Basic Linear Algebra, Metric Spaces, Differential Calculus and Nonlinear Programming[1]
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Part I

Basic Linear Algebra

Chapter 1

Systems of linear equations

1.1 Linear equations and solutions

Definition 1 *A*[1] *linear equation in the unknowns* $x_1, x_2, ..., x_n$ *is an equation of the form*

$$a_1x_1 + a_2x_2 + ...a_nx_n = b, \tag{1.1}$$

where $b \in \mathbb{R}$ *and* $\forall j \in \{1, ..., n\}$, $a_j \in \mathbb{R}$ *. The real number* a_j *is called the coefficient of* x_j *and* b *is called the constant of the equation.* a_j *for* $j \in \{1, ..., n\}$ *and* b *are also called parameters of system* (1.1)*.*

Definition 2 *A solution to the linear equation* (1.1) *is an ordered n-tuple* $(\overline{x}_1, ..., \overline{x}_n) := (\overline{x}_j)_{j=1}^n$ *such*[2] *that the following statement (obtained by substituting* $\overline{x}_j$ *in the place of* x_j *for any* j *) is true:*

$$a_1\overline{x}_1 + a_2\overline{x}_2 + ...a_n\overline{x}_n = b,$$

The set of all such solutions is called the solution set or the general solution or, simply, the solution of equation (1.1)*.*

The following fact is well known.

Proposition 3 *Let the linear equation*

$$ax = b \tag{1.2}$$

in the unknown (variable) $x \in \mathbb{R}$ *and parameters* $a, b \in \mathbb{R}$ *be given. Then,*

1. *if* $a \neq 0$*, then* $x = \frac{b}{a}$ *is the unique solution to* (1.2)*;*
2. *if* $a = 0$ *and* $b \neq 0$*, then* (1.2) *has no solutions;*
3. *if* $a = 0$ *and* $b = 0$*, then any real number is a solution to* (1.2)*.*

Definition 4 *A linear equation* (1.1) *is said to be degenerate if* $\forall j \in \{1, ..., n\}$, $a_j = 0$*, i.e., it has the form*

$$0x_1 + 0x_2 + ...0x_n = b, \tag{1.3}$$

Clearly,

1. if $b \neq 0$, then equation (1.3) has no solution,
2. if $b = 0$, any n-tuple $(\overline{x}_j)_{j=1}^n$ is a solution to (1.3).

Definition 5 *Let a non-degenerate equation of the form* (1.1) *be given. The leading unknown of the linear equation* (1.1) *is the first unknown with a nonzero coefficient, i.e.,* x_p *is the leading unknown if*

$$\forall j \in \{1, ..., p-1\}, a_j = 0 \quad \text{and} \quad a_p \neq 0.$$

For any $j \in \{1, ..., n\} \setminus \{p\}$, x_j *is called a free variable - consistently with the following obvious result.*

[1] In this part, I often follow Lipschutz (1991).
[2] ":=" means "equal by definition".

Proposition 6 *Consider a non-degenerate linear equation $a_1x_1 + a_2x_2 + ...a_nx_n = b$ with leading unknown x_p. Then the set of solutions to that equation is*

$$\left\{ (x_k)_{k=1}^n : \forall j \in \{1, ..., n\} \setminus \{p\} ,\ x_j \in \mathbb{R} \text{ and } x_p = \frac{b - \sum_{j \in \{1,...,n\} \setminus \{,p\}} a_j x_j}{a_p} \right\}$$

1.2 Systems of linear equations, equivalent systems and elementary operations

Definition 7 *A system of m linear equations in the n unknowns $x_1, x_2, ..., x_n$ is a system of the form*

$$\begin{cases} a_{11}x_1 + ... + a_{1j}x_j + ... + a_{1n}x_n = b_1 \\ ... \\ a_{i1}x_1 + ... + a_{ij}x_j + ... + a_{in}x_n = b_i \\ ... \\ a_{m1}x_i + ... + a_{mj}x_j + ... + a_{mn}x_n = b_m \end{cases} \tag{1.4}$$

where $\forall i \in \{1, ..., m\}$ and $\forall j \in \{1, ..., n\}$, $a_{ij} \in \mathbb{R}$ and $\forall i \in \{1, ..., m\}$, $b_i \in \mathbb{R}$. We call L_i the $i-th$ linear equation of system (1.4).

A solution to the above system is an ordered n-tuple $(\overline{x}_j)_{j=1}^n$ which is a solution of each equation of the system. The set of all such solutions is called the solution set of the system.

Definition 8 *Systems of linear equations are equivalent if their solutions set is the same.*

The following fact is obvious.

Proposition 9 *Assume that a system of linear equations contains the degenerate equation*

$$L : \qquad 0x_1 + 0x_2 + ...0x_n = b.$$

1. *If $b = 0$, then L may be deleted from the system without changing the solution set;*
2. *if $b \neq 0$, then the system has no solutions.*

A way to solve a system of linear equations is to transform it in an equivalent system whose solution set is "easy" to be found. In what follows we make precise the above sentence.

Definition 10 *An elementary operation on a system of linear equations* (1.4) *is one of the following operations:*

[E_1] *Interchange L_i with L_j, an operation denoted by $L_i \leftrightarrow L_j$ (which we can read "put L_i in the place of L_j and L_j in the place of L_i");*

[E_2] *Multiply L_i by $k \in \mathbb{R} \setminus \{0\}$, denoted by $kL_i \to L_i$, $k \neq 0$ (which we can read "put kL_i in the place of L_i, with $k \neq 0$");*

[E_3] *Replace L_i by (k times L_j plus L_i), denoted by $(L_i + kL_j) \to L_i$ (which we can read "put $L_i + kL_j$ in the place of L_i").*

Sometimes we apply [E_2] and [E_3] in one step, i.e., we perform the following operation

[E] Replace L_i by (k' times L_j and $k \in \mathbb{R} \setminus \{0\}$ times L_i), denoted by $(k'L_j + kL_i) \to L_i, k \neq 0$.

Elementary operations are important because of the following obvious result.

Proposition 11 *If S_1 is a system of linear equations obtained from a system S_2 of linear equations using a finite number of elementary operations, then system S_1 and S_2 are equivalent.*

In what follows, first we define two types of "simple" systems (triangular and echelon form systems), and we see why those systems are in fact "easy" to solve. Then, we show how to transform any system in one of those "simple" systems.

1.3 Systems in triangular and echelon form

Definition 12 *A linear system* (1.4) *is in triangular form if the number* n *of equations is equal to the number* n *of unknowns and* $\forall i \in \{1, ..., n\}$, x_i *is the leading unknown of equation* i, *i.e., the system has the following form:*

$$\begin{cases} a_{11}x_1 & +a_{12}x_2 & +... & +a_{1,n-1}x_{n-1} & +a_{1n}x_n & = & b_1 \\ & a_{22}x_2 & +... & +a_{2,n-1}x_{n-1} & +a_{2n}x_n & = & b_2 \\ & & & & & ... & \\ & & & a_{n-1,n-1}x_{n-1} & +a_{n-1n}x_n & = & b_{n-1} \\ & & & & a_{nn}x_n & = & b_n \end{cases} \tag{1.5}$$

where $\forall i \in \{1, ..., n\}$, $a_{ii} \neq 0$.

Proposition 13 *System* (1.5) *has a unique solution.*

Proof. We can compute the solution of system (1.5) using the following procedure, known as back-substitution.

First, since by assumption $a_{nn} \neq 0$, we solve the last equation with respect to the last unknown, i.e., we get

$$x_n = \frac{b_n}{a_{nn}}.$$

Second, we substitute that value of x_n in the next-to-the-last equation and solve it for the next-to-the-last unknown, i.e.,

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n} \cdot \frac{b_n}{a_{nn}}}{a_{n-1,n-1}}$$

and so on. The process ends when we have determined the first unknown, x_1.

Observe that the above procedure shows that the solution to a system in triangular form is unique since, at each step of the algorithm, the value of each x_i is uniquely determined, as a consequence of Proposition 3, conclusion 1. ■

Definition 14 *A linear system* (1.4) *is said to be in echelon form if*

1. *no equation is degenerate, and*
2. *the leading unknown in each equation is to the right of the leading unknown of the preceding equation.*

In other words, the system is of the form

$$\begin{cases} a_{11}x_1 & +... & +a_{1j_2}x_{j_2} & +... & & & & +a_{1s}x_s & =b_1 \\ & & a_{2j_2}x_{j_2} & +... & +a_{2,j_3}x_{j_3} & +... & & +a_{2s}x_s & =b_2 \\ & & & & a_{3,j_3}x_{j_3} & +... & & +a_{3s}x_s & =b_3 \\ & & & & & & & & ... \\ & & & & & a_{r,j_r}x_{j_r} & +a_{r,j_r+1}+... & +a_{rs}x_s & =b_r \end{cases} \tag{1.6}$$

with $j_1 := 1 < j_2 < ... < j_r$ *and* $a_{11}, a_{2j_2}, ..., a_{rj_r} \neq 0$. *Observe that the above system has* r *equations and* s *variables and that* $s \geq r$. *The leading unknown in equation* $i \in \{1, ..., r\}$ *is* x_{j_i}.

Remark 15 *Systems with no degenerate equations are the "interesting" ones. If an equation is degenerate and the right hand side term is zero, then you can erase it; if the right hand side term is not zero, then the system has no solutions.*

Definition 16 *An unknown* x_k *in system* (1.6) *is called a free variable if* x_k *is not the leading unknown in any equation, i.e.,* $\forall i \in \{1, ..., r\}$, $x_k \neq x_{j_i}$.

In system (1.6), there are r leading unknowns, r equations and $s - r \geq 0$ free variables.

Proposition 17 *Let a system in echelon form with* r *equations and* s *variables be given. Then, the following results hold true.*

1. *If $s = r$, i.e., the number of unknowns is equal to the number of equations, then the system has a unique solution;*

2. *if $s > r$, i.e., the number of unknowns is greater than the number of equations, then we can arbitrarily assign values to the $n - r > 0$ free variables and obtain solutions of the system.*

Proof. We prove the theorem by induction on the number r of equations of the system.

Step 1. $r = 1$.

In this case, we have a single, non-degenerate linear equation, to which Proposition 6 applies if $s > r = 1$, and Proposition 3 applies if $s = r = 1$.

Step 2.

Assume that $r > 1$ and the desired conclusion is true for a system with $r - 1$ equations. Consider the given system in the form (1.6) and erase the first equation, so obtaining the following system:

$$\begin{cases} a_{2j_2}x_{j_2} & +... & +a_{2,j_3}x_{j_3} & +... & & & = & b_2 \\ & & a_{3,j_3}x_{j_3} & +... & & & & \\ & & & ... & & & ... & \\ & & & a_{r,j_r}x_{j_r} & +a_{r,j_r+1} & +a_{rs}x_s & = & b_r \end{cases} \tag{1.7}$$

in the unknowns $x_{j_2}, ..., x_s$. First of all observe that the above system is in echelon form and has $r - 1$ equation; therefore we can apply the induction argument distinguishing the two case $s > r$ and $s = r$.

If $s > r$, then we can assign arbitrary values to the free variables, whose number is (the "old" number minus the erased ones)

$$s - r - (j_2 - j_1 - 1) = s - r - j_2 + 2$$

and obtain a solution of system (1.7). Consider the first equation of the original system

$$a_{11}x_1 \quad +a_{12}x_2 \quad +... \quad +a_{1,j_2-1}x_{j_2-1} \quad +a_{1j_2}x_{j_2} \quad +... \quad = \quad b_1 \ . \tag{1.8}$$

We immediately see that the above found values together with arbitrary values for the additional

$$j_2 - 2$$

free variable of equation (1.8) yield a solution of that equation, as desired. Observe also that the values given to the variables $x_1, ..., x_{j_{2-1}}$from the first equation do satisfy the other equations simply because their coefficients are zero there.

If $s = r$, the system in echelon form, in fact, becomes a system in triangular form and then the solution exists and it is unique. ■

Remark 18 *From the proof of the previous Proposition, if the echelon system* (1.6) *contains more unknowns than equations, i.e., $s > r$, then the system has an infinite number of solutions since each of the $s - r \geq 1$ free variables may be assigned an arbitrary real number.*

1.4 Reduction algorithm

The following algorithm (sometimes called row reduction) reduces system (1.4) of m equation and n unknowns to either echelon form, or triangular form, or shows that the system has no solution. The algorithm then gives a proof of the following result.

Proposition 19 *Any system of linear equations has either*

1. *infinite solutions, or*

2. *a unique solution, or*

3. *no solutions.*

Reduction algorithm.

Consider a system of the form (1.4) such that

$$\forall j \in \{1, ..., n\}, \quad \exists i \in \{1, .., m\} \text{ such that } a_{ij} \neq 0, \tag{1.9}$$

i.e., a system in which each variable has a nonzero coefficient in at least one equation. If that is not the case, the remaining variables can renamed in order to have (1.9) satisfied.

Step 1. Interchange equations so that the first unknown, x_1, appears with a nonzero coefficient in the first equation; i.e., rearrange the equations in the system in order to have $a_{11} \neq 0$.

Step 2. Use a_{11} as a "pivot" to eliminate x_1 from all equations but the first equation. That is, for each $i > 1$, apply the elementary operation

$$[E_3] : -\left(\frac{a_{i1}}{a_{11}}\right) L_1 + L_i \to L_i$$

or

$$[E] : -a_{i1} L_1 + a_{11} L_i \to L_i.$$

Step 3. Examine each new equation L :

1. If L has the form
$$0x_1 + 0x_2 + ... + 0x_n = 0,$$
or if L is a multiple of another equation, then delete L from the system.[3]
2. If L has the form
$$0x_1 + 0x_2 + ... + 0x_n = b,$$
with $b \neq 0$, then exit the algorithm. The system has no solutions.

Step 4. Repeat Steps 1, 2 and 3 with the subsystem formed by all the equations, excluding the first equation.

Step 5. Continue the above process until the system is in echelon form or a degenerate equation is obtained in Step 3.2.

Summarizing, our method for solving system (1.4) consists of two steps:

Step A. Use the above reduction algorithm to reduce system (1.4) to an equivalent simpler system (in triangular form, system (1.5) or echelon form (1.6)).

Step B. If the system is in triangular form, use back-substitution to find the solution; if the system is in echelon form, bring the free variables on the right hand side of each equation, give them arbitrary values (say, the name of the free variable with an upper bar), and then use back-substitution.

Example 20

$$\begin{cases} x_1 & + & 2x_2 & + & (-3)x_3 & = & -1 \\ 3x_1 & + & (-1)\,x_2 & + & 2x_3 & = & 7 \\ 5x_1 & + & 3x_2 & + & (-4)\,x_3 & = & 2 \end{cases}$$

Step A.

Step 1. Nothing to do.

Step 2. Apply the operations

$$-3L_1 + L_2 \to L_2$$

and

$$-5L_1 + L_3 \to L_3,$$

to get

$$\begin{cases} x_1 & + & 2x_2 & + & (-3)x_3 & = & -1 \\ & & (-7)\,x_2 & + & 11x_3 & = & 10 \\ & & (-7)\,x_2 & + & 11x_3 & = & 7 \end{cases}$$

[3] The justification of Step 3 is Propositon 9 and the fact that if $L = kL'$ for some other equation L' in the system, then the operation $-kL' + L \to L$ replace L by $0x_1 + 0x_2 + ... + 0x_n = 0$, which again may be deleted by Propositon 9.

Step 3. Examine each new equations L_2 and L_3:

1. L_2 and L_3 do not have the form
$$0x_1 + 0x_2 + ... + 0x_n = 0;$$
L_2 is not a multiple L_3;
2. L_2 and L_3 do not have the form
$$0x_1 + 0x_2 + ... + 0x_n = b,$$

Step 4.

Step 1.1 Nothing to do.

Step 2.1 Apply the operation
$$-L_2 + L_3 \to L_3$$
to get
$$\begin{cases} x_1 & + & 2x_2 & + & (-3)x_3 & = & -1 \\ & & (-7)\,x_2 & + & 11x_3 & = & 10 \\ 0x_1 & + & 0x_2 & + & 0x_3 & = & -3 \end{cases}$$

Step 3.1 L_3 has the form
$$0x_1 + 0x_2 + ... + 0x_n = b,$$

1. with $b = -3 \neq 0$, then exit the algorithm. The system has no solutions.

1.5 Matrices

Definition 21 *Given $m, n \in \mathbb{N}$, a matrix (of real numbers) of order $m \times n$ is a table of real numbers with m rows and n columns as displayed below.*
$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1j} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2j} & ... & a_{2n} \\ ... \\ a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{m1} & a_{m2} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}$$

For any $i \in \{1, ..., m\}$ and any $j \in \{1, ..., n\}$ the real numbers a_{ij} are called entries of the matrix; the first subscript i denotes the row the entries belongs to, the second subscript j denotes the column the entries belongs to. We will usually denote matrices with capital letters and we will write $A_{m\times n}$ to denote a matrix of order $m \times n$. Sometimes it is useful to denote a matrix by its "typical" element and we write $[a_{ij}]_{\substack{i\in\{1,...,m\} \\ j\in\{1,...,n\}}}$, or simply $[a_{ij}]$ if no ambiguity arises about the number of rows and columns. For $i \in \{1, ..., m\}$,
$$\begin{bmatrix} a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \end{bmatrix}$$
is called the $i-th$ row of A and it denoted by $R^i(A)$. For $j \in \{1, ..., n\}$,
$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ ... \\ a_{ij} \\ ... \\ a_{mj} \end{bmatrix}$$
is called the $j-th$ column of A and it denoted by $C^j(A)$.

We denote the set of $m \times n$ matrices by $\mathcal{M}_{m,n}$, and we write, in an equivalent manner, $A_{m\times n}$ or $A \in \mathcal{M}_{m,n}$.

Definition 22 *The matrix*

$$A_{m\times 1}=\begin{bmatrix} a_1 \\ ... \\ a_m \end{bmatrix}$$

is called column vector and the matrix

$$A_{1\times n}=\begin{bmatrix} a_1, & ... & a_n \end{bmatrix}$$

is called row vector. We usually denote row or column vectors by small Latin letters.

Definition 23 *The first nonzero entry in a row R of a matrix $A_{m\times n}$ is called the leading nonzero entry of R. If R has no leading nonzero entries, i.e., if every entry in R is zero, then R is called a zero row. If all the rows of A are zero, i.e., each entry of A is zero, then A is called a zero matrix, denoted by $0_{m\times n}$ or simply 0, if no confusion arises.*

In the previous sections, we defined triangular and echelon systems of linear equations. Below, we define triangular, echelon matrices and a special kind of echelon matrices. In Section (1.6), we will see that there is a simple relationship between systems and matrices.

Definition 24 *A matrix $A_{m\times n}$ is square if $m = n$. A square matrix A belonging to $\mathcal{M}_{m,m}$ is called square matrix of order m.*

Definition 25 *Given $A = [a_{ij}] \in \mathcal{M}_{m,m}$, the main diagonal of A is made up by the entries a_{ii} with $i \in \{1,..,m\}$.*

Definition 26 *A square matrix $A = [a_{ij}] \in \mathcal{M}_{m,m}$ is an upper triangular matrix or simply a triangular matrix if all entries below the main diagonal are equal to zero, i.e., $\forall i, j \in \{1,..,m\}$, if $i > j$, then $a_{ij} = 0$.*

Definition 27 *$A \in \mathcal{M}_{mm}$ is called diagonal matrix of order m if any element outside the principal diagonal is equal to zero, i.e., $\forall i, j \in \{1,...,m\}$ such that $i \neq j$, $a_{ij} = 0$.*

Definition 28 *A matrix $A \in \mathcal{M}_{m,n}$ is called an echelon (form) matrix, or it is said to be in echelon form, if the following two conditions hold:*

1. *All zero rows, if any, are on the bottom of the matrix.*
2. *The leading nonzero entry of each row is to the right of the leading nonzero entry in the preceding row.*

Definition 29 *If a matrix A is in echelon form, then its leading nonzero entries are called pivot entries, or simply, pivots*

Remark 30 *If a matrix $A \in \mathcal{M}_{m,n}$ is in echelon form and r is the number of its pivot entries, then $r \leq \min\{m, n\}$. In fact, $r \leq m$, because the matrix may have zero rows and $r \leq n$, because the leading nonzero entries of the first row maybe not in the first column, and the other leading nonzero entries may be "strictly to the right" of previous leading nonzero entry.*

Definition 31 *A matrix $A \in \mathcal{M}_{m,n}$ is called in row canonical form if*

1. *it is in echelon form,*
2. *each pivot is 1, and*
3. *each pivot is the only nonzero entry in its column.*

Example 32 *1. All the matrices below are echelon matrices; only the fourth one is in row canonical form.*

$$\begin{bmatrix} 0 & 7 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 2 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

2. Any zero matrix is in row canonical form.

Remark 33 *Let a matrix $A_{m\times n}$ in row canonical form be given. As a consequence of the definition, we have what follows.*

1. *If some rows from A are erased, the resulting matrix is still in row canonical form.*
2. *If some columns of zeros are added, the resulting matrix is still in row canonical form.*

Definition 34 *Denote by R^i the $i-th$ row of a matrix A. An elementary row operation is one of the following operations on the rows of A:*

$[E_1]$ *(Row interchange) Interchange R^i with R^j, an operation denoted by $R^i \leftrightarrow R^j$ (which we can read "put R^i in the place of R^j and R^j in the place of R^i");;*

$[E_2]$ *(Row scaling) Multiply R^i by $k \in \mathbb{R}\backslash\{0\}$, denoted by $kR^i \to R^i$, $k \neq 0$ (which we can read "put kR^i in the place of R^i, with $k \neq 0$");*

$[E_3]$ *(Row addition) Replace R^i by (k times R^j plus R^i), denoted by $\left(R^i + kR^j\right) \to R^i$ (which we can read "put $R^i + kR^j$ in the place of R^i").*

Sometimes we apply $[E_2]$ and $[E_3]$ in one step, i.e., we perform the following operation

$[E]$ Replace R^i by (k' times R^j and $k \in \mathbb{R}\backslash\{0\}$ times R^i), denoted by $\left(k'R^j + kR^i\right) \to R^i, k \neq 0$.

Definition 35 *A matrix $A \in \mathcal{M}_{m,n}$ is said to be row equivalent to a matrix $B \in \mathcal{M}_{m,n}$ if B can be obtained from A by a finite number of elementary row operations.*

It is hard not to recognize the similarity of the above operations and those used in solving systems of linear equations.

We use the expression "row reduce" as having the meaning of "transform a given matrix into another matrix using row operations". The following algorithm "row reduces" a matrix A into a matrix in echelon form.

Row reduction algorithm to echelon form.

Consider a matrix $A = [a_{ij}] \in \mathcal{M}_{m,n}$.

Step 1. Find the first column with a nonzero entry. Suppose it is column j_1.

Step 2. Interchange the rows so that a nonzero entry appears in the first row of column j_1, i.e., so that $a_{1j_1} \neq 0$.

Step 3. Use a_{1j_1} as a "pivot" to obtain zeros below a_{1j_1}, i.e., for each $i > 1$, apply the row operation

$$[E_3] : -\left(\frac{a_{ij_1}}{a_{1j_1}}\right) R^1 + R^i \to R^i$$

or

$$[E] : -a_{ij_1} R^1 + a_{11} R^i \to R^i.$$

Step 4. Repeat Steps 1, 2 and 3 with the submatrix formed by all the rows, excluding the first row.

Step 5. Continue the above process until the matrix is in echelon form.

Example 36 *Let's apply the above algorithm to the following matrix*

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{bmatrix}$$

Step 1. Find the first column with a nonzero entry: that is C^1, and therefore $j_1 = 1$.

Step 2. Interchange the rows so that a nonzero entry appears in the first row of column j_1, i.e., so that $a_{1j_1} \neq 0$: $a_{1j_1} = a_{11} = 1 \neq 0$.

Step 3. Use a_{11} as a "pivot" to obtain zeros below a_{11}. Apply the row operations

$$-3R^1 + R^2 \to R^2$$

and

$$-5R^1 + R^3 \to R^3,$$

to get

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{bmatrix}$$

Step 4. Apply the operation

$$-R^2 + R^3 \to R^3$$

to get

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

which is is in echelon form.

Row reduction algorithm from echelon form to row canonical form.
Consider a matrix $A = [a_{ij}] \in \mathcal{M}_{m,n}$ in echelon form, say with pivots

$$a_{1j_1}, a_{2j_2}, ..., a_{rj_r}.$$

Step 1. Multiply the last nonzero row R^r by $\frac{1}{a_{rj_r}}$,so that the leading nonzero entry of that row becomes 1.

Step 2. Use a_{rj_r} as a "pivot" to obtain zeros above the pivot, i.e., for each $i \in \{r-1, r-2, ..., 1\}$, apply the row operation

$$[E_3] : -a_{i,j_r} R^r + R^i \to R^i.$$

Step 3. Repeat Steps 1 and 2 for rows $R^{r-1}, R^{r-2}, ..., R^2$.

Step 4. Multiply R^1 by $\frac{1}{a_{1j_1}}$.

Example 37 *Consider the matrix*

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

in echelon form, with leading nonzero entries

$$a_{11} = 1, a_{22} = -7, a_{34} = -3.$$

Step 1. Multiply the last nonzero row R^3 by $\frac{1}{-3}$,so that the leading nonzero entry becomes 1:

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2. Use $a_{rj_r} = a_{34}$ as a "pivot" to obtain zeros above the pivot, i.e., for each $i \in \{r-1, r-2, ..., 1\} = \{2, 1\}$, apply the row operation

$$[E_3] : -a_{i,j_r} R^r + R^i \to R^i,$$

which in our case are

$$-a_{2,4}R^3 + R^2 \to R^2 \quad \text{i.e.,} \quad -10R^3 + R^2 \to R^2,$$
$$-a_{1,4}R^3 + R^1 \to R^1 \quad \text{i.e.,} \quad R^3 + R^1 \to R^1.$$

Then, we get

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -7 & 11 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3. Multiply R^2 by $\frac{1}{-7}$, and get

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -\frac{11}{7} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use a_{23} as a "pivot" to obtain zeros above the pivot, applying the operation:

$$-2R^2 + R^1 \to R^1,$$

to get

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} & 0 \\ 0 & 1 & -\frac{11}{7} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is in row reduced form.

Proposition 38 *Any matrix $A \in \mathcal{M}_{m,n}$ is row equivalent to a matrix in row canonical form.*

Proof. The two above algorithms show that any matrix is row equivalent to at least one matrix in row canonical form. ■

Remark 39 *In fact, in Proposition 157, we will show that: Any matrix $A \in \mathcal{M}_{m,n}$ is row equivalent to a* ***unique*** *matrix in row canonical form.*

1.6 Systems of linear equations and matrices

Definition 40 *Given system* (1.4)*, i.e., a system of m linear equation in the n unknowns $x_1, x_2, ..., x_n$*

$$\begin{cases} a_{11}x_1 + ... + a_{1j}x_j + ... + a_{1n}x_n = b_1 \\ ... \\ a_{i1}x_1 + ... + a_{ij}x_j + ... + a_{in}x_n = b_i \\ ... \\ a_{m1}x_i + ... + a_{mj}x_j + ... + a_{mn}x_n = b_m, \end{cases}$$

the matrix

$$\begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} & b_1 \\ ... \\ a_{i1} & ... & a_{ij} & ... & a_{in} & b_i \\ ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix M of system (1.4)*.*

Each row of M corresponds to an equation of the system, and each column of M corresponds to the coefficients of an unknown, except the last column which corresponds to the constant of the system.

In an obvious way, given an arbitrary matrix M, we can find a unique system whose associated matrix is M; moreover, given a system of linear equations, there is only one matrix M associated with it. We can therefore identify system of linear equations with (augmented) matrices.

The coefficient matrix of the system is

$$A = \begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... \\ a_{i1} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}$$

One way to solve a system of linear equations is as follows:

1. Reduce its augmented matrix M to echelon form, which tells if the system has solution; if M has a row of the form $(0, 0, ..., 0, b)$ with $b \neq 0$, then the system has no solution and you can stop. If the system admits solutions go to the step below.

2. Reduce the matrix in echelon form obtained in the above step to its row canonical form. Write the corresponding system. In each equation, bring the free variables on the right hand side, obtaining a triangular system. Solve by back-substitution.

The simple justification of this process comes from the following facts:

1. Any elementary row operation of the augmented matrix M of the system is equivalent to applying the corresponding operation on the system itself.
2. The system has a solution if and only if the echelon form of the augmented matrix M does not have a row of the form $(0, 0, ..., 0, b)$ with $b \neq 0$ - simply because that row corresponds to a degenerate equation.
3. In the row canonical form of the augmented matrix M (excluding zero rows) the coefficient of each non-free variable is a leading nonzero entry which is equal to one and is the only nonzero entry in its respective column; hence the free variable form of the solution is obtained by simply transferring the free variable terms to the other side of each equation.

Example 41 *Consider the system presented in Example 20:*

$$\begin{cases} x_1 & + & 2x_2 & + & (-3)x_3 & = & -1 \\ 3x_1 & + & (-1)\,x_2 & + & 2x_3 & = & 7 \\ 5x_1 & + & 3x_2 & + & (-4)\,x_3 & = & 2 \end{cases}$$

The associated augmented matrix is:

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{bmatrix}$$

In example 36, we have see that the echelon form of the above matrix is

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

which has its last row of the form $(0, 0, ..., 0, b)$ *with* $b = -3 \neq 0$*, and therefore the system has no solution.*

1.7 Exercises

Chapter 1 in Lipschutz:
1,2,3,4,6,7,8,10, 11,12,14,15,16.

Chapter 2

The Euclidean Space $\mathbb{R}^n$

2.1 Sum and scalar multiplication

It is well known that the real line is a representation of the set $\mathbb{R}$ of real numbers. Similarly, a ordered pair (x, y) of real numbers can be used to represent a point in the plane and a triple (x, y, z) or (x_1, x_2, x_3) a point in the space. In general, if $n \in \mathbb{N} := \{1, 2, ..., \}$, we can define $(x_1, x_2, ..., x_n)$ or $(x_i)_{i=1}^n$ as a point in the $n-$ *space*.

Definition 42 $\mathbb{R}^n := \mathbb{R} \times ... \times \mathbb{R}$.

In other words, $\mathbb{R}^n$ is the Cartesian product of $\mathbb{R}$ multiplied n times by itself.

Definition 43 *The elements of $\mathbb{R}^n$ are ordered n-tuple of real numbers, are usually called vectors and are denoted by*

$$x = (x_1, x_2, ..., x_n) \text{ or } x = (x_i)_{i=1}^n.$$

x_i is called $i-th$ component of $x \in \mathbb{R}^n$.

Definition 44 *$x = (x_i)_{i=1}^n \in \mathbb{R}^n$ and $y = (y_i)_{i=1}^n$ are equal if*

$$\forall i \in \{1, ..., n\} \ , \ \ x_i = y_i.$$

In that case we write $x = y$.

Let us introduce two operations on $\mathbb{R}^n$ and analyze some properties they satisfy.

Definition 45 *Given $x \in \mathbb{R}^n, y \in \mathbb{R}^n$, we call addition or sum of x and y the element denoted by $x + y \in \mathbb{R}^n$ obtained as follows*

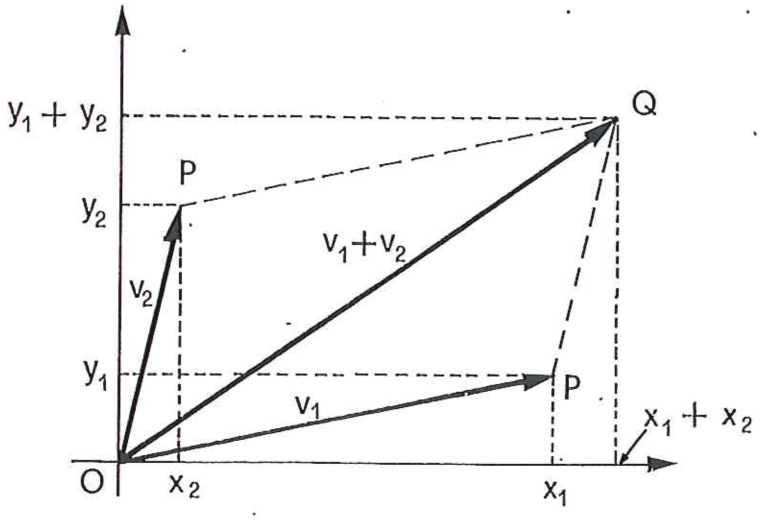
$$x + y := (x_i + y_i)_{i=1}^n.$$

Definition 46 *An element $\lambda \in \mathbb{R}$ is called scalar.*

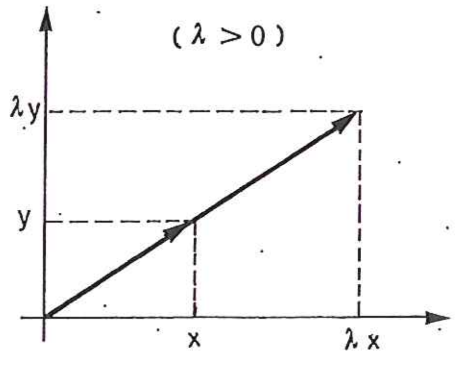
Definition 47 *Given $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we call scalar multiplication of x by λ the element $\lambda x \in \mathbb{R}^n$ obtained as follows*

$$\lambda x := (\lambda x_i)_{i=1}^n.$$

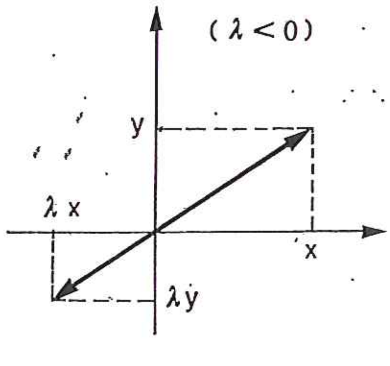
Geometrical interpretation of the two operations in the case $n = 2$.



