(x) \\ g'(x)\, e^{g(x)} + 1 \end{pmatrix},$$

$$[dh_0] = Dh(0) = \begin{pmatrix} 1 + g'(0) \\ g'(0)\, e^{g(0)} + 1 \end{pmatrix},$$

b.

Let us define $l : \mathbb{R} \to \mathbb{R}^2, \quad l(x) = (x, g(x))$. Then $h = f \circ l$. As l is differentiable on $\mathbb{R}$ and f is differentiable on $\mathbb{R}^2$ we may apply the "chain rule".

$$dh_0 = df_{l(0)} \circ dl_0$$

$$[dl_x] = Dl(x) = \begin{bmatrix} 1 \\ g'(x) \end{bmatrix}$$

$$[df_{(x,y)}] = Dl(x,y) = \begin{bmatrix} e^x & 1 \\ 1 & e^y \end{bmatrix}$$

Then

$$[dh_0] = \begin{bmatrix} e^0 & 1 \\ 1 & e^{g(0)} \end{bmatrix} \begin{bmatrix} 1 \\ g'(0) \end{bmatrix} = \begin{bmatrix} 1 + g'(0) \\ 1 + g'(0)e^{g(0)} \end{bmatrix}$$

14 .

$$D_x(b \circ a)(x) = D_y b(y)_{|y=a(x)} \cdot D_x a(x).$$

$$D_y b(y) = \begin{bmatrix} D_{y_1} g(y) & D_{y_2} g(y) & D_{y_3} g(y) \\ D_{y_1} f(y) & D_{y_2} f(y) & D_{y_3} f(y) \end{bmatrix}_{|y=a(x)}$$

$$D_x a(x) = \begin{bmatrix} D_{x_1} f(x) & D_{x_2} f(x) & D_{x_3} f(x) \\ D_{x_1} g(x) & D_{x_2} g(x) & D_{x_3} g(x) \\ 1 & 0 & 0 \end{bmatrix}$$

$$D_x(b \circ a)(x) =$$

$$\begin{bmatrix} D_{y_1} g(f(x),g(x),x_1) & D_{y_2} g(f(x),g(x),x_1) & D_{y_3} g(f(x),g(x),x_1) \\ D_{y_1} f(f(x),g(x),x_1) & D_{y_2} f(f(x),g(x),x_1) & D_{y_3} f(f(x),g(x),x_1) \end{bmatrix} \begin{bmatrix} D_{x_1} f(x) & D_{x_2} f(x) & D_{x_3} f(x) \\ D_{x_1} g(x) & D_{x_2} g(x) & D_{x_3} g(x) \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} D_{y_1} g & D_{y_2} g & D_{y_3} g \\ D_{y_1} f & D_{y_2} f & D_{y_3} f \end{bmatrix} \begin{bmatrix} D_{x_1} f & D_{x_2} f & D_{x_3} f \\ D_{x_1} g & D_{x_2} g & D_{x_3} g \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} D_{y_1}g\cdot D_{x_1}f + D_{y_2}gD_{x_1}g + D_{y_3}g & D_{y_1}g\cdot D_{x_2}f + D_{y_2}g\cdot D_{x_2}g & D_{y_1}g\cdot D_{x_3}f + D_{y_2}g\cdot D_{x_3}g \\ D_{y_1}f\cdot D_{x_1}f + D_{y_2}f\cdot D_{x_1}g + D_{y_3}f & D_{y_1}f\cdot D_{x_2}f + D_{y_2}f\cdot D_{x_2}g & D_{y_1}f\cdot D_{x_3}f + D_{y_2}f\cdot D_{x_3}g \end{bmatrix}.$$

15 .

By the sufficient condition of differentiability, it is enough to show that the function $f \in C^1$. Partial derivatives are $\frac{\partial f}{\partial x} = 2x + y$ and $\frac{\partial f}{\partial y} = -2y + x$ – both are indeed continuous, so f is differentiable.

16 .

i) $f \in C^1$ as $Df(x,y,z) = (1+4xy^2+3yz, 3y^2+4x^2y+3z, 1+3xy+3z^2)$ has continuous entries (everywhere, in particular around (x_0,y_0,z_0))).
ii) $f(x_0,y_0,z_0) = 0$ by direct calculation.
iii) $f'_z = \frac{\partial f}{\partial z}|_{(x_0,y_0,z_0)} = 7 \neq 0$, $f'_y = \frac{\partial f}{\partial y}|_{(x_0,y_0,z_0)} = 10 \neq 0$ and $f'_x = \frac{\partial f}{\partial x}|_{(x_0,y_0,z_0)} = 8 \neq 0$.
Therefore we can apply Implicit Function Theorem around $(x_0,y_0,z_0) = (1,1,1)$ to get

$$\frac{\partial x}{\partial z} = -\frac{f'_z}{f'_x} = -7/8,$$

$$\frac{\partial y}{\partial z} = -\frac{f'_z}{f'_y} = -7/10.$$

17 .

a)

$$Df = \begin{bmatrix} 2x_1 & -2x_2 & 2 & 3 \\ x_2 & x_1 & 1 & -1 \end{bmatrix}$$

and each entry of Df is continuous; then f is C^1. $\det D_x f(x,t) = 2x_1^2 + 2x_2^2 \neq 0$ except for $x_1 = x_2 = 0$. Finally,

$$Dg(t) = -\begin{bmatrix} 2x_1 & -2x_2 \\ x_2 & x_1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = -\frac{1}{2x_1^2 + 2x_2^2} \begin{bmatrix} 2x_1 + 2x_2 & 3x_1 - 2x_2 \\ -2x_2 + 2x_1 & -2x_1 - 3x_2 \end{bmatrix}.$$

b)

$$Df = \begin{bmatrix} 2x_2 & 2x_1 & 1 & 2t_2 \\ 2x_1 & 2x_2 & 2t_1 - 2t_2 & -2t_1 + 2t_2 \end{bmatrix}$$

continuous, $\det D_x f(x,t) = 4x_2^2 - 4x_1^2 \neq 0$ except for $|x_1| = |x_2|$. Finally

$$Dg(t) = -\begin{bmatrix} 2x_2 & 2x_1 \\ 2x_1 & 2x_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2t_2 \\ 2t_1 - 2t_2 & -2t_1 + 2t_2 \end{bmatrix} =$$

$$= -\frac{1}{4x_2^2 - 4x_1^2} \begin{bmatrix} -4x_1t_1 + 4x_1t_2 + 2x_2 & 4x_1t_1 - 4x_1t_2 + 4x_2t_2 \\ -2x_1 + 4x_2t_1 - 4x_2t_2 & -4x_1t_2 - 4x_2t_1 + 4x_2t_2 \end{bmatrix}.$$

c)

$$Df = \begin{bmatrix} 2 & 3 & 2t_1 & -2t_2 \\ 1 & -1 & t_2 & t_1 \end{bmatrix}$$

continuous, $\det D_x f(x,t) = -5 \neq 0$ always. Finally

$$Dg(t) = -\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2t_1 & -2t_2 \\ t_2 & t_1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2t_1 - 3t_2 & 2t_2 - 3t_1 \\ -2t_1 + 2t_2 & 2t_2 + 2t_1 \end{bmatrix}.$$

18.