$y_1 + y_2$
Q
P
y_2
$v_1 + v_2$
v_2
y_1
P
v_1
$x_1 + x_2$
O
x_2
x_1



($\lambda > 0$)
λy
y
x
λx



($\lambda < 0$)
y
λx
x
λy

From the well known properties of the sum and product of real numbers it is possible to verify that the following properties of the above operations do hold true.

Properties of addition.

A1. (Associative) $\forall x, y \in \mathbb{R}^n, \quad (x+y)+z = x+(y+z)$;

A2. (existence of null element) there exists an element e in $\mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$, $x+e=x$; in fact such element is unique and it is denoted by 0;

A3. (existence of inverse element) $\forall x \in \mathbb{R}^n \ \exists y \in \mathbb{R}^n$ such that $x+y=0$; in fact, that element is unique and denoted by $-x$;

A4. (Commutative) $\forall x, y \in \mathbb{R}^n$, $x+y=y+x$.

Properties of multiplication.

M1. (distributive) $\forall \alpha \in \mathbb{R},\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^n \quad \alpha(x+y) = \alpha x + \alpha y$;

M2. (distributive) $\forall \alpha \in \mathbb{R},\ \beta \in \mathbb{R},\ x \in \mathbb{R}^n,\ (\alpha+\beta)x = \alpha x + \beta x$

M3. $\forall \alpha \in \mathbb{R},\ \beta \in \mathbb{R},\ x \in \mathbb{R}^n, \quad (\alpha\beta)x = \alpha(\beta x)$;

M4. $\forall x \in \mathbb{R}^n$, $1x = x$.

2.2 Scalar product

Definition 48 *Given $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in \mathbb{R}^n$, we call dot, scalar or inner product of x and y, denoted by xy or $x \cdot y$, the scalar*

$$\sum_{i=1}^{n} x_i \cdot y_i \in \mathbb{R}.$$

Remark 49 *The scalar product of elements of $\mathbb{R}^n$ satisfies the following properties.*

1. $\forall x, y \in \mathbb{R}^n \quad x \cdot y = y \cdot x$;
2. $\forall \alpha, \beta \in \mathbb{R}, \forall x, y, z \in \mathbb{R}^n \quad (\alpha x + \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z)$;
3. $\forall x \in \mathbb{R}^n, \quad x \cdot x \geq 0$;
4. $\forall x \in \mathbb{R}^n, \quad x \cdot x = 0 \iff x = 0$.

Definition 50 *The set $\mathbb{R}^n$ with above described three operations (addition, scalar multiplication and dot product) is usually called Euclidean space of dimension n.*

Definition 51 *Given $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, we denote the (Euclidean) norm or length of x by*

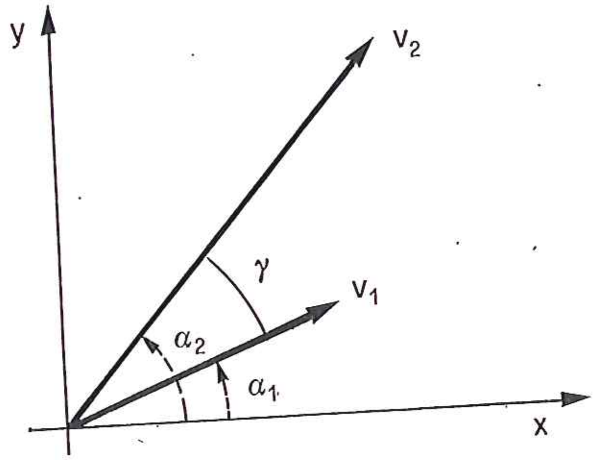
$$\|x\| := (x \cdot x)^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{n} (x_i)^2}.$$

Geometrical Interpretation of scalar products in $\mathbb{R}^2$.

Given $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$, from elementary trigonometry we know that

$$x = (\|x\| \cos\alpha, \|x\| \sin\alpha) \tag{2.1}$$

where α is the measure of the angle between the positive part of the horizontal axes and x itself.



y
v_2
γ
v_1
α_2
α_1
x

Using the above observation we can verify that given $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2 \setminus \{0\}$,

$$xy = \|x\| \cdot \|y\| \cdot \cos(\gamma)$$

where γ is an[1] angle between x and y.
`scan and insert picture (Marcellini-Sbordone page 179)`
From the picture and (2.1), we have

$$x = (\|x\| \cos(\alpha_1), \|x\| \sin(\alpha_1))$$

and

$$y = (\|y\| \cos(\alpha_2), \|y\| \sin(\alpha_2)).$$

Then[2]

$$xy = \|x\| \|y\| (\cos(\alpha_1) \cdot \cos(\alpha_2) + \sin(\alpha_1) \cdot \sin(\alpha_2)) = \|x\| \|y\| \cos(\alpha_2 - \alpha_1).$$

Taken x and y not belonging to the same line, define $\theta^* :=$ (angle between x and y with minimum measure). From the above equality, it follows that

$$\begin{array}{lcl} \theta^* = \frac{\pi}{2} & \Leftrightarrow & x \cdot y = 0 \\ \theta^* < \frac{\pi}{2} & \Leftrightarrow & x \cdot y > 0 \\ \theta^* > \frac{\pi}{2} & \Leftrightarrow & x \cdot y < 0. \end{array}$$

Definition 52 *$x, y \in \mathbb{R}^n \setminus \{0\}$ are orthogonal if $xy = 0$.*

2.3 Norms and Distances

Proposition 53 *(Properties of the norm). Let $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.*
1. $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$,
2. $\|\alpha x\| = |\alpha| \cdot \|x\|$,
3. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality),
4. $|xy| \leq \|x\| \cdot \|y\|$ (Cauchy-Schwarz inequality).

Proof. 1. By definition $\|x\| = \sqrt{\sum_{i=1}^n (x_i)^2} \geq 0$. Moreover, $\|x\| = 0 \Leftrightarrow \|x\|^2 = 0 \Leftrightarrow \sum_{i=1}^n (x_i)^2 = 0 \Leftrightarrow x = 0$.
2. $\|\alpha x\| = \sqrt{\sum_{i=1}^n \alpha^2 (x_i)^2} = |\alpha| \sqrt{\sum_{i=1}^n (x_i)^2} = |\alpha| \cdot \|x\|$.
4. (3 is proved using 4)
We want to show that $|xy| \leq \|x\| \cdot \|y\|$ or $|xy|^2 \leq \|x\|^2 \cdot \|y\|^2$, i.e.,

$$\left(\sum_{i=1}^n x_i y_i\right)^2 \leq \left(\sum_{i=1}^n x_i^2\right) \cdot \left(\sum_{i=1}^n y_i^2\right)$$

Defined $X := \sum_{i=1}^n x_i^2, Y := \sum_{i=1}^n y_i^2$ and $Z := \sum_{i=1}^n x_i y_i$, we have to prove that

$$Z^2 \leq XY. \tag{2.2}$$

[1] Recall that $\forall x \in \mathbb{R}$, $\cos x = \cos(-x) = \cos(2\pi - x)$.

[2] Recall that for any $x_1, x_2 \in \mathbb{R}$

$$\cos(x_1 \pm x_2) = \cos(x_1) \cdot \cos(x_2) \mp \sin(x_1) \cdot \sin(x_2),$$

and

$$\cos(x_1) = \cos(-x_1)$$

Observe that

$$\forall a \in \mathbb{R}, \quad \begin{array}{l} 1.\ \sum_{i=1}^{n} (ax_i + y_i)^2 \geq 0, \quad \text{and} \\ 2.\ \sum_{i=1}^{n} (ax_i + y_i)^2 = 0 \quad \Leftrightarrow \quad \forall i \in \{1, ..., n\}, ax_i + y_i = 0 \end{array}$$

Moreover,

$$\sum_{i=1}^{n} (ax_i + y_i)^2 = a^2 \sum_{i=1}^{n} x_i^2 + 2a \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 = a^2 X + 2aZ + Y \geq 0 \tag{2.3}$$

If $X > 0$, we can take $a = -\frac{Z}{X}$, and from (2.3), we get

$$0 \leq \frac{Z^2}{X^2} X - 2\frac{Z^2}{X} + Y$$

or

$$Z^2 \leq XY,$$

as desired.

If $X = 0$, then $x = 0$ and $Z = 0$, and (2.2) is true simply because $0 \leq 0$.

3. It suffices to show that $\|x + y\|^2 \leq (\|x\| + \|y\|)^2$.

$$\|x + y\|^2 = \sum_{i=1}^{n} (x_i + y_i)^2 = \sum_{i=1}^{n} \left((x_i)^2 + 2x_i \cdot y_i + (y_i)^2 \right) =$$

$$= \|x\|^2 + 2xy + \|y\|^2 \leq \|x\|^2 + 2|xy| + \|y\|^2 \overset{(4 \text{ above})}{\leq} \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

■

Proposition 54 *For any $x, y \in \mathbb{R}^n$ and any $\lambda, \mu \in \mathbb{R}$, we have*
1. $|\|x\| - \|y\|| \leq \|x - y\|$, and
2. $\|\lambda x + \mu y\|^2 = \lambda^2 \|x\|^2 + 2\lambda\mu x \cdot y + \mu^2 \|y\|^2$.

Proof. 1. Recall that $\forall a, b \in \mathbb{R}$

$$-b \leq a \leq b \Leftrightarrow |a| \leq b.$$

From Proposition 53.3,identifying x with $x - y$ and y with y, we get $\|x - y + y\| \leq \|x - y\| + \|y\|$, i.e.,

$$\|x\| - \|y\| \leq \|x - y\|$$

From Proposition 53.3, identifying x with $y - x$ and y with x, we get $\|y - x + x\| \leq \|y - x\| + \|x\|$, i.e.,

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\|$$

and

$$-\|x - y\| \leq \|x\| - \|y\|,$$

as desired.

2.

$$\|\lambda x + \mu y\|^2 = \sum_{i=1}^{n} (\lambda_i x_i + \mu y_i)^2 = \sum_{i=1}^{n} \left(\lambda^2 (x_i)^2 + 2\lambda\mu (x_i)(y_i) + \mu^2 (y_i)^2 \right) =$$

$$= \lambda^2 \sum_{i=1}^{n} (x_i)^2 + 2\lambda\mu \sum_{i=1}^{n} (x_i)(y_i) + \mu^2 \sum_{i=1}^{n} (y_i)^2 = \lambda^2 \|x\|^2 + 2\lambda\mu x \cdot y + \mu^2 \|y\|^2.$$

■

Definition 55 *For any $n \in \mathbb{N} \backslash \{0\}$ and for any $i \in \{1, ..., n\}$, $e_n^i := \left(e_{j,n}^i\right)_{j=1}^n \in \mathbb{R}^n$ with*

$$e_{n,j}^i = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases}$$

In other words, e_n^i is an element of $\mathbb{R}^n$ whose components are all zero, but the $i - th$ component which is equal to 1. The vector e_n^i is called the $i - th$ canonical vector in $\mathbb{R}^n$.

Remark 56 *$\forall x \in \mathbb{R}^n$,*

$$\|x\| \leq \sum_{i=1}^{n} |x_i|,$$

as verified below.

$$\|x\| = \left\| \sum_{i=1}^{n} x_i e^i \right\| \overset{(1)}{\leq} \sum_{i=1}^{n} \left\| x_i e^i \right\| \overset{(2)}{=} \sum_{i=1}^{n} |x_i| \cdot \left\| e^i \right\| = \sum_{i=1}^{n} |x_i|,$$

where (1) *follows from the triangle inequality, i.e., Proposition 53.3, and* (2) *from Proposition 53.2.*

Definition 57 *Given $x, y \in \mathbb{R}^n$,we denote the (Euclidean) distance between x and y by*

$$d(x, y) := \|x - y\|$$

Proposition 58 *(Properties of the distance). Let $x, y, z \in \mathbb{R}^n$.*
1. $d(x, y) \geq 0$, and $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality).

Proof. 1. It follows from property 1 of the norm.
2. It follows from the definition of the distance as a norm.
3. Identifying x with $x - y$ and y with $y - z$ in property 3 of the norm, we get
$\|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\|$, i.e., the desired result. ∎

2.4 Exercises

From Lipschutz (1991), starting from page 53: 2.1 → 2.4, 2.12 → 2.19, 2.26, 2.27.

Chapter 3

Matrices

We presented the concept of matrix in Definition 21. In this chapter, we study further properties of matrices.

Definition 59 *Given two matrices* $A = [a_{ij}]_{\substack{i\in\{1,...,m\}\\ j\in\{1,...,n\}}}$*,* $A = [a_{ij}]_{\substack{i\in\{1,...,m\}\\ j\in\{1,...,n\}}} \in \mathcal{M}_{m,n}$*, we say that* $A = B$ *if*

$$\forall i \in \{1,...,m\}, j \in \{1,...,n\}, \quad a_{ij} = b_{ij}.$$

Definition 60 *The transpose of a matrix* $A \in \mathcal{M}_{m,n}$*, denoted by* A^T *belongs to* $M_{n,m}$ *and it is the matrix obtained by writing the rows of* A*, in order, as columns:*

$$A^T = \begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... \\ a_{i1} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & ... & a_{i1} & ... & a_{m1} \\ ... \\ a_{1j} & ... & a_{ij} & ... & a_{mj} \\ ... \\ a_{1n} & ... & a_{in} & ... & a_{mn} \end{bmatrix}.$$

In other words, row 1 of the matrix A becomes column 1 of A^T, row 2 of A becomes column 2 of A^T, and so on, up to row m which becomes column m of A^T. Same results is obtained proceeding as follows: column 1of A becomes row 1 of A^T, column 2 of A becomes row 2 of A^T, and so on, up to column n which becomes row n of A^T. More formally, given $A = [a_{ij}]_{\substack{i\in\{1,...,m\}\\ j\in\{1,...,n\}}} \in \mathrm{M}_{m,n}$, then

$$A^T = [a_{ji}]_{\substack{j\in\{1,...,n\}\\ i\in\{1,...,m\}}} \in \mathcal{M}_{n,m}.$$

Definition 61 *A matrix* $A \in M_{n,n}$ *is said to be symmetric if* $A = A^T$ *, i.e.,* $\forall i, j \in \{1, ..., n\}$*,* $a_{ij} = a_{ji}$*.*

Remark 62 *We can write a matrix* $A_{m\times n} = [a_{ij}]$ *as*

$$A = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix} = \left[C^1(A), ..., C^j(A), ..., C^n(A)\right]$$

where

$$R^i(A) = [a_{i1}, ..., a_{ij}, ...a_{in}] := \left[R^{i1}(A), ..., R^{ij}(A), ...R^{in}(A)\right] \in \mathbb{R}^n \quad \text{for } i \in \{1, ..., m\} \quad \text{and}$$

$$C^j(A) = \begin{bmatrix} a_{1j} \\ \\ a_{ij} \\ \\ a_{mj} \end{bmatrix} := \begin{bmatrix} C^{j1}(A) \\ \\ C^{ji}(A) \\ \\ C^{jm}(A) \end{bmatrix} \in \mathbb{R}^n \qquad \text{for } j \in \{1, ..., n\}.$$

In other words, $R^i(A)$ denotes row i of the matrix A and $C^j(A)$ denotes column j of matrix A.

3.1 Matrix operations

Definition 63 *Given the matrices $A_{m\times n} := [a_{ij}]$ and $B_{m\times n} := [b_{ij}]$, the sum of A and B, denoted by $A+B$ is the matrix $C_{m\times n} = [c_{ij}]$ such that*

$$\forall i \in \{1, ..., m\}, j \in \{1, ..., n\}, \qquad c_{ij} = a_{ij} + b_{ij}$$

Definition 64 *Given the matrices $A_{m\times n} := [a_{ij}]$ and the scalar α, the product of the matrix A by the scalar α, denoted by $\alpha \cdot A$ or αA, is the matrix obtained by multiplying each entry A by α :*

$$\alpha A := [\alpha a_{ij}]$$

Remark 65 *It is easy to verify that the set of matrices $\mathcal{M}_{m,n}$ with the above defined sum and scalar multiplication satisfies all the properties listed for elements of $\mathbb{R}^n$ in Section 2.1.*

Definition 66 *Given $A = [a_{ij}] \in \mathcal{M}_{m,n}$, $B = [b_{jk}] \in \mathcal{M}_{n,p}$, the product $A \cdot B$ is a matrix $C = [c_{ik}] \in \mathcal{M}_{m,p}$ such that*

$$\forall i \in \{1, ...m\}, \forall k \in \{1, ..., p\}, \qquad c_{ik} := \sum_{j=1}^{n} a_{ij} b_{jk} = R^i(A) \cdot C^k(B)$$

i.e., since

$$A = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix}, B = \left[C^1(B), ..., C^k(B), ..., C^p(B)\right] \tag{3.1}$$

$$AB = \begin{bmatrix} R^1(A)\cdot C^1(B) & ... & R^1(A)\cdot C^k(B) & ... & R^1(A)\cdot C^p(B) \\ ... \\ R^i(A)\cdot C^1(B) & ... & R^i(A)\cdot C^k(B) & ... & R^i(A)\cdot C^p(B) \\ ... \\ R^m(A)\cdot C^1(B) & ... & R^m(A)\cdot C^k(B) & ... & R^m(A)\cdot C^p(B) \end{bmatrix} \tag{3.2}$$

Remark 67 *If $A \in \mathcal{M}_{1n}$, $B \in \mathcal{M}_{n,1}$, the above definition coincides with the definition of scalar product between elements of $\mathbb{R}^n$. In what follows, we often identify an element of $\mathbb{R}^n$ with a row or a column vectors (- see Definition 22) consistently with we what write. In other words $A_{m\times n}x = y$ means that x and y are column vector with n entries, and $wA_{m\times n} = z$ means that w and z are row vectors with m entries.*

Definition 68 *If two matrices are such that a given operation between them is well defined, we say that they are conformable with respect to that operation.*

Remark 69 *If $A, B \in M_{m,n}$, they are conformable with respect to matrix addition. If $A \in M_{m,n}$ and $B \in M_{n,p}$, they are conformable with respect to multiplying A on the left of B. We often say the two matrices are conformable and let the context define precisely the sense in which conformability is to be understood.*

Remark 70 *(For future use) $\forall k \in \{1, ..., p\}$,*

$$A \cdot C^k(B) = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix} \cdot C^k(B) = \begin{bmatrix} R^1(A)\cdot C^k(B) \\ ... \\ R^i(A)\cdot C^k(B) \\ ... \\ R^m(A)\cdot C^k(B) \end{bmatrix} \tag{3.3}$$

Then, just comparing (3.2) and (3.3), we get

$$AB = \begin{bmatrix} A\cdot C^1(B) & ... & A\cdot C^k(B) & ... & A\cdot C^p(B) \end{bmatrix}, \tag{3.4}$$

i.e.,

$$C^k(AB) = A \cdot C^k(B).$$

Similarly, $\forall i \in \{1, ..., m\}$,

$$R^i(A) \cdot B = R^i(A) \cdot \begin{bmatrix} C^1(B) & ... & C^k(B) & ... & C^p(B) \end{bmatrix} = \\ = \begin{bmatrix} R^i(A) \cdot C^1(B) & ... & R^i(A) \cdot C^k(B) & ... & R^i(A) \cdot C^p(B) \end{bmatrix}. \tag{3.5}$$

Then, just comparing (3.2) *and* (3.5)*, we get*

$$AB = \begin{bmatrix} R^1(A)B \\ ... \\ R^i(A)B \\ ... \\ R^m(A)B \end{bmatrix}, \tag{3.6}$$

i.e.,

$$R^i(AB) = R^i(A) \cdot B.$$

Definition 71 *A submatrix of a matrix* $A \in M_{m,n}$ *is a matrix obtained from* A *erasing some rows and columns.*

Definition 72 *A matrix* $A \in M_{m,n}$ *is partitioned in blocks if it is written as submatrices using a system of horizontal and vertical lines.*

Example 73 *The matrix*

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \end{bmatrix}$$

can be partitioned in block submatrices in several ways. For example as follows

$$\begin{bmatrix} 1 & 2 & | & 3 & 4 & 5 \\ 6 & 7 & | & 8 & 9 & 0 \\ - & - & | & - & - & - \\ 1 & 2 & | & 3 & 4 & 5 \\ 6 & 7 & | & 8 & 9 & 0 \end{bmatrix},$$

whose blocks are

$$\begin{bmatrix} 1 & 2 \\ 6 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 5 \\ 8 & 9 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 6 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 5 \\ 8 & 9 & 0 \end{bmatrix}.$$

The reason of the partition into blocks is that the result of operations on block matrices can obtained by carrying out the computation with blocks, just as if they were actual scalar entries of the matrices, as described below.

Remark 74 *We verify below that for matrix multiplication, we do not commit an error if, upon conformably partitioning two matrices, we proceed to regard the partitioned blocks as real numbers and apply the usual rules.*

1. Take $a := (a_i)_{i=1}^{n_1} \in \mathbb{R}^{n_1}, b := (b_j)_{j=1}^{n_2} \in \mathbb{R}^{n_2}, c := (c_i)_{i=1}^{n_1} \in \mathbb{R}^{n_1}, d := (d_j)_{j=1}^{n_2} \in \mathbb{R}^{n_2}$,

$$\begin{bmatrix} a & | & b \end{bmatrix}_{1 \times (n_1+n_2)} \begin{bmatrix} c \\ - \\ d \end{bmatrix}_{(n_1+n_2)\times 1} = \sum_{i=1}^{n_1} a_i c_i + \sum_{j=1}^{n_2} b_j d_j = a \cdot c + b \cdot d. \tag{3.7}$$

2.

Take $A \in M_{m,n_1}, B \in M_{m,n_2}, C \in M_{n_1,p}, D \in M_{n_2,p}$*, with*

$$A = \begin{bmatrix} R^1(A) \\ ... \\ R^m(A) \end{bmatrix}, B = \begin{bmatrix} R^1(B) \\ ... \\ R^m(B) \end{bmatrix},$$

$$C = \left[C^1(C), ..., C^p(C)\right],$$

$$D = \left[C^1(D), ..., C^p(D)\right]$$

Then,

$$\begin{bmatrix} A & B \end{bmatrix}_{m\times(n_1+n_2)} \begin{bmatrix} C \\ D \end{bmatrix}_{(n_1+n_2)\times p} = \begin{bmatrix} R^1(A) & R^1(B) \\ ... & ... \\ R^m(A) & R^m(B) \end{bmatrix} \begin{bmatrix} C^1(C), ..., C^p(C) \\ C^1(D), ..., C^p(D) \end{bmatrix} =$$

$$= \begin{bmatrix} R^1(A)\cdot C^1(C) + R^1(B)\cdot C^1(D) & ... & R^1(A)\cdot +R^1(B)\cdot C^p(D) \\ ... & & \\ R^m(A)\cdot C^1(C) + R^m(B)\cdot C^1(D) & & R^m(A)\cdot C^p(C) + R^m(B)\cdot C^p(D) \end{bmatrix} =$$

$$= \begin{bmatrix} R^1(A)\cdot C^1(C) & ... & R^1(A)\cdot C^p(C) \\ ... & & \\ R^m(A)\cdot C^1(C) & & R^m(A)\cdot C^p(C) \end{bmatrix} + \begin{bmatrix} R^1(B)\cdot C^1(D) & ... & R^1(B)\cdot C^p(D) \\ ... & & \\ R^m(B)\cdot C^1(D) & & R^m(B)\cdot C^p(D) \end{bmatrix} =$$

$$= AC + BD.$$

Definition 75 *Let the matrices $A_i \in \mathcal{M}(n_i, n_i)$ for $i \in \{1, ..., K\}$, then the matrix*

$$A = \begin{bmatrix} A_1 & & & & & \\ & A_2 & & & & \\ & & \ddots & & & \\ & & & A_i & & \\ & & & & \ddots & \\ & & & & & A_K \end{bmatrix} \in \mathbb{M}\left(\sum_{i=1}^K n_i, \sum_{i=1}^K n_i\right)$$

is called block diagonal matrix.

Very often having information on the matrices A_i gives information on A.

Remark 76 *It is easy, but cumbersome, to verify the following properties.*

1. (associative property) $\forall A \in \mathcal{M}_{m,n}$, $\forall B \in \mathcal{M}_{np}$, $\forall C \in \mathcal{M}_{pq}$, $A(BC) = (AB)C$;
2. (distributive property) $\forall A \in \mathcal{M}_{m,n}$, $\forall B \in \mathcal{M}_{m,n}$, $\forall C \in \mathcal{M}_{np}$, $(A+B)C = AC + BC$.
3. (linearity) $\forall x, y \in \mathbb{R}^n$ and $\forall \alpha, \beta \in \mathbb{R}$,

$$A(\alpha x + \beta y) = A(\alpha x) + B(\beta y) = \alpha Ax + \beta Ay$$

It is false that:
1. (commutative property) $\forall A \in \mathcal{M}_{m,n}$, $\forall B \in \mathcal{M}_{np}$, $AB = BA$;
2. (cancellation law) $\forall A \in \mathcal{M}_{m,n}$, $\forall B, C \in \mathcal{M}_{np}$, $\langle A \neq 0, AB = AC\rangle \Longrightarrow \langle B = C\rangle$;
3. $\forall A \in \mathcal{M}_{m,n}$, $\forall B \in \mathcal{M}_{np}$, $\langle A \neq 0, AB = 0\rangle \Longrightarrow \langle B = 0\rangle$.
Let's show why the above statements are false.
1.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 5 \\ -1 & 1 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$CD = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 2 & 2 \end{bmatrix}$$

$$DC = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 8 \end{bmatrix}$$

Observe that since the commutative property does not hold true, we have to distinguish between "left factor out" and "right factor out" and also between "left multiplication or pre-multiplication" and "right multiplication or post-multiplication":

$$\begin{array}{c} AB + AC = A(B + C) \\ EF + GF = (E + G) F \\ AB + CA \neq A(B + C) \\ AB + CA \neq (B + C) A \end{array}$$

2.
Given

$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 \\ -5 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 14 & 18 \end{bmatrix}$$

$$AC = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 14 & 18 \end{bmatrix}$$

3.
Observe that 3. $\Rightarrow$ 2. and therefore $\neg$2. $\Rightarrow$ $\neg$3. Otherwise, you can simply observe that 3. follows from 2., choosing A in 3. equal to A in 2., and B in 3. equal to $B - C$ in 2.:

$$A(B - C) = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 4 & 1 \\ -5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since the associative property of the product between matrices does hold true we can give the following definition.

Definition 77 *Given $A \in \mathcal{M}_{m,m}$,*

$$A^k := \underset{1}{A} \cdot \underset{2}{A} \cdot \ldots \cdot \underset{k\ times}{A}.$$

Observe that if $A \in \mathcal{M}_{m,m}$ and $k, l \in \mathbb{N} \setminus \{0\}$, then

$$A^k \cdot A^l = A^{k+l}.$$

Remark 78 ***Properties of transpose matrices.*** *For any $m, n, p \in \mathbb{N}$,*

1.	$\forall A \in \mathcal{M}_{m,n}$	$(A^T)^T = A$
2.	$\forall A, B \in \mathcal{M}_{m,n}$	$(A + B)^T = A^T + B^T$
3.	$\forall \alpha \in \mathbb{R}, \forall A \in \mathcal{M}_{m,n}$	$(\alpha A)^T = \alpha A^T$
4.	$\forall A \in \mathcal{M}_{m,n}, \forall B \in \mathcal{M}_{n,p}$	$(AB)^T = B^T A^T$

Matrices and linear systems.
In Section 1.6, we have seen that a system of m linear equation in the n unknowns $x_1, x_2, ..., x_n$ and parameters a_{ij}, for $i \in \{1, ..., m\}$, $j \in \{1, ..., n\}$, $(b_i)_{i=1}^n \in \mathbb{R}^n$ is displayed below:

$$\begin{cases} a_{11}x_1 + ... + a_{1j}x_j + ... + a_{1n}x_n = b_1 \\ ... \\ a_{i1}x_1 + ... + a_{ij}x_j + ... + a_{in}x_n = b_i \\ ... \\ a_{m1}x_i + ... + a_{mj}x_j + ... + a_{mn}x_n = b_m \end{cases} \tag{3.8}$$

Moreover, the matrix

$$\begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} & b_1 \\ ... \\ a_{i1} & ... & a_{ij} & ... & a_{in} & b_i \\ ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix M of system (1.4). The coefficient matrix A of the system is

$$A = \begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... \\ a_{i1} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}$$

Using the notations we described in the present section, we can rewrite linear equations and systems of linear equations in a convenient and short manner, as described below.

The linear equation in the unknowns $x_1, ..., x_n$ and parameters $a_1, ..., a_i, ..., a_n$, $b \in \mathbb{R}$

$$a_1 x_1 + ... + a_i x_i + ... + a_n x_n = b$$

can be rewritten as

$$\sum_{i=1}^{n} a_i x_i = b$$

or

$$a \cdot x = b$$

where $a = [a_1, ..., a_n]$ and $x = \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix}$.

The linear system (3.8) can be rewritten as

$$\begin{cases} \sum_{j=1}^{n} a_{1j} x_j = b_1 \\ ... \\ \sum_{j=1}^{n} a_{mj} x_j = b_m \end{cases}$$

or

$$\begin{cases} R^1(A)\, x = b_1 \\ ... \\ R^m(A)\, x = b_m \end{cases}$$

or

$$Ax = b$$

where $A = [a_{ij}]$.

Definition 79 *The trace of $A \in \mathcal{M}_{mm}$, written* tr A, *is the sum of the diagonal entries, i.e.,*

$$\text{tr } A = \sum_{i=1}^{m} a_{ii}.$$

Definition 80 *The identity matrix I_m is a diagonal matrix of order m with each element on the principal diagonal equal to* 1. *If no confusion arises, we simply write I in the place of I_m.*

Remark 81 ***1.*** $\forall n \in \mathbb{N} \setminus \{0\}, \ (I_m)^n = I_m$;
2. $\forall A \in \mathcal{M}_{m,n}, \ I_m A = A I_n = A$.

Proposition 82 *Let $A, B \in \mathcal{M}(m, m)$ and $k \in \mathbb{R}$. Then*
1. tr $(A + B) =$ tr $A +$ tr B;
2. tr $kA = k \cdot$ tr A;
3. tr $AB =$ tr BA.

Proof. Exercise. ■

3.2 Inverse matrices

Definition 83 *Given a matrix $A_{n\times n}$, , a matrix $B_{n\times n}$ is called an inverse of A if*

$$AB = BA = I_n.$$

We then say that A is invertible, or that A admits an inverse.

Proposition 84 *If A admits an inverse, then the inverse is unique.*

Proof. Let the inverse matrices B and C of A be given. Then

$$AB = BA = I_n \tag{3.9}$$

and

$$AC = CA = I_n \tag{3.10}$$

Left multiplying the first two terms in the equality (3.9) by C, we get

$$(CA)\,B = C\,(BA)$$

and from (3.10) and (3.9) we get $B = C$, as desired. ■
Thanks to the above Proposition, we can present the following definition.

Definition 85 *If the inverse of A does exist, then it is denoted by A^{-1}.*

Example 86 *1. Assume that for $i \in \{1, ..., n\}$, $\lambda_i \neq 0$. The diagonal matrix*

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

is invertible and its inverse is

$$\begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{bmatrix}.$$

*2. It is easy to check that the following matrix is **not** invertible.*

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Remark 87 *If a row or a column of A is zero, then A is not invertible, as verified below.*
Without loss of generality, assume the first row of A is equal to zero. Assume that B is the inverse of A. But then, since $I = AB$, we wold have $1 = R^1(A) \cdot C^1(B) = 0$, a contradiction.