As an application of the Implicit Function Theorem, we have that

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial(z^3 - xz - y)}{\partial x}}{\frac{\partial(z^3 - xz - y)}{\partial z}} = -\frac{-z}{3z^2 - x}$$

if $3z^2 - x \neq 0$. Then,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial\left(\frac{z(x,y)}{3(z(x,y))^2-x}\right)}{\partial y} = \frac{\frac{\partial z}{\partial y}\left(3z^2 - x\right) - 6\frac{\partial z}{\partial y}\cdot z^2}{\left(3z^2-x\right)^2}$$

Since,

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial\left(z^3-xz-y\right)}{\partial y}}{\frac{\partial\left(z^3-xz-y\right)}{\partial z}} = -\frac{-1}{3z^2-x},$$

we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\frac{1}{3z^2-x}\left(3z^2 - x\right) - 6\frac{1}{3z^2-x}\cdot z^2}{\left(3z^2-x\right)^2} = \frac{-3z^2-x}{\left(3z^2-x\right)^3}$$

19.

As an application of the Implicit Function Theorem, we have that the Marginal Rate of Substitution in (x_0, y_0) is

$$\frac{dy}{dx}_{|(x,y)=(x_0,y_0)} = -\frac{\frac{\partial(u(x,y)-k)}{\partial x}}{\frac{\partial(u(x,y)-k)}{\partial y}}_{|(x,y)=(x_0,y_0)} < 0$$

$$\frac{d^2y}{dx^2} = -\frac{\partial\left(\frac{D_x u(x,y(x))}{D_y u(x,y(x))}\right)}{\partial x} = -\frac{\left(\overset{(-)}{D_{xx}u} + \overset{(+)}{D_{xy}u}\overset{(-)}{\frac{dy}{dx}}\right)\overset{(+)}{D_y u} - \left(\overset{(+)}{D_{xy}u} + \overset{(-)}{D_{yy}u}\overset{(-)}{\frac{dy}{dx}}\right)\overset{(+)}{D_x u}}{\underset{(+)}{\left(D_y u\right)^2}} > 0$$

and therefore the function $y(x)$ describing indifference curves is convex.

20.

Adapt the proof for the case of the derivative of the product of functions from $\mathbb{R}$ to $\mathbb{R}$.

21.

Differentiate both sides of

$$f(ax_1, ax_2) = a^n f(x_1, x_2)$$

with respect to a and then replace a with 1.

22.4 Nonlinear Programming

1.

(*a*)

If $\beta = 0$, then $f(x) = \alpha$. The constant function is concave and therefore pseudo-concave, quasi-concave, not strictly concave.

If $\beta > 0$, $f'(x) = \alpha\beta x^{\beta-1}$, $f''(x) = \alpha\beta(\beta-1)x^{\beta-2}$.

$f''(x) \leq 0 \Leftrightarrow \alpha\beta(\beta-1) \geq 0 \overset{\alpha\geq 0,\beta\geq 0}{\Leftrightarrow} 0 \leq \beta \leq 1 \Leftrightarrow f$ concave $\Rightarrow f$ quasi-concave.

$f''(x) < 0 \Leftrightarrow (\alpha > 0$ and $\beta \in (0,1)) \Rightarrow f$ strictly concave.

(*b*)

The Hessian matrix of f is

$$D^2 f(x) = \begin{bmatrix} \alpha_1\beta_1(\beta_1-1)x^{\beta_1-2} & & 0 \\ & \ddots & \\ 0 & & \alpha_n\beta_n(\beta_n-1)x^{\beta_n-2} \end{bmatrix}$$

.

$D^2 f(x)$ is negative semidefinite $\Leftrightarrow (\forall i,\ \beta_i \in [0,1]) \Rightarrow f$ is concave.

$D^2 f(x)$ is negative definitive $\Leftrightarrow (\forall i,\ \alpha_i > 0$ and $\beta_i \in (0,1)) \Rightarrow f$ is strictly concave.

The border Hessian matrix is

$$B(f(x)) = \begin{bmatrix} 0 & \alpha_1\beta_1 x^{\beta_1-1} & - & \alpha_n\beta_n x^{\beta_n-1} \\ \alpha_1\beta_1 x^{\beta_1-1} & \alpha_1\beta_1(\beta_1-1)x^{\beta_1-2} & & 0 \\ | & & \ddots & \\ \alpha_n\beta_n x^{\beta-1} & 0 & & \alpha_n\beta_n(\beta_n-1)x^{\beta_n-2} \end{bmatrix}$$

.

The determinant of the significant leading principal minors are

$$\det\begin{bmatrix} 0 & \alpha_1\beta_1 x^{\beta_1-1} \\ \alpha_1\beta_1 x^{\beta_1-1} & \alpha_1\beta_1(\beta_1-1)x^{\beta_1-2} \end{bmatrix} = -\alpha_1^2\beta_1^2\left(x^{\beta_1-1}\right)^2 < 0$$

$$\det\begin{bmatrix} 0 & \alpha_1\beta_1 x^{\beta_1-1} & \alpha_2\beta_2 x^{\beta_2-1} \\ \alpha_1\beta_1 x^{\beta_1-1} & \alpha_1\beta_1(\beta_1-1)x^{\beta_1-2} & 0 \\ \alpha_2\beta_2 x^{\beta_1-1} & 0 & \alpha_2\beta_2(\beta_2-1)x^{\beta_2-2} \end{bmatrix} =$$

$$= -\left[\alpha_1\beta_1 x^{\beta_1-1}\alpha_1\beta_1 x^{\beta_1-1}\alpha_2\beta_2(\beta_2-1)x^{\beta_2-2}\right] - \left[\alpha_2\beta_2 x^{\beta_1-1}\alpha_2\beta_2 x^{\beta_2-1}\alpha_1\beta_1(\beta_1-1)x^{\beta_1-2}\right] =$$

$$= -\left(\alpha_1\beta_1\alpha_2\beta_2 x^{\beta_1+\beta_2-4}\right)\left[\alpha_1\beta_1 x^{\beta_2}(\beta_2-1) + \alpha_2\beta_2 x^{\beta_1}(\beta_1-1)\right] =$$

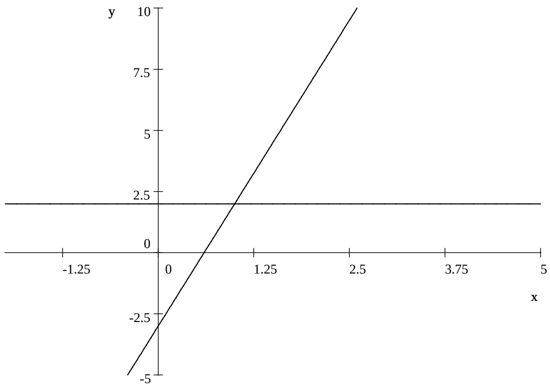
$$= -\alpha_1\beta_1\alpha_2\beta_2 x^{\beta_1+\beta_2-4}\left[\alpha_1\beta_1 x^{\beta_2}(\beta_2-1) + \alpha_2\beta_2 x^{\beta_1}(\beta_1-1)\right] > 0$$

iff for $i = 1, 2,\ \alpha_i > 0$ and $\beta_i \in (0,1)$.

(c)

If $\beta = 0$, then $f(x) = \min\{\alpha, -\gamma\} = 0$.

If $\beta > 0$, we have



The intersection of the two line has coordinates $x^* : \left(= \frac{\alpha+\gamma}{\beta}, \alpha\right)$.

f is clearly not strictly concave, because it is constant in a subset of its domain. Let's show it is concave and therefore pseudo-concave and quasi-concave.

Given $x', x'' \in X$, 3 cases are possible.

Case 1. $x', x'' \leq x^*$.

Case 2. $x', x'' \geq x^*$.

Case 3. $x' \leq x^*$ and $x'' \geq x^*$.

The most difficult case is case 3: we want to show that $(1-\lambda)f(x') + \lambda f(x'') \leq f((1-\lambda)x' + \lambda x'')$.

Then, we have

$$(1-\lambda)f(x') + \lambda f(x'') = (1-\lambda)\min\{\alpha, \beta x' - \gamma\} + \lambda\min\{\alpha, \beta x'' - \gamma\} = (1-\lambda)(\beta x' - \gamma) + \lambda\alpha.$$

Since, by construction $\alpha \geq \beta x' - \gamma$,

$$(1-\lambda)(\beta x' - \gamma) + \lambda\alpha \leq \alpha;$$

since, by construction $\alpha \leq \beta x'' - \gamma$,

$$(1-\lambda)(\beta x' - \gamma) + \lambda\alpha \leq (1-\lambda)(\beta x' - \gamma) + \lambda((\beta x'' - \gamma) = \beta[(1-\lambda)x' + \lambda x''] - \gamma.$$

Then

$$(1-\lambda)f(x') + \lambda f(x'') \leq \min\{\alpha, \beta[(1-\lambda)x' + \lambda x''] - \gamma\} = f((1-\lambda)x' + \lambda x''),$$

as desired.

2.

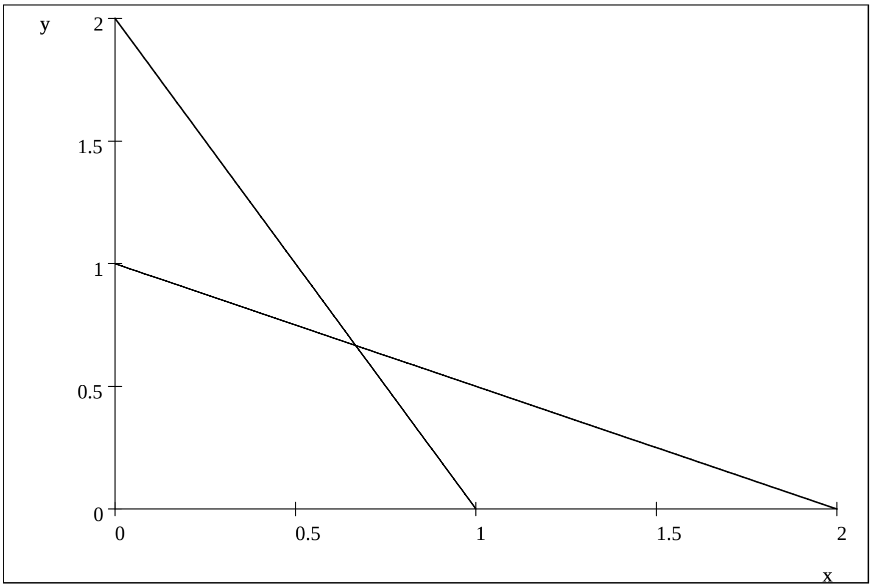
a.

i. Canonical form.

For given $\pi \in (0,1)$, $a \in (0,+\infty)$,

$$\max_{(x,y)\in\mathbb{R}^2} \quad \pi \cdot u(x) + (1-\pi) u(y) \quad \text{s.t.} \quad \begin{array}{ll} a - \frac{1}{2}x - y \geq 0 & \lambda_1 \\ 2a - 2x - y \geq 0 & \lambda_2 \\ x \geq 0 & \lambda_3 \\ y \geq 0 & \lambda_4 \end{array}$$

$$\left\{ \begin{array}{l} y = a - \frac{1}{2}x \\ y = 2a - 2x \end{array} \right., \text{ solution is } \left[x = \frac{2}{3}a, y = \frac{2}{3}a \right]$$



ii. The set X and the functions f and g (X open and convex; f and g at least differentiable).

a. The domain of all function is $\mathbb{R}^2$. Take $X = \mathbb{R}^2$ which is open and convex.

b. $Df(x,y) = (\pi \cdot u'(x), (1-\pi) u'(y))$.The Hessian matrix is

$$\left[\begin{array}{cc} \pi \cdot u''(x) & 0 \\ 0 & (1-\pi) u''(y) \end{array} \right]$$

Therefore, f and g are C^2 functions and f is strictly concave and the functions g^j are affine.

iii. Existence.

C is closed and bounded below by $(0,0)$ and above by (a,a) :

$y \leq a - \frac{1}{2}x \leq a$

$2x \leq 2a - y \leq 2a.$

iv. Number of solutions.

The solution is unique because f is strictly concave and the functions g^j are affine and therefore concave.

v. Necessity of K-T conditions.

The functions g^j are affine and therefore concave.

$x^{++} = \left(\frac{1}{2}a, \frac{1}{2}a\right)$

$$\begin{array}{l} a - \frac{1}{2}\frac{1}{2}a - \frac{1}{2}a = \frac{1}{4}a > 0 \\ 2a - 2\frac{1}{2}a - \frac{1}{2}a = \frac{1}{2}a > 0 \\ \frac{1}{2}a > 0 \\ \frac{1}{2}a > 0 \end{array}$$

vi. Sufficiency of K-T conditions.

The objective function is strictly concave and the functions g^j are affine
vii. K-T conditions.

$$\mathcal{L}(x, y, \lambda_1, ..., \lambda_4; \pi, a) = \pi \cdot u(x) + (1-\pi) u(y) + \lambda_1 \left(a - \frac{1}{2}x - y\right) + \lambda_2 (2a - 2x - y) + \lambda_3 x + \lambda_4 y.$$

$$\begin{cases} \pi \cdot u'(x) - \frac{1}{2}\lambda_1 - 2\lambda_2 + \lambda_3 & = 0 \\ (1-\pi) u'(y) - \lambda_1 - \lambda_2 + \lambda_4 & = 0 \\ \min\left\{\lambda_1, a - \frac{1}{2}x - y\right\} & = 0 \\ \min\{\lambda_2, 2a - 2x - y\} & = 0 \\ \min\{\lambda_3, x\} & = 0 \\ \min\{\lambda_4, y\} & = 0 \end{cases}$$

b. Inserting $(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{2}{3}a, \frac{2}{3}a, \lambda_1, 0, 0, 0\right)$, with $\lambda_1 > 0$, in the Kuhn-Tucker conditions we get:

$$\begin{cases} \pi \cdot u'\left(\frac{2}{3}a\right) - \frac{1}{2}\lambda_1 & = 0 \\ (1-\pi) u'\left(\frac{2}{3}a\right) - \lambda_1 & = 0 \\ a - \frac{1}{2}\frac{2}{3}a - \frac{2}{3}a & = 0 \\ \min\left\{0, 2a - 2\frac{2}{3}a - \frac{2}{3}a\right\} & = 0 \\ \min\left\{0, \frac{2}{3}a\right\} & = 0 \\ \min\left\{0, \frac{2}{3}a\right\} & = 0 \end{cases}$$

and

$$\begin{cases} \lambda_1 & = 2\pi \cdot u'\left(\frac{2}{3}a\right) > 0 \\ \lambda_1 & = (1-\pi) u'\left(\frac{2}{3}a\right) \geq 0 \\ a - \frac{1}{2}\frac{2}{3}a - \frac{2}{3}a & = 0 \\ \min\{0, 0\} & = 0 \\ \min\left\{0, \frac{2}{3}a\right\} & = 0 \\ \min\left\{0, \frac{2}{3}a\right\} & = 0 \end{cases}$$