Proposition 88 *If $A \in \mathcal{M}_{m,m}$ and $B \in \mathcal{M}_{m,m}$ are invertible matrices, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

$$(AB)\,B^{-1}A^{-1} = A\left(BB^{-1}\right)A^{-1} = AIA^{-1} = AA^{-1} = I.$$
$$B^{-1}A^{-1}\,(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}IB = B^{-1}B = I.$$

■

Remark 89 *The existence of the inverse matrix gives an obvious way of solving systems of linear equations with the same number of equations and unknowns.*
Given the system

$$A_{n\times n}x = b,$$

if A^{-1} exists, then

$$x = A^{-1}b.$$

Proposition 90 *(Some other properties of the inverse matrix)*
Let the invertible matrix A be given.
1. A^{-1} in invertible and $\left(A^{-1}\right)^{-1} = A$;
2. A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$;

Proof. 1. We want to verify that the inverse of A^{-1} is A, i.e.,

$$A^{-1}A = I \text{ and } AA^{-1} = I,$$

which is obvious.
2. Observe that

$$A^T \cdot \left(A^{-1}\right)^T = \left(A^{-1} \cdot A\right)^T = I^T = I,$$

and

$$\left(A^{-1}\right)^T \cdot A^T = \left(A \cdot A^{-1}\right)^T = I.$$

■

3.3 Elementary matrices

Below, we recall the definition of elementary row operations on a matrix $A \in \mathrm{M}_{m\times n}$ presented in Definition 34.

Definition 91 *An elementary row operation on a matrix $A \in M_{m\times n}$ is one of the following operations on the rows of A:*

$[\mathcal{E}_1]$ *(Row interchange) Interchange R^i with R^j, denoted by $R^i \leftrightarrow R^j$;*

$[\mathcal{E}_2]$ *(Row scaling) Multiply R^i by $k \in \mathbb{R}\setminus\{0\}$, denoted by $kR^i \to R^i$, $k \neq 0$;*

$[\mathcal{E}_3]$ *(Row addition) Replace R^i by (k times R^j plus R^i), denoted by $\left(R^i + kR^j\right) \to R^i$.*

Sometimes we apply $[\mathcal{E}_2]$ and $[\mathcal{E}_3]$ in one step, i.e., we perform the following operation

$[\mathcal{E}']$ Replace R^i by (k' times R^j and $k \in \mathbb{R}\setminus\{0\}$ times R^i), denoted by $\left(k'R^j + kR^i\right) \to R^i, k \neq 0$.

Definition 92 *Let $\mathfrak{E}$ be the set of functions $\mathcal{E} : \mathcal{M}_{m,n} \to \mathcal{M}_{m,n}$ which associate with any matrix $A \in \mathcal{M}_{m,n}$ a matrix $\mathcal{E}(A)$ obtained from A via an elementary row operation presented in Definition 91. For $i \in \{1, 2, 3\}$, let $\mathfrak{E}_i \subseteq \mathfrak{E}$ be the set of elementary row operation functions of type i presented in Definition 91.*

Definition 93 *For any $\mathcal{E} \in \mathfrak{E}$, define*

$$E_{\mathcal{E}} = \mathcal{E}(I_m) \in \mathcal{M}_{m,m}.$$

$E_{\mathcal{E}}$ is called the elementary matrix corresponding to the elementary row operation function $\mathcal{E}$.

With some abuse of terminology, we call any $\mathcal{E} \in \mathfrak{E}$ an elementary row operation (omitting the word "function"), and we sometimes omit the subscript $\mathcal{E}$.

Proposition 94 *Each elementary row operations $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ has an inverse, and that inverse is of the same type, i.e., for $i \in \{1, 2, 3\}$, $\mathcal{E} \in \mathfrak{E}_i \Leftrightarrow \mathcal{E}^{-1} \in \mathfrak{E}_i$.*

Proof. 1. The inverse of $R^i \leftrightarrow R^j$ is $R^j \leftrightarrow R^i$.
2. The inverse of $kR^i \to R^i$, $k \neq 0$ is $k^{-1}R^i \to R^i$.
3. The inverse of $\left(R^i + kR^j\right) \to R^i$ is $\left(-kR^j + R^i\right) \to R^i$. ■

Remark 95 *Given the row, canonical[1] vectors e_m^i, for $i \in \{1, ..., m\}$,*

$$I_m = \begin{bmatrix} e_m^1 \\ ... \\ e_m^i \\ ... \\ e_m^j \\ ... \\ e_m^n \end{bmatrix}$$

[1] See Definition 55.

The following Proposition shows that the result of applying an elementary row operation $\mathcal{E}$ to a matrix A can be obtained by premultiplying A by the corresponding elementary matrix $E_{\mathcal{E}}$.

Proposition 96 *For any* $A \in M_{m,n}$ *and for any* $\mathcal{E} \in \mathfrak{E}$,

$$\mathcal{E}(A) = \mathcal{E}(I_m) \cdot A := E_{\mathcal{E}} A. \tag{3.11}$$

Proof. *Recall that*

$$A = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^j(A) \\ ... \\ R^m(A) \end{bmatrix}$$

We have to prove that (3.11) *does hold true* $\forall \mathcal{E} \in \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$.

1. $\mathcal{E} \in \mathfrak{E}_1$.

First of all observe that

$$\mathcal{E}(I_m) = \begin{bmatrix} e_m^1 \\ ... \\ e_m^j \\ ... \\ e_m^i \\ ... \\ e_m^m \end{bmatrix} \quad \text{and} \quad \mathcal{E}(A) = \begin{bmatrix} R^1(A) \\ ... \\ R^j(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix}.$$

From (3.6),

$$\mathcal{E}(I_m) \cdot A = \begin{bmatrix} e_m^1 \cdot A \\ ... \\ e_m^j \cdot A \\ ... \\ e_m^i \cdot A \\ ... \\ e_m^m \cdot A \end{bmatrix} = \begin{bmatrix} R^1(A) \\ ... \\ R^j(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix},$$

as desired.

2. $\mathcal{E} \in \mathfrak{E}_2$.

Observe that

$$\mathcal{E}(I_m) = \begin{bmatrix} e_m^1 \\ ... \\ k \cdot e_m^i \\ ... \\ e_m^j \\ ... \\ e_m^m \end{bmatrix} \quad \text{and} \quad \mathcal{E}(A) = \begin{bmatrix} R^1(A) \\ ... \\ k \cdot R^j(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix}.$$

$$\mathcal{E}(I_m) \cdot A = \begin{bmatrix} e_m^1 \cdot A \\ ... \\ k \cdot e_m^i \cdot A \\ ... \\ e_m^j \cdot A \\ ... \\ e_m^m \cdot A \end{bmatrix} = \begin{bmatrix} R^1(A) \\ ... \\ k \cdot R^i(A) \\ ... \\ R^j(A) \\ ... \\ R^m(A) \end{bmatrix},$$

as desired.

3. $\mathcal{E} \in \mathfrak{E}_3$.

Observe that

$$\mathcal{E}(I_m) = \begin{bmatrix} e_m^1 \\ ... \\ e_m^i + k \cdot e_m^j \\ ... \\ e_m^j \\ ... \\ e_m^m \end{bmatrix} \quad \text{and} \quad \mathcal{E}(A) = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) + k \cdot R^j(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix}.$$

$$\mathcal{E}(I_m) \cdot A = \begin{bmatrix} e_m^1 \\ ... \\ e_m^i + k \cdot e_m^j \\ ... \\ e_m^j \\ ... \\ e_m^m \end{bmatrix} A = \begin{bmatrix} e_m^1 \cdot A \\ ... \\ \left(e_m^i + k \cdot e_m^j\right) \cdot A \\ ... \\ e_m^j \cdot A \\ ... \\ e_m^m \cdot A \end{bmatrix} = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) + k \cdot R^j(A) \\ ... \\ R^j(A) \\ ... \\ R^m(A) \end{bmatrix},$$

as desired. ■

Corollary 97 *If A is row equivalent to B, then there exist $k \in \mathbb{N}$ and elementary matrices $E_1, ..., E_k$ such that*

$$B = E_1 \cdot E_2 \cdot ... \cdot E_k \cdot A$$

Proof. It follows from the definition of row equivalence and Proposition 96. ■

Proposition 98 *Every elementary matrix $E_{\mathcal{E}}$ is invertible and $(E_{\mathcal{E}})^{-1}$ is an elementary matrix. In fact, $(E_{\mathcal{E}})^{-1} = E_{\mathcal{E}^{-1}}$.*

Proof. Given an elementary matrix E, from Definition 93, $\exists \mathcal{E} \in \mathfrak{E}$ such that

$$E = \mathcal{E}(I) \tag{3.12}$$

Define

$$E' = \mathcal{E}^{-1}(I).$$

Then

$$I \overset{\text{def. inv. func.}}{=} \mathcal{E}^{-1}(\mathcal{E}(I)) \overset{(3.12)}{=} \mathcal{E}^{-1}(E) \overset{\text{Prop. (96).}}{=} \mathcal{E}^{-1}(I) \cdot E \overset{\text{def. } E'}{=} E'E$$

and

$$I \overset{\text{def. inv.}}{=} \mathcal{E}\left(\mathcal{E}^{-1}(I)\right) \overset{\text{def. } E'}{=} \mathcal{E}(E') \overset{\text{Prop. (96).}}{=} \mathcal{E}(I) \cdot E' \overset{(3.12)}{=} EE'.$$

■

Corollary 99 *If $E_1, ..., E_k$ are elementary matrices, then*

$$P := E_1 \cdot E_2 \cdot ... \cdot E_k$$

is an invertible matrix.

Proof. It follows from Proposition 88 and Proposition 98. In fact, $\left(E_k^{-1} \cdot ... E_2^{-1} \cdot E_1^{-1}\right)$ is the inverse of P. ■

Proposition 100 *Let $A \in M_{m \times n}$ be given. Then, there exist a matrix $B \in M_{m \times n}$ in row canonical form, $k \in \mathbb{N}$ and elementary matrices $E_1, ..., E_k$ such that*

$$B = E_1 \cdot E_2 \cdot ... \cdot E_k \cdot A.$$

Proof. From Proposition 38, there exist $k \in \mathbb{N}$ elementary operations $\mathcal{E}^1, ..., \mathcal{E}^k$ such that

$$\left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^k\right)(A) = B.$$

From Proposition 96, $\forall j \in \{1, ..., k\}$,

$$\mathcal{E}^j(M) = \mathcal{E}^j(I) \cdot M := E_j \cdot M.$$

Then,

$$\left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^k\right)(A) = \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^{k-1}\right)\left(\mathcal{E}^k(A)\right) = \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^{k-1}\right)(E_k \cdot A) =$$

$$= \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^{k-2}\right) \circ \mathcal{E}^{k-1}(E_k \cdot A) = \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ... \circ \mathcal{E}^{k-2}\right)(E_{k-1} \cdot E_k \cdot A) =$$

$$= \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ...\mathcal{E}^{k-3}\right) \circ \mathcal{E}^{k-2}(E_{k-1} \cdot E_k \cdot A) = \left(\mathcal{E}^1 \circ \mathcal{E}^2 \circ ...\mathcal{E}^{k-3}\right)(E_{k-2} \cdot E_{k-1} \cdot E_k \cdot A) =$$

$$... = E_1 \cdot E_2 \cdot ... \cdot E_k \cdot A,$$

as desired. ■

Remark 101 *In fact, in Proposition 157, we will show that the matrix B of the above Corollary is unique.*

Proposition 102 *To be row equivalent is an equivalence relation.*

Proof. Obvious. ■

Proposition 103 $\forall n \in \mathbb{N} \setminus \{0\}$, *$A_{n \times n}$ is in row canonical form and it is invertible $\Leftrightarrow A = I$.*

Proof. [⇐]Obvious.
[⇒]
We proceed by induction on n.
Case 1. $n = 1$.
The case $n = 1$ is obvious. To try to better understand the logic of the proof, take $n = 2$, i.e., suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is in row canonical form and invertible. Observe that $A \neq 0$.
1. $a_{11} = 1$. Suppose $a_{11} = 0$. Then, from 1. in the definition of matrix in echelon form - see Definition 28 - $a_{12} \neq 0$ (otherwise, you would have a zero row not on the bottom of the matrix). Then, from 2. in that definition, we must have $a_{21} = 0$. But then the first column is zero, contradicting the fact that A is invertible - see Remark 87. Since $a_{11} \neq 0$, then from 2. in the Definition of row canonical form matrix - see Definition 31 - we get $a_{11} = 1$.
2. $a_{21} = 0$. It follows from the fact that $a_{11} = 1$ and 3. in Definition 31.
3. $a_{22} = 1$. Suppose $a_{22} = 0$, but then the last row would be zero, contradicting the fact that A is invertible and a_{22} is the leading nonzero entry of the second row, i.e., $a_{22} \neq 0$. Then from 2. in the Definition of row canonical form matrix, we get $a_{22} = 1$.
4. $a_{12} = 0$. It follows from the fact that $a_{22} = 1$ and 3. in Definition 31.
Case 2. Assume that statement is true for $n - 1$.
Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1j} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2j} & ... & a_{2n} \\ ... \\ a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{n1} & a_{n2} & ... & a_{nj} & ... & a_{nn} \end{bmatrix}$$

is in row canonical form and invertible.
1. $a_{11} = 1$. Suppose $a_{11} = 0$. Then, from 1. in the definition of matrix in echelon form - see Definition 28 -

$$(a_{12}, ..., a_{1n}) \neq 0.$$

Then, from 2. in that definition, we must have

$$\begin{bmatrix} a_{21} \\ ... \\ a_{i1} \\ ... \\ a_{n1} \end{bmatrix} = 0.$$

But then the first column is zero, contradicting the fact that A is invertible - see Remark 87. Since $a_{11} \neq 0$, then from 2. in the Definition of row canonical form matrix - see Definition 31 - we get $a_{11} = 1$.

2. Therefore, we can rewrite the matrix as follows

$$A = \begin{bmatrix} 1 & a_{12} & ... & a_{1j} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2j} & ... & a_{2n} \\ ... \\ 0 & a_{i2} & ... & a_{ij} & ... & a_{in} \\ ... \\ 0 & a_{n2} & ... & a_{nj} & ... & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & A_{22} \end{bmatrix} \tag{3.13}$$

with obvious definitions of a and A_{22}. Since, by assumption, A is invertible, there exists B which we can partition in the same we partitioned A, i.e.,

$$B = \begin{bmatrix} b_{11} & b \\ c & B_{22} \end{bmatrix}.$$

and such that B is invertible. Then,

$$I_n = BA = \begin{bmatrix} b_{11} & b \\ c & B_{22} \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{11} + bB_{22} \\ c & ca + B_{22}A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \end{bmatrix};$$

then $c = 0$ and $A_{22}B_{22} = I_{n-1}$.

Moreover,

$$I_n = AB = \begin{bmatrix} 1 & a \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b \\ c & B_{22} \end{bmatrix} = \begin{bmatrix} b_{11} + ac & b + aB_{22} \\ c & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \end{bmatrix}. \tag{3.14}$$

Therefore, A_{22} is invertible. From 3.13, A_{22}can be obtained from A erasing the first row and then erasing a column of zero, from Remark 33, A_{22} is a row reduced form matrix. Then, we can apply the assumption of the induction argument to conclude that $A_{22} = I_{n-1}$. Then, from 3.13,

$$A = \begin{bmatrix} 1 & a \\ 0 & I \end{bmatrix}.$$

Since, by assumption, $A_{n\times n}$ is in row canonical form, from 3. in Definition 31, $a = 0$, and, as desired $A = I$. ■

Proposition 104 *Let A belong to $M_{m,m}$. Then the following statements are equivalent.*

1. *A is invertible;*
2. *A is row equivalent to I_m;*
3. *A is the product of elementary matrices.*

Proof. 1. ⇒ 2.

From Proposition 100, there exist a matrix $B \in \mathrm{M}_{m\times m}$ in row canonical form, $k \in \mathbb{N}$ and elementary matrices $E_1, ..., E_k$ such that

$$B = E_1 \cdot E_2 \cdot ... \cdot E_k \cdot A.$$

Since A is invertible and, from Corollary 99, $E_1 \cdot E_2 \cdot ... \cdot E_k$ is invertible as well, from Proposition 88, B is invertible as well. Then, from Proposition 103, $B = I$.

2. ⇒ 3.

By assumption and from Corollary 97, there exist $k \in \mathbb{N}$ and elementary matrices $E_1, ..., E_k$ such that

$$A = E_1 \cdot E_2 \cdot ... \cdot E_k \cdot I,$$

Since $\forall i \in \{1, ..., k\}$, E_i is an elementary matrix, the desired result follows.
3. $\Rightarrow$ 1.
By assumption, there exist $k \in \mathbb{N}$ and elementary matrices $E_1, ..., E_k$ such that

$$A = E_1 \cdot E_2 \cdot ... \cdot E_k.$$

Since, from Proposition 98, $\forall i \in \{1, ..., k\}$, E_i is invertible, A is invertible as well, from Proposition 88.
■

Proposition 105 *Let $A_{m\times n}$ be given.*
1. $B_{m\times n}$ is row equivalent to $A_{m\times n}$ $\Leftrightarrow$ there exists an invertible $P_{m\times m}$ such that $B = PA$.
2. $P_{m\times m}$ is an invertible matrix $\Rightarrow$ PA is row equivalent to A.

Proof. 1.
[$\Rightarrow$] From Corollaries 99 and 97, $B = E_1 \cdot ... \cdot E_k \cdot A$ with $(E_1 \cdot ... \cdot E_k)$ invertible matrix. Then, it suffices to take $P = E_1 \cdot ... \cdot E_k$.
[$\Leftarrow$] From Proposition 104, P is row equivalent to I, i.e., there exist $E_1, ..., E_k$ such that $P = E_1 \cdot ... \cdot E_k \cdot I$. Then by assumption $B = E_1 \cdot ... \cdot E_k \cdot I \cdot A$, i.e., B is row equivalent to A.
2.
From Proposition 104, P is the product of elementary matrices. Then, the desired result follows from Proposition 96. ■

Proposition 106 *If A is row equivalent to a matrix with a zero row, then A is not invertible.*

Proof. Suppose otherwise, i.e., A is row equivalent to a matrix C with a zero row and A is invertible. From Proposition 105, there exists an invertible P such that $A = PC$ and then $P^{-1}A = C$. Since A and $P^{-1}$are invertible, then, from Proposition 88, $P^{-1}A$ is invertible, while C, from Remark 87, C is not invertible, a contradiction. ■

Remark 107 *From Proposition 104, we know that if $A_{m\times m}$ is invertible, then there exist $E_1, ..., E_k$ such that*

$$I = E_1 \cdot ... \cdot E_k \cdot A \tag{3.15}$$

or

$$A^{-1} = E_1 \cdot ... \cdot E_k \cdot I. \tag{3.16}$$

Then, from (3.15)and (3.16), if A is invertible then A^{-1} is equal to the finite product of those elementary matrices which "transform" A in I, or, equivalently, can be obtained applying a finite number of corresponding elementary operations to the identity matrix I. That observation leads to the following (Gaussian elimination) algorithm, which either show that an arbitrary matrix $A_{m\times m}$ is not invertible or finds the inverse of A.

An algorithm to find the inverse of a matrix $A_{m\times m}$ or to show the matrix is not invertible.

Step 1. Construct the following matrix $M_{m\times(2m)}$:

$$\begin{bmatrix} A & I_m \end{bmatrix}$$

Step 2. Row reduce M to echelon form. If the process generates a zero row in the part of M corresponding to A, then stop: A is not invertible : A is row equivalent to a matrix with a zero row and therefore, from Proposition 106 is not invertible. Otherwise, the part of M corresponding to A is a triangular matrix.

Step 3. Row reduce M to the row canonical form

$$\begin{bmatrix} I_m & B \end{bmatrix}$$

Then, from Remark 107, $A^{-1} = B$.

Example 108 *We find the inverse of*

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix},$$

applying the above algorithm.
Step 1.

$$M = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{bmatrix}.$$

Step 2.

$$\begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -2 & 1 & 0 \\ 0 & 1 & 0 & | & -4 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & -6 & 1 & 1 \end{bmatrix}$$

The matrix is invertible.
Step 3.

$$\begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & -6 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & 6 & -1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -11 & 2 & 2 \\ 0 & -1 & 0 & | & 4 & 0 & -1 \\ 0 & 0 & 1 & | & 6 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & | & -11 & 2 & 2 \\ 0 & 1 & 0 & | & -4 & 0 & 1 \\ 0 & 0 & 1 & | & 6 & -1 & -1 \end{bmatrix}$$

Then

$$A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}.$$

Example 109

$$\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 4 & 2 & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 0 & -10 & | & -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 0 & 1 & | & \frac{4}{10} & -\frac{1}{10} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -\frac{2}{10} & \frac{0}{3} \\ 0 & 1 & | & \frac{4}{10} & -\frac{1}{10} \end{bmatrix}$$

3.4 Elementary column operations

This section repeats some of the discussion of the previous section using column instead of rows of a matrix.

Definition 110 *An elementary column operation is one of the following operations on the columns of* $A_{m\times n}$*:*

$[\mathcal{F}_1]$ *(Column interchange) Interchange* C_i *with* C_j*, denoted by* $C_i \leftrightarrow C_j$*;*

$[\mathcal{F}_2]$ *(Column scaling) Multiply* C_i *by* $k \in \mathbb{R}\setminus\{0\}$*, denoted by* $kC_i \to C_i$*,* $k \neq 0$*;*

$[\mathcal{F}_3]$ *(Column addition) Replace* C_i *by (k times* C_j *plus* C_i *), denoted by* $(C_i + kC_j) \to C_i$*..*

Each of the above column operation has an inverse operation of the same type just like the corresponding row operations.

Definition 111 *Let* $\mathcal{F}$ *be an elementary column operation on a matrix* $A_{m\times n}$*. We denote the resulting matrix by* $\mathcal{F}(A)$*. We define also*

$$F_{\mathcal{F}} = \mathcal{F}(I_n) \in \mathcal{M}_{n,n}.$$

$F_{\mathcal{F}}$ *is then called an elementary matrix corresponding to the elementary column operation* $\mathcal{F}$*. We sometimes omit the subscript* $\mathcal{F}$*.*

Definition 112 *Given an elementary row operation $\mathcal{E}$, define $\mathcal{F}_{\mathcal{E}}$, if it exists*[2]*, as the column operation obtained by $\mathcal{E}$ substituting the word row with the word column. Similarly, given an elementary column operation $\mathcal{F}$ define $\mathcal{E}_{\mathcal{F}}$, if it exists, as the row operation obtained by $\mathcal{F}$ substituting the word column with the word row.*

In what follows, $\mathcal{F}$ and $\mathcal{E}$ are such that $\mathcal{F} = \mathcal{F}_{\mathcal{E}}$ and $\mathcal{E}_{\mathcal{F}} = \mathcal{E}$.

Proposition 113 *Let a matrix $A_{m\times n}$ be given. Then*

$$\mathcal{F}(A) = \left[\mathcal{E}\left(A^T\right)\right]^T.$$

Proof. The above fact is equivalent to $\mathcal{E}\left(A^T\right) = (\mathcal{F}(A))^T$ and it is a consequence of the fact that the columns of A are the rows of A^T and vice versa. As an exercise, carefully do the proof in the case of each of the three elementary operation types. ■

Remark 114 *The above Proposition says that applying the column operation $\mathcal{F}$ to a matrix A gives the same result as applying the corresponding row operation $\mathcal{E}_{\mathcal{F}}$ to A^T and then taking the transpose.*

Proposition 115 *Let a matrix $A_{m\times n}$ be given. Then*
1.

$$\mathcal{F}(A) = A \cdot (\mathcal{E}(I))^T = A \cdot \mathcal{F}(I),$$

or, since $E := \mathcal{E}(I)$ and $F := \mathcal{F}(I)$,

$$\mathcal{F}(A) = A \cdot E^T = A \cdot F. \tag{3.17}$$

2. $F = E^T$ and F is invertible.

Proof. 1.

$$\mathcal{F}(A) \overset{Lemma\ 113}{=} \left[\mathcal{E}\left(A^T\right)\right]^T \overset{Lemma\ 96}{=} \left(\mathcal{E}(I) \cdot A^T\right)^T = A \cdot (\mathcal{E}(I))^T \overset{Lemma\ 113}{=} A \cdot \mathcal{F}(I).$$

2. From (3.17), we then get

$$F := \mathcal{F}(I) = I \cdot E^T = E^T.$$

From Proposition 90 and Proposition 98, it follows that F is invertible. ■

Remark 116 *The above Proposition says that says that the result of applying an elementary column operation $\mathcal{F}$ on a matrix A can be obtained by postmultiplying A by the corresponding elementary matrix F.*

Definition 117 *A matrix $B_{m\times n}$ is said column equivalent to a matrix $A_{m\times n}$ if B can be obtained from A using a finite number of elementary column operations.*

Remark 118 *By definition of row equivalent, column equivalent and transpose of a matrix, we have that*

$$A \text{ and } B \text{ are row equivalent} \Leftrightarrow \quad A^T \text{ and } B^T \text{ are column equivalent,}$$

and

$$A \text{ and } B \text{ are column equivalent} \Leftrightarrow \quad A^T \text{ and } B^T \text{ are row equivalent.}$$

Proposition 119 *1. $B_{m\times n}$ is column equivalent to $A_{m\times n}$ $\Leftrightarrow$ there exists an invertible $Q_{n\times n}$ such that $B_{m\times n} = A_{m\times n}Q_{n\times n}$.*
2. $Q_{n\times n}$ in invertible matrix $\Rightarrow$ AQ is column equivalent to A.

Proof. It is very similar to the proof of Proposition 105. ■

Definition 120 *A matrix $B_{m\times n}$ is said equivalent to a matrix $A_{m\times n}$ if B can be obtained from A using a finite number of elementary row and column operations.*

---

[2] Of course, if you exchange the first and the third row, and the matrix has only two columns, you cannot exchange the first and the third column.

Proposition 121 *A matrix $B_{m\times n}$ is equivalent to a matrix $A_{m\times n}$ $\Leftrightarrow$ there exist invertible matrices $P_{m\times m}$ and $Q_{n\times n}$ such that $B_{m\times n} = P_{m\times m}A_{m\times n}Q_{n\times n}$.*

Proof. [$\Rightarrow$]
By assumption $B = E_1 \cdot ... \cdot E_k \cdot A \cdot F_1 \cdot ... \cdot F_h$.
[$\Leftarrow$]
Similar to the proof of Proposition 105. ■

Proposition 122 *For any matrix $A_{m\times n}$ there exists a number $r \in \{0, 1, ..., \min\{m, n\}\}$ such that A is equivalent to the block matrix of the form*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.18}$$

Proof. The proof is constructive in the form of an algorithm.

Step 1. Row reduce A to row canonical form, with leading nonzero entries $a_{11}, a_{2j_2}, ..., a_{j_{j_r}}$.

Step 2. Interchange C^2 and C_{j_2}, C^3 and C_{j_3} and so on up to C_r and C_{j_r}.You then get a matrix of the form

$$\begin{bmatrix} I_r & B \\ 0 & 0 \end{bmatrix}.$$

Step 3. Use column operations to replace entries in B with zeros.

■

Remark 123 *From Proposition 157 the matrix in Step 2 is unique and therefore the resulting matrix in Step 3, i.e., matrix (3.18) is unique.*

Proposition 124 *For any $A \in M_{m,n}$, there exists invertible matrices $P \in M_{m,m}$ and $Q \in M_{n,n}$ and $r \in \{0, 1, ..., \min\{m, n\}\}$ such that*

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Proof. It follows immediately from Propositions 122 and 121. ■

Remark 125 *From Proposition 157 the number r in the statement of the previous Proposition is unique.*

Example 126 *Take*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix} \overset{R^3-(R^1+R^2)\to R^3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

$$\overset{R^2-4R^1\to R^2}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frac{-1}{3}R^2\to R^2}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

$$\overset{C^2-2C^1\to C^2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \overset{C^3-3C^1\to C^3}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \overset{C^3-2C^2\to C^3}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can find matrices P and Q as follows.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Then

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{4}{3} & -\frac{1}{3} & 0 \\ -1 & -1 & 1 \end{bmatrix};$$

indeed

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{4}{3} & -\frac{1}{3} & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

d
Then

$$Q = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix};$$

indeed,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Summarizing

$$PAQ = \begin{bmatrix} 1 & 0 & 0 \\ \frac{4}{3} & -\frac{1}{3} & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Proposition 127 *If $A_{m\times m}B_{m\times m} = I$, then $BA = I$ and therefore A is invertible and $A^{-1} = B$.*

Proof. Suppose A is not invertible, the from Proposition 104 is not row equivalent to I_m and from Proposition 122, A is equivalent to a block matrix of the form displayed in (3.18) with $r < m$. Then, from Proposition 121, there exist invertible matrices $P_{m\times m}$ and $Q_{m\times m}$ such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

and from $AB = I$, we get

$$P = PAQQ^{-1}B$$

and

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \left(Q^{-1}B\right) = P$$

Therefore, P has some zero rows and columns, contradicting that P is invertible. ■

Remark 128 *The previous Proposition says that to verify that A in invertible it is enough to check that $AB = I$.*

Remark 129 *We will come back to the analysis of further properties of the inverse and on another way of computing it in Section 5.3.*

3.5 Exercises

From Lipschutz (1991),
starting from page 81: 3.1 - 3.11, 3.14 - 3.16;
starting from page 111: 4.1, 4.4, 4.5, 4.7, 4.8.

Chapter 4

Vector spaces

4.1 Definition

Definition 130 *Let a nonempty set F with the operations of*
addition which assigns to any $x, y \in F$ an element denoted by $x \oplus y \in F$, and
multiplication which assigns to any $x, y \in F$ an element denoted by $x \odot y \in F$
be given. $(F, \oplus, \odot)$ is called a field, if the following properties hold true.

1. *(Commutative) $\forall x, y \in F$, $x \oplus y = y \oplus x$ and $x \odot y = y \odot x$;*
2. *(Associative) $\forall x, y, z \in F, (x \oplus y) \oplus z = x \oplus (y \oplus z)$ and $(x \odot y) \odot z = x \odot (y \odot z)$;*
3. *(Distributive) $\forall x, y, z \in F$, $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$;*
4. *(Existence of null elements) $\exists f_0, f_1 \in F$ such that $\forall x \in F$, $f_0 \oplus x = x$ and $f_1 \odot x = x$;*
5. *(Existence of a negative element) $\forall x \in F$ $\exists y \in F$ such that $x \oplus y = f_0$;*

   From the above properties, it follows that f_0 and f_1 are unique.[1] *We denote f_0 and f_1 by 0 and 1, respectively.*
6. *(Existence of an inverse element) $\forall x \in F \setminus \{0\}$, $\exists y \in F$ such that $x \odot y = 1$.*

   Elements of a field are called scalars.

Example 131 *The set $\mathbb{R}$ of real numbers with the standard addition and multiplication is a field. From the above properties all the rules of "elementary" algebra can be deduced.*[2] *The set $\mathbb{C}$ of complex numbers is a field .*

The sets $\mathbb{N} := \{1, 2, , 3, ..., n, ...\}$ and $\mathbb{Z}$ of positive integers and integers, respectively, with the standard addition and multiplication are not fields.

Definition 132 *Let $(F, \oplus, \odot)$ be a field and V be a nonempty set with the operations of*
addition which assigns to any $u, v \in V$ an element denoted by $u + v \in V$, and
scalar multiplication which assigns to any $u \in V$ and any $\alpha \in F$ an element $\alpha \cdot u \in V$.

Then $(V, +, \cdot)$ is called a vector space on the field $(F, \oplus, \odot)$ and its elements are called vectors if the following properties are satisfied.

A1. (Associative) $\forall u, v, w \in V, \quad (u + v) + w = u + (v + w)$;
A2. (existence of zero element) there exists an element 0 in V such that $\forall u \in V,\ u + 0 = u$;
A3. (existence of inverse element) $\forall u \in V$ $\exists v \in V$ such that $u + v = 0$;
A4. (Commutative) $\forall u, v \in V,\ u + v = v + u$;
M1. (distributive) $\forall \alpha \in F$ and $\forall u, v \in V$, $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$;
M2. (distributive) $\forall \alpha, \beta \in F$ and $\forall u \in V$, $(\alpha \oplus \beta) \cdot u = \alpha \cdot u + \beta \cdot u$;
M3. $\forall \alpha, \beta \in F$ and $\forall u \in V$, $(\alpha \odot \beta) \cdot u = \alpha \cdot (\beta \cdot u)$;
M4. $\forall u \in V$, $1 \cdot u = u$.
Elements of a vector space are called vectors.

[1] The proof of that result is very similar to the proof of Proposition 134.1 and 2.
[2] See, for example, Apostol (1967), Section 13.2, page 17.

Remark 133 *In the remainder of these notes, if no confusion arises, for ease of notation, we will denote a field simply by F and a vector space by V. Moreover, we will write $+$ in the place of $\oplus$, and we will omit $\odot$ and $\cdot$, i.e., we will write xy instead of $x \odot y$ and αv instead of $\alpha \cdot v$.*

Proposition 134 *If V is a vector space, then (as a consequence of the first four properties)*
1. *The zero vector is unique and it is denoted by 0.*
2. *$\forall u \in V$, the inverse element of u is unique and it is denoted by $-u$.*
3. *(cancellation law) $\forall u, v, w \in V$,*

$$u + w = v + w \Rightarrow u = v.$$

Proof. 1. Assume that there exist $0_1, 0_2 \in V$ which are zero vectors. Then by definition of zero vector - see $(A2)$ - we have that

$$0_1 + 0_2 = 0_1 \quad \text{and} \quad 0_2 + 0_1 = 0_2.$$

From $(A.4)$,

$$0_1 + 0_2 = 0_2 + 0_1,$$

and therefore $0_1 = 0_2$.
2. Given $u \in V$, assume there exist $v^1, v^2 \in V$ such that

$$u + v^1 = 0 \quad \text{and} \quad u + v^2 = 0.$$

Then

$$v^2 = v^2 + 0 = v^2 + \left(u + v^1\right) = \left(v^2 + u\right) + v^1 = \left(u + v^2\right) + v^1 = 0 + v^1 = v^1.$$

3.

$$u + w = v + w \overset{(1)}{\Rightarrow} u + w + (-w) = v + w + (-w) \overset{(2)}{\Rightarrow} u + 0 = v + 0 \overset{(3)}{\Rightarrow} u = v,$$

where (1) follows from the definition of operation, (2) from the definition of $-w$ and (3) from the definition of 0. ■

Remark 135 *From A2. in Definition 132, we have that for any vector space V, $0 \in V$.*

Proposition 136 *If V is a vector space over a field F, then*

1. *For $0 \in F$ and $\forall u \in V$, $0u = 0$.*
2. *For $0 \in V$ and $\forall \alpha \in F$, $\alpha 0 = 0$.*
3. *If $\alpha \in F$, $u \in V$ and $\alpha u = 0$, then either $\alpha = 0$ or $u = 0$ or both.*
4. *$\forall \alpha \in F$ and $\forall u \in V$, $(-\alpha) u = \alpha (-u) = -(\alpha u) := -\alpha u$.*

Proof. 1. From $(M1)$,

$$0u + 0u = (0 + 0) u = 0u.$$

Then, adding $-(0u)$ to both sides,

$$0u + 0u + (-(0u)) = 0u + (-(0u))$$

and, using $(A3)$,

$$0u + 0 = 0$$

and, using $(A2)$, we get the desired result.
2. From $(A2)$,

$$0 + 0 = 0;$$

then multiplying both sides by α and using $(M1)$,

$$\alpha 0 = \alpha (0 + 0) = \alpha 0 + \alpha 0;$$

and, using $(A3)$,

$$\alpha 0 + (-(\alpha 0)) = \alpha 0 + \alpha 0 + (-(\alpha 0))$$

and, using $(A2)$, we get the desired result.

3. Assume that $\alpha u = 0$ and $\alpha \neq 0$. Then

$$u = 1u = \left(\alpha^{-1} \cdot \alpha\right) u = \alpha^{-1} (\alpha u) = \alpha^{-1} \cdot 0 = 0.$$

Taking the contropositive of the above result, we get $\langle u \neq 0 \rangle \Rightarrow \langle \alpha u \neq 0 \vee \alpha = 0 \rangle$. Therefore $\langle u \neq 0 \wedge \alpha u = 0 \rangle \Rightarrow \langle \alpha = 0 \rangle$.

4. From $u + (-u) = 0$, we get $\alpha (u + (-u)) = \alpha 0$, and then $\alpha u + \alpha (-u) = 0$, and therefore $-(\alpha u) = \alpha (-u)$.

From $\alpha + (-\alpha) = 0$, we get $(\alpha + (-\alpha)) u = 0u$, and then $\alpha u + (-\alpha) u = 0$, and therefore $-(\alpha u) = (-\alpha) u$.
■

Remark 137 *From Proposition 136.4, and $(M4)$in Definition 132, we have*

$$(-1) u = 1 (-u) = -(1u) = -u.$$

We also define subtraction as follows:

$$v - u := v + (-u)$$

4.2 Examples

Euclidean spaces.[3]

The Euclidean space $\mathbb{R}^n$ with sum and scalar product defined in Chapter 2 is a vector space over the field $\mathbb{R}$ with the standard addition and multiplication

Matrices on $\mathbb{R}$.

For any $m, n \in \mathbb{N}$, the set $\mathcal{M}_{m,n}$ of matrices with elements belonging to the field $\mathbb{R}$ with the operation of addition and scalar multiplication, as defined in Section 2.3 is a vector space on the field $\mathbb{R}$ and it is denoted by

$$\mathbb{M}(m, n).$$

Matrices on a field F.

For any $m, n \in \mathbb{N}$, we can also consider the set of matrices whose entries are elements belonging to an arbitrary field F. It is easy to check that set is a vector space on the field F, with the operation of addition and scalar multiplication inherited by F, is a vector space and it is denoted by

$$\mathbb{M}_F(m, n).$$

We do set

$$\mathbb{M}_{\mathbb{R}}(m, n) = \mathbb{M}(m, n).$$

Polynomials

The set of *all* polynomials

$$a_0 + a_1 t + a_2 t^2 + ... + a_n t^n$$

with $n \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, a_2, ..., a_n \in \mathbb{R}$ is a vector space on $\mathbb{R}$ with respect to the standard sum between polynomials and scalar multiplication.

Function space $\mathcal{F}(X)$.

Given a nonempty set X, the set of all functions $f : X \to \mathbb{R}$ with obvious sum and scalar multiplication is a a vector space on $\mathbb{R}$. More generally we can consider the vector space of functions $f : X \to F$, where X is a nonempty set and F is an arbitrary field, on the same field F.

Sets which are not vector spaces.

$(0, +\infty)$ and $[0, +\infty)$ are **not** a vector spaces in $\mathbb{R}$.

For any $n \in \mathbb{N}$, the set of all polynomials of degree n is not a vector space on $\mathbb{R}$.

`On the role of the field: put Exercise 5.29 from Lipschutz 2nd edition.`

[3] A detailed proof of some of the statements below is contained, for example, in Hoffman and Kunze (1971), starting from page 28.

4.3 Vector subspaces

In what follows, if no ambiguity may arise, we will say "vector space" instead of "vector space on a field".

Definition 138 *Let W be a subset of a vector space V. W is called a vector subspace of V if W is a vector space with respect to the operation of vector addition and scalar multiplication defined on V restricted to W. In other words, given a vector space $(V,+,\cdot)$ on a field $(F,\oplus,\odot)$ and a subset W of V, we say that W is a vector subspace of V if, defined*

$$+_{|W\times W}: W\times W\to W, \qquad (w_1,w_2)\mapsto w_1+w_2$$

and

$$\cdot_{|F\times W}: F\times W\to W, \qquad (\alpha,w_2)\mapsto \alpha\cdot w_2,$$

then $\left(W,+_{|W\times W},\cdot_{|F\times W}\right)$ is a vector space on the field $(F,\oplus,\odot)$.

Proposition 139 *Let W be a subset of a vector space V. The following three statements are equivalent.*

1. *W is a vector subspace of V.*
2. *a. $W\neq\varnothing$;*
   b. $\forall u,v\in W,\ u+v\in W$;
   c. $\forall u\in W,\alpha\in F,\ \alpha u\in W$.
3. *a. $W\neq\varnothing$;*
   b. $\forall u,v\in W,\ \forall\alpha,\beta\in F,\ \alpha u+\beta v\in W$.

Proof. 2. $\Rightarrow$ 3.
From 2c., $\alpha u\in W$ and $\beta v\in W$. Then, from 2b., $\alpha u+\beta v\in W$, as desired.
2. $\Leftarrow$ 3.
From 3b., identifying α,u,β,v with $1,u,1,v$, we get $u+v\in W$, i.e., 2b.
From 3b., identifying α,u,β,v with $\frac{1}{2}\alpha,u,\frac{1}{2}\alpha,u$, we get $\frac{1}{2}\alpha u+\frac{1}{2}\alpha u\in W$. Moreover, since $W\subseteq V$, then $u\in V$ and from the distributive property $(M1)$ for V, we get $\frac{1}{2}\alpha u+\frac{1}{2}\alpha u=\left(\frac{1}{2}\alpha+\frac{1}{2}\alpha\right)u=\alpha u$.
1. $\Rightarrow$ 2.
All properties follow immediately from the definition of vector space. Let's check the commutative property. We want to show that given $w^1,w^2\in W$, then $w^1+w^2=w^2+w^1$. Indeed,

$$w^1,w^2\in W \overset{W\subseteq V}{\Rightarrow} w^1,w^2\in V,$$

and then

$$w^1+_{|W\times W}w^2 \overset{\text{def. restriction}}{=} w^1+w^2 \overset{\text{prop. in V}}{=} w^2+w^1=w^2+_{|W\times W}w^1.$$

1. $\Leftarrow$ 2.
2a., 2b. and 2c. insure that the "preliminary properties" of a vector space are satisfied: $W\neq\varnothing$ and the operations are well defined.
Properties $A1,A4,M1,M2,M3$ and $M4$ are satisfied because $W\subseteq V$. Let's check the remaining two properties.
$(A2)$: We want to show that there exists $0_W\in W$ such that $\forall w\in W$, we have $w+0_W=w$.
Since $W\neq\varnothing$, we can take $w\in W$. Since $W\subseteq V$, then $w\in V$ and from Proposition 136.1,

$$0_F w=0_V. \tag{4.1}$$

Moreover, from 2c., identifying α,u with $0_F,w$, we get $0_V\in W$. Then $\forall w\in W\subseteq V$, $w+0_V=w$, i.e.,

$$0_W=0_V:=0. \tag{4.2}$$

$(A3)$: We want to show that $\forall w\in W\ \ \exists w'\in W$ such that $w+w'=0$.
From 2.c, we have $(-1)\,w\in W$ and $1w\in W$. Then

$$(-1)\,w+w \overset{(M4)}{=} (-1)\,w+1w \overset{(M1)}{=} (-1+1)\,w=0_F w \overset{(136.1)}{=} 0.$$

Taking $w'=(-1)\,w$, we have shown the desired result. ∎

Remark 140 ***("A Recipe to try to show that a set is a vector space")*** *An often successful way to show that a set S is a vector space is the following one:*

1. find a vector space V such that $S \subseteq V$; in Section 4.2 above, we provide a list of "commonly used" vector spaces;

2. use Proposition 139.

Example 141 *1. Given an arbitrary vector space V, $\{0\}$ and V are vector subspaces of V.*

2. Given $\mathbb{R}^3$,

$$W := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\right\}$$

is a vector subspace of $\mathbb{R}^3$. Observe that $w \in W$ if and only if there exist $w_1, w_2 \in \mathbb{R}$ such that $w = (w_1, w_2, 0)$.

3. Given the space V of polynomials, the set W of all polynomials of degree $\leq n$ is a vector subspace of V.

4. The set of all bounded or continuous or differentiable or integrable functions $f : X \to \mathbb{R}$ is a vector subspace of $\mathcal{F}(X)$.

5. If V and W are vector spaces, then $V \cap W$ is a vector subspace of V and W.

6. $[0, +\infty)$ is not a vector subspace of $\mathbb{R}$.

7. Let $V = \{0\} \times \mathbb{R} \subseteq \mathbb{R}^2$ and $W = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. Then $V \cup W$ is not a vector subspace of $\mathbb{R}^2$.

4.4 Linear combinations

Notation convention. Unless otherwise stated, a greek (or Latin) letter with a subscript denotes a scalar; a Latin letter with a superscript denotes a vector.

Definition 142 *Let a vector space V on a field F, $m \in \mathbb{N}$ and vectors $v^1, v^2, ..., v^m \in V$ be given. The linear combination of the vectors $v^1, v^2, ..., v^m$ via scalars $\alpha_1, \alpha_2, ..., \alpha_m \in F$ is the vector*

$$\sum_{i=1}^{m} \alpha_i v^i.$$

The set of all such combinations

$$\left\{v \in V : \exists (\alpha_i)_{i=1}^m \in F^m \text{ such that } v = \sum_{i=1}^m \alpha_i v^i\right\}$$

is called span of $v^1, v^2, ..., v^m$ and it is denoted by

$$\operatorname{span}\left(v^1, v^2, ..., v^m\right).$$

`add examples`

Definition 143 *Let V be a vector space and S a subset of V. $\operatorname{span}(S)$ is the set of all linear combinations of vectors in S, i.e.,*

$$\operatorname{span}(S) = \left\{v \in V : \exists m \in \mathbb{N}, \quad \exists v^1, v^2, ..., v^m \in S, \quad \exists (\alpha_i)_{i=1}^m \in F^m \text{ such that } v = \sum_{i=1}^m \alpha_i v^i\right\}.$$

$\operatorname{span}(\varnothing) := \{0\}$.

Proposition 144 *Let V be a vector space and $S \neq \varnothing, S \subseteq V$. Then, "$\operatorname{span}(S)$ is the smallest vector space containing S", i.e.,*

1. a. $S \subseteq \operatorname{span}(S)$ and b. $\operatorname{span}(S)$ is a vector subspace of V.

2. If W is a subspace of V and $S \subseteq W$, then $\operatorname{span}(S) \subseteq W$.

Proof. 1a. Given $v \in S$, $1v = v \in \text{span}(S)$. 1b. Since $S \neq \varnothing$, then $\text{span}(S) \neq \varnothing$. Given $\alpha, \beta \in F$ and $v, w \in span\ S$. Then $\exists \alpha_1, ..., \alpha_n \in F$, $v^1, ..., v^n \in S$ and $\beta_1, ..., \beta_m \in F$, $w^1, ..., w^m \in S$, such that $v = \sum_{i=1}^n \alpha_i v^i$ and $w = \sum_{j=1}^m \beta_j w^j$. Then

$$\alpha v + \beta w = \sum_{i=1}^n (\alpha \alpha_i) v^i + \sum_{j=1}^m (\beta \beta_j) w^j \in \text{span} S.$$

2. Take $v \in \text{span}\ S$. Then $\exists \alpha_1, ..., \alpha_n \in F$, $v^1, ..., v^n \in S \subseteq W$ such that $v = \sum_{i=1}^n \alpha_i v^i \in W$, as desired. ■

Definition 145 *Let V be a vector space and $v^1, v^2, ..., v^m \in V$. If $V = \text{span}\left(v^1, v^2, ..., v^m\right)$, we say that V is the vector space generated or spanned by the vectors $v^1, v^2, ..., v^m$.*

Example 146 *1.* $\mathbb{R}^3 = \text{span}\left(\{(1,0,0), (0,1,0), (0,0,1)\}\right)$;
2. $\text{span}\left(\{(1,1)\}\right) = \text{span}\left(\{(1,1), (2,2)\}\right) = \left\{(x,y) \in \mathbb{R}^2 : x = y\right\}$;
3. $\text{span}\left(\{(1,1), (0,1)\}\right) = \mathbb{R}^2$.
2. $\text{span}\left(\{t^n\}_{n \in \mathbb{N}}\right)$ *is equal to the vector space of all polynomials.*

Exercise 147 *1. If $A \subseteq B$, then* $\text{span}(A) \subseteq \text{span}(B)$;
2. if W is a vector subspace of V, then $\text{span}(W) = W$.

4.5 Row and column space of a matrix

Definition 148 *Given $A \in \mathbb{M}(m,n)$,*

$$\text{row span} A := \text{span}\left(R^1(A), .., R^i(A), ..., R^m(A)\right)$$

is called the row space of A or row span *of A.*
The column space of A or col span$A$ *is*

$$\text{col span} A := \text{span}\left(C^1(A), .., C^j(A), ..., C^n(A)\right).$$

Remark 149 *Given $A \in \mathcal{M}(m,n)$*

$$\text{col span} A = \text{row span} A^T.$$

Remark 150 ***Linear combinations of columns and rows of a matrix.***
Let $A \in \mathbb{M}(m,n), x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Then, $\forall j \in \{1,..,n\}$, $C^j(A) \in \mathbb{R}^m$ and $\forall i \in \{1,...,m\}$, $R^i(A) \in \mathbb{R}^n$. Then,

Remark 151

$$Ax = \sum_{j=1}^n x_j \cdot C^j(A),$$

as verified below.

$$Ax = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1j} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2j} & ... & a_{2n} \\ ... \\ a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{m1} & a_{m2} & ... & a_{mj} & ... & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_j \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n a_{1j} x_j \\ \sum_{j=1}^n a_{2j} x_j \\ ... \\ \sum_{j=1}^n a_{ij} x_j \\ ... \\ \sum_{j=1}^n a_{mj} x_j \end{bmatrix};$$

$$\sum_{j=1}^n x_j \cdot C^j(A) = \sum_{j=1}^n x_j \cdot \begin{bmatrix} a_{1j} \\ a_{2j} \\ \\ a_{ij} \\ \\ a_{mj} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n x_j a_{1j} \\ \sum_{j=1}^n x_j a_{2j} \\ \\ \sum_{j=1}^n x_j a_{ij} \\ \\ \sum_{j=1}^n x_j a_{mj} \end{bmatrix}.$$

Therefore,

Ax is a linear combination of the <u>columns</u> of A via the components of the vector x.

Moreover,

$$yA = \sum_{i=1}^{m} y_i \cdot R^i(A),$$

as verified below.

$$yA = [y_1, ..., y_i, ..., y_m] \begin{bmatrix} a_{11} & a_{12} & ... & a_{1j} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2j} & ... & a_{2n} \\ ... \\ a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \\ ... \\ a_{m1} & a_{m2} & ... & a_{mj} & ... & a_{mn} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{i=1}^m y_i \cdot a_{i1} & \sum_{i=1}^m y_i \cdot a_{i2} & ... & \sum_{i=1}^m y_i \cdot a_{ij} & ... & \sum_{i=1}^m y_i \cdot a_{in} \end{bmatrix};$$

$$\sum_{i=1}^m y_i \cdot R^i(A) = \sum_{i=1}^m y_i \cdot \begin{bmatrix} a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{i=1}^m y_i \cdot a_{i1} & \sum_{i=1}^m y_i \cdot a_{i2} & ... & \sum_{i=1}^m y_i \cdot a_{ij} & ... & \sum_{i=1}^m y_i \cdot a_{in} \end{bmatrix}.$$

Therefore,

yA is a linear combination of the <u>rows</u> of A via the components of the vector y.

As a consequence of the above observation, we have what follow.

1.

$$\text{row span } A = \{w \in \mathbb{R}^n : \exists \;\; y \in \mathbb{R}^m \text{ such that } w = yA\}.$$

2.

$$\text{col span } A = \{z \in \mathbb{R}^m : \exists \;\; x \in \mathbb{R}^n \text{ such that } z = Ax\};$$

Proposition 152 *Given $A, B \in \mathbb{M}(m, n)$,*
1. if A is row equivalent to B, then $\text{row span}A = \text{row span}B$*;*
2. if A is column equivalent to B, then $\text{col span}A = \text{col span}B$*.*

Proof. 1. B is obtained by A via elementary row operations. Therefore, $\forall i \in \{1, ..., m\}$, either
i. $R^i(B) = R^i(A)$, or
ii. $R^i(B)$ is a linear combination of rows of A.
Therefore, $\text{row span}B \subseteq \text{row span}A$. Since A is obtained by B via elementary row operations, $\text{row span}B \supseteq \text{row span}A$.
2. if A is column equivalent to B, then A^T is row equivalent to B^T and therefore, from i. above, $\text{row span}A^T = \text{row span}B^T$. Then the result follows from Remark 149. ■

Remark 153 *Let $A \in \mathbb{M}(m, n)$ be given and assume that*

$$b := (b_j)_{j=1}^n = \sum_{i=1}^m c_i \cdot R^i(A),$$

i.e., b is a linear combination of the rows of A. Then,

$$\forall j \in \{1, ..., n\}, \quad b_j = \sum_{i=1}^m c_i \cdot R^{ij}(A),$$

where $\forall i \in \{1, ...m\}$ and $\forall j \in \{1, ...n\}$, $R^{ij}(A)$ is the $j-th$ component of the $i-th$ row $R^i(A)$ of A.

Lemma 154 *Assume that $A, B \in \mathbb{M}(m,n)$ are in echelon form with pivots*

$$a_{1j_1}, ..., a_{ij_i}, ..., a_{rj_r},$$

and

$$b_{1k_1}, ..., b_{ik_i}, ..., b_{sk_s},$$

respectively, and[4] $r, s \leq \min\{m, n\}$. *Then*

$$\langle \text{row span} A = \text{row span} B \rangle \Rightarrow \langle s = r \text{ and for } i \in \{1, ..., s\}, \quad j_i = k_i \rangle .$$

Proof. Preliminary remark 1. If $A = 0$, then $A = B$ and $s = r = 0$.

Preliminary remark 2. Assume that $A, B \neq 0$ and then $s, r \geq 1$. We want to verify that $j_1 = k_1$. Suppose $j_1 < k_1$. Then, by definition of echelon matrix, $C_{j_1}(B) = 0$, otherwise you would contradict Property 2 of the Definition 28 of echelon matrix. Then, from the assumption that row span$A$ = row span$B$,we have that $R^1(A)$ is a linear combination of the rows of B, via some coefficients $c_1, ..., c_m$, and from Remark 153 and the fact that $C^{j_1}(B) = 0$, we have that $a_{1j_1} = c_1 \cdot 0 + ...c_m \cdot 0 = 0$, contradicting the fact that a_{1j_1} is a pivot for A. Therefore, $j_1 \geq k_1$. A perfectly symmetric argument shows that $j_1 \leq k_1$.

We can now prove the result by induction on the number m of rows.

Step 1. $m = 1$.

It is basically the proof of Preliminary Remark 2.

Step 2.

Given $A, B \in \mathbb{M}(m,n)$, define $A', B' \in \mathbb{M}(m-1,n)$ as the matrices obtained erasing the first row in matrix A and B respectively. From Remark 33, A' and B' are still in echelon form. If we show that row spanA' = row spanB', from the induction assumption, and using Preliminary Remark 2, we get the desired result.

Let $R = (a_1, ..., a_n)$ be any row of A'. Since $R \in$ row span$B$, $\exists (d_i)_{i=1}^m$ such that

$$R = \sum_{i=1}^{m} d_i R^i(B).$$

Since A is in echelon form and we erased its first row, we have that if $i \leq j_1 = k_1$, then $a_i = 0$, otherwise you would contradict the definition of j_1. Since B is in echelon form, each entry in its $k_1 - th$ column are zero, but b_{1k_1} which is different from zero. Then,

$$a_{1k_1} = 0 = \sum_{i=1}^{m} d_i \cdot b_{ik_1} = d_1 \cdot b_{ik_1},$$

and therefore $d_1 = 0$, i.e., $R = \sum_{i=2}^{m} d_i R^i(B)$, or $R \in$ row spanB', as desired. Symmetric argument shows the other inclusion. ■

Remark 155 *Given*

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix},$$

clearly $A \neq B$ and row span$A$ = row span$B$.

Proposition 156 *Assume that $A, B \in \mathbb{M}(m,n)$ are in row canonical form. Then,*

$$\langle \text{row span} A = \text{row span} B \rangle \Leftrightarrow \langle A = B \rangle .$$

Proof. [⇐] Obvious.

[⇒] From Lemma 154, the number of pivots in A and B is the same. Therefore, A and B have the same number s of nonzero rows, which in fact are the first s rows. Take $i \in \{1, ..., s\}$. Since row span$A$ = row span$B$, there exists $(c_h)_{h=1}^s$ such that

$$R^i(A) = \sum_{h=1}^{s} c_h \cdot R^h(B). \tag{4.3}$$

[4] See Remark 30.

We want then to show that $c_i = 1$ and $\forall l \in \{1, ..., s\} \setminus \{i\}$, $c_l = 0$.

Let a_{ij_i} be the pivot of $R^i(A)$, i.e., a_{ij_i} is the nonzero $j_i - th$ component of $R^i(A)$. Then, from Remark 153,

$$a_{ij_i} = \sum_{h=1}^{s} c_h \cdot R^{hj_i}(B) = \sum_{h=1}^{s} c_h \cdot b_{hj_i}. \tag{4.4}$$

From Lemma 154, for $i \in \{1, ..., s\}$, $\quad j_i = k_i$, and therefore b_{ij_i} is a pivot entry for B, and since B is in row reduced form, b_{ij_i} is the only nonzero element in the j_i column of B. Therefore, from (4.4),

$$a_{ij_i} = \sum_{h=1}^{s} c_h \cdot R^{hj_i}(B) = c_i \cdot b_{ij_i}.$$

Since A and B are in row canonical form $a_{ij_i} = b_{ij_i} = 1$ and therefore

$$c_i = 1.$$

Now take $l \in \{1, ..., s\} \setminus \{i\}$ and consider the pivot element b_{lj_l} in $R^l(B)$. From (4.3) and Remark 153,

$$a_{ij_l} = \sum_{h=1}^{s} c_h \cdot b_{hj_l} = c_l, \tag{4.5}$$

where the last equalities follow from the fact that B is in row reduced form and therefore b_{lj_l} is the only nonzero element in the $j_l - th$ column of B, in fact, $b_{lj_l} = 1$. From Lemma 154, since b_{lj_l} is a pivot element for B, a_{lj_l} is a pivot element for A. Since A is in row reduced form, a_{lj_l} is the only nonzero element in column j_l of A. Therefore, since $l \neq i$, $a_{ij_l} = 0$, and from (4.5), the desired result,

$$\forall l \in \{1, ..., s\} \setminus \{i\}, \quad c_l = 0.,$$

does follow. ■

Proposition 157 *For every $A \in \mathbb{M}(m,n)$, there exists a unique $B \in \mathbb{M}(m,n)$ which is in row canonical form and row equivalent to A.*

Proof. The existence of at least one matrix with the desired properties is the content of Proposition 38. Suppose that there exists B_1 and B_2 with those properties. Then from Proposition 152, we get

$$\text{row span} A = \text{row span} B_1 = \text{row span} B_2.$$

From Proposition 156,

$$B_1 = B_2.$$

■

Corollary 158 *1. For any matrix $A \in \mathbb{M}(m,n)$ there exists a unique number $r \in \{0, 1, ..., \min\{m, n\}\}$ such that A is <u>equivalent</u> to the block matrix of the form*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

2. For any $A \in \mathbb{M}(m,n)$, there exist invertible matrices $P \in \mathbb{M}(m,m)$ and $Q \in \mathbb{M}(n,n)$ and a unique number $r \in \{0, 1, ..., \min\{m, n\}\}$ such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Proof. 1.

From Step 1 in the proof of Proposition 122 and from Proposition 157, there exists a unique matrix A^* which is row equivalent to A and it is in row canonical form.

From Step 2 and 3 in the proof of Proposition 122 and from Proposition 157, there exist a unique matrix

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}^T$$

which is row equivalent to A^{*T} and it is in row canonical form. Therefore the desired result follows.

2.

From Proposition 105.2, PA is row equivalent to A; from Proposition 119.2, PAQ is column equivalent to PA. Therefore, PAQ is equivalent to A.From Proposition 124 ,

$$\begin{bmatrix} I_{r'} & 0 \\ 0 & 0 \end{bmatrix}$$

is equivalent to A. From part 1 of the present Proposition, the desired result then follows. ∎

Example 159 *Let* $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ *be given. Then,*

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \overset{\frac{1}{2}R^1 \to R^1}{\longrightarrow} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \overset{R^2 - R^1 \to R^2}{\longrightarrow} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \overset{C^2 - C^1 \to C^1}{\longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Indeed,

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

4.6 Linear dependence and independence

Definition 160 *Let a vector space V on a field F and a number $m \in \mathbb{N}$ be given. The vectors $v^1, v^2, ..., v^m \in V$ are linearly dependent if*

either $m = 1$ and $v^1 = 0$,

or $m > 1$ and $\exists\ k \in \{1, ...m\}$ and there exist $(m-1) \geq 1$ coefficients $\alpha_j \in F$ with $j \in \{1, ...m\} \setminus \{k\}$ such that

$$v^k = \sum_{j \in \{1,...m\} \setminus \{k\}} \alpha_j v^j$$

or, shortly,

$$v^k = \sum_{j \neq k} \alpha_j v^j$$

i.e., there exists a vector equal to a linear combination of the other vectors.

`Geometrical interpretation of linear (in)dependence in` $\mathbb{R}^2$.

Proposition 161 *Let a vector space V on a field F and vectors $v^1, v^2, ..., v^m \in V$ be given. The vectors $v^1, v^2, ..., v^m \in V$ are linearly dependent vectors if and only if*

$$\exists\ (\beta_1, ..., \beta_i, ..., \beta_m) \in F^m \setminus \{0\} \text{ such that } \sum_{i=1}^{m} \beta_i \cdot v^i = 0, \tag{4.6}$$

i.e., there exists a linear combination of the vectors equal to the null vector and with some nonzero coefficient.

Proof. [⇒]
If $m=1$, any $\beta \in \mathbb{R}\backslash\{0\}$ is such that $\beta \cdot 0 = 0$. Assume then that $m>1$. Take

$$\beta_i = \begin{cases} \alpha_i & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}$$

[⇐]
If $m = 1, \exists \beta \in \mathbb{R}\backslash\{0\}$ is such that $\beta \cdot v^1 = 0$,then from Proposition 58.3 $v^1 = 0$. Assume then that $m > 1$. Without loss of generality take $\beta_1 \neq 0$. Then,

$$\beta_1 v^1 + \sum_{i \neq 1} \beta_i v^i = 0$$

and

$$v^1 = \sum_{i \neq 1} \frac{\beta_i}{\beta_1} v^i.$$

■

Proposition 162 *Let $m \geq 2$ and $v^1, ..., v^m$ be nonzero linearly dependent vectors. Then, one of the vectors is a linear combination of the preceding vectors, i.e., $\exists\ k > 1$ and $\left(\alpha^i\right)_{i=1}^{k-1}$ such that $v^k = \sum_{i=1}^{k-1} \alpha_i v^i$.*

Proof. Since $v^1, ..., v^m$ are linearly dependent, $\exists\left(\beta_i\right)_{i=1}^m \in \mathbb{R}^m\backslash\{0\}$ such that $\sum_{i=1}^m \beta_i v^i = 0$. Let k be the largest i such that $\beta_i \neq 0$, i.e.,

$$\exists k \in \{1, ..., m\} \text{ such that } \beta_k \neq 0 \text{ and } \forall i \in \{k+1, ..., m\}, \quad \beta_i = 0. \tag{4.7}$$

Consider the case $k = 1$. Then we would have $\beta_1 \neq 0$ and $\forall i > 1$, $\beta_i = 0$ and therefore $0 = \sum_{i=1}^m \beta_i v^i = \beta_1 v^1$, contradicting the assumption that $v^1, ..., v^m$ are nonzero vectors. Then, we must have $k > 1$, and from (4.7) , we have

$$0 = \sum_{i=1}^m \beta_i v^i = \sum_{i=1}^k \beta_i v^i$$

and

$$\beta_k v^k = \sum_{i=1}^{k-1} \beta_i v^i,$$

or, as desired,

$$v^k = \sum_{i=1}^{k-1} \frac{-\beta^i}{\beta_k} v^i.$$

It is then enough to choose $\alpha_i = \frac{-\beta^i}{\beta_k}$ for any $i \in \{1, ..., k-1\}$. ■

Example 163 *Take the vectors $x^1 = (1,2)$, $x^2 = (-1,-2)$ and $x^3 = (0,4)$. x^1, x^2, x^3 are linearly dependent vectors: $x^1 = -1 \cdot x^2 + 0 \cdot x^3$. Observe that there are no $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $x^3 = \alpha_1 \cdot x^1 + \alpha_2 \cdot x^2$.*

Definition 164 *Let a vector space V be given. The vectors $v^1, v^2, ..., v^m \in V$ are linearly independent vectors if they are not linearly dependent.*

Remark 165 *The vectors $v^1, v^2, ..., v^m$ are linearly independent if $\langle\ \neg(4.6)\rangle$ holds true, i.e., if*

$$\forall\ (\beta_1, ..., \beta_i, ..., \beta_m) \in F^m\backslash\{0\} \text{ it is the case that } \textstyle\sum_{i=1}^m \beta_i \cdot v^i \neq 0,$$

or

$$(\beta_1, ..., \beta_i, ..., \beta_m) \in F^m\backslash\{0\} \quad \Rightarrow \quad \sum_{i=1}^m \beta_i \cdot v^i \neq 0$$

or

$$\sum_{i=1}^m \beta_i \cdot v^i = 0 \quad \Rightarrow \quad (\beta_1, ..., \beta_i, ..., \beta_m) = 0$$

or the only linear combination of the vectors which is equal to the null vector has each coefficient equal to zero.