Therefore, the proposed vector is a solution if

$$2\pi \cdot u'\left(\frac{2}{3}a\right) = (1-\pi) u'\left(\frac{2}{3}a\right) > 0,$$

i.e.,

$$2\pi = 1 - \pi \quad \text{or} \quad \pi = \frac{1}{3} \text{ and for any } a \in \mathbb{R}_{++}.$$

c. If the first, third and fourth constraint hold with a strict inequality, and the multiplier associated with the second constraint is strictly positive, Kuhn-Tucker conditions become:

$$\begin{cases} \pi \cdot u'(x) - 2\lambda_2 & = 0 \\ (1-\pi) u'(y) - \lambda_2 & = 0 \\ a - \frac{1}{2}x - y & > 0 \\ 2a - 2x - y & = 0 \\ x & > 0 \\ y & > 0 \end{cases}$$

$$\begin{cases} \pi \cdot u'(x) - 2\lambda_2 & = 0 \\ (1-\pi) u'(y) - \lambda_2 & = 0 \\ 2a - 2x - y & = 0 \end{cases}$$

	x	y	λ_2	π	a
$\pi \cdot u'(x) - 2\lambda_2$	$\pi \cdot u''(x)$	0	-2	$u'(x)$	0
$(1-\pi) u'(y) - \lambda_2$	0	$(1-\pi) u''(y)$	-1	$-u'(y)$	0
$2a - 2x - y$	-2	-1	0	0	2

$$\det\begin{bmatrix} \pi\cdot u''(x) & 0 & -2 \\ 0 & (1-\pi)\,u''(y) & -1 \\ -2 & -1 & 0 \end{bmatrix} =$$

$$= \pi u''(x)\det\begin{bmatrix} (1-\pi)\,u''(y) & -1 \\ -1 & 0 \end{bmatrix} - 2\det\begin{bmatrix} 0 & -2 \\ (1-\pi)\,u''(y) & -1 \end{bmatrix} =$$

$$= -\pi u''(x) - 4(1-\pi)\,u''(y) > 0$$

$$D_{(\pi,a)}(x,y,\lambda_2) = -\begin{bmatrix} \pi\cdot u''(x) & 0 & -2 \\ 0 & (1-\pi)\,u''(y) & -1 \\ -2 & -1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} u'(x) & 0 \\ -u'(y) & 0 \\ 0 & 2 \end{bmatrix}$$

Using maple:

$$\begin{bmatrix} \pi\cdot u''(x) & 0 & -2 \\ 0 & (1-\pi)\,u''(y) & -1 \\ -2 & -1 & 0 \end{bmatrix}^{-1} =$$

$$= \frac{1}{-\pi u''(x)-4(1-\pi)u''(y)}\begin{bmatrix} -1 & 2 & 2u''(y)-2\pi u''(y) \\ 2 & -4 & \pi u''(x) \\ 2u''(y)-2\pi u''(y) & \pi u''(x) & \pi u''(x)\,u''(y)-\pi^2 u''(x)\,u''(y) \end{bmatrix}$$

$$D_{(\pi,a)}(x,y,\lambda_2) =$$

$$= -\frac{1}{-\pi u''(x)-4(1-\pi)u''(y)}\begin{bmatrix} -1 & 2 & 2u''(y)(1-\pi) \\ 2 & -4 & \pi u''(x) \\ 2u''(y)(1-\pi) & \pi u''(x) & \pi u''(x)\,u''(y)(1-\pi) \end{bmatrix}\begin{bmatrix} u'(x) & 0 \\ -u'(y) & 0 \\ 0 & 2 \end{bmatrix} =$$

$$= \frac{1}{\pi u''(x)+4(1-\pi)u''(y)}\begin{bmatrix} -u'(x)-2u'(y) & 4u''(y)(1-\pi) \\ 2u'(x)-4u'(y) & 2\pi u''(x) \\ 2u''(y)(1-\pi)\cdot u'(x)+\pi u''(x)(-u'(y)) & 2\pi u''(x)\,u''(y)(1-\pi) \end{bmatrix}$$

3.

i. Canonical form.

For given $\pi \in (0,1)$, $w_1, w_2 \in \mathbb{R}_{++}$,

$$\max_{(x,y,m)\in\mathbb{R}^2_{++}\times\mathbb{R}} \quad \pi\log x + (1-\pi)\log y \quad s.t.$$
$$w_1 - m - x \geq 0 \quad \lambda_x$$
$$w_2 + m - y \geq 0 \quad \lambda_y$$

where λ_x and λ_y are the multipliers associated with the first and the second constraint respectively.

ii. The set X and the functions f and g (X open and convex; f and g at least differentiable)

The set $X = \mathbb{R}^2_{++}\times\mathbb{R}$ is open and convex. The constraint functions are affine and therefore $\mathcal{C}^2$. The gradient and the hessian matrix of the objective function are computed below:

	x	y	m
$\pi\log x + (1-\pi)\log y$	$\frac{\pi}{x}$	$\frac{1-\pi}{y}$	0

	x	y	m
$\frac{\pi}{x}$	$-\frac{\pi}{x^2}$	0	0
$\frac{1-\pi}{y}$	0	$-\frac{1-\pi}{y^2}$	0
0	0	0	0

Therefore, the objective function is $\mathcal{C}^2$ and concave, but not strictly concave.

iii. Existence.

The problem has the same solution set as the following problem:

$$\begin{array}{llll} \max_{(x,y,m)\in\mathbb{R}^2_{++}\times\mathbb{R}} & \pi\log x+(1-\pi)\log y & s.t & \\ & & w_1-m-x\geq 0 & \lambda_x \\ & & w_2+m-y\geq 0 & \lambda_y \\ & & \pi\log x+(1-\pi)\log y\geq \pi\log w_1+(1-\pi)\log w_2 & \end{array}$$

whose constraint set is compact (details left to the reader).

iv. Number of solutions.

The objective function is concave and the constraint functions are affine; uniqueness is not insured on the basis of the sufficient conditions presented in the notes.

v. Necessity of K-T conditions.

Constraint functions are affine and therefore pseudo-concave. Choose $(x,y,m)^{++}=\left(\frac{w_1}{2},\frac{w_2}{2},0\right)$.

vi. Sufficiency of Kuhn-Tucker conditions.

f is concave and therefore pseudo-concave and constraint functions are affine and therefore quasi-concave..

vii. K-T conditions.

$$\begin{array}{ll} D_xL=0\Rightarrow & \frac{\pi}{x}-\lambda_x=0 \\ D_yL=0\Rightarrow & \frac{1-\pi}{y}-\lambda_y=0 \\ D_mL=0\Rightarrow & -\lambda_x+\lambda_y=0 \\ & \min\{w_1-m-x,\lambda_x\}=0 \\ & \min\{w_2+m-y,\lambda_y\}=0 \end{array}$$

viii. Solve the K-T conditions.