Example 166 *The vectors* $(1,2)$, $(1,5)$ *are linearly independent.*

Remark 167 *From Remark 150, we have what follows:*

$$\langle Ax = 0 \Rightarrow x = 0 \rangle \Leftrightarrow \langle \text{the column vectors of } A \text{ are linearly independent} \rangle \tag{4.8}$$

$$\langle yA = 0 \Rightarrow y = 0 \rangle \Leftrightarrow \langle \text{the row vectors of } A \text{ are linearly independent} \rangle \tag{4.9}$$

Example 168 *Let* V *be the vector space of all functions* $f : \mathbb{R} \to \mathbb{R}$. f, g, h *defined below are linearly independent:*

$$f(x) = e^x, \qquad g(x) = x^2, \qquad h(x) = x.$$

Assume that $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ *are such that*

$$\alpha_1 \cdot f + \alpha_2 \cdot g + \alpha_3 \cdot h = 0_V,$$

which means that

$$\forall x \in \mathbb{R}, \quad \alpha_1 \cdot f(x) + \alpha_2 \cdot g(x) + \alpha_3 \cdot h(x) = 0.$$

We want to show that $(\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$. *The trick is to find appropriate values of* x *to get the desired value of* $(\alpha_1, \alpha_2, \alpha_3)$. *Choose* x *to take values* $0, 1, -1$. *We obtain the following system of equations*

$$\begin{cases} \alpha_1 & & & & & = & 0 \\ e \cdot \alpha_1 & + & \alpha_2 & + & \alpha_3 & = & 0 \\ e^{-1}\alpha_1 & + & \alpha_2 & + & (-1)\,\alpha_3 & = & 0 \end{cases}$$

It then follows that $(\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$, *as desired.*

Example 169 *1. The vectors* $v^1, ..., v^m$ *are linearly dependent if* $\exists k \in \{1, ..., m\}$ *such that* $v^k = 0$:

$$v^k + \sum_{i \neq k} 0 \cdot v^i = 0$$

2. The vectors $v^1, ..., v^m$ *are linearly dependent if* $\exists k, k' \in \{1, ..., m\}$ *and* $\alpha \in \mathbb{R} \setminus \{1\}$ *such that* $v^{k'} = \alpha v^k$:

$$v^{k'} - \alpha v^k + \sum_{i \neq k, k'} 0 \cdot v^i = 0$$

Proposition 170 *The nonzero rows of a matrix* A *in echelon form are linearly independent.*

Proof. We will show that each row of A starting from the first one is not a linear combination of the subsequent rows. Then, as a consequence of Proposition 162, the desired result will follow.

Since A is in echelon form, the first row has a pivot below which all the elements are zero. Then that row cannot be a linear combination of the following rows. Similar argument applies to the other rows. ■

Remark 171 *(Herstein (1975), page 178) "We point out that linear dependence is a function not only of the vectors but also of the field. For instance, the field of complex numbers is a vector space over the field of real numbers and it is also a vector space over the field of complex numbers. The elements* $v_1 = 1, v_2 = i$ *in it are linearly independent over the reals, but linearly dependent over the complexes, since* $iv_1 + (-1)\,v_2 = 0$."

Proposition 172 *Let* V *be a vector space and* $v^1, v^2, ..., v^m \in V$. *If* $v^1, ..., v^m$ *are linearly dependent vectors and* $v^{m+1}, ..., v^{m+k} \in V$ *are arbitrary vectors, then*

$$v^1, ..., v^m, v^{m+1}, ..., v^{m+k}$$

are linearly dependent vectors.

Proof. From the assumptions $\exists i^* \in \{1, ..., m\}$ and $(a_j)_{j \neq i^*}$ such that

$$v^j = \sum_{j \neq i^*} \alpha_j v^j$$

But then

$$v^j = \sum_{j \neq i^*} \alpha_j v^j + 0 \cdot v^{m+1} + 0 \cdot v^{m+k},$$

as desired ■

Proposition 173 *If* $v^1, ..., v^m \in V$ *are linearly independent vectors,* $S := \{v^1, ..., v^m\}$ *and* $S' \subseteq S$, *then vectors in* S' *are linearly independent.*

Proof. Suppose otherwise, but then you contradict the previous proposition. ■

4.7 Basis and dimension

Definition 174 *Given a nonempty set X, a sequence on X is a function $x : N \to X$, where $N \subseteq \mathbb{N}$. A subsequence of x is the restriction of x to a subset of N, i.e., $x_{|N}$. If the cardinality of N is finite, then the sequence is called a finite sequence. If the cardinality of N is infinite, the the sequence is called an infinite sequence.*

Remark 175 *Very commonly, for any $i \in N$, the value $x(i)$ is denoted by x_i; the sequence is denoted by $(x_i)_{i \in N}$; the subsequence of x equal to $x_{|N}$ with $N \subseteq \{1, ..., n\}$ is denoted by $(x_i)_{i \in N}$.*

Remark 176 *Observe that given $x, y \in X$, the two finite sequences (x, y) and (y, x) are different: in other words, for (finite) sequences, "order matters".*

Remark 177 *Observe that Proposition 173 can be rephrased as follows: A finite sequence of vectors is linearly independent if and only if any subsequence is linearly independent. That observation motivates and makes the following definition consistent with the definition of linear independent finite sequences.*

Definition 178 *Let $N \subseteq \mathbb{N}$ be given. A sequence $(v_n)_{n \in N}$ of vectors in a vector space V is linearly independent if any finite subsequence of $(v_n)_{n \in N}$ is linearly independent. A sequence $(v_n)_{n \in N}$ of vectors in a vector space V is linearly dependent if it not linearly independent, i.e., if there exists a finite subsequence of $(v_n)_{n \in N}$ which is linearly dependent.*

Remark 179 *We consider the following sentences as having the same meaning:*
"the vectors v^i, with $i \in N$, in the vectors space V are linearly dependent (independent) vectors";
"the sequence $(v_n)_{n \in N}$ in the vectors space V is linearly dependent (independent)";
" $(v_n)_{n \in N}$ in the vectors space V is linearly dependent (independent)".

Definition 180 *Let $N \subseteq \mathbb{N}$ be given. A sequence $(v_n)_{n \in N}$ of vectors in a vector space V on a field F is called a basis of V if*

1. *$(v_n)_{n \in N}$ is linearly independent;*
2. *$\operatorname{span}\left((v_n)_{n \in N}\right) = V$.*

Lemma 181 *Suppose that given a vector space V, $\operatorname{span}\left(v^1, ..., v^m\right) = V$.*

1. *If $w \in V$, then $w, v^1, ..., v^m$ are linearly dependent and $\operatorname{span}\left(w, v^1, ..., v^m\right) = V$;*
2. *If v^i is a linear combination of $v^1, ..., v^{i-1}$, then $\operatorname{span}\left(v^1, ..., v^{i-1}, v^{i+1}, ..., v^m\right) = V$.*

Proof. Obvious. ■

Lemma 182 *(Replacement Lemma) Let a vector space V and vectors*

$$v^1, ..., v^n \in V$$

and

$$w^1, ..., w^m \in V$$

be given. If

1. *$\operatorname{span}\left(v^1, ..., v^n\right) = V$,*
2. *$w^1, ..., w^m$ are linearly independent,*

then

1. *$n \geq m$,*
2. *a. If $n = m$, then $\operatorname{span}\left(w^1, ..., w^m\right) = V$.*
   b. if $n > m$, there exists $\left\{v^{i_1}, ..., v^{i_{n-m}}\right\} \subseteq \left\{v^1, ..., v^n\right\}$ such that

$$\operatorname{span}\left(w^1, ..., w^m, v^{i_1}, ..., v^{i_{n-m}}\right) = V.$$

Proof. Observe preliminary that since $w^1, ..., w^m$ are linearly independent, for any $j \in \{1, ..., m\}$, $w^j \neq 0$.

We now distinguish 2 cases: Case 1. For any $i \in \{1, ..., n\}$, $v^i \neq 0$, and Case 2. There exists $i \in \{1, ..., n\}$ such that $v^i = 0$.

Case 1.

Now consider the case $n = 1$. $w^1, ..., w^m \in V$ implies that there exists $(\alpha_j)_{j=1}^m \in \mathbb{R}^m$ such that $\forall j$, $\alpha_j \neq 0$ and $w^j = \alpha^j v^1$, then it has to be $m = 1$ (and conclusion 1 holds) and since $w^1 = \alpha_1 v_1$, $\text{span}\left(w^1\right) = V$ (and conclusion 2 holds).

Consider now the case $n \geq 2$.

First of all, observe that from Lemma 181.1, $w^1, v^1, ..., v^n$ are linearly dependent and

$$\text{span}\left(w^1, v^1, ..., v^n\right) = V.$$

By Lemma 162, there exists $k_1 \in \{1, ..., n\}$ such that v^k is a linear combination of the preceding vectors. Then from Lemma 181.2, we have

$$\text{span}\left(w^1, \left(v^i\right)_{i \neq k_1}\right) = V.$$

Then again from Lemma 181.1, $w^1, w^2, \left(v^i\right)_{i \neq k_1}$ are linearly dependent and

$$\text{span}\left(w^1, w^2, \left(v^i\right)_{i \neq k_1}\right) = V.$$

By Lemma 162, there exists $k_2 \in \{2, ..., n\} \setminus \{k_1\}$ such that v^{k_2} or w^2 is a linear combination of the preceding vectors. That vector cannot be w^2 because of assumption 2. Therefore,

$$\text{span}\left(w^1, w^2, \left(v^i\right)_{i \neq k_1, k_2}\right) = V.$$

We can now distinguish three cases: $m < n$, $m = n$ and $m > n$.

Now if $m < n$, after m steps of the above procedure we get

$$\text{span}\left(w^1, ..., w^m, \left(v^i\right)_{i \neq k_1, k_2, .., k_m}\right) = V,$$

which shows 2.a. If $m = n$,we have

$$\text{span}\left(w^1, ..., w^m\right) = V,$$

which shows 2.b.

Let's now show that it cannot be $m > n$. Suppose that is the case. Then, after n of the above steps, we get $\text{span}\left(w^1, ..., w^n\right) = V$ and therefore w^{n+1} is a linear combination of $\left(w^1, ..., w^n\right)$, contradicting assumption 2.

Case 2.

In the present case, we assume that there exists a set I_0 such that $I_0 \neq \varnothing$ and $I_0 \subseteq \{1, ..., n\}$. Define also $I_1 = \{1, ..., n\} \setminus I_0$ and $n_1 = \# I_1$. Clearly,

$$\text{span}\left(v^i : i \in I_1\right) = \text{span}\left(v^1, ..., v^n\right) = V$$

and $n_1 < n$. Then, repeating the "replacement" argument described in Case 1 applied to

$$v^i \text{ such that } i \in I_1$$

and

$$w^1, ..., w^m,$$

we get

(i) $n > n_1 \geq m$, i.e., Conclusion 1 holds and Conclusion 2a does not hold true, and

(ii) there exists

$$\left\{v^{i_1}, ..., v^{i_{\# I_1 - m}}\right\} \subseteq \left\{v^i : i \in I_1\right\} \subseteq \left\{v^1, ..., v^n\right\}$$

such that

$$\text{span}\left(w^1, ..., w^m, v^{i_1}, ..., v^{i_{\# I_1 - m}}\right) = V,$$

and therefore Conclusion 2b does hold true. ■

Proposition 183 *Assume that* $(u^1, u^2, ..., u^n)$ *and* $(v^1, v^2, ..., v^m)$ *are bases of* V*. Then* $n = m$*.*

Proof. By definition of basis we have that

$$\text{span}\left(u^1, u^2, ..., u^n\right) = V \quad \text{and} \quad v^1, v^2, ..., v^m \text{ are linearly independent.}$$

Then from Lemma 182, $m \leq n$. Similarly,

$$\text{span}\left(v^1, v^2, ..., v^m\right) = V \quad \text{and} \quad u^1, u^2, ..., u^n \text{ are linearly independent,}$$

and from Lemma 182, $n \leq m$. ■

The above Proposition allows to give the following Definition.

Definition 184 *A vector space* V *has dimension* $n \in \mathbb{N}$ *if there exists a basis of* V *with cardinality* n*. In that case, we say that* V *has finite dimension (equal to* n*) and we write* $\dim V = n$*. If a vector space does not have finite dimension, it is said to be of infinite dimension.*

Definition 185 *The vector space* $\{0\}$ *has dimension* 0*. A (indeed, the) basis of* $\{0\}$ *is the empty set.*

Example 186 *1. A basis of* $\mathbb{R}^n$ *is* $(e^1, ..., e^i, ..., e^n)$*, where* e^i *is defined in Definition 55. That basis is called canonical basis. Then* $\dim \mathbb{R}^n = n$*.*

2. Consider the vector space $\mathbb{P}_n(t)$ *of polynomials of degree* $\leq n$*.* $t^0, t^1, ...t^n$ *is a basis of* $\mathbb{P}_n(t)$ *and therefore* $\dim \mathbb{P}_n(t) = n + 1$*.*

Example 187 `Put the example of infinite dimensional space from Hoffman and Kunze, page 43.`

Proposition 188 *Let* V *be a vector space of dimension* n*.*

1. *$m > n$ vectors in V are linearly dependent;*
2. *If $u^1, ..., u^n \in V$ are linearly independent, then $(u^1, ..., u^n)$ is a basis of V;*
3. *If $\text{span}(u^1, ..., u^n) = V$, then $(u^1, ..., u^n)$ is a basis of V.*

Proof. Let $(w^1, ..., w^n)$ be a basis of V.

1. We want to show that $v^1, ..., v^m$ arbitrary vectors in V are linearly dependent. Suppose otherwise, then by Lemma 182, we would have $m \leq n$, a contradiction.
2. It is the content of Lemma 182.2.a.
3. We have to show that $u^1, ..., u^n$ are linearly independent. Suppose otherwise. Then there exists $k \in \{1, ..., n\}$ such that $\text{span}\left(\left(u^i\right)_{i \neq k}\right) = V$, but since $w^1, ..., w^n$ are linearly independent, from Proposition 182 (the Replacement Lemma), we have $n \leq n - 1$, a contradiction.

■

Remark 189 *The above Proposition 188 shows that in the case of finite dimensional vector spaces, one of the two conditions defining a basis is sufficient to obtain a basis.*

Proposition 190 *(Completion Lemma) Let* V *be a vector space of dimension* n *and* $w^1, ..., w^m \in V$ *be linearly independent, with*[5] $m \leq n$*. If* $m < n$*, then, there exist* $u^1, ..., u^{n-m}$ *such that*

$$\left(w^1, ..., w^m, u^1, ..., u^{n-m}\right)$$

is a basis of V*.*

Proof. Take a basis $v^1, ..., v^n$ of V. Then from Conclusion 2.b in Lemma 182,

$$\text{span}\left(w^1, ..., w^m, v^{i_1}, ..., v^{i_{n-m}}\right) = V.$$

Then from Proposition 188.3, we get the desired result. ■

[5] The inequality $m \leq n$ follows from Proposition 188.1.

Proposition 191 *Let W be a subspace of an $n-$dimensional vector space V. Then*

1. $\dim W \leq n$;

2. *If* $\dim W = n$, *then* $W = V$.

Proof. 1. From Proposition 188.1, $m > n$ vectors in V are linearly dependent. Since the vectors in a basis of W are linearly independent, then $\dim W \leq n$.

2. If $\left(w^1, ..., w^n\right)$ is a basis of W, then $\operatorname{span}\left(w^1, ..., w^n\right) = W$. Moreover, those vectors are n linearly independent vectors in V. Therefore from Proposition 188.2., $\operatorname{span}\left(w^1, ..., w^n\right) = V$. ■

Remark 192 *As a trivial consequence of Proposition 190,* $V = \operatorname{span}\left\{u^1, ..., u^r\right\} \Rightarrow \dim V \leq r$.

Example 193 *Let W be a subspace of $\mathbb{R}^3$, whose dimension is 3. Then from the previous Proposition,* $\dim W \in \{0, 1, 2, 3\}$. *In fact,*

1. *If* $\dim W = 0$, *then* $W = \{0\}$, *i.e., a point,*
2. *if* $\dim W = 1$, *then W is a straight line trough the origin,*
3. *if* $\dim W = 2$, *then W is a plane trough the origin,*
4. *if* $\dim W = 3$, *then* $W = \mathbb{R}^3$.

4.8 Coordinates

Proposition 194 *Let $\left(u^1, u^2, ..., u^n\right)$ be a basis of V on a field F if and only if $\forall v \in V$ there exists a unique $(\alpha_i)_{i=1}^n \in F^n$ such that $v = \sum_{i=1}^n \alpha_i u^i$.*

Proof. [⇒] Suppose there exist $(\alpha_i)_{i=1}^n, (\beta_i)_{i=1}^n \in \mathbb{R}^n$ such that $v = \sum_{i=1}^n \alpha_i u^i = \sum_{i=1}^n \beta_i u^i$. Then

$$0 = \sum_{i=1}^n \alpha_i u^i - \sum_{i=1}^n \beta_i u^i = \sum_{i=1}^n (\alpha_i - \beta_i) u^i.$$

Since $u^1, u^2, ..., u^n$ are linearly independent,

$$\forall i \in \{1, ..., n\}, \ \alpha_i - \beta_i = 0,$$

as desired.

[⇐]

Clearly $V = \operatorname{span}(S)$; we are left with showing that $u^1, u^2, ..., u^n$ are linearly independent. Consider $\sum_{i=1}^n \alpha_i u^i = 0$. Moreover, $\sum_{i=1}^n 0 \cdot u^i = 0$. But since there exists a unique $(\alpha_i)_{i=1}^n \in \mathbb{R}^n$ such that $v = \sum_{i=1}^n \alpha_i u^i$, it must be the case that $\forall i \in \{1, ..., n\}, \ \alpha_i = 0$. ■

The above proposition allows to give the following definition.

Definition 195 *Given a vector space V on a field F with a basis $\mathcal{V} = \left(v^1, ..., v^n\right)$, the associated coordinate function is*

$$cr_{\mathcal{V}} : V \to F^n, \quad v \mapsto cr_{\mathcal{V}}(v) := [v]_{\mathcal{V}}$$

where $cr_{\mathcal{V}}(v)$ is the unique vectors of coefficients which give v as a linear combination of the element of the basis, i.e., $cr_{\mathcal{V}}(v) := (\alpha_i)_{i=1}^n$ such that $v = \sum_{i=1}^n \alpha_i v^i$.

4.9 Row and column span

We start our analysis with a needed lemma.

Lemma 196 *Given a vector space V, if*

1. *the vectors $v^1, ..., v^k \in V$ are linearly independent, and*
2. *$v^{k+1} \in V$ and the vectors $v^1, ..., v^k, v^{k+1}$ are linearly dependent,*

then

v^{k+1} is a linear combination of the vectors $v^1, ..., v^k$.

Proof. Since $v^1, ..., v^k, v^{k+1}$ are linearly dependent, then

$$\exists i \in \{1, ..., k+1\}, \left(\beta_j\right)_{j \in \{1,...,k+1\} \setminus \{i\}} \text{ such that } v^i = \sum_{j \in \{1,...,k+1\} \setminus \{i\}} \beta_j v^j.$$

If $i = k+1$,we are done. If $i \neq k+1$, without loss of generality, take $i = 1$. Then

$$\exists \left(\gamma_j\right)_{j \in \{1,...,k+1\} \setminus \{1\}} \text{ such that } v^1 = \sum_{j=2}^{k+1} \gamma_j v^j.$$

If $\gamma_{k+1} = 0$, we would have $v^1 - \sum_{j=2}^{k} \gamma_j v^j = 0$, contradicting Assumption 1. Then

$$v^{k+1} = \frac{1}{\gamma_{k+1}} \left(v^1 - \sum_{j=2}^{k} \gamma_j v \right).$$

■

Remark 197 *Let V be a vector space and $S \subseteq T \subseteq V$. Then,*
1. span $S \subseteq$ span T*;*
2. span (span S) = span S*.*

Definition 198 *For any $A \in \mathbb{M}(m, n)$,*
the row rank of A is the dim(row span *of A*)*;*
the column rank of A is the dim(col span *of A*)*.*

Proposition 199 *For any $A \in \mathbb{M}(m, n)$, row rank of A is equal to column rank of A.*

Proof. Let A be an arbitrary $m \times n$ matrix

$$\begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... & & & & \\ a_{i1} & ... & a_{ij} & ... & a_{in} \\ ... & & & & \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}$$

Suppose the row rank is $r \leq m$ and the following r vectors form a basis of the row space:

$$\begin{bmatrix} S_1 & = & [b_{11} & ... & b_{1j} & ... & b_{1n}] \\ & & ... & & & & \\ S_k & = & [b_{k1} & ... & b_{kj} & ... & b_{kn}] \\ & & ... & & & & \\ S_r & = & [b_{r1} & ... & b_{rj} & ... & b_{rn}] \end{bmatrix}$$

Then, each row vector of A is a linear combination of the above vectors, i.e., we have

$$\forall i \in \{1, ..., m\}, \quad R^i = \sum_{k=1}^{r} \alpha_{ik} S_k,$$

or

$$\forall i \in \{1, ..., m\}, \quad \begin{bmatrix} a_{i1} & ... & a_{ij} & ... & a_{in} \end{bmatrix} = \sum_{k=1}^{r} \alpha_{ik} \begin{bmatrix} b_{k1} & ... & b_{kj} & ... & b_{kn} \end{bmatrix},$$

and setting the j component of each of the above vector equations equal to each other, we have

$$\forall j \in \{1, .., n\} \quad \text{and} \quad \forall i \in \{1, ..., m\}, \qquad a_{ij} = \sum_{k=1}^{r} \alpha_{ik} b_{kj},$$

and

$$\forall j \in \{1,..,n\} \ , \quad \begin{cases} a_{1j} & = \sum_{k=1}^{r} \alpha_{1k} b_{kj}, \\ .... \\ a_{ij} & = \sum_{k=1}^{r} \alpha_{ik} b_{kj}, \\ ... \\ a_{mj} & = \sum_{k=1}^{r} \alpha_{mk} b_{kj}, \end{cases}$$

or

$$\forall j \in \{1,..,n\} \ , \begin{bmatrix} a_{1j} \\ ... \\ a_{ij} \\ ... \\ a_{mj} \end{bmatrix} = \sum_{k=1}^{r} b_{kj} \begin{bmatrix} \alpha_{1k} \\ ... \\ \alpha_{ik} \\ ... \\ \alpha_{mk} \end{bmatrix},$$

i.e., each column of A is a linear combination of the r vectors

$$\left\{ \begin{pmatrix} \alpha_{11} \\ ... \\ \alpha_{i1} \\ ... \\ \alpha_{m1} \end{pmatrix}, ..., \begin{pmatrix} \alpha_{1k} \\ ... \\ \alpha_{ik} \\ ... \\ \alpha_{mk} \end{pmatrix}, ..., \begin{pmatrix} \alpha_{1r} \\ ... \\ \alpha_{ir} \\ ... \\ \alpha_{mr} \end{pmatrix} \right\}.$$

Then, from Remark 192,

$$\dim \operatorname{col span} \ A \leq r = \operatorname{row rank} \ A, \tag{4.10}$$

i.e.,

$$\operatorname{col rank} \ A \leq \operatorname{row rank} \ A.$$

From (4.10) ,which holds for arbitrary matrix A, we also get

$$\dim \operatorname{col span} \ A^T \leq \operatorname{row rank} \ A^T. \tag{4.11}$$

Moreover,

$$\dim \operatorname{col} \ \operatorname{span} \ A^T \overset{Rmk.\ 149}{=} \dim \operatorname{row span} \ A := \operatorname{row rank} \ A$$

and

$$\operatorname{row rank} \ A^T := \dim \operatorname{row span} \ A^T \overset{Rmk.\ 149}{=} \dim \operatorname{col span} \ A.$$

Then, from the two above equalities and (4.11), we get

$$\operatorname{row rank} \ A \leq \dim \operatorname{col span} \ A, \tag{4.12}$$

and (4.10) and (4.12) gives the desired result. ■

Proposition 200 *For any $A \in \mathbb{M}(m,n)$,*

1. the row rank of A is equal to the maximum number of linearly independent rows of A;

2. the column rank of A is equal to the maximum number of linearly independent columns of A.

Proof. 1.

Let $k_1 :=$ row rank of A and k_2 :=maximum number of linearly independent rows of A. Assume our claim is false and therefore, either a. $k_1 > k_2$, or *b.* $k_1 < k_2$.

a.

Let $(v^1, ..., v^{k_2})$ a finite sequence of linearly independent rows of A.Since, by assumption, k_2 is the maximum number of linearly independent rows of A, then, from Lemma 196, the other rows of A are a linear combination of $v^1, .., , v^{k_2}$ and from Lemma 181, $\operatorname{span}(v^1, ..., v^{k_2}) = \operatorname{row span} A$. Then, $(v^1, ..., v^{k_2})$ is a basis of span A and therefore $k_1 := \operatorname{row rank} A := \dim \operatorname{row span} A = k_2$, contradicting the assumption of case a.

b.

In this case we would have k_2 linearly independent vectors in a vector space of dimension k_1, with $k_2 > k_1$, contradicting Proposition 188.1 .

2.

The proof is basically the same as the above one. ■

Corollary 201 *For every $A \in \mathbb{M}(m,n)$,*

$$\begin{aligned}&\text{row rank } A = \textit{maximum number of linearly independent rows of } A = \\ &= \textit{maximum number of linearly independent columns of } A = \text{col rank } A.\end{aligned} \tag{4.13}$$

Proof. It follows from Propositions 199 and 200. ■

Exercise 202 *Check the above result on the following matrix*

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 1 & 6 \\ 3 & 7 & 10 \end{bmatrix}.$$

4.10 Exercises

Problem sets: 1,2,3,4,5,6,7.

From Lipschutz (1991), starting from page 162:

5.3, 5.7, 5.8, 5.9, 5.10, 5.12 → 5.15, 5.17 → 5.23, 5.24 → 5.29, 5.31 → 5.34, 5.46 → 5.49.

Chapter 5

Determinant and rank of a matrix

In this chapter we are going to introduce the definition of determinant, an useful tool to study linear dependence, invertibility of matrices and solutions to systems of linear equations.

5.1 Definition and properties of the determinant of a matrix

To motivate the definition of determinant, we present an informal discussion of a way to find solutions to the linear system with two equations and two unknowns, shown below.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \tag{5.1}$$

The system can be rewritten as follows

$$Ax = b$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Let's informally discuss how to find solutions to system (5.1). If $a_{22} \neq 0$ and $a_{12} \neq 0$, multiplying both sides of the first equation by a_{22}, of the second equation by $-a_{12}$ and adding up, we get

$$a_{11}a_{22}x_1 + a_{12}a_{22}x_2 - a_{12}a_{21}x_1 - a_{12}a_{22}x_2 = a_{22}b_1 - a_{12}b_2$$

Therefore, if

$$a_{11}a_{22} - a_{12}a_{21} \neq 0$$

we have

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \tag{5.2}$$

In a similar manner[1] we have

$$x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \tag{5.3}$$

We can then the following preliminary definition: given $A \in \mathcal{M}_{22}$, the determinant of A is

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} := a_{11}a_{22} - a_{12}a_{21}$$

Using the definition of determinant, we can rewrite (5.2) and (5.3)as follows.

[1] Assuming $a_{21} \neq 0$ and $a_{11} \neq 0$, multiply both sides of the first equation by a_{21}, of the second equation by $-a_{11}$ and then add up.

$$x_1 = \frac{\det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det A} \quad \text{and} \quad x_2 = \frac{\det \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}}{\det A}$$

We can now present the definition of the determinant of a **square** matrix $A_{n\times n}$ for arbitrary $n \in \mathbb{N}$.

Definition 203 *Given $n > 1$ and $A \in \mathbb{M}(n,n)$, $\forall i,j \in \{1,...,n\}$, we call $A_{ij} \in \mathbb{M}(n-1,n-1)$ the matrix obtained from A erasing the $i-th$ row and the $j-th$ column.*

Definition 204 *Given $A \in \mathbb{M}(1,1)$, i.e., $A = [a]$ with $a \in \mathbb{R}$. The determinant of A is denoted by $\det A$ and we let $\det A := a$. For $\mathbb{N}\setminus\{1\}$, given $A \in \mathbb{M}(n,n)$, we define the determinant of A as*

$$\det A := \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Observe that $[a_{1j}]_{j=1}^{n}$ is the first row of A, i.e.,

$$\det A := \sum_{j=1}^{n} (-1)^{1+j} R^{1j}(A) \det A_{1j}$$

Example 205 *For $n = 2$, we have*

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \sum_{j=1}^{2} (-1)^{1+j} a_{1j} \det A_{1j} = (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} =$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

and we get the informal definition given above.

Example 206 *For $n = 3$, we have*

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} + (-1)^{1+3} a_{13} \det A_{13}$$

Definition 207 *Given $A = [a_{ij}] \in \mathbb{M}(n,n)$, , $\forall i,j$, $\det A_{ij}$ is called minor of a_{ij} in A;*

$$(-1)^{i+j} \det A_{ij}$$

is called cofactor of a_{ij} in A.

Theorem 208 *Given $A \in \mathbb{M}(n,n)$, $\det A$ is equal to the sum of the products of the elements of <u>any</u> rows or column for the corresponding cofactors, i.e.,*

$$\forall i \in \{1,...n\}, \ \det A = \sum_{j=1}^{n} (-1)^{i+j} R^{ij}(A) \det A_{ij} \tag{5.4}$$

and

$$\forall j \in \{1,...n\}, \ \det A = \sum_{i=1}^{n} (-1)^{i+j} C^{ji}(A) \det A_{ij} \tag{5.5}$$

Proof. Omitted. We are going to omit several proofs about determinants. There are different ways of introducing the concept of determinant of a square matrix. One of them uses the concept of permutations - see, for example, Lipschutz (1991), Chapter 7. Another one is an axiomatic approach - see, for example, Lang (1971) - he introduces (three) properties that a function $f : \mathbb{M}(n,n) \to \mathbb{R}$ has to satisfy and then shows that there exists a unique such function, called determinant. Following the first approach the proof of the present theorem can be found on page 252, in Lipschutz (1991) Theorem 7.8, or following the second approach, in Lang (1971), page 128, Theorem 4. ■

Definition 209 *The expression used above for the computation of* $\det A$ *in (5.4) is called "(Laplace) expansion" of the determinant by row* i.

The expression used above for the computation of $\det A$ *in (5.5) is called "(Laplace) expansion" of the determinant by column* j.

Definition 210 *Consider a matrix* $A_{n\times n}$. *Let* $1 \leq k \leq n$. *A* $k-th$ *order principal submatrix (minor) of* A *is the (determinant of the) square submatrix of* A *obtained deleting* $(n-k)$ *rows and* $(n-k)$ *columns in the same position.*

Theorem 211 *(Properties of determinants)*

Let the matrix $A = [a_{ij}] \in \mathbb{M}(n,n)$ *be given. Properties presented below hold true even if words "column, columns" are substituted by the words "row, rows".*

1. $\det A = \det A^T$.
2. *if two columns are interchanged, the determinant changes its sign,.*
3. *if there exists* $j \in \{1,..,n\}$ *such that* $C^j(A) = \sum_{k=1}^{p} \beta_k b^k$, *then*

$$\det \left[C^1(A), ..., \sum_{k=1}^{p} \beta_k b^k, ..., C^n(A) \right] = \sum_{k=1}^{p} \beta_k \det \left[C^1(A), ..., b^k, ..., C^n(A) \right],$$

i.e., the determinant of a matrix which has a column equal to the linear combination of some vectors is equal to the linear combination of the determinants of the matrices in which the column under analysis is each of the vector of the initial linear combination, and, therefore, $\forall \beta \in \mathbb{R}$ *and* $\forall j \in \{1,...,n\}$,

$$\det \left[C^1(A), ..., \beta C^j(A), ..., C^n(A) \right] = \beta \det \left[C^1(A), ..., C^j(A), ..., C^n(A) \right] = \beta \det A.$$

4. *if* $\exists j \in \{1,...,n\}$ *such that* $C^j(A) = 0$, *then* $\det A = 0$, *i.e., if a matrix has column equal to zero, then the determinant is zero.*
5. *if* $\exists\ j,k \in \{1,...,n\}$ *and* $\beta \in \mathbb{R}$*such that* $C^j(A) = \beta C^k(A)$, *then* $\det A = 0$, *i.e., the determinant of a matrix with two columns proportional one to the other is zero.*
6. *If* $\exists k \in \{1,...,n\}$ *and* $\beta_1, ..., \beta_{k-1}, \beta_{k+1}, ..., \beta_n \in \mathbb{R}$ *such that* $C^k(A) = \sum_{j\neq k} \beta_j C^j(A)$, *then* $\det A = 0$, *i.e., if a column is equal to a linear combination of the other columns, then* $\det A = 0$.
7.

$$\det \left[C^1(A), ..., C^k(A) + \sum_{j\neq k} \beta_j \cdot C^j(A), ..., C^n(A) \right] = \det A$$

8. $\forall j, j^* \in \{1,...,n\}$, $\sum_{i=1}^{n} a_{ij} \cdot (-1)^{i+j^*} \det A_{ij^*} = 0$, *i.e., the sum of the products of the elements of a column times the cofactor of the analogous elements of another column is equal to zero.*
9. *If* A *is triangular,* $\det A = a_{11} \cdot ... \cdot a_{22} \cdot ... a_{nn}$, *i.e., if* A *is triangular (for example, diagonal), the determinant is the product of the elements on the diagonal.*

Proof. 1.