Constraints are binding: $\lambda_x=\frac{\pi}{x}>0$ and $\lambda_y=\frac{1-\pi}{y}>0$. Then, we get

$$\begin{array}{l} \lambda_x=\frac{\pi}{x} \text{ and } x=\frac{\pi}{\lambda_x} \\ \lambda_y=\frac{1-\pi}{y}\text{and } y=\frac{1-\pi}{\lambda_y} \\ \lambda_x=\lambda_y:=\lambda \\ w_1-m-x=0 \\ w_2+m-y=0 \end{array}$$

$$\begin{array}{l} \lambda_x=\frac{\pi}{x} \text{ and } x=\frac{\pi}{\lambda_x} \\ \lambda_y=\frac{1-\pi}{y}\text{and } y=\frac{1-\pi}{\lambda_y} \\ \lambda_x=\lambda_y \\ w_1-m-\frac{\pi}{\lambda}=0 \\ w_2+m-\frac{1-\pi}{\lambda}=0 \end{array}$$

Then $w_1-\frac{\pi}{\lambda}=-w_2+\frac{1-\pi}{\lambda}$ and $\lambda=\frac{1}{w_1+w_2}$.Therefore

$$\begin{array}{l} \lambda_x=\frac{1}{w_1+w_2} \\ \lambda_y=\frac{1}{w_1+w_2} \\ x=\pi(w_1+w_2) \\ y=(1-\pi)(w_1+w_2) \end{array}$$

b., c.

Computations of the desired derivatives are straightforward.

4.

i. Canonical form.

$$\begin{array}{llll} \max_{(x,y)\in\mathbb{R}^2} & -x^2-y^2+4x+6y & s.t. & -x-y+6\geq 0 & \lambda_1 \\ & & & 2-y\geq 0 & \lambda_2 \\ & & & x\geq 0 & \mu_x \\ & & & y\geq 0. & \mu_y \end{array}$$

ii. The set X and the functions f and g (X open and convex; f and g at least differentiable)

$X = \mathbb{R}^2$ is open and convex. The constraint functions are affine and therefore C^2. The gradient and hessian matrix of the objective function are computed below.

	x	y
$-x^2 - y^2 + 4x + 6y$	$-2x+4$	$-2y+6$

,

	x	y
$-2x+4$	-2	0
$-2y+6$	0	-2

Therefore the objective function is C^2 and strictly concave.

iii. Existence.

The constraint set C is nonempty (0 belongs to it) and closed. It is bounded below by 0. y is bounded above by 2. x is bounded above because of the first constraint:: $x \leq 6 - y \overset{y \geq 0}{\leq} 6$. Therefore C is compact.

iv. Number of solutions.

Since the objective function is strictly concave (and therefore strictly quasi-concave) and the constraint function are affine and therefore quasi-concave, the solution is unique.

v. Necessity of K-T conditions.

Constraints are affine and therefore pseudo-concave. Take $(x^{++}, y^{++}) = (1, 1)$.

vi. Sufficiency of K-T conditions.

The objective function is strictly concave and therefore pseudo-concave. Constraints are affine and therefore quasi-concave.

vii. K-T conditions.

$\mathcal{L}(x, y, \lambda_1, \lambda_2, \mu_x, \mu_y) = -x^2 - y^2 + 4x + 6y + \lambda_1 \cdot (-x - y + 6) + +\lambda_2 \cdot (2 - y) + \mu_x x + \mu_y y.$

$$
\begin{array}{l}
-2x + 4 - \lambda_1 + \mu_x = 0 \\
-2y + 6 - \lambda_1 - \lambda_2 + \mu_y = 0 \\
\min\{-x - y + 6, \lambda_1\} = 0 \\
\min\{2 - y, \lambda_2\} = 0 \\
\min\{x, \mu_x\} = 0 \\
\min\{y, \mu_y\} = 0
\end{array}
$$

b.

$$
\begin{array}{l}
+4 - \lambda_1 + \mu_x = 0 \\
-2y + 6 - \lambda_2 + \mu_y = 0 \\
\min\{-y + 6, \lambda_1\} = 0 \\
\min\{2 - y, \lambda_2\} = 0 \\
\min\{0, \mu_x\} = 0 \\
\min\{y, \mu_y\} = 0
\end{array}
$$

Since $y \leq 2$, we get $-y + 6 > 0$ and therefore $\lambda_1 = 0$. But then $\mu_x = -4$, which contradicts the Kuhn-Tucker conditions above.

c.

$$
\begin{array}{l}
-4 + 4 - \lambda_1 + \mu_x = 0 \\
-4 + 6 - \lambda_2 + \mu_y = 0 \\
\min\{-4 + 6, \lambda_1\} = 0 \\
\min\{2 - 2, \lambda_2\} = 0 \\
\min\{2, \mu_x\} = 0 \\
\min\{2, \mu_y\} = 0
\end{array}
$$

$$
\begin{array}{l}
-\lambda_1 + \mu_x = 0 \\
+2 - \lambda_2 + \mu_y = 0 \\
\lambda_1 = 0 \\
\min\{0, \lambda_2\} = 0 \\
\mu_x = 0 \\
\mu_y = 0
\end{array}
$$

$$\begin{array}{l} \mu_x = 0 \\ +2 - \lambda_2 = 0 \\ \lambda_1 = 0 \\ \min\{0, \lambda_2\} = 0 \\ \mu_x = 0 \\ \mu_y = 0 \end{array}$$

$$\begin{array}{l} \mu_x = 0 \\ \lambda_2 = 2 \\ \lambda_1 = 0 \\ \mu_x = 0 \\ \mu_y = 0 \end{array}$$

Therefore $(x^*, y^*, \lambda_1^*, \lambda_2^*, \mu_x^*, \mu_y^*) = (2, 2, 0, 2, 0)$ is a solution to the Kuhn-Tucker conditions
5 .[2]

5. a.

i. Canonical form.

For given $\delta \in (0,1), e \in \mathbb{R}_{++}$,

$$\begin{array}{lll} \max_{(c_1,c_2,k)\in\mathbb{R}^3} & u(c_1) + \delta u(c_2) & \\ & s.t. & \\ & e - c_1 - k \geq 0 & \lambda_1 \\ & f(k) - c_2 \geq 0 & \lambda_2 \\ & c_1 \geq 0 & \mu_1 \\ & c_2 \geq 0 & \mu_2 \\ & k \geq 0. & \mu_3 \end{array}$$

ii. The set X and the functions f and g (X open and convex; f and g at least differentiable).

$X = \mathbb{R}^3$ is open and convex. Let's compute the gradient and the Hessian matrix of the second constraint:

	c_1	c_2	k
$f(k) - c_2$	0	-1	$f'(k)$

	c_1	c_2	k
0	0	0	0
-1	0	0	0
$f'(k)$	0	0	$f''(k) < 0$

Therefore the second constraint function is C^2 and concave; the other constraint functions are affine.

Let's compute the gradient and the Hessian matrix of the objective functions:

	c_1	c_2	k
$u(c_1) + \delta u(c_2)$	$u'(c_1)$	$\delta u'(c_2)$	0

	c_1	c_2	k
$u'(c_1)$	$u''(c_1)$	0	0
$\delta u'(c_2)$	0	$\delta u''(c_2)$	0
0	0	0	0

Therefore the objective function is C^2 and concave.

[2] Exercise 7 and 8 are taken from David Cass' problem sets for his Microeconomics course at the University of Pennsylvania.

iii. Existence.

The objective function is continuous on $\mathbb{R}^3$.

The constraint set is closed because inverse image of closed sets via continuous functions. It is bounded below by 0. It is bounded above: suppose not then

if $c_1 \to +\infty$, then from the first constraint it must be $k \to -\infty$, which is impossible;

if $c_2 \to +\infty$, then from the second constraint and the fact that $f' > 0$, it must be $k \to +\infty$, violating the first constraint;

if $k \to +\infty$, then the first constraint is violated.

Therefore, as an application of the Extreme Value Theorem, a solution exists.

iv. Number of solutions.

Since the objective function is concave and the constraint functions are either concave or affine, uniqueness is not insured on the basis of the sufficient conditions presented in the notes.

v. Necessity of K-T conditions.

The constraints are affine or concave. Take $\left(c_1^{++}, c_2^{++}, k^{++}\right) = \left(\frac{e}{4}, \frac{1}{2} f\left(\frac{e}{4}\right), \frac{e}{4}\right)$. Then the constraints are verified with strict inequality.

vi. Sufficiency of K-T conditions.

The objective function is concave and therefore pseudo-concave. The constraint functions are either concave or affine and therefore quasi-concave.

vii. K-T conditions.

$\mathcal{L}(c_1, c_2, k, \lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3) := u(c_1) + \delta u(c_2) + \lambda_1 (e - c_1 - k) + \lambda_2 (f(k) - c_2) + \mu_1 c_1 + \mu_2 c_2 + \mu_3 k$.

$$\begin{cases} u'(c_1) - \lambda_1 + \mu_1 = 0 \\ \delta u'(c_2) - \lambda_2 + \mu_2 = 0 \\ -\lambda_1 + \lambda_2 f'(k) + \mu_3 = 0 \\ \min\{e - c_1 - k, \lambda_1\} = 0 \\ \min\{f(k) - c_2, \lambda_2\} = 0 \\ \min\{c_1, \mu_1\} = 0 \\ \min\{c_2, \mu_2\} = 0 \\ \min\{k, \mu_3\} = 0 \end{cases}$$

viii. Solve the K-T conditions.

Since we are looking for positive solution we get

$$\begin{cases} u'(c_1) - \lambda_1 = 0 \\ \delta u'(c_2) - \lambda_2 = 0 \\ -\lambda_1 + \lambda_2 f'(k) = 0 \\ \min\{e - c_1 - k, \lambda_1\} = 0 \\ \min\{f(k) - c_2, \lambda_2\} = 0 \\ \mu_1 = 0 \\ \mu_2 = 0 \\ \mu_3 = 0 \end{cases}$$

$$\begin{cases} u'(c_1) - \lambda_1 = 0 \\ \delta u'(c_2) - \lambda_2 = 0 \\ -\lambda_1 + \lambda_2 f'(k) = 0 \\ e - c_1 - k = 0 \\ f(k) - c_2 = 0 \end{cases}$$

Observe that from the first two equations of the above system, $\lambda_1, \lambda_2 > 0$.

5. b.

i. Canonical form.

For given $p > 0$, $w > 0$ and $\bar{l} > 0$,

$$\begin{array}{ll} \max_{(x,l) \in \mathbb{R}^2} & u(x, l) \\ & s.t. \\ & -px - wl + w\bar{l} \geq 0 \\ & \bar{l} - l \geq 0 \\ & x \geq 0 \\ & l \geq 0 \end{array}$$

ii. The set X and the functions f and g (X open and convex; f and g at least differentiable).

$X = \mathbb{R}^2$ is open and convex.

The constraint functions are affine and therefore C^2. The objective function is C^2 and differentiably strictly quasi concave by assumption.

iii. Existence.

The objective function is continuous on $\mathbb{R}^3$.

The constraint set is closed because inverse image of closed sets via continuous functions. It is bounded below by 0. It is bounded above: suppose not then

if $x \to +\infty$, then from the first constraint ($px+wl = w\bar{l}$),it must be $l \to -\infty$, which is impossible. Similar case is obtained, if $l \to +\infty$.

Therefore, as an application of the Extreme Value Theorem, a solution exists.

iv. Number of solutions.

The budget set is convex. The function is differentiably strictly quasi concave and therefore strictly quasi-concave and the solution is unique.

v. Necessity of K-T conditions.

The constraints are pseudo-concave $(x^{++}, l^{++}) = (\frac{w\bar{l}}{3p}, \frac{\bar{l}}{3})$ satisfies the constraints with strict inequalities.

vi. Sufficiency of K-T conditions.

The objective function is differentiably strictly quasi-concave and therefore pseudo-concave. The constraint functions are quasi-concave

vii. K-T conditions.

$\mathcal{L}\left(c_1, c_2, k, \lambda_1, \lambda_2, \lambda_3, \lambda_4; p, w, \bar{l}\right) := u\left(x, l\right) + \lambda_1\left(-px - wl + w\bar{l}\right) + \lambda_2\left(\bar{l} - l\right) + \lambda_3 x + \lambda_4 l.$

$$\begin{array}{l} D_x u - \lambda_1 p + \lambda_3 = 0 \\ D_l u - \lambda_1 w - \lambda_2 + \lambda_4 = 0 \\ \min\left\{-px - wl + w\bar{l}, \lambda_1\right\} = 0 \\ \min\left\{\bar{l} - l, \lambda_2\right\} = 0 \\ \min\left\{x, \lambda_3\right\} = 0 \\ \min\left\{l, \lambda_4\right\} = 0 \end{array}$$

viii. Solve the K-T conditions.

Since we are looking for solutions at which $x > 0$ and $0 < l < \bar{l}$, we get

$$\begin{array}{l} D_x u - \lambda_1 p + \lambda_3 = 0 \\ D_l u - \lambda_1 w - \lambda_2 + \lambda_4 = 0 \\ \min\left\{-px - wl + w\bar{l}, \lambda_1\right\} = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \\ \lambda_4 = 0 \end{array}$$

$$\begin{array}{l} D_x u - \lambda_1 p = 0 \\ D_l u - \lambda_1 w = 0 \\ \min\left\{-px - wl + w\bar{l}, \lambda_1\right\} = 0 \end{array}$$

and then $\lambda_1 > 0$ and

$$\left\{\begin{array}{l} D_x u - \lambda_1 p = 0 \\ D_l u - \lambda_1 w = 0 \\ -px - wl + w\bar{l} = 0 \end{array}\right. .$$

7.

a.

Let's apply the Implicit Function Theorem (:= IFT) to the conditions found in Exercise 7.(a). Writing them in the usual informal way we have:

	c_1	c_2	k	λ_1	λ_2	e	a
$u'(c_1)-\lambda_1=0$	$u''(c_1)$			-1			
$\delta u'(c_2)-\lambda_2=0$		$\delta u''(c_2)$			-1		
$-\lambda_1+\lambda_2 f'(k)=0$			$\lambda_2 f''(k)$	-1	$f'(k)$		$\lambda_2 \alpha k^{\alpha-1}$
$e-c_1-k=0$	-1		-1			1	
$f(k)-c_2=0$		-1	$f'(k)$				k^{α}

To apply the IFT, we need to check that the following matrix has full rank

$$M:=\begin{bmatrix} u''(c_1) & & & -1 & \\ & \delta u''(c_2) & & & -1 \\ & & \lambda_2 f''(k) & -1 & f'(k) \\ -1 & & -1 & & \\ & -1 & f'(k) & & \end{bmatrix}$$

.

Suppose not then there exists $\Delta := (\Delta c_1, \Delta c_2, \Delta k, \Delta\lambda_1, \Delta\lambda_2) \neq 0$ such that $M\Delta = 0$, i.e.,

$$\begin{cases} u''(c_1)\Delta c_1 + & & & -\Delta\lambda_1 & & =0 \\ & \delta u''(c_2)\Delta c_2 + & & & -\Delta\lambda_2 & =0 \\ & & \lambda_2 f''(k)\Delta k + & -\Delta\lambda_1 + & f'(k)\Delta\lambda_2 & =0 \\ -\Delta c_1 + & & -\Delta k & & & =0 \\ & -\Delta c_2 + & f'(k)\Delta k + & & & =0 \end{cases}$$

Recall that

$$[M\Delta = 0 \Rightarrow \Delta = 0] \quad iff \quad M \quad \text{has full rank.}$$

The idea of the proof is either you prove directly $[M\Delta = 0 \Rightarrow \Delta = 0]$, or you 1. assume $M\Delta = 0$ and $\Delta \neq 0$ and you get a contradiction.