Consider the expansion of the determinant by the first row for the matrix A and the expansion of the determinant by the first column for the matrix A^T.

2.

We proceed by induction. Let A be the starting matrix and $A'$the matrix with the interchanged columns.

$\mathcal{P}(2)$ is obviously true.

$\mathcal{P}(n-1) \Rightarrow \mathcal{P}(n)$

Expand $\det A$ and $\det A'$ by a column which is not any of the interchanged ones:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} C_j^i(A) \det A_{ij}$$

$$\det A' = \sum_{i=1}^{n} (-1)^{i+k} C_j^i(A) \det A'_{ij}$$

Since $\forall k \in \{1, ..., n\}$, A_{kj}, $A'_{kj} \in \mathbb{M}(n-1, n-1)$, and they have interchanged column, by the induction argument, $\det A_{kj} = -\det A'_{kj}$, and the desired result follows.

3.

Observe that

$$\sum_{k=1}^{p} \beta_k b^k = \left(\sum_{k=1}^{p} \beta_k b_i^k \right)_{i=1}^{n}.$$

Then,

$$\det \left[C^1(A), ..., \textstyle\sum_{k=1}^{p} \beta_k b^k, ..., C^n(A) \right] =$$

$$= \det \left[C^1(A), ..., \sum_{k=1}^{p} \beta_k \begin{bmatrix} b_1^k \\ ... \\ b_i^k \\ ... \\ b_n^k \end{bmatrix}, ..., C^n(A) \right] =$$

$$= \det \left[C^1(A), ..., \begin{bmatrix} \sum_{k=1}^{p} \beta_k b_1^k \\ ... \\ \sum_{k=1}^{p} \beta_k b_i^k \\ ... \\ \sum_{k=1}^{p} \beta_k b_n^k \end{bmatrix}, ..., C^n(A) \right] =$$

$$= \sum_{i=1}^{n} (-1)^{i+j} \left(\sum_{k=1}^{p} \beta_k b_i^k \right) \det A_{ij} =$$

$$= \textstyle\sum_{k=1}^{p} \beta_k \sum_{i=1}^{n} (-1)^{i+j} b_i^k \det A_{ij} = \sum_{k=1}^{p} \beta_k \det \left[C^1(A), ..., b^k, ..., C^n(A) \right].$$

4.

It is sufficient to expand the determinant by the column equal to zero.

5.

Let $A := \left[C^1(A), \beta C^1(A), C^3(A), ..., C^n(A) \right]$ and $\widetilde{A} := \left[C^1(A), C^1(A), C^3(A), ..., C^n(A) \right]$ be given. Then $\det A = \beta \det \left[C^1(A), C^1(A), C^3(A), ..., C^n(A) \right] = \beta \det \widetilde{A}$. Interchanging the first column with the second column of the matrix $\widetilde{A}$, from property 2, we have that $\det \widetilde{A} = -\det \widetilde{A}$ and therefore $\det \widetilde{A} = 0$, and $\det A = \beta \det \widetilde{A} = 0$.

6.

It follows from 3 and 5.

7.

It follows from 3 and 6.

8.

It follows from the fact that the obtained expression is the determinant of a matrix with two equal columns.

9.

It can be shown by induction and expanding the determinant by the firs row or column, choosing one which has all the elements equal to zero excluding at most the first element. In other words, in the case of an upper triangular matrix, we can say what follows.

$$\det \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = a_{11} \cdot a_{22}.$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1n} \\ 0 & a_{22} & a_{23} & ... & a_{2n} \\ 0 & 0 & a_{33} & & a_{3n} \\ ... & & & \ddots & \\ 0 & ... & 0 & ... & a_{nn} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & a_{23} & ... & a_{2n} \\ 0 & a_{33} & & a_{3n} \\ & & \ddots & \\ ... & 0 & ... & a_{nn} \end{bmatrix} = a_{11} \cdot a_{22} \cdot a_{33} \cdot ... a_{nn}.$$

■

Theorem 212 *For any* $A, B \in \mathbb{M}(n,n)$, $\det(AB) = \det A \cdot \det B$.

Proof. Exercise. ■

Definition 213 $A \in \mathbb{M}(n,n)$ *is called nonsingular if* $\det A \neq 0$.

5.2 Rank of a matrix

Definition 214 *Given* $A \in \mathbb{M}(m,n)$, *a square submatrix of* A *of order* $k \leq \min\{m,n\}$ *is a matrix obtained considering the elements belonging to* k *rows and* k *columns of* A.

Definition 215 *Given* $A \in \mathbb{M}(m,n)$, *the* rank *of* A *is the greatest order of square nonsingular submatrices of* A.

Remark 216 $\text{rank}A \leq \min\{m,n\}$.

To compute rank A, with $A \in \mathbb{M}(m,n)$, we can proceed as follows.

1. Consider $k = \min\{m,n\}$, and the set of square submatrices of A of order k. If there exists a nonsingular matrix among them, then rank $A = k$. If all the square submatrices of A of order k are singular, go to step 2 below.

2. Consider $k-1$, and then the set of the square submatrices of A of order $k-1$. If there exists a nonsingular matrix among them, then rank $A = k-1$. If all square submatrices of order $k-1$are singular, go to step 3.

3. Consider $k-2$ ...

and so on.

Remark 217 *1.* rank $I_n = n$.

2. The rank of a matrix with a zero row or column is equal to the rank of that matrix without that row or column, i.e.,

$$\text{rank}\begin{bmatrix} A \\ 0 \end{bmatrix} = \text{rank}\begin{bmatrix} A & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \text{rank } A$$

That result follows from the fact that the determinant of any square submatrix of A *involving that zero row or columns is zero.*

3. From the above results, we also have that

$$\text{rank}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = r$$

We now describe an easier way to the compute the rank of A, which in fact involves elementary row and column operations we studied in Chapter 1.

Proposition 218 *Given* $A, A' \in \mathbb{M}(m,n)$,

$$\langle\ A \text{ is equivalent to } A'\rangle \Leftrightarrow \langle \text{rank } A = \text{rank } A' \rangle$$

Proof. $[\Rightarrow]$ Since A is equivalent to A', it is possible to go from A to A' through a finite number of elementary row or column operations. In each step, in any square submatrix A^* of A which has been changed accordingly to those operations, the elementary row or column operations 1, 2 and 3 (i.e., 1. row or column interchange, 2. row or column scaling and 3. row or column addition) are such that the determinant of A^* remains unchanged or changes its sign (Property 2, Theorem 211), it is multiplied by a nonzero constant (Property 3), remains unchanged (Property 7), respectively.

Therefore, each submatrix A^* whose determinant is different from zero remains with determinant different from zero and any submatrix A^* whose determinant is zero remains with zero determinant.

$[\Leftarrow]$

From Corollary 158.2 [2], we have that there exist unique $\widehat{r}$ and $\widehat{r}'$ such that

$$A \text{ is equivalent to } \begin{bmatrix} I_{\widehat{r}} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$A' \text{ is equivalent to } \begin{bmatrix} I_{\widehat{r}'} & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, from [⇒] part of the present proposition, and Remark 217

$$rankA = \text{rank} \begin{bmatrix} I_{\widehat{r}} & 0 \\ 0 & 0 \end{bmatrix} = \widehat{r}$$

and

$$rankA' = \text{rank} \begin{bmatrix} I_{\widehat{r}'} & 0 \\ 0 & 0 \end{bmatrix} = \widehat{r}'.$$

Then, by assumption, $\widehat{r} = \widehat{r}' := r$, and A and A' are equivalent to

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

and therefore A is equivalent to A'. ■

Example 219 *Given*

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

we can perform the following elementary rows and column operations, and cancellation of zero row and columns on the matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the rank of the matrix is 2.

5.3 Inverse matrices (continued)

Using the notion of determinant, we can find another way of analyzing the problems of i. existence and ii. computation of the inverse matrix. To do that, we introduce the concept of adjoint matrix.

Definition 220 *Given a matrix $A_{n\times n}$, we call adjoint matrix of A, and we denote it by Adj A, the matrix whose elements are the cofactors*[3] *of the corresponding elements of A^T.*

[2] That results says what follows:
For any $A \in \mathbb{M}(m,n)$, there exist invertible matrices $P \in \mathbb{M}(m,m)$ and $Q \in \mathbb{M}(n,n)$ and a unique number $r \in \{0,1,...,\min\{m,n\}\}$ such that A is equivalent to

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

[3] From Definition 207, recall that given $A = [a_{ij}] \in \mathbb{M}(m,m)$, $\forall i,j$,

$$(-1)^{i+j} \det A_{ij}$$

is called cofacor of a_{ij} in A.

Remark 221 *In other words to construct Adj A,*

1. *construct* A^T,
2. *consider the cofactors of each element of* A^T.

Example 222

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \; A^T = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, \; Adj\ A = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$$

Observe that

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = (\det A) \cdot I.$$

Example 223

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \; A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}, \; Adj\ A = \begin{bmatrix} -4 & 6 & 1 \\ 2 & -3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

Proposition 224 *Given* $A_{n \times n}$, *we have*

$$A \cdot Adj\ A = Adj\ A \cdot A = \det A \cdot I \tag{5.6}$$

Proof. Making the product $A \cdot Adj\ \ A := B$, we have

1. $\forall i \in \{1, ..., n\}$, the $i-th$ element on the diagonal of B is the expansion of the determinant by the $i-th$ row and therefore is equal to $\det A$.

2. any element not on the diagonal of B is the product of the elements of a row times the corresponding cofactor a parallel row and it is therefore equal to zero due to Property 8 of the determinants stated in Theorem 211). ■

Example 225

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = -3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -4 & 6 & 1 \\ 2 & -3 & -2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = -3 \cdot I$$

Proposition 226 *Given an* $n \times n$ *matrix* A, *the following statements are equivalent:*

1. $\det A \neq 0$, *i.e.,* A *is nonsingular;*
2. A^{-1} *exists, i.e.,* A *is invertible;*
3. *rank* $A = n$;
4. $\text{row rank} A = n$;
5. *the column vectors of the matrix* A *are linearly independent;*
6. $\text{col rank} A = n$;
7. *the row vectors of the matrix* A *are linearly independent;*

Proof. $1 \Rightarrow 2$

From (5.6) and from the fact that $\det A \neq 0$, we have

$$A \cdot \frac{Adj\ A}{\det A} = \frac{Adj\ A}{\det A} \cdot A = I$$

and therefore

$$A^{-1} = \frac{Adj\ A}{\det A} \tag{5.7}$$

$1 \Leftarrow 2$
$AA^{-1} = I \Rightarrow \det\left(AA^{-1}\right) = \det I \Rightarrow \det A \cdot \det A^{-1} = 1 \Rightarrow \det A \neq 0$ (and $\det A^{-1} \neq 0$).

$1 \Leftrightarrow 3$
It follows from the definition of rank and the fact that A is $n \times n$ matrix.
$2 \Rightarrow 5$
From (4.8), it suffices to show that $\langle Ax = 0 \Rightarrow x = 0 \rangle$. Since A^{-1} exists, $Ax = 0 \Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow x = 0$.
$2 \Leftarrow 5$
From Proposition 188.2, the n linearly independent column vectors $\left[C^1(A), ..., C_i(A), ..., C^n(A)\right]$ are a basis of $\mathbb{R}^n$. Therefore, each vector in $\mathbb{R}^n$ is equal to a linear combination of those vectors. Then $\forall k \in \{1, ..., n\}$ $\exists b^k \in \mathbb{R}^n$ such that the $k-th$ vector e^k in the canonical basis is equal to $\left[C^1(A), ..., C_i(A), ..., C^n(A)\right] \cdot b^k = Ab^k$, i.e.,

$$\left[\begin{array}{ccccc} e^1 & ... & e^k & ... & e^n \end{array}\right] = \left[\begin{array}{ccccc} Ab^1 & ... & Ab^k & ... & Ab^n \end{array}\right]$$

or, from (3.4) in Remark 70, defined

$$B := \left[\begin{array}{ccccc} b^1 & ... & b^k & ... & b^n \end{array}\right]$$

$$I = AB$$

i.e., A^{-1} exists (and it is equal to B).
The remaining equivalences follow from Corollary 201. ■

Remark 227 *From the proof of the previous Proposition, we also have that, if* $\det A \neq 0$, *then* $\det A^{-1} = (\det A)^{-1}$.

Remark 228 *The previous theorem gives a way to compute the inverse matrix as explained in* (5.7).

Example 229 *1.*

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{array}\right]^{-1} = -\frac{1}{3}\left[\begin{array}{ccc} -4 & 6 & 1 \\ 2 & -3 & -2 \\ -1 & 0 & 1 \end{array}\right]$$

2.

$$\left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right]^{-1} = \left[\begin{array}{ccc} -1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{array}\right]$$

3.

$$\left[\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{array}\right]^{-1} \textit{ does not exist because } \det\left[\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{array}\right] = 0$$

4.

$$\left[\begin{array}{ccc} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{array}\right]^{-1} = \left[\begin{array}{ccc} 1 & \frac{3}{4} & -\frac{1}{2} \\ 1 & 0 & 0 \\ -1 & -\frac{1}{4} & \frac{1}{2} \end{array}\right]$$

5.

$$\left[\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right]^{-1} = \left[\begin{array}{ccc} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{array}\right]$$

if $a, b, c \neq 0$.

5.4 Span of a matrix, linearly independent rows and columns, rank

Proposition 230 *Given $A \in \mathbb{M}(m,n)$, then*

$$\text{row rank } A = \text{rank } A.$$

Proof. First proof.

The following result which is the content of Corollary 158.5 plays a crucial role in the proof:

For any $A \in \mathbb{M}(m,n)$, there exist invertible matrices $P \in \mathbb{M}(m,m)$ and $Q \in \mathbb{M}(n,n)$ and a unique number $r \in \{0,1,...,\min\{m,n\}\}$ such that

$$A \text{ is equivalent to } PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

From the above result, Proposition 218 and Remark 217 , we have that

$$\text{rank } A = \text{rank } \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = r. \tag{5.8}$$

From Propositions 105 and 152, we have

$$\text{row span } A = \text{row span } PA;$$

from Propositions 119 and 152, we have

$$\text{col span } PA = \text{col span } PAQ = \text{col span } \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

From Corollary 201,

$$\dim \text{row span } PA = \dim \text{col span } PA.$$

Therefore

$$\dim \text{row span } A = \dim \text{col span } \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = r, \tag{5.9}$$

where the last equality follows simply because

$$\text{col span } \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \text{col span } \begin{bmatrix} I_r \\ 0 \end{bmatrix},$$

and the r column vectors of the matrix $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ are linearly independent and therefore, from Proposition **??** , they are a basis of span $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$.

From (5.8) and (5.9), the desired result follows.

Second proof.

We are going to show that row rank A = rank A.

Recall that

$$\text{row rank } A := \left\langle \begin{array}{l} r \in \mathbb{N} \text{ such that} \\ \text{i . } r \text{ row vectors of } A \text{ are linearly independent,} \\ \text{ii. if } m > r \text{, any finite sequence of rows of } A \text{ of cardinality} > r \text{ is linearly dependent.} \end{array} \right\rangle$$

We ant to show that

1. if row rank $A = r$, then rank $A = r$, and
2. if rank $A = r$, then row rank $A = r$.

1.

Consider the r linearly independent row vectors of A. Since r is the maximal number of linearly independent row vectors, from Lemma 196, each of the remaining $(m-r)$ row vectors is a linear combination of the

r linearly independent ones. Then, up to reordering of the rows of A, which do not change either row rank A or rank A, there exist matrices $A_1 \in \mathbb{M}(r,n)$ and $A_2 \in \mathbb{M}(m-r,n)$ such that

$$\text{rank } A = \text{rank} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \text{rank } A_1$$

where the last equality follows from Proposition 218. Then r is the maximum number of linearly independent row vectors of A_1 and therefore, from Proposition 199, the maximum number of linearly independent column vectors of A_1. Then, again from Lemma 196, we have that there exist $A_{11} \in \mathbb{M}(r,r)$ and $A_{12} \in \mathbb{M}(r,n-r)$ such that

$$\text{rank } A_1 = \text{rank} \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} = \text{rank } A_{11}$$

Then the square $r \times r$ matrix A_{11} contains r linearly independent vectors. Then from Proposition 226, the result follows.

2.

By Assumption, up to reordering of rows, which do not change either row rank A or rank A,

$$A = \begin{bmatrix} & s & r & n-r-s \\ \\ r & A_{11} & A_{12} & A_{13} \\ m-r & A_{21} & A_{22} & A_{23} \end{bmatrix}$$

with

$$\text{rank } A_{11} = r.$$

Then from Proposition 226, row, and column, vectors of A_{12} are linearly independent. Then from Corollary **??**, the r row vectors of

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \end{bmatrix}$$

are linearly independent

Now suppose that the maximum number of linearly independent row vectors of A are $r' > r$ (and the other $m - r'$ row vectors of A are linear combinations of them). Then from part 1 of the present proof, *rank* $A = r' > r$, contradicting the assumption. ■

Remark 231 *From Corollary 201 and the above Proposition, we have for any matrix $A_{m \times n}$, the following numbers are equal.*

1. rank A :=*greatest order of square nonsingular submatrices of A,*

2. row rank $A :=$ dim row span A*,*

3. max *number of linear independent rows of A,*

4. col rank $A :=$ dim col span A*,*

5. max *number of linear independent columns of A.*

Corollary 232 *For any matrix $A \in \mathbb{M}(m,n)$, there exists $r \in \{0,...,\min\{m,n\}\}$ such that A is equivalent*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where $r =$ rank A.

5.5 Exercises

Problem sets: 9,10

From Lipschutz (1991), starting from page 115:

4.13, 4.14;

starting from page 258:

7.1 → 7.10, 7.14 → 7.16, 7.44, 7.48.

Chapter 6

Linear functions

6.1 Definition

Definition 233 *Given the vector spaces V and U over the same field F, a function $l : V \to U$ is linear if*

1. $\forall v, w \in V$, $l(v+w) = l(v) + l(w)$, *and*

2. $\forall \alpha \in F$, $\forall v \in V$, $l(\alpha v) = \alpha l(v)$.

Call $L(V,U)$ the set of all such functions. Any time we write $L(V,U)$, we implicitly assume that V and U are vector spaces on the same field F.

In other words, l is linear if it "preserves" the two basic operations of vector spaces.

Remark 234 *1. $l \in L(V,U)$ iff $\forall v, w \in V$ and $\forall \alpha, \beta \in F, l(\alpha v + \beta w) = \alpha l(v) + \beta l(w)$;*
2. If $l \in L(V,U)$, then $l(0) = 0$: for arbitrary $x \in V$, $l(0) = l(0x) = 0 l(x) = 0$.

Example 235 *Let V and U be vector spaces. The following functions are linear.*
1. (identity function)

$$l_1 : V \to V, \qquad l_1(v) = v.$$

2. (null function)

$$l_2 : V \to U, \qquad l_2(v) = 0.$$

3.

$$\forall a \in F, \qquad l_a : V \to V, \qquad l_a(v) = av.$$

4. (projection function)
$\forall n \in \mathbb{N}, \forall k \in \mathbb{N}$,

$$proj_{n+k,n} : \mathbb{R}^{n+k} \to \mathbb{R}^n, \qquad proj_{n+k,n} : (x_i)_{i=1}^{n+k} \mapsto (x_i)_{i=1}^{n};$$

5. (immersion function)
$\forall n \in \mathbb{N}, \forall k \in \mathbb{N}$,

$$i_{n,n+k} : \mathbb{R}^n \to \mathbb{R}^{n+k}, \qquad i_{n,n+k} : (x_i)_{i=1}^{n} \mapsto ((x_i)_{i=1}^{n}, 0) \qquad \text{with } 0 \in \mathbb{R}^k.$$

Example 236 *Taken $A \in \mathbb{M}(m,n)$, then*

$$l : \mathbb{R}^n \to \mathbb{R}^m, \qquad l(x) = Ax$$

is a linear function, as shown in part 3 in Remark 76.

Remark 237 *Let V and U be vector spaces. If $l \in L(V,U)$, then*

$$graph\ l := \{(v,u) \in V \times U : u = l(v)\}$$

is a vector subspace of $V \times U$.

Example 238 *Let V be the vector space of polynomials in the variable t. The following functions are linear.*

1. The derivative defines a function $\mathbb{D} : V \to V$ as

$$\mathbb{D} : p \mapsto p'$$

where p' is the derivative function of p.

2. The definite integral from 0 to 1 defines a function $i : V \to \mathbb{R}$ as

$$i : p \mapsto \int_0^1 p(t)\, dt.$$

Proposition 239 *If $l \in L(V,U)$ is invertible, then its inverse l^{-1} is linear.*

Proof. Take arbitrary $u, u' \in U$ and $\alpha, \beta \in F$. Then, since l is onto, there exist $v, v' \in V$ such that

$$l(v) = u \quad \text{and} \quad l(v') = u'$$

and by definition of inverse function

$$l^{-1}(u) = v \quad \text{and} \quad l^{-1}(u') = v' \ .$$

Then

$$\alpha u + \beta u' = \alpha l(v) + \beta l(v') = l(\alpha v + \beta v')$$

where last equality comes from the linearity of l. Then again by definition of inverse,

$$l^{-1}(\alpha u + \beta u') = \alpha v + \beta v' = \alpha l^{-1}(u) + \beta l^{-1}(u').$$

■

6.2 Kernel and Image of a linear function

Definition 240 *Assume that $l \in L(V,U)$. The kernel of l, denoted by $\ker l$ is the set*

$$\{v \in V : l(v) = 0\} = l^{-1}(0).$$

The Image of l, denoted by $\operatorname{Im} l$ is the set

$$\{u \in U : \exists v \in V \text{ such that } f(v) = u\} = l(V).$$

Proposition 241 *Given $l \in L(V,U)$, then*

1. $\ker l$ is a vector subspace of V, and

2. $\operatorname{Im} l$ is a vector subspace of U.

Proof. 1.

Since $l(0) = 0$, then $0 \in \ker l$.

Take $v^1, v^2 \in \ker l$ and $\alpha, \beta \in F$. Then

$$l\left(\alpha v^1 + \beta v^2\right) = \alpha l\left(v^1\right) + \beta l\left(v^2\right) = 0$$

i.e., $\alpha v^1 + \beta v^2 \in \ker l$.

2.

Since $0 \in V$ and $l(0) = 0$, then $0 \in \operatorname{Im} l$.

Take $w^1, w^2 \in \operatorname{Im} l$ and $\alpha, \beta \in F$. Then for $i \in \{1, 2\}$, $\exists v^i \in V$ such that $l\left(v^i\right) = w_i$. Moreover,

$$\alpha w^1 + \beta w^2 = \alpha l(v_i) + \beta l\left(v^2\right) = l\left(\alpha v^1 + \beta v^2\right)$$

i.e., $\alpha w^1 + \beta w^2 \in \operatorname{Im} l$. ■

Proposition 242 *If $\operatorname{span}\left(v^1, ..., v^n\right) = V$ and $l \in L(V,U)$, then $\operatorname{span}\left(l\left(v^1\right), ..., l(v^n)\right) = \operatorname{Im} l$.*

Proof. Taken $u \in \operatorname{Im} l$, there exists $v \in V$ such that $l(v) = u$. Moreover, $\exists\ (\alpha_i)_{i=1}^n \in \mathbb{R}^n$ such that $v = \sum_{i=1}^n \alpha_i v^i$. Then

$$u = l(v) = l\left(\sum_{i=1}^n \alpha_i v^i\right) = \sum_{i=1}^n \alpha_i l\left(v^i\right),$$

as desired. ■

Remark 243 *From the previous proposition, we have that if $\left(v^1, ..., v^n\right)$ is a basis of V, then*

$$n \geq \dim \operatorname{span}\left(l\left(v^1\right), ..., l\left(v^n\right)\right) = \dim \operatorname{Im} l.$$

Example 244 *Let V the vector space of polynomials and $\mathbb{D}^3 : V \to V$, $p \mapsto p'''$, i.e., the third derivative of p. Then*

$$\ker \mathbb{D}^3 = \text{ set of polynomials of degree } \leq 2,$$

since $\mathbb{D}^3\left(at^2 + bt + c\right) = 0$ and $\mathbb{D}^3\left(t^n\right) \neq 0$ for $n > 2$. Moreover,

$$\operatorname{Im} \mathbb{D}^3 = V,$$

since every polynomial is the third derivative of some polynomial.

Proposition 245 *(**Dimension Theorem**)If V is a finite dimensional vector space and $l \in L(V, U)$, then*

$$\dim V = \dim \ker l + \dim \operatorname{Im} l$$

Proof. (Idea of the proof.

1. Using a basis of $\ker l$ (with n_1 elements) and a basis of $\operatorname{Im} l$ (with n_2 elements), we construct $n_1 + n_2$ vectors which generate V.

2. We show those vectors are linearly independent (by contradiction), and therefore a basis of V, and therefore $\dim V = n_1 + n_2$.)

Since $\ker l \subseteq V$ and from Remark 243, $\ker l$ and $\operatorname{Im} l$ have finite dimension. Therefore, we can define $n_1 = \dim \ker l$ and $n_2 = \dim \operatorname{Im} l$.

Take an arbitrary $v \in V$. Let

$$\left(w^1, .., w^{n_1}\right) \text{ be a basis of } \ker l \tag{6.1}$$

and

$$\left(u^1, .., u^{n_2}\right) \text{ be a basis of } \operatorname{Im} l \tag{6.2}$$

Then,

$$\forall i \in \{1, .., n_2\},\ \exists v^i \in V \text{ such that } u^i = l\left(v^i\right) \tag{6.3}$$

From (6.2),

$$\exists c = (c_i)_{i=1}^{n_2} \text{ such that } l(v) = \sum_{i=1}^{n_2} c_i u^i \tag{6.4}$$

Then, from (6.4) and (6.3) ,we get

$$l(v) = \sum_{i=1}^{n_2} c_i u^i = \sum_{i=1}^{n_2} c_i l\left(v^i\right)$$

and from linearity of l

$$0 = l(v) - \sum_{i=1}^{n_2} c_i l\left(v^i\right) = l(v) - l\left(\sum_{i=1}^{n_2} c_i v^i\right) = l\left(v - \sum_{i=1}^{n_2} c_i v^i\right)$$

i.e.,

$$v - \sum_{i=1}^{n_2} c_i v^i \in \ker l \tag{6.5}$$

From (6.1),

$$\exists \left(d_j\right)_{j=1}^{n} \text{ such that } v-\sum_{i=1}^{n_2} c_i v^i=\sum_{j=1}^{n_1} d_j w^j$$

Summarizing, we have

$$\forall v \in V, \exists \left(c_i\right)_{i=1}^{n_2} \text{ and } \left(d_j\right)_{j=1}^{n_1} \text{ such that } v=\sum_{i=1}^{n_2} c_i v^i+\sum_{j=1}^{n_1} d_j w^j$$

Therefore, we found n_1+n_2 vectors which generate V; if we show that they are linearly independent .i., then, by definition, they are a basis and therefore $n=n_1+n_2$ as desired.

We want to show that

$$\sum_{i=1}^{n_2} \alpha_i v^i+\sum_{j=1}^{n_1} \beta_j w^j=0 \Rightarrow\left(\left(\alpha_i\right)_{i=1}^{n_2},\left(\beta_j\right)_{j=1}^{n_1}\right)=0 \tag{6.6}$$

Then

$$0=l\left(\sum_{i=1}^{n_2} \alpha_i v^i+\sum_{j=1}^{n_1} \beta_j w^j\right)=\sum_{i=1}^{n_2} \alpha_i l\left(v^i\right)+\sum_{j=1}^{n_1} \beta_j l\left(w^j\right)$$

From (6.1), i.e., $\left(w^1, .., w^{n_1}\right)$ is a basis of $\ker l$, and from (6.3), we get

$$\sum_{i=1}^{n_2} \alpha_i u_i=0$$

From (6.2), i.e., $\left(u^1, .., u^{n_2}\right)$ is a basis of $\operatorname{Im} l$,

$$\left(\alpha_i\right)_{i=1}^{n_2}=0 \tag{6.7}$$

But from the assumption in (6.6) and (6.7) we have that

$$\sum_{j=1}^{n_1} \beta_j w^j=0$$

and since $\left(w^1, .., w^{n_1}\right)$ is a basis of $\ker l$, we get also that

$$\left(\beta_j\right)_{j=1}^{n_1}=0,$$

as desired. ■

Example 246 *Let V and U be vector spaces, with $\dim V=n$.*

In 1. and 2. below, we verify the statement of the Dimension Theorem: in 3. and 4., we use that statement.

1. Identity function id_V.

$$\begin{aligned}&\dim \operatorname{Im} id_V=n\\&\dim \ker l=0.\end{aligned}$$

2. Null function $0 \in L(V, U)$

$$\begin{aligned}&\dim \operatorname{Im} 0=0\\&\dim \ker 0=n.\end{aligned}$$

3. $l \in L\left(\mathbb{R}^2, \mathbb{R}\right)$,

$$l\left(\left(x_1, x_2\right)\right)=x_1.$$

Since $\ker l=\left\{\left(x_1, x_2\right) \in \mathbb{R}^2: x_1=0\right\}$, $\{(0,1)\}$ *is a basis*[1] *of* $\ker l$ *and*

$$\begin{aligned}&\dim \ker l=1,\\&\dim \operatorname{Im} l=2-1=1.\end{aligned}$$

[1] In Remark 315 we will present an algorithm to compute a basis of $\ker l$.

4. $l \in L\left(\mathbb{R}^3, \mathbb{R}^2\right)$,

$$l\left((x_1, x_2, x_3)\right) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Defined

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix},$$

since

$$\operatorname{Im} l = \left\{y \in \mathbb{R}^2 : \exists x \in \mathbb{R}^3 \text{ such that } Ax = y\right\} = \operatorname{span}\operatorname{col} A = rank\ A,$$

and since the first two column vectors of A are linearly independent, we have that

$$\begin{aligned} &\dim \operatorname{Im} l = 2 \\ &\dim \ker l = 3 - 2 = 1. \end{aligned}$$

Exercise 247 *Let $l \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ such that*

$$l \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{pmatrix}$$

be given. Find $\ker l$ and $\operatorname{Im} l$ (Answer: $\ker l = \{0\}$; $\operatorname{Im}\ l = \mathbb{R}^3$).

6.3 Nonsingular functions and isomorphisms

Definition 248 *$l \in L(V, U)$ is singular if $\exists v \in V \setminus \{0\}$ such that $l(v) = 0$.*

Remark 249 *Thus $l \in L(V, U)$ is nonsingular if $\forall v \in V \setminus \{0\}$, $l(v) \neq 0$ i.e., $\ker l = \{0\}$. Briefly,*

$$l \in L(V, U) \text{ is nonsingular } \Leftrightarrow \quad \ker l = \{0\}.$$

Remark 250 *In Remark 289, we will discuss the relationship between singular matrices and singular linear functions.*

Example 251 *1. Let $l \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ be the projection mapping into the xy plane, i.e.,*

$$\begin{array}{cccc} l: & \mathbb{R}^3 & \to & \mathbb{R}^3 \\ & \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \end{array}$$

Then l is singular, since for any $a \in \mathbb{R}$, $l(0,0,a) = (0,0,0)$.
2. $l \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ defined in Example 247 is nonsingular, since $\ker l = \{0\}$.