If we define $\Delta c := (\Delta c_1, \Delta c_2)$, $\Delta\lambda := (\Delta\lambda_1, \Delta\lambda_2)$, $D^2 := \begin{bmatrix} u''(c_1)\Delta c_1 & \\ & \delta u''(c_2)\Delta c_2 \end{bmatrix}$, the above system can be rewritten as

$$\begin{cases} D^2\Delta c + & & -\Delta\lambda & =0 \\ & \lambda_2 f''(k)\Delta k + & [-1, f'(k)]\Delta\lambda & =0 \\ -\Delta c + & \begin{bmatrix} -1 \\ f'(k) \end{bmatrix}\Delta k & & =0 \end{cases}.$$

$$\begin{cases} \Delta c^T D^2 \Delta c + & & -\Delta c^T \Delta\lambda & =0 & (1) \\ & \Delta k \lambda_2 f''(k)\Delta k + & \Delta k[-1, f'(k)]\Delta\lambda & =0 & (2) \\ -\Delta\lambda^T\Delta c + & \Delta\lambda^T \begin{bmatrix} -1 \\ f'(k) \end{bmatrix}\Delta k & & =0 & (3) \end{cases}$$

.

$\Delta c^T D^2 \Delta c \overset{(1)}{=} \Delta c^T \Delta\lambda = -\Delta\lambda^T \Delta c \overset{(3)}{=} \Delta\lambda^T \begin{bmatrix} -1 \\ f'(k) \end{bmatrix} \Delta k \overset{(2)}{=} \Delta k[-1, f'(k)]\Delta\lambda =$
$= \Delta k \lambda_2 f''(k)\Delta k > 0,$

while $\Delta c^T D^2 \Delta c = (\Delta c_1)^2 u''(c_1) + (\Delta c_2)^2 \delta u''(c_1) < 0$. since we got a contradiction, M has full rank.

Therefore, in a neighborhood of the solution we have

$$D_{(e,a)}(c_1, c_2, k, \lambda_1, \lambda_2) = -\begin{bmatrix} u''(c_1) & & & -1 & \\ & \delta u''(c_2) & & & -1 \\ & & \lambda_2 f''(k) & -1 & f'(k) \\ -1 & & -1 & & \\ & -1 & f'(k) & & \end{bmatrix}^{-1} \begin{bmatrix} & \\ & \\ & \lambda_2\alpha k^{\alpha-1} \\ 1 & \\ & k^{\alpha} \end{bmatrix}.$$

To compute the inverse of the above matrix, we can use the following fact about the inverse of partitioned matrix (see Goldberger, (1964), page 27:

Let A be an $n \times n$ nonsingular matrix partitioned as

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

where $E_{n_1 \times n_1}$, $F_{n_1 \times n_2}$, $G_{n_2 \times n_1}$, $H_{n_2 \times n_2}$ and $n_1 + n_2 = n$. Suppose that E and $D := H - GE^{-1}F$ are non singular. Then

$$A^{-1} = \begin{bmatrix} E^{-1}\left(I + FD^{-1}GE^{-1}\right) & -E^{-1}FD^{-1} \\ -D^{-1}GE^{-1} & D^{-1} \end{bmatrix}.$$

In fact, using Maple, with obviously simplified notation, we get

$$\begin{bmatrix} u_1 & 0 & 0 & -1 & 0 \\ 0 & \delta u_2 & 0 & 0 & -1 \\ 0 & 0 & \lambda_2 f_2 & -1 & f_1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & f_1 & 0 & 0 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix}
\frac{1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{\delta u_2 f_1^2+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\delta u_2 \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} \\
-\frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \frac{f_1^2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -f_1 \frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{u_1+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} \\
-\frac{1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \frac{1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \delta u_2 \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} \\
-\frac{\delta u_2 f_1^2+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -f_1 \frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -u_1 \frac{\delta u_2 f_1^2+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\delta u_2 f_1 \frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} \\
-\delta u_2 \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{u_1+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \delta u_2 \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\delta u_2 f_1 \frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\delta u_2 \frac{u_1+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2}
\end{bmatrix}$$

Therefore,

$$\begin{bmatrix} D_e c_1 & D_a c_1 \\ D_e c_2 & D_a c_2 \\ D_e k & D_a k \\ D_e \lambda_1 & D_a \lambda_1 \\ D_e, \lambda_2 & D_a, \lambda_2 \end{bmatrix} = \begin{bmatrix}
\frac{\delta u_2 f_1+\lambda_2 f_2}{u_1+\delta u_2 f_1+\lambda_2 f_2} & \frac{\lambda_2 \alpha k^{\alpha-1}+\delta u_2 f_1 k^{\alpha}}{u_1+\delta u_2 f_1+\lambda_2 f_2} \\
f_1 \frac{u_1}{u_1+\delta u_2 f_1+\lambda_2 f_2} & \frac{-f_1 \lambda_2 \alpha k^{\alpha-1}+k^{\alpha} u_1+k^{\alpha}\lambda_2 f_2}{u_1+\delta u_2 f_1+\lambda_2 f_2} \\
\frac{u_1}{u_1+\delta u_2 f_1+\lambda_2 f_2} & -\frac{\lambda_2 \alpha k^{\alpha-1}+\delta u_2 f_1 k^{\alpha}}{u_1+\delta u_2 f_1+\lambda_2 f_2} \\
u_1 \frac{\delta u_2 f_1+\lambda_2 f_2}{u_1+\delta u_2 f_1+\lambda_2 f_2} & u_1 \frac{\lambda_2 \alpha k^{\alpha-1}+\delta u_2 f_1 k^{\alpha}}{u_1+\delta u_2 f_1+\lambda_2 f_2} \\
\delta u_2 f_1 \frac{u_1}{u_1+\delta u_2 f_1+\lambda_2 f_2} & \delta u_2 \frac{-f_1 \lambda_2 \alpha k^{\alpha-1}+k^{\alpha} u_1+k^{\alpha}\lambda_2 f_2}{u_1+\delta u_2 f_1+\lambda_2 f_2}
\end{bmatrix}.$$

Then $D_e c_1 = \frac{\delta u_2 f_1+\lambda_2 f_2}{u_1+\delta u_2 f_1+\lambda_2 f_2} := \frac{\overset{+}{\delta}\,\overset{-}{u''}(c_2)\overset{+}{f'}+\overset{+}{\lambda_2}\overset{-}{f''}}{\underset{-}{u''}(c_1)+\underset{+}{\delta}\,\underset{-}{u''}(c_2)\underset{+}{f'}+\underset{+}{\lambda_2}\underset{-}{f''}}" = \frac{-}{-}" > 0$

$D_e c_2 = f_1 \frac{u_1}{u_1+\delta u_2 f_1+\lambda_2 f_2} := \frac{\overset{+}{f'}\overset{-}{u''}(c_1)}{\underset{-}{u''}(c_1)+\underset{+}{\delta}\,\underset{-}{u''}(c_2)\underset{+}{f'}+\underset{+}{\lambda_2}\underset{-}{f''}}" = \frac{-}{-}" > 0$

$D_a k = -\frac{\lambda_2 \alpha k^{\alpha-1}+\delta u_2 f_1 k^{\alpha}}{u_1+\delta u_2 f_1+\lambda_2 f_2} := -\frac{\overset{+}{\lambda_2}\alpha \overset{+}{k^{\alpha-1}}+\overset{+}{\delta}\,\overset{-}{u''}(c_2)\overset{+}{f'}\overset{+}{k^{\alpha}}}{\underset{-}{u''}(c_1)+\underset{+}{\delta}\,\underset{-}{u''}(c_2)\underset{+}{f'}+\underset{+}{\lambda_2}\underset{-}{f''}}$,

which has sign equal to $sign\left(\overset{+}{\lambda_2}\alpha \overset{+}{k^{\alpha-1}}+\overset{+}{\delta}\,\overset{-}{u''}(c_2)\overset{+}{f'}\overset{+}{k^{\alpha}}\right)$.

b.