Proposition 252 *If $l \in L(V, U)$ is nonsingular, then the images of linearly independent vectors are linearly independent.*

Proof. Suppose $v^1, ..., v^n$ are linearly independent. We want to show that $l\left(v^1\right), ..., l\left(v^n\right)$ are linearly independent as well. Suppose

$$\sum_{i=1}^{n} \alpha_i \cdot l\left(v^i\right) = 0.$$

Then

$$l\left(\sum_{i=1}^{n} \alpha_i \cdot v^i\right) = 0.$$

and therefore $\left(\sum_{i=1}^{n} \alpha_i \cdot v^i\right) \in \ker l = \{0\}$, where the last equality comes from the fact that l is nonsingular. Then $\sum_{i=1}^{n} \alpha_i \cdot v^i = 0$ and, since $\left\{v^1, ..., v^n\right\}$ are linearly independent, $(\alpha_i)_{i=1}^{n} = 0$, as desired. ■

Definition 253 *Let two vector spaces V and U be given. U is isomorphic to V if there exists a function $l \in L(V, U)$ which is one-to-one and onto. l is called an isomorphism from V to U.*

Remark 254 *By definition of isomorphism , if l is an isomorphism, the l is invertible and therefore, from Proposition 239, $l^{-1}$is linear.*

Remark 255 *"To be isomorphic" is an equivalence relation.*

Proposition 256 *Any n-dimensional vector space Von a field F is isomorphic to F^n.*

Proof. Since V and F^n are vector spaces, we are left with showing that there exists an isomorphism between them. Let $\mathcal{V} = \left\{v^1, ..., v^n\right\}$ be a basis of V. Recall that we define

$$cr : V \to F^n, \quad v \mapsto [v]_{\mathcal{V}},$$

where cr stands for "coordinates".

1. cr is linear. Given $v, w \in V$, suppose

$$v = \sum_{i=1}^{n} a_i v^i \quad \text{and} \quad w = \sum_{i=1}^{n} b_i v^i$$

i.e.,

$$[v]_{\mathcal{V}} = [a_i]_{i=1}^n \quad \text{and} \quad [w]_{\mathcal{V}} = [b_i]_{i=1}^n \ .$$

$\forall \alpha, \beta \in F$ and $\forall v^1, v^2 \in V$,

$$\alpha v + \beta w = \alpha \sum_{i=1}^{n} a_i v^i + \beta \sum_{i=1}^{n} b_i v^i = \sum_{i=1}^{n} (\alpha a_i + \beta b_i) v^i$$

i.e.,

$$[\alpha v + \beta w]_{\mathcal{V}} = \alpha [a_i]_{i=1}^n + \beta [b_i]_{i=1i}^n = \alpha \ [v]_{\mathcal{V}} + \beta [w]_{\mathcal{V}}.$$

2. cr is onto. $\forall (a)_{i=1}^n \in \mathbb{R}^n$, $cr\left(\sum_{i=1}^n a_i v^i\right) = (a)_{i=1}^n$.

3. cr is one-to-one. $cr(v) = cr(w)$ implies that $v = w$, simply because $v = \sum_{i=1}^n cr_i(v) u^i$ and $w = \sum_{i=1}^n cr_i(w) u^i$. ■

Remark 257 *If two spaces S and C are isomorphic, then we can use the isomorphism between the two spaces to infer properties about one the two, knowing properties of the other one. Indeed, sometimes it is possible to show a "complicated space" C is isomorphic to a "simple space" S. Then, we can first show properties about S, and then, using the isomorphism, infer properties of the complicated space C.*

Proposition 258 *Let V and U be finite dimensional vectors spaces on the same field F such that $S = \left(v^1, ..., v^n\right)$ is a basis of V and $u^1, ..., u^n$ are arbitrary vectors in U. Then there exists a unique linear function $l : V \to U$ such that $\forall i \in \{1, ..., n\}$, $l\left(v^i\right) = u^i$.*

Proof. The proof goes the following three steps.

1. Define l;

2. Show that l is linear;

3. Show that l is unique.

1. Using the definition of coordinates, $\forall v \in V$, define

$$l : V \to U, \quad v \mapsto \sum_{i=1}^{n} [v]_s^i \cdot u^i,$$

where $[v]_S^i$ denotes the $i-th$ component of the vector $[v]_S$. Recall that $e_n^j \in \mathbb{R}^n$ is the $j-th$ element in the canonical basis of $\mathbb{R}^n$ and defined $e_n^j := \left(e_{n,i}^j\right)_{i=1}^n$, we have

$$e_{n,i}^j = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j. \end{cases}$$

Then $\forall j \in \{1, ..., n\}$, $\left[v^j\right]_S = e_n^j$ and

$$l\left(v^j\right) = \sum_{i=1}^{n} \left[v^j\right]_s^i \cdot u^i = \sum_{i=1}^{n} e_{n,i}^j \cdot u^i = u^j.$$

2. Let $v, w \in V$ and $\alpha, \beta \in F$. Then

$$l\left(\alpha v + \beta w\right) = \sum_{i=1}^{n} \left[\alpha v + \beta w\right]_S^i \cdot u^i = \sum_{i=1}^{n} \left(\alpha \left[v\right]_S^i + \beta \left[w\right]_S^i\right) \cdot u^i = \alpha \sum_{i=1}^{n} \left[v\right]_S^i \cdot u^i + \beta \sum_{i=1}^{n} \left[w\right]_S^i \cdot u^i,$$

where the before the last equality follows from the linearity of [.] - see the proof of Proposition 256

3. Suppose $g \in L\left(V, U\right)$ and $\forall i \in \{1, ..., n\}$, $g\left(v^i\right) = u^i$. Then $\forall v \in V$,

$$g\left(v\right) = g\left(\sum_{i=1}^{n} \left[v\right]_S^i \cdot v^i\right) = \sum_{i=1}^{n} \left[v\right]_S^i \cdot g\left(v^i\right) = \sum_{i=1}^{n} \left[v\right]_S^i \cdot u^i = l\left(v\right)$$

where the last equality follows from the definition of l. ■

Remark 259 *Observe that if V and U are finite(nonzero) dimensional vector spaces, there is a multitude of functions from V to U. The above Proposition says that linear functions are completely determined by what "they do to the elements of a basis" of V.*

Example 260 *We want to find the unique linear mapping $l : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$l(1, 2) = (2, 3) \quad \textit{and} \quad l(0, 1) = (1, 4).$$

Observe that $\mathcal{B} := \{(1, 2), (0, 1)\}$ is a basis of $\mathbb{R}^2$. For any $(a, b) \in \mathbb{R}^2$, there exist $x, y \in \mathbb{R}$ such that

$$(a, b) = x(1, 2) + y(0, 1) = (x, 2x + y),$$

i.e., $a = x$ and $b = 2x + y$ and therefore $x = a$ and $y = -2a + b$. Then,

$$l(a, b) = l(x(1, 2) + y(0, 1)) \overset{l \text{ linear}}{=} xl(1, 2) + yl(0, 1) = a(2, 3) + (-2a + b)(1, 4) = (b, -5a + 4b),$$

i.e.,

$$l : \mathbb{R}^2 \to \mathbb{R}^2, \quad (a, b) \mapsto (b, -5a + 4b).$$

Proposition 261 *Assume that $l \in L\left(V, U\right)$. Then,*

$$l \textit{ is one-to-one} \Leftrightarrow l \textit{ is nonsingular}$$

Proof. [⇒]
Take $v \in \ker l$. Then

$$l\left(v\right) = 0 = l\left(0\right)$$

where last equality follows from Remark 234. Since l is one-to-one, $v = 0$.
[⇐] If $l\left(v\right) = l\left(w\right)$, then $l\left(v - w\right) = 0$ and, since l is nonsingular, $v - w = 0$. ■

Proposition 262 *Assume that V and U are finite dimensional vector spaces and $l \in L\left(V, U\right)$. Then,*

1. *l is one-to-one $\Rightarrow \dim V \leq \dim U$;*
2. *l is onto $\Rightarrow \dim V \geq \dim U$;*
3. l is invertible $\Rightarrow \dim V = \dim U$.

Proof. The main ingredient in the proof is Proposition 245 , i.e., the Dimension Theorem.

1. Since l is one-to-one, from the previous Proposition, $\dim \ker l = 0$. Then, from Proposition 245 (the Dimension Theorem), $\dim V = \dim \operatorname{Im} l$. Since $\operatorname{Im} l$ is a subspace of U, then $\dim \operatorname{Im} l \leq \dim U$.

2. Since l is onto iff $\operatorname{Im} l = U$, from Proposition 245 (the Dimension Theorem), we get

$$\dim V = \dim \ker l + \dim U \geq \dim U.$$

3. l is invertible iff l is one-to-one and onto. ■

Proposition 263 *Let V and U be finite dimensional vector space on the same field F. Then,*

$$U \text{ and } V \text{ are isomorphic} \Leftrightarrow \dim U = \dim V.$$

Proof. $[\Rightarrow]$
It follows from the definition of isomorphism and part 3 in the previous Proposition.
$[\Leftarrow]$
Assume that V and U are vector spaces such that $\dim V = \dim U = n$. Then, from Proposition 256, V and U are isomorphic to F^n and from Remark 255, the result follows. ■

Proposition 264 *Suppose V and U are vector spaces such that $\dim V = \dim U = n$ and $l \in L(V, U)$. Then the following statements are equivalent.*
1. l is nonsingular, i.e., $\ker l = \{0\}$,
2. l is one-to-one,
3. l is onto,
4. l is an isomorphism.

Proof. $[1 \Leftrightarrow 2]$.
It is the content of Proposition 261.
$[1 \Rightarrow 3]$
Since l is nonsingular, then $\ker l = \{0\}$ and $\dim \ker l = 0$. Then, from Proposition 245 (the Dimension Theorem), i.e., $\dim V = \dim \ker l + \dim \operatorname{Im} l$, and the fact $\dim V = \dim U$, we get $\dim U = \dim \operatorname{Im} l$. Since $\operatorname{Im} l \subseteq U$ and U is finite dimensional, from Proposition 191, $\operatorname{Im} l = U$, i.e., l is onto, as desired.
$[3 \Rightarrow 1]$
Since l is onto, $\dim \operatorname{Im} l = \dim V$ and from Proposition 245 (the Dimension Theorem), $\dim V = \dim \ker t + \dim V$, and therefore $\dim \ker l = 0$, i.e., l is nonsingular.
$[1 \Rightarrow 4]$
It follows from the definition of isomorphism and the facts that $[1 \Leftrightarrow 2]$ and $[1 \Rightarrow 3]$.
$[4 \Rightarrow 1]$
It follows from the definition of isomorphism and the facts that $[2 \Leftrightarrow 1]$. ■

Definition 265 *A vector space endomorphism is a linear function from a vector space V into itself.*

6.4 Exercises

From Lipschutz (1991), starting from page 325:
9.3, 9.6, 9.9 → 9.11, 9.16 → 9.21, 9.26, 9.27, 9.31 → 9.35, 9.42 → 9.44; observe that Lipschutz denotes $\mathcal{L}(V, U)$ by $Hom(V, U)$.

Chapter 7

Linear functions and matrices

In Remark 65 we have seen that the set of $m \times n$ matrices with the standard sum and scalar multiplication is a vector space, called $\mathbb{M}(m,n)$. We are going to show that:

1. the set $L(V,U)$ with naturally defined sum and scalar multiplication is a vector space, called $\mathcal{L}(V,U)$;
2. If $\dim V = n$ and $\dim U = m$, then $\mathcal{L}(V,U)$ and $\mathbb{M}(m,n)$ are isomorphic.

7.1 From a linear function to the associated matrix

Definition 266 *Suppose V and U are vector spaces over a field F and $l_1, l_2 \in L(V,U)$ and $\alpha \in F$.*

$$l_1 + l_2 : V \to U, \qquad v \mapsto l_1(v) + l_2(v)$$

$$\alpha l_1 : V \to U, \qquad v \mapsto \alpha l_1(v).$$

Proposition 267 *$L(V,U)$ with the above defined operations is a vector space on F, denoted by $\mathcal{L}(V,U)$.*

Proof. Exercise.[1] ■

Proposition 268 *Compositions of linear functions are linear.*

Proof. Suppose V, U, W are vector spaces over a field F, $l_1 \in \mathcal{L}(V,U)$ and $l_2 \in \mathcal{L}(U,W)$. We want to show that $l := l_2 \circ l_1 \in \mathcal{L}(V,W)$. Indeed, for any$\alpha_1, \alpha_2 \in F$ and for any $v^1, v^2 \in V$, we have that

$$l\left(\alpha_1 v^1 + \alpha_2 v^2\right) := (l_2 \circ l_1)\left(\alpha_1 v^1 + \alpha_2 v^2\right) = l_2\left(l_1\left(\alpha_1 v^1 + \alpha_2 v^2\right)\right) =$$

$$= l_2\left(\alpha_1 l_1\left(v^1\right) + \alpha_2 l_1\left(v^2\right)\right) = \alpha_1 l_2\left(l_1\left(v^1\right)\right) + \alpha_2 l_2\left(l_1\left(v^2\right)\right) = \alpha_1 l\left(v^1\right) + \alpha_2 l\left(v^2\right),$$

as desired. ■

Definition 269 *Suppose $l \in \mathcal{L}(V,U)$, $\mathcal{V} = \left(v^1, ..., v^n\right)$ is a basis of V, $\mathcal{U} = \left(u^1, ..., u^m\right)$ is a basis of U. Then,*

$$[l]_{\mathcal{V}}^{\mathcal{U}} := \left[\ \left[l\left(v^1\right)\right]_{\mathcal{U}} \quad ... \quad \left[l\left(v^j\right)\right]_{\mathcal{U}} \quad ... \quad \left[l\left(v^n\right)\right]_{\mathcal{U}}\ \right] \in \mathbb{M}(m,n), \tag{7.1}$$

where for any $j \in \{1, ..., n\}$, $\left[l\left(v^j\right)\right]_{\mathcal{U}}$ is a column vector, is called the matrix representation of l relative to the basis $\mathcal{V}$ and $\mathcal{U}$. In words, $[l]_{\mathcal{V}}^{\mathcal{U}}$ is the matrix whose columns are the coordinates relative to the basis of the codomain of l of the images of each vector in the basis of the domain of l.

Remark 270 *Observe that by definition of coordinates, there is a unique matrix representation of a linear function relative to the basis $\mathcal{V}$ and $\mathcal{U}$.*

Definition 271 *Suppose $l \in \mathcal{L}(V,U)$, $\mathcal{V}: = \left(v^1, ..., v^n\right)$ is a basis of V, $\mathcal{U} := \left(u^1, ..., u^m\right)$ is a basis of U.*

$$\varphi_{\mathcal{V}}^{\mathcal{U}} : \mathcal{L}(V,U) \to \mathbb{M}(m,n), \quad l \mapsto [l]_{\mathcal{V}}^{\mathcal{U}} \text{ defined in } (7.1).$$

If no confusion may arise, we will denote $\varphi_{\mathcal{V}}^{\mathcal{U}}$ simply by φ.

[1] For a detailed proof see Lipschutz (1989), page 270.

Example 272 *Given* $id_V \in \mathcal{L}(V,V)$ *and a basis* $\mathcal{V} = \left(v^1, ..., v^n\right)$ *of* V *, then* $[id_V]_{\mathcal{V}}^{\mathcal{V}} = I_n$.

The proposition below shows that multiplying the coordinate vector of v relative to the basis $\mathcal{V}$ by the matrix $[l]_{\mathcal{V}}^{\mathcal{U}}$, we get the coordinate vector of $l(v)$ relative to the basis $\mathcal{U}$.

Proposition 273 $\forall v \in V$,

$$[l]_{\mathcal{V}}^{\mathcal{U}} \cdot [v]_{\mathcal{V}} = [l(v)]_{\mathcal{U}} \tag{7.2}$$

Proof. Assume $v \in V$. First of all observe that

$$[l]_{\mathcal{V}}^{\mathcal{U}} \cdot [v]_{\mathcal{V}} = \left[\begin{array}{ccccc} \left[l\left(v^1\right)\right]_{\mathcal{U}} & ... & \left[l\left(v^j\right)\right]_{\mathcal{U}} & ... & \left[l\left(v^n\right)\right]_{\mathcal{U}} \end{array}\right] \left[\begin{array}{c} [v]_{\mathcal{V}}^1 \\ ... \\ [v]_{\mathcal{V}}^j \\ ... \\ [v]_{\mathcal{V}}^n \end{array}\right] = \sum_{j=1}^n [v]_{\mathcal{V}}^j \cdot \left[l\left(v^j\right)\right]_{\mathcal{U}}.$$

Moreover, from the linearity of the function $cr_{\mathcal{U}} := [.]_{\mathcal{U}}$, and using the fact that the composition of linear functions is a linear function, we get:

$$[l(v)]_{\mathcal{U}} = cr_{\mathcal{U}}(l(v)) = (cr_{\mathcal{U}} \circ l)\left(\sum_{j=1}^n [v]_{\mathcal{V}}^j \cdot v^j\right) = \sum_{j=1}^n [v]_{\mathcal{V}}^j \cdot (cr_{\mathcal{U}} \circ l)\left(v^j\right) = \sum_{j=1}^n [v]_{\mathcal{V}}^j \cdot \left[l\left(v^j\right)\right]_{\mathcal{U}}.$$

■

Example 274 *Let's verify equality* (7.2) *in the case in which*

a.

$$l : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto \left(\begin{array}{c} x_1 + x_2 \\ x_1 - x_2 \end{array}\right)$$

b. the basis $\mathcal{V}$ *of the domain of* l *is*

$$\left(\left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \end{array}\right)\right),$$

c. the basis $\mathcal{U}$ *of the codomain of* l *is*

$$\left(\left(\begin{array}{c} 1 \\ 1 \end{array}\right), \left(\begin{array}{c} 2 \\ 1 \end{array}\right)\right),$$

d.

$$v = \left(\begin{array}{c} 3 \\ 4 \end{array}\right).$$

The main needed computations are presented below.

$$[l]_{\mathcal{V}}^{\mathcal{U}} := \left[\left[l\left(\begin{array}{c} 1 \\ 0 \end{array}\right)\right]_{\mathcal{U}}, \left[l\left(\begin{array}{c} 0 \\ 1 \end{array}\right)\right]_{\mathcal{U}}\right] = \left[\left[\begin{array}{c} 1 \\ 1 \end{array}\right]_{\mathcal{U}}, \left[\begin{array}{c} 1 \\ -1 \end{array}\right]_{\mathcal{U}}\right] = \left[\begin{array}{cc} 1 & -3 \\ 0 & 2 \end{array}\right],$$

$$[v]_{\mathcal{V}} = \left[\begin{array}{c} 3 \\ 4 \end{array}\right],$$

$$[l(v)]_{\mathcal{U}} = \left[\left(\begin{array}{c} 7 \\ -1 \end{array}\right)\right]_{\mathcal{U}} = \left[\begin{array}{c} -9 \\ 8 \end{array}\right],$$

$$[l]_{\mathcal{V}}^{\mathcal{U}} \cdot [v]_{\mathcal{V}} = \left[\begin{array}{cc} 1 & -3 \\ 0 & 2 \end{array}\right]\left[\begin{array}{c} 3 \\ 4 \end{array}\right] = \left[\begin{array}{c} -9 \\ 8 \end{array}\right].$$

7.2 From a matrix to the associated linear function

Given $A \in \mathbb{M}(m,n)$, recall that $\forall i \in \{1,..,m\}$, $R^i(A)$ denotes the $i-th$ row vector of A, i.e.,

$$A = \begin{bmatrix} R^1(A) \\ ... \\ R^i(A) \\ ... \\ R^m(A) \end{bmatrix}_{m \times n}$$

Definition 275 *Consider vector spaces V and U with bases $\mathcal{V} = (v^1, ..., v^n)$ and $\mathcal{U} = (u^1, ..., u^m)$, respectively. Given $A \in \mathbb{M}(m,n)$, define*

$$l^{\mathcal{U}}_{A,\mathcal{V}} : V \to U, \quad v \mapsto \sum_{i=1}^{m} \left(R^i(A) \cdot [v]_{\mathcal{V}}\right) \cdot u^i.$$

Example 276 *Take $U = V = \mathbb{R}^2$, $\mathcal{V} = \mathcal{E}_2$, $\mathcal{U} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ and $A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$. Then,*

$$l^{\mathcal{U}}_{A,\mathcal{V}}(x_1,x_2) := \sum_{i=1}^{2} \left(R^i(A) \cdot [v]_{\mathcal{E}_2}\right) \cdot u^i = \left(\begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

$$= (x_1 - 3x_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 + 4x_2 \\ x_1 - 3x_2 + 2x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

Proposition 277 *$l^{\mathcal{U}}_{A,\mathcal{V}}$ defined above is linear, i.e., $l^{\mathcal{U}}_{A,\mathcal{V}} \in \mathcal{L}(V,U)$.*

Proof. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall v^1, v^2 \in V$,

$$l^{\mathcal{U}}_{A,\mathcal{V}}\left(\alpha v^1 + \beta v^2\right) = \sum_{i=1}^{m} R^i(A) \cdot \left[\alpha v^1 + \beta v^2\right]_{\mathcal{V}} \cdot u^i = \sum_{i=1}^{m} R^i(A) \cdot \left(\alpha \left[v^1\right]_{\mathcal{V}} + \beta \left[v^2\right]_{\mathcal{V}}\right) \cdot u^i =$$

$$= \alpha \sum_{i=1}^{m} R^i(A) \cdot \left[v^1\right]_{\mathcal{V}} \cdot u^i + \beta \sum_{i=1}^{m} R^i(A) \cdot \left[v^2\right]_{\mathcal{V}} \cdot u^i = \alpha l^{\mathcal{U}}_{A,\mathcal{V}}\left(v^1\right) + \beta l^{\mathcal{U}}_{A,\mathcal{V}}\left(v^2\right).$$

where the second equality follows from the proof of Proposition 256. ■

Definition 278 *Given the vector spaces V and U with bases $\mathcal{V} = (v^1, ..., v^n)$ and $\mathcal{U} = (u^1, ..., u^m)$, respectively, define*

$$\psi^{\mathcal{U}}_{\mathcal{V}} : \mathbb{M}(m,n) \to \mathcal{L}(V,U) \qquad : A \mapsto l^{\mathcal{U}}_{A,\mathcal{V}}.$$

If no confusion may arise, we will denote $\psi^{\mathcal{U}}_{\mathcal{V}}$ simply by ψ.

Proposition 279 *ψ defined above is linear.*

Proof. We want to show that $\forall \alpha, \beta \in \mathbb{R}$ and $\forall A, B \in \mathbb{M}(m,n)$,

$$\psi(\alpha A + \beta B) = \alpha \psi(A) + \beta \psi(B)$$

i.e.,

$$l^{\mathcal{V}}_{\alpha A + \beta B, \mathcal{U}} = \alpha l^{\mathcal{U}}_{A,\mathcal{V}} + \beta l^{\mathcal{V}}_{B,\mathcal{U}}$$

i.e., $\forall v \in V$,

$$l^{\mathcal{V}}_{\alpha A + \beta B, \mathcal{U}}(v) = \alpha l^{\mathcal{U}}_{A,\mathcal{V}}(v) + \beta l^{\mathcal{V}}_{B,\mathcal{U}}(v).$$

Now,

$$l^{\mathcal{V}}_{\alpha A + \beta B, \mathcal{U}}(v) = \sum_{i=1}^{m} \left(\alpha \cdot R^i(A) + \beta \cdot R^i(B)\right) \cdot [v]_{\mathcal{V}} \cdot u^i =$$

$$= \alpha \sum_{i=1}^{m} R^i(A) \cdot [v]_{\mathcal{V}} \cdot u^i + \beta \sum_{i=1}^{m} R^i(B) \cdot [v]_{\mathcal{V}} \cdot u^i = \alpha l^{\mathcal{U}}_{A,\mathcal{V}}(v) + \beta l^{\mathcal{V}}_{B,\mathcal{U}}(v),$$

where the first equality come from Definition 275. ■

7.3 $\mathbb{M}(m,n)$ and $\mathcal{L}(V,U)$ are isomorphic

Proposition 280 *Given the vector space V and U with dimension n and m, respectively,*

$$\mathbb{M}(m,n) \text{ and } \mathcal{L}(V,U) \text{ are isomorphic,}$$

and

$$\dim \mathcal{L}(V,U) = mn.$$

Proof. Linearity of the two spaces was proved above. We want now to check that ψ presented in Definition 278 is an isomorphism, i.e., ψ is linear, one-to-one and onto. In fact, thanks to Proposition 279, it is enough to show that ψ is invertible.

First proof.

1. ψ is one-to-one: see Theorem 2, page 105 in Lang (1971);

2. ψ is onto: see bottom of page 107 in Lang (1970).

Second proof.

1. $\psi \circ \varphi = id_{\mathcal{L}(V,U)}$.

Given $l \in \mathcal{L}(V,U)$, we want to show that $\forall v \in V$,

$$l(v) = ((\psi \circ \varphi)(l))(v)$$

i.e., from Proposition 256,

$$[l(v)]_{\mathcal{U}} = [((\psi \circ \varphi)(l))(v)]_{\mathcal{U}}.$$

First of all, observe that from (7.2), we have

$$[l(v)]_{\mathcal{U}} = [l]_{\mathcal{V}}^{\mathcal{U}} [v]_{\mathcal{V}}.$$

Moreover,

$$[((\psi \circ \varphi)(l))(v)]_{\mathcal{U}} \overset{(1)}{=} \left[\psi\left([l]_{\mathcal{V}}^{\mathcal{U}}\right)(v)\right]_{\mathcal{U}} \overset{(2)}{=} \left[\sum_{i=1}^{n}\left(i-th \text{ row of } [l]_{\mathcal{V}}^{\mathcal{U}}\right)\cdot [v]_{\mathcal{V}} \cdot u^i\right]_{\mathcal{U}} \overset{(3)}{=}$$

$$= \left[\left(i-th \text{ row of } [l]_{\mathcal{V}}^{\mathcal{U}}\right)\cdot [v]_{\mathcal{V}}\right]_{i=1}^{n} \overset{(4)}{=} [l]_{\mathcal{V}}^{\mathcal{U}} \cdot [v]_{\mathcal{V}}$$

where (1) comes from the definition of φ, (2) from the definition of ψ, (3) from the definition of $[.]_{\mathcal{U}}$, (4) from the definition of product between matrices.

2. $\varphi \circ \psi = id_{\mathbb{M}(m,n)}$.

Given $A \in \mathbb{M}(m,n)$, we want to show that $(\varphi \circ \psi)(A) = A$. By definition of ψ,

$$\psi(A) = l_{A,\mathcal{V}}^{\mathcal{U}} \text{ such that } \forall v \in V,\ l_{A,\mathcal{V}}^{\mathcal{U}}(v) = \sum_{i=1}^{m} R^i(A)\cdot [v]_{\mathcal{V}} \cdot u^i. \tag{7.3}$$

By definition of φ,

$$\varphi(\psi(A)) = \left[l_{A,\mathcal{V}}^{\mathcal{U}}\right]_{\mathcal{V}}^{\mathcal{U}}.$$

Therefore, we want to show that $\left[l_{A,\mathcal{V}}^{\mathcal{U}}\right]_{\mathcal{V}}^{\mathcal{U}} = A$. Observe that from 7.3,

$$\begin{array}{lllllll}
l_{A,\mathcal{V}}^{\mathcal{U}}(v^1) & = & \sum_{i=1}^{m} R^i(A)\cdot[v^1]_{\mathcal{V}}\cdot u^i & = & \sum_{i=1}^{m}[a_{i1},...,a_{ij},...,a_{in}]\cdot[1,...,0,...,0]\cdot u^i & = & a_{11}u^1+...+a_{i1}u^i+....+a_{m1}u^m \\
.... & & & & & & \\
l_{A,\mathcal{V}}^{\mathcal{U}}(v^j) & = & \sum_{i=1}^{m} R^i(A)\cdot[v^j]_{\mathcal{V}}\cdot u^i & = & \sum_{i=1}^{m}[a_{i1},...,a_{ij},...,a_{in}]\cdot[0,...,1,...,0]\cdot u^i & = & a_{1j}u^1+...+a_{ij}u^i+...+a_{mj}u^m \\
... & & & & & & \\
l_{A,\mathcal{V}}^{\mathcal{U}}(v^n) & = & \sum_{i=1}^{m} R^i(A)\cdot[v^n]_{\mathcal{V}}\cdot u^i & = & \sum_{i=1}^{m}[a_{i1},...,a_{ij},...,a_{in}]\cdot[0,...,0,...,1]\cdot u^i & = & a_{1n}u^1+...+a_{in}u^i+...+a_{mn}u^m
\end{array}$$

(From the above, it is clear why in definition 269 we take the transpose.) Therefore,

$$\left[l_{A,\mathcal{V}}^{\mathcal{U}}\right]_{\mathcal{V}}^{\mathcal{U}} = \begin{bmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... & & & & \\ a_{i1} & & a_{ij} & & a_{in} \\ ... & & & & \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{bmatrix} = A,$$

as desired.

The fact that $\dim \mathcal{L}(V,U)$ follows from Proposition 263. ■

Proposition 281 *Let the following objects be given.*

1. Vector spaces V with basis $\mathcal{V}=\left(v^1,...,v^j,...,v^n\right)$, U with basis $\mathcal{U}=\left(u^1,...,u^i,...,u^m\right)$ and W with basis $\mathcal{W}=\left(w^1,...,w^k,...,w^p\right)$;

2. $l_1\in\mathcal{L}(V,U)$ and $l_2\in\mathcal{L}(U,W)$.

Then

$$[l_2\circ l_1]_{\mathcal{V}}^{\mathcal{W}}=[l_2]_{\mathcal{U}}^{\mathcal{W}}\cdot[l_1]_{\mathcal{V}}^{\mathcal{U}},$$

or

$$\varphi_{\mathcal{V}}^{\mathcal{W}}(l_2\circ l_1)=\varphi_{\mathcal{U}}^{\mathcal{W}}(l_2)\cdot\varphi_{\mathcal{V}}^{\mathcal{U}}(l_1).$$

Proof. By definition[2]

$$[l_1]_{\mathcal{V}}^{\mathcal{U}}=\left[\ \left[l_1\left(v^1\right)\right]_{\mathcal{U}}\ \ ...\ \ \left[l_1\left(v^j\right)\right]_{\mathcal{U}}\ \ ...\ \ \left[l_1\left(v^n\right)\right]_{\mathcal{U}}\ \right]_{m\times n}:=$$

$$:=\begin{bmatrix} l_1^1\left(v^1\right) & ... & l_1^1\left(v^j\right) & ... & l_1^1\left(v^n\right)\\ ... & & & & \\ l_1^i\left(v^1\right) & & l_1^i\left(v^j\right) & & l_1^i\left(v^n\right)\\ ... & & & & \\ l_1^m\left(v^1\right) & ... & l_1^m\left(v^j\right) & ... & l_1^m\left(v^n\right)\end{bmatrix}:=\begin{bmatrix} l_1^{11} & ... & l_1^{1j} & ... & l_1^{1n}\\ ... & & & & \\ l_1^{i1} & & l_1^{ij} & & l_1^{in}\\ ... & & & & \\ l_1^{m1} & ... & l_1^{mj} & ... & l_1^{mn}\end{bmatrix}:=$$

$$:=\left[l_1^{ij}\right]_{i\in\{1,...,m\},j\in\{1,...,n\}}:=A\in\mathbb{M}(m,n),$$

and therefore $\forall j\in\{1,...,n\}$, $l_1\left(v^j\right)=\sum_{i=1}^m l_1^{ij}\cdot u^i$.

Similarly,

$$[l_2]_{\mathcal{U}}^{\mathcal{W}}=\left[\ \left[l_2\left(u^1\right)\right]_{\mathcal{W}}\ \ ...\ \ \left[l_2\left(u^i\right)\right]_{\mathcal{W}}\ \ ...\ \ \left[l_2\left(u^m\right)\right]_{\mathcal{W}}\ \right]_{p\times m}:=$$

$$:=\begin{bmatrix} l_2^1\left(u^1\right) & ... & l_2^1\left(u^i\right) & ... & l_2^1\left(u^m\right)\\ ... & & & & \\ l_2^k\left(u^1\right) & & l_2^k\left(u^i\right) & & l_2^k\left(u^m\right)\\ ... & & & & \\ l_2^p\left(u^1\right) & ... & l_2^p\left(u^i\right) & ... & l_2^p\left(u^m\right)\end{bmatrix}:=\begin{bmatrix} l_2^{11} & ... & l_2^{1i} & ... & l_2^{1m}\\ ... & & & & \\ l_2^{k1} & & l_2^{ki} & & l_2^{km}\\ ... & & & & \\ l_2^{p1} & ... & l_2^{pi} & ... & l_2^{pm}\end{bmatrix}:=$$

$$:=\left[l_2^{ki}\right]_{k\in\{1,...,p\},i\in\{1,...,m\}}:=B\in\mathbb{M}(p,m),$$

and therefore $\forall i\in\{1,...,m\}$, $l_2\left(u^i\right)=\sum_{k=1}^p l_2^{ki}\cdot w^k$.