Let's apply the Implicit Function Theorem to the conditions found in a previous exercise. Writing them in the usual informal way we have:

$$\begin{array}{lccccccc}
 & x & l & \lambda_1 & p & w & \bar{l} & \\
 & & & & & & & \\
D_x u - \lambda_1 p = 0 & D_x^2 & D_{xl}^2 & -p & -\lambda_1 & & & \\
D_l u - \lambda_1 w = 0 & D_{xl}^2 & D_l^2 & -w & & -\lambda_1 & & \\
-px - wl + w\bar{l} = 0 & -p & -w & & -1 & \bar{l} - l & w &
\end{array}$$

To apply the IFT, we need to check that the following matrix has full rank

$$M := \begin{bmatrix} D_x^2 & D_{xl}^2 & -p \\ D_{xl}^2 & D_l^2 & -w \\ -p & -w & \end{bmatrix}$$

.

Defined $D^2 := \begin{bmatrix} D_x^2 & D_{xl}^2 \\ D_{xl}^2 & D_l^2 \end{bmatrix}$, $q := \begin{bmatrix} -p \\ -w \end{bmatrix}$, we have $M := \begin{bmatrix} D^2 & -q \\ -q^T & \end{bmatrix}$.

Suppose not then there exists $\Delta := (\Delta y, \Delta\lambda) \in \left(\mathbb{R}^2 \times \mathbb{R}\right) \setminus \{0\}$ such that $M\Delta = 0$, i.e.,

$$\begin{cases} D^2\Delta y & -q\Delta\lambda & = 0 & (1) \\ -q^T\Delta y & & = 0 & (2) \end{cases}$$

We are going to show

Step 1. $\Delta y \neq 0$; Step 2. $Du \cdot \Delta y = 0$; Step 3. It is not the case that $\Delta y^T D^2 \Delta y < 0$.

These results contradict the assumption about u.

Step 1.

Suppose $\Delta y = 0$. Since $q \gg 0$, from (1), we get $\Delta\lambda = 0$, and therefore $\Delta = 0$, a contradiction.

Step 2.

From the First Order Conditions, we have

$Du - \lambda_1 q = 0 \quad (3)$.

$Du\Delta y \overset{(3)}{=} \lambda_1 q \Delta y \overset{(2)}{=} 0.$

Step 3.

$\Delta y^T D^2 \Delta y \overset{(1)}{=} \Delta y^T q \Delta\lambda \overset{(2)}{=} 0.$

Therefore, in a neighborhood of the solution we have

$$D_{(p,w,\bar{l})}(x, l, \lambda_1) = -\begin{bmatrix} D_x^2 & D_{xl}^2 & -p \\ D_{xl}^2 & D_l^2 & -w \\ -p & -w & \end{bmatrix}^{-1} \begin{bmatrix} -\lambda_1 & & \\ & -\lambda_1 & \\ -1 & \bar{l} - l & w \end{bmatrix}.$$

Unfortunately, here we cannot use the formula seen in the Exercise 4 (a) because the Hessian of the utility function is not necessarily nonsingular. We can invert the matrix using the definition of inverse. (For the inverse of a partitioned matrix with this characteristics see also Dhrymes, P. J., (1978), *Mathematics for Econometrics*, 2nd edition, Springer-Verlag, New York, NY, Addendum pages 142-144.

With obvious notation and using Maple, we get

$$\begin{bmatrix} d_x & d & -p \\ d & d_l & -w \\ -p & -w & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{w^2}{d_x w^2 - 2dpw + p^2 d_l} & -p\frac{w}{d_x w^2 - 2dpw + p^2 d_l} & -\frac{-dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} \\ -p\frac{w}{d_x w^2 - 2dpw + p^2 d_l} & \frac{p^2}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} \\ -\frac{-dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x d_l + d^2}{d_x w^2 - 2dpw + p^2 d_l} \end{bmatrix}$$

Therefore,

$D_{(p,w,\bar{l})}(x, l, \lambda_1) =$

$$-\begin{bmatrix} \frac{w^2}{d_x w^2 - 2dpw + p^2 d_l} & -p\frac{w}{d_x w^2 - 2dpw + p^2 d_l} & -\frac{-dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} \\ -p\frac{w}{d_x w^2 - 2dpw + p^2 d_l} & \frac{p^2}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} \\ -\frac{-dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x d_l + d^2}{d_x w^2 - 2dpw + p^2 d_l} \end{bmatrix} \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & 0 \\ -1 & \bar{l} - l & w \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{-w^2\lambda_1 - dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} & -\frac{pw\lambda_1 - ldw + lpd_l}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} w \\ \frac{-pw\lambda_1 - d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} & \frac{p^2\lambda_1 - ld_x w + ldp}{d_x w^2 - 2dpw + p^2 d_l} & -\frac{-d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} w \\ -\frac{-\lambda_1 dw + \lambda_1 pd_l + d_x d_l - d^2}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-\lambda_1 d_x w + \lambda_1 dp - ld_x d_l + ld^2}{d_x w^2 - 2dpw + p^2 d_l} & -\frac{-d_x d_l + d^2}{d_x w^2 - 2dpw + p^2 d_l} w \end{bmatrix}$$

$$D_p l = \frac{-pw\lambda_1 - d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l}$$

$$D_w l = \frac{p^2\lambda_1 - ld_x w + ldp}{d_x w^2 - 2dpw + p^2 d_l}.$$

The sign of these expressions is ambiguous, unless other assumptions are made.

7.

[⇒]

Since f is concave, then

$$f\left(\frac{1}{2}x + \frac{1}{2}y\right) \geq \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

Since f is homogenous of degree 1, then

$$f\left(\frac{1}{2}(x+y)\right)=\frac{1}{2}f(x+y).$$

Therefore,

$$f(x+y)\geq f(x)+f(y).$$

[⇐]
Since f is homogenous of degree 1, then for any $z\in\mathbb{R}^2$ and any $a\in\mathbb{R}_+$, we have

$$f(az)=af(z). \tag{22.12}$$

By assumption, we have that

$$\text{for any } x,y\in\mathbb{R}^2,\ f(x+y)\geq f(x)+f(y). \tag{22.13}$$

Then, for any $\lambda\in[0,1]$,

$$f((1-\lambda)x+\lambda y)\overset{(22.13)}{\geq} f((1-\lambda)x)+f(\lambda y)\overset{(22.12)}{=}(1-\lambda)f(x)+\lambda f(y),$$

as desired.

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