Moreover, defined $l:=(l_2\circ l_1)$, we get

$$[l_2\circ l_1]_{\mathcal{V}}^{\mathcal{W}}=\left[\ \left[l\left(v^1\right)\right]_{\mathcal{W}}\ \ ...\ \ \left[l\left(v^j\right)\right]_{\mathcal{W}}\ \ ...\ \ \left[l\left(v^n\right)\right]_{\mathcal{W}}\ \right]_{p\times n}:=$$

$$:=\begin{bmatrix} l^1\left(v^1\right) & ... & l^1\left(v^j\right) & ... & l^1\left(v^n\right)\\ ... & & & & \\ l^k\left(v^1\right) & & l^k\left(v^j\right) & & l^k\left(v^n\right)\\ ... & & & & \\ l^p\left(v^1\right) & ... & l^p\left(v^j\right) & ... & l^p\left(v^n\right)\end{bmatrix}:=\begin{bmatrix} l^{11} & ... & l^{1j} & ... & l^{1n}\\ ... & & & & \\ l^{k1} & & l^{kj} & & l^{kn}\\ ... & & & & \\ l^{p1} & ... & l^{pj} & ... & l^{pn}\end{bmatrix}:=$$

$$:=\left[l^{kj}\right]_{k\in\{1,...,p\},j\in\{1,...,n\}}:=C\in\mathbb{M}(p,n),$$

and therefore $\forall j\in\{1,...,n\}$, $l\left(v^j\right)=\sum_{k=1}^p l^{kj}\cdot w^k$.

Now, $\forall j\in\{1,...,n\}$

$$l\left(v^j\right)=(l_2\circ l_1)\left(v^j\right)=l_2\left(l_1\left(v^j\right)\right)=l_2\left(\sum_{i=1}^m l_1^{ij}\cdot u^i\right)=$$
$$\sum_{i=1}^m l_1^{ij}\cdot l_2\left(u^i\right)=\sum_{i=1}^m l_1^{ij}\cdot\sum_{k=1}^p l_2^{ki}\cdot w^k=\sum_{k=1}^p\sum_{i=1}^m l_2^{ki}\cdot l_1^{ij}\cdot w^k.$$

[2] $\left[l_1\left(v^1\right)\right]_{\mathbf{u}}, .., \left[l_1\left(v^n\right)\right]_{\mathbf{u}}$ .are column vectors.

The above says that $\forall j \in \{1, ..., n\}$, the $j-th$ column of C is

$$\begin{bmatrix} \sum_{i=1}^m l_2^{1i} \cdot l_1^{ij} \\ \sum_{i=1}^m l_2^{ki} \cdot l_1^{ij} \\ \sum_{i=1}^m l_2^{pi} \cdot l_1^{ij} \end{bmatrix}$$

On the other hand, the $j-th$ column of $B \cdot A$ is

$$\begin{bmatrix} [1st \text{ row of } B] \cdot [j-th \text{ column of } A] \\ ... \\ [k-th \text{ row of } B] \cdot [j-th \text{ column of } A] \\ ... \\ [p-th \text{ row of } B] \cdot [j-th \text{ column of } A] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} l_2^{11} & ... & l_2^{1i} & ... & l_2^{1m} \end{bmatrix} \cdot \begin{bmatrix} l_1^{1j} \\ l_1^{ij} \\ l_1^{mj} \end{bmatrix} \\ ... \\ \begin{bmatrix} l_2^{k1} & l_2^{ki} & l_2^{km} \end{bmatrix} \cdot \begin{bmatrix} l_1^{1j} \\ l_1^{ij} \\ l_1^{mj} \end{bmatrix} \\ ... \\ \begin{bmatrix} l_2^{p1} & ... & l_2^{pi} & ... & l_2^{pm} \end{bmatrix} \cdot \begin{bmatrix} l_1^{1j} \\ l_1^{ij} \\ l_1^{mj} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m l_2^{1i} \cdot l_1^{ij} \\ \sum_{i=1}^m l_2^{ki} \cdot l_1^{ij} \\ \sum_{i=1}^m l_2^{pi} \cdot l_1^{ij} \end{bmatrix}$$

as desired. ■

7.4 Some related properties of a linear function and associated matrix

In this section, the following objects will be given: a vector space V with a basis $\mathcal{V} = \left(v^1, ..., v^n\right)$; a vector space U with a basis $\mathcal{U} = \left(u^1, ..., u^n\right)$; $l \in \mathcal{L}(V, U)$ and $A \in \mathbb{M}(m, n)$.

From Remark 150, recall that

$$\operatorname{col\,span} A = \{z \in \mathbb{R}^m : \exists \;\; x \in \mathbb{R}^n \text{ such that } z = Ax\};$$

Lemma 282 $cr_{\mathcal{U}}(\operatorname{Im} l) = \operatorname{col\,span}[l]_{\mathcal{V}}^{\mathcal{U}}$.

Proof. [$\subseteq$]

$y \in cr_{\mathcal{U}}(\operatorname{Im} l) \overset{\text{def } cr}{\Rightarrow} \exists u \in \operatorname{Im} l$ such that $cr_{\mathcal{U}}(u) = [u]_{\mathcal{U}} = y \overset{\text{def Im}\, l}{\Rightarrow} \exists v \in V$ such that $l(v) = u \Rightarrow \exists v \in V$ such that $[l(v)]_{\mathcal{U}} = y \overset{\text{Prop. 273}}{\Rightarrow} \exists v \in V$ such that $[l]_{\mathcal{V}}^{\mathcal{U}} \cdot [v]_{\mathcal{V}} = y \Rightarrow y \in \operatorname{col\,span}[l]_{\mathcal{V}}^{\mathcal{U}}$.

[$\supseteq$]

We want to show that $y \in \operatorname{col\,span}[l]_{\mathcal{V}}^{\mathcal{U}} \Rightarrow y \in cr_{\mathcal{U}}(\operatorname{Im} l)$, i.e., $\exists u \in \operatorname{Im} l$ such that $y = [u]_{\mathcal{U}}$.

$y \in \operatorname{col\,span}[l]_{\mathcal{V}}^{\mathcal{U}} \Rightarrow \exists x_y \in \mathbb{R}^n$ such that $[l]_{\mathcal{V}}^{\mathcal{U}} \cdot x_y = y \overset{\text{def } cr}{\Rightarrow} \exists v = \sum_{j=1}^n x_{y,j} v^j$ such that $[l]_{\mathcal{V}}^{\mathcal{U}} \cdot [v]_{\mathcal{V}} = y \overset{\text{Prop. 273}}{\Rightarrow} \exists v \in V$ such that $[l(v)]_{\mathcal{U}} = y \overset{u=l(v)}{\Rightarrow} \exists u \in \operatorname{Im} l$ such that $y = [u]_{\mathcal{U}}$, as desired. ■

Lemma 283 $\dim \operatorname{Im} l = \dim \operatorname{col\,span}[l]_{\mathcal{V}}^{\mathcal{U}} = \operatorname{rank}\ [l]_{\mathcal{V}}^{\mathcal{U}}$.

Proof. It follows from the above Lemma, the proof of Proposition 256, which says that $cr_{\mathcal{U}}$ is an isomorphism and Proposition 263, which says that isomorphic spaces have the same dimension. ■

Proposition 284 *Given $l \in \mathcal{L}(V, U)$,*

1. *l onto* $\Leftrightarrow \operatorname{rank}[l]_{\mathcal{V}}^{\mathcal{U}} = \dim U$;
2. *l one-to-one* $\Leftrightarrow \operatorname{rank}[l]_{\mathcal{V}}^{\mathcal{U}} = \dim V$;
3. *l invertible* $\Leftrightarrow [l]_{\mathcal{V}}^{\mathcal{U}}$ *invertible, and in that case* $\left[l^{-1}\right]_{\mathcal{U}}^{\mathcal{V}} = \left[[l]_{\mathcal{V}}^{\mathcal{U}}\right]^{-1}$.

Proof. Recall that from Remark 249 and Proposition 261, l one-to-one $\Leftrightarrow$ l nonsingular $\Leftrightarrow \ker l = \{0\}$.

1. l onto $\Leftrightarrow \operatorname{Im} l = U \Leftrightarrow \dim \operatorname{Im} l = \dim U \overset{\text{Lemma 283}}{\Leftrightarrow} rank\ [l]_{\mathcal{V}}^{\mathcal{U}} = \dim U$.
2. l one-to-one $\overset{\text{Proposition 245}}{\Leftrightarrow} \dim \operatorname{Im} l = \dim V \overset{\text{Lemma 283}}{\Leftrightarrow} rank\ [l]_{\mathcal{V}}^{\mathcal{U}} = \dim V$.
3. The first part of the statement follows from 1. and 2. above. The second part is proven below. First of all observe that for any vector space W with a basis $\mathcal{W} = \left(w^1, .., w^k\right)$, we have that $id_W \in \mathcal{L}(W, W)$ and
$$[id_W]_{\mathcal{W}}^{\mathcal{W}} = \left[\left[id_W\left(w^1\right)\right]_{\mathcal{W}}, .., \left[id_W\left(w^k\right)\right]_{\mathcal{W}}\right] = I_k.$$
Moreover, if l is invertible
$$l^{-1} \circ l = id_V$$
and
$$\left[l^{-1} \circ l\right]_{\mathcal{V}}^{\mathcal{V}} = [id_U]_{\mathcal{V}}^{\mathcal{V}} = I_m.$$
Since
$$\left[l^{-1} \circ l\right]_{\mathcal{V}}^{\mathcal{V}} = \left[l^{-1}\right]_{\mathcal{U}}^{\mathcal{V}} \cdot [l]_{\mathcal{V}}^{\mathcal{U}},$$
the desired result follows.

■

Remark 285 *From the definitions of φ and ψ, we have what follows:*

1. $$l = \psi(\varphi(l)) = \psi\left([l]_{\mathcal{V}}^{\mathcal{U}}\right) = l_{[l]_{\mathcal{V}}^{\mathcal{U}}, \mathcal{V}}^{\mathcal{U}}.$$
2. $$A = \varphi(\psi(A)) = \varphi\left(l_{A,\mathcal{V}}^{\mathcal{U}}\right) = \left[l_{A,\mathcal{V}}^{\mathcal{U}}\right]_{\mathcal{V}}^{\mathcal{U}}.$$

Lemma 286 $cr_{\mathcal{U}}\left(\operatorname{Im} l_{A,\mathcal{V}}^{\mathcal{U}}\right) = \operatorname{col\,span} A$.

Proof. Recall that $\varphi(l) = [l]_{\mathcal{V}}^{\mathcal{U}}$ and $\psi(A) = l_{A,\mathcal{V}}^{\mathcal{U}}$. For any $l \in \mathcal{L}(V, U)$,
$$cr_{\mathcal{U}}(\operatorname{Im} l) \overset{Lemma\ 282}{=} \operatorname{col\,span}[l]_{\mathcal{V}}^{\mathcal{U}} \overset{Def.\ 271}{=} \operatorname{col\,span}\ \ \varphi(l) \tag{7.4}$$
Take $l = l_{A,\mathcal{V}}^{\mathcal{U}}$. Then from (7.4) ,we have
$$cr_{\mathcal{U}}\left(\operatorname{Im} l_{A,\mathcal{V}}^{\mathcal{U}}\right) = \operatorname{col\,span}\ \ \varphi\left(l_{A,\mathcal{V}}^{\mathcal{U}}\right) \overset{Rmk.\ 285.2}{=} \operatorname{col\,span}\ \ A.$$

■

Lemma 287 $\dim \operatorname{Im} l_{A,\mathcal{V}}^{\mathcal{U}} = \dim \operatorname{col\,span} A = \operatorname{rank} A$.

Proof. Since Lemma 283 holds for any $l \in \mathcal{L}(V, U)$ and $l_{A,\mathcal{V}}^{\mathcal{U}} \in \mathcal{L}(V, U)$, we have that
$$\dim \operatorname{Im} l_{A,\mathcal{V}}^{\mathcal{U}} = \dim \operatorname{col\,span}\left[l_{A,\mathcal{V}}^{\mathcal{U}}\right]_{\mathcal{V}}^{\mathcal{U}} \overset{Rmk.\ 285.2}{=} \dim \operatorname{col\,span}\ A \overset{Rmk.\ 231.}{=} \operatorname{rank}\ A.$$

■

Proposition 288 *Let $A \in \mathbb{M}(m, n)$ be given.*

1. $\text{rank}A = m \Leftrightarrow l^{\mathcal{U}}_{A,\mathcal{V}}$ *onto;*

2. $\text{rank}A = n \Leftrightarrow l^{\mathcal{U}}_{A,\mathcal{V}}$ *one-to-one;*

3. *A invertible* $\Leftrightarrow l^{\mathcal{U}}_{A,\mathcal{V}}$ *invertible, and in that case* $l^{\mathcal{V}}_{A^{-1},\mathcal{U}} = \left(l^{\mathcal{U}}_{A,\mathcal{V}}\right)^{-1}$.

Proof. 1. $\text{rank}A = m \overset{\text{Lemma 287}}{\Leftrightarrow} \dim \operatorname{Im} l^{\mathcal{U}}_{A,\mathcal{V}} = m \Leftrightarrow l^{\mathcal{U}}_{A,\mathcal{V}}$ onto;
2. $\text{rank}A = n \overset{(1)}{\Leftrightarrow} \dim \ker l^{\mathcal{U}}_{A,\mathcal{V}} = 0 \overset{\text{Proposition 261}}{\Leftrightarrow} l^{\mathcal{U}}_{A,\mathcal{V}}$ one-to-one,
where (1) the first equivalence follows form the fact that $n = \dim \ker l^{\mathcal{U}}_{A,\mathcal{V}} + \dim \operatorname{Im} l^{\mathcal{U}}_{A,\mathcal{V}}$, and Lemma 287.
3. First statement: A invertible $\overset{\text{Prop. 226}}{\Leftrightarrow} \text{rank}A = m = n \overset{\text{1 and 2 above}}{\Leftrightarrow} l^{\mathcal{U}}_{A,\mathcal{V}}$ invertible.
Second statement: Since $l^{\mathcal{U}}_{A,\mathcal{V}}$ invertible, there exists $\left(l^{\mathcal{U}}_{A,\mathcal{V}}\right)^{-1} : U \to V$ such that

$$id_V = \left(l^{\mathcal{U}}_{A,\mathcal{V}}\right)^{-1} \circ l^{\mathcal{U}}_{A,\mathcal{V}}.$$

Then

$$I = \varphi^{\mathcal{V}}_{\mathcal{V}}(id_V) \overset{\text{Prop. 281}}{=} \varphi^{\mathcal{V}}_{\mathcal{U}}\left(\left(l^{\mathcal{U}}_{A,\mathcal{V}}\right)^{-1}\right) \cdot \varphi^{\mathcal{U}}_{\mathcal{V}}\left(l^{\mathcal{U}}_{A,\mathcal{V}}\right) \overset{\text{Rmk. 285}}{=} \varphi^{\mathcal{V}}_{\mathcal{U}}\left(\left(l^{\mathcal{U}}_{A,\mathcal{V}}\right)^{-1}\right) \cdot A.$$

Then, by definition of inverse matrix,

$$A^{-1} = \varphi^{\mathcal{V}}_{\mathcal{U}}\left(\left(l^{\mathcal{U}}_{A,\mathcal{V}}\right)^{-1}\right)$$

and

$$\psi^{\mathcal{V}}_{\mathcal{U}}\left(A^{-1}\right) = \left(\psi^{\mathcal{V}}_{\mathcal{U}} \circ \varphi^{\mathcal{V}}_{\mathcal{U}}\right)\left(\left(l^{\mathcal{U}}_{A,\mathcal{V}}\right)^{-1}\right) = id_{\mathcal{L}(U,U)}\left(\left(l^{\mathcal{U}}_{A,\mathcal{V}}\right)^{-1}\right) = \left(l^{\mathcal{U}}_{A,\mathcal{V}}\right)^{-1}.$$

Finally, from the definition of $\psi^{\mathcal{V}}_{\mathcal{U}}$, we have

$$\psi^{\mathcal{V}}_{\mathcal{U}}\left(A^{-1}\right) = l^{\mathcal{V}}_{A^{-1},\mathcal{U}},$$

as desired. ■

Remark 289 *Consider $A \in \mathbb{M}(n,n)$. Then from Proposition 288,*

$$A \textit{ invertible} \Leftrightarrow l^{\mathcal{U}}_{A,\mathcal{V}} \textit{ invertible;}$$

from Proposition 226,

$$A \textit{ invertible} \Leftrightarrow A \textit{ nonsingular;}$$

from Proposition 264,

$$l^{\mathcal{U}}_{A,\mathcal{V}} \textit{ invertible} \Leftrightarrow l^{\mathcal{U}}_{A,\mathcal{V}} \textit{ nonsingular.}$$

Therefore,

$$A \textit{ nonsingular} \Leftrightarrow l^{\mathcal{U}}_{A,\mathcal{V}} \textit{ nonsingular.}$$

Symmetrically,

$$[l]^{\mathcal{U}}_{\mathcal{V}} \textit{ invertible} \Leftrightarrow [l]^{\mathcal{U}}_{\mathcal{V}} \textit{ nonsingular} \Leftrightarrow l \textit{ invertible} \Leftrightarrow l \textit{ nonsingular.}$$

Proposition 290 *Let $l \in \mathcal{L}(V,U)$ be given. Then there exists a basis $\mathcal{V}$ of V and a basis $\mathcal{U}$ of U such that*

$$[l]^{\mathcal{U}}_{\mathcal{V}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where I is the r-square identity matrix and $r = rank[l]^{\mathcal{U}}_{\mathcal{V}}$.

Proof. Suppose $\dim V = m$ and $\dim U = n$. Let $W = \ker l$ and $U' = Iml$. By assumption, $rank[l]^{\mathcal{U}}_{\mathcal{V}} = r$. Then, by Lemma 283,

$$\dim Iml = rank[l]^{\mathcal{U}}_{\mathcal{V}} = r.$$

Therefore, from the Dimension Theorem

$$\dim \ker = \dim V - \dim Iml = m - r.$$

Let $\mathcal{W} = \left(w^1, ..., w^{m-r}\right)$ be a basis of W. Then, from the Completion Lemma, there exist vectors $v^1, ..., v^r \in V$ such that $\mathcal{V} := \left(v^1, ..., v^r, w^1, ..., w^{m-r}\right)$ is a basis of V. For any $i \in \{1, ..., r\}$, set $u^i = l(v^i)$. Then, $\left(u^1, ..., u^r\right)$ is a basis of U'. Then, again from the Completion Lemma, there exists $\left(u^{r+1}, ..., u^n\right) \in U$ such that $\mathcal{U} := \left(u^1, ..., u^r, u^{r+1}, ..., u^n\right)$ is a basis of U. Then,

$$\begin{array}{l}
l(v^1) = u^1 = 1u^1 + 0u^2 + ... + 0u^r + 0u^{r+1} + ... + 0u^n, \\
l(v^2) = u^2 = 0u^1 + 1u^2 + ... + 0u^r + 0u^{r+1} + ... + 0u^n, \\
\cdots\cdots\cdots \\
l(v^r) = u^r = 0u^1 + 0u^2 + ... + 1u^r + 0u^{r+1} + ... + 0u^n, \\
l(w^1) = 0 = 0u^1 + 0u^2 + ... + 0u^r + 0u^{r+1} + ... + 0u^n, \\
\cdots\cdots\cdots \\
l(w^{m-r}) = 0 = 0u^1 + 0u^2 + ... + 0u^r + 0u^{r+1} + ... + 0u^n,
\end{array}$$

i.e.,

$$[l]_{\mathcal{V}}^{\mathcal{U}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

■

7.5 Some facts on $\mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right)$

In this Section, we specialize (basically repeat) the content of the previous Section in the important case in which

$$\begin{array}{ll}
V = \mathbb{R}^n, & \mathcal{V} = \left(e_n^j\right)_{j=1}^n := \mathbf{e}_n \\
U = \mathbb{R}^m & \mathcal{U} = \left(e_m^i\right)_{i=1}^m := \mathbf{e}_m \\
& v = x
\end{array} \tag{7.5}$$

and therefore

$$l \in \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right).$$

From $\mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right)$ to $\mathbb{M}(m, n)$.
From Definition 269, we have

$$\begin{array}{l}
[l]_{\mathbf{e}_n}^{\mathbf{e}_m} = \left[\begin{array}{ccccc} \left[l\left(e_n^1\right)\right]_{\mathbf{e}_m} & ... & \left[l\left(e_n^j\right)\right]_{\mathbf{e}_m} & ... & \left[l\left(e_n^n\right)\right]_{\mathbf{e}_m} \end{array}\right] = \\
= \left[\begin{array}{ccccc} l\left(e_n^1\right) & ... & l\left(e_n^j\right) & ... & l\left(e_n^n\right) \end{array}\right] := [l];
\end{array} \tag{7.6}$$

from Definition 271,

$$\varphi := \varphi_{\mathbf{e}_n}^{\mathbf{e}_m} : \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right) \to \mathbb{M}(m, n), \quad l \mapsto [l];$$

from Proposition 273,

$$[l] \cdot x = l(x). \tag{7.7}$$

From $\mathbb{M}(m, n)$ to $\mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^m\right)$.
From Definition 275,

$$l_A := l_{A, \mathbf{e}_m}^{\mathbf{e}_m} : \mathbb{R}^n \to \mathbb{R}^m, \tag{7.8}$$

$$\begin{array}{l}
l_A(x) = \sum_{i=1}^m \left(R^i(A) \cdot [x]_{\mathbf{e}_n}\right) \cdot e_m^i = \\
= \begin{bmatrix} R^1(A) \cdot x \\ ... \\ R^i(A) \cdot x \\ ... \\ R^m(A) \cdot x \end{bmatrix} = Ax.
\end{array} \tag{7.9}$$

From Definition 278,

$$\psi := \psi_{\mathbf{e}_n}^{\mathbf{e}_m} : \mathbb{M}(m,n) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \qquad : A \mapsto l_A. \ .$$

From Proposition 280,

$$\mathbb{M}(m,n) \text{ and } \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \text{ are isomorphic.}$$

From Proposition 281, if $l_1 \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $l_2 \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$, then

$$[l_2 \circ l_1] = [l_2] \cdot [l_1] . \tag{7.10}$$

Some related properties.
From Proposition 284, given $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$,
1. l onto $\Leftrightarrow$ rank $[l] = m$;
2. l one-to-one $\Leftrightarrow$ rank $[l] = n$;
3. l invertible $\Leftrightarrow$ $[l]$ invertible, and in that case $\left[l^{-1}\right] = [l]^{-1}$.

From Remark 285,
1.
$$l = \psi(\varphi(l)) = \psi([l]) = l_{[l]}.$$
2.
$$A = \varphi(\psi(A)) = \varphi(l_A) = [l_A] .$$

From Proposition 288, given $A \in \mathbb{M}(m,n)$,
1. rank$A = m \Leftrightarrow l_A$ onto;
2. rank$A = n \Leftrightarrow l_A$ one-to-one;
3. A invertible $\Leftrightarrow l_A$ invertible, and in that case $l_{A^{-1}} = (l_A)^{-1}$.

Remark 291 *From* (7.7) *and Remark 150,*

$$\operatorname{Im} l := \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ such that } y = [l] \cdot x\} = \operatorname{col span}\ [l] . \tag{7.11}$$

Then, from the above and Remark 231,

$$\dim \operatorname{Im} l = \text{rank }\ [l] = \max \# \text{ linearly independent columns of } [l] . \tag{7.12}$$

Similarly, from (7.9) *and Remark 231, we get*

$$\operatorname{Im} l_A := \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ such that } y = l_A \cdot x = Ax\} = \operatorname{col span}\ A,$$

and

$$\dim \operatorname{Im} l_A = \text{rank }\ A = \max \# \text{ linearly independent columns of } A.$$

Remark 292 ***Recipe to find a basis of*** $\operatorname{Im} l$ ***for*** $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$***.***

Assume that $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$*. The above Remark gives a way of finding a basis of* $\operatorname{Im} l$*: it is enough to consider a number equal to* rank $[l]$ *of linearly independent vectors among the column vectors of* $[l]$*. In a more detailed way, we have what follows.*

1. Compute $[l]$*.*

2. Compute $\dim \operatorname{Im} l = \text{rank }\ [l] := r$*.*

3. To find a basis of $\operatorname{Im} l$*, we have to find* r *vectors which are a. linearly independent, and b. elements of* $\operatorname{Im} l$*. Indeed, it is enough to take* r *linearly independent columns of* $[l]$*. Observe that for any* $i \in \{1, ..., m\}$*,* $C^i([l]) \in \operatorname{Im} l = \operatorname{col span}\ [l]$*.*

To get a "simpler" basis, you can make elementary operations on those column. Recall that elementary operations on linearly independent vectors lead to linearly independent vectors, and that elementary operations on vectors lead to vector belonging to the span of the starting vectors.

Example 293 *Given* $n \in \mathbb{N}$ *with* $n \geq 3$*, and the linear function*

$$l : \mathbb{R}^n \to \mathbb{R}^3, x := (x_i)_{i=1}^n \mapsto \begin{cases} \sum_{i=1}^n x_i \\ x_2 \\ x_2 + x_3 \end{cases} .$$

find a basis forIm l.

1.

$$[l] = \begin{bmatrix} 1 & 1 & 1 & 1 & ... & 1 \\ 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 1 & 0 & ... & 0 \end{bmatrix}$$

2.

$$\text{rank} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3$$

3. A basis of $\operatorname{Im} l_a$ is given by the column vectors of

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark 294 *From* (7.7)*, we have that*

$$\ker l = \{x \in \mathbb{R}^n : [l]\, x = 0\},$$

i.e., ker l *is the set, in fact the vector space, of solution to the systems* $[l]\, x = 0$*. In Remark 315, we will describe an algorithm to find a basis of the kernel of an arbitrary linear function.*

Remark 295 *From Remark 291 and Proposition 245 (the Dimension Theorem), given* $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$*,*

$$\dim \mathbb{R}^n = \dim \ker l + rank\ \ [l],$$

and given $A \in \mathbb{M}(m, n)$*,*

$$\dim \mathbb{R}^n = \dim \ker l_A + rank\ A.$$

7.6 Examples of computation of $[l]_{\mathcal{V}}^{\mathcal{U}}$

1. $id \in \mathcal{L}(V, V)$.

$$[id]_{\mathcal{V}}^{\mathcal{V}} = \left[\left[id\left(v^1\right)\right]_{\mathcal{V}}, ..., \left[id\left(v^j\right)\right]_{\mathcal{V}}, ..., \left[id\left(v^n\right)\right]_{\mathcal{V}}\right] =$$

$$= \left[\left[v^1\right]_{\mathcal{V}}, ..., \left[v^j\right]_{\mathcal{V}}, ..., \left[v^n\right]_{\mathcal{V}}\right] = \left[e_n^1, ..., e_n^j, ..., e_n^n\right] = I_n.$$

2. $0 \in \mathcal{L}(V, U)$.

$$[0]_{\mathcal{V}}^{\mathcal{U}} = \left[[0]_{\mathcal{V}}, ..., [0]_{\mathcal{V}}, ..., [0]_{\mathcal{V}}\right] = 0 \in \mathbb{M}(m, n).$$

3. $l_\alpha \in \mathcal{L}(V, V)$, with $\alpha \in F$.

$$[l_\alpha]_{\mathcal{V}}^{\mathcal{V}} = \left[\left[\alpha \cdot v^1\right]_{\mathcal{V}}, ..., \left[\alpha \cdot v^j\right]_{\mathcal{V}}, ..., \left[\alpha \cdot v^n\right]_{\mathcal{V}}\right] = \left[\alpha \cdot \left[v^1\right]_{\mathcal{V}}, ..., \alpha \cdot \left[v^j\right]_{\mathcal{V}}, ..., \alpha \cdot \left[v^n\right]_{\mathcal{V}}\right] =$$

$$= \left[\alpha \cdot e_n^1, ..., \alpha \cdot e_n^j, ..., \alpha \cdot e_n^n\right] = \alpha \cdot I_n.$$

4. $l_A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, with $A \in \mathbb{M}(m, n)$.

$$[l_A] = \left[A \cdot e_n^1, ..., A \cdot e_n^j, ..., A \cdot e_n^n\right] = A \cdot \left[e_n^1, ..., e_n^j, ..., e_n^n\right] = A \cdot I_n = A.$$

5. (projection function) $proj_{n+k,n} \in \mathcal{L}\left(\mathbb{R}^{n+k}, \mathbb{R}^n\right)$, $proj_{n+k,n} : (x_i)_{i=1}^{n+k} \mapsto (x_i)_{i=1}^n$. Defined $proj_{n+k,n} := p$, we have

$$[p] = \left[p\left(e_{n+k}^1\right), ..., p\left(e_{n+k}^n\right), p\left(e_{n+k}^{n+1}\right), ..., p\left(e_{n+k}^{n+k}\right)\right] = \left[e_n^1, ..., e_n^n, 0, ..., 0\right] = [I_n | 0],$$

where $0 \in \mathbb{M}(n, k)$.

6. (immersion function) $i_{n,n+k} \in \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^{n+k}\right)$, $i_{n,n+k} : (x_i)_{i=1}^n \mapsto \left((x_i)_{i=1}^n, 0\right)$ with $0 \in \mathbb{R}^k$.Defined $i_{n,n+k} := i$, we have

$$[i] = \left[i\left(e_n^1\right), ..., i\left(e_n^n\right)\right] = \begin{bmatrix} e_n^1 & ... & e_n^n \\ 0 & ... & 0 \end{bmatrix} = \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

where $0 \in \mathbb{M}(k, n)$.

Remark 296 *Point 4. above implies that if $l : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$, then $[l] = A$. In other words, to compute $[l]$ you do not have to take the image of each element in the canonical basis; the first line of $[l]$ is the vector of the coefficient of the first component function of l and so on. For example, if $l : \mathbb{R}^2 \to \mathbb{R}^3$,*

$$x \mapsto \begin{cases} a_{11}x_1 & + & a_{12}x_2 \\ a_{21}x_1 & + & a_{22}x_2 \\ a_{31}x_1 & + & a_{32}x_2 \end{cases},$$

then

$$[l] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

7.7 Exercises

Problem sets: 15,16,17,18,19,21,22.

From Lipschutz (1991), starting from page 352:

10.1 → 10.9, 10.29 → 10.33.

Chapter 8

Solutions to systems of linear equations

8.1 Some preliminary basic facts

Let's recall some basic definition from Section 1.6.

Definition 297 *Consider the following linear system with m equations and n unknowns*

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

which can be rewritten as

$$Ax = b$$

$A_{m\times n}$ is called matrix of the coefficients (or coefficient matrix) associated with the system and $M_{m\times(n+1)} = \begin{bmatrix} A & | & b \end{bmatrix}$ is called augmented matrix associated with the system.

Recall the following definition.

Definition 298 *Two linear system are said to be equivalent if they have the same solutions.*

Let's recall some basic facts we discussed in previous chapters.

Remark 299 *It is well known that the following operations applied to a system of linear equations lead to an equivalent system:*
I) interchange two equations;
II) multiply both sides of an equation by a nonzero real number;
III) add left and right hand side of an equation to the left and right hand side of another equation;
IV) change the place of the unknowns.
The transformations I), II), III) and IV) are said elementary transformations, *and, as it is well known, they do not change the solution set of the system they are applied to.*
Those transformations correspond to elementary operations on rows of M or columns of A in the way described below
I) interchange two rows of M;
II) multiply a row of M by a nonzero real number;
III) sum a row of M to another row of M ;
IV) interchange two columns of A.
The above described operations do not change the rank of A and they do not change the rank of M - see Proposition 218.

Homogenous linear system.

Definition 300 *A linear system for which* $b = 0$*, i.e., of the type*

$$Ax = 0$$

with $A \in \mathbb{M}(m,n)$*, is called homogenous system.*

Remark 301 *Obviously,* 0 *is a solution of the homogenous system. The set of solution of a homogeneous system is* $\ker l_A$*. From Remark 295,*

$$\dim \ker l_A = n - \text{rank } A.$$

8.2 A solution method: Rouchè-Capelli's and Cramer's theorems

The solution method presented in this section is based on two basic theorems.

1. Rouchè-Capelli's Theorem, which gives necessary and sufficient condition for the existence of solutions;
2. Cramer's Theorem, which gives a method to compute solutions - if they exist.

Theorem 302 $(Rouch\grave{e} - Capelli)$ *A system with* m *equations and* n *unknowns*

$$A_{m\times n}x = b \tag{8.1}$$

has solutions

$\Leftrightarrow$

$$\text{rank } A = \text{rank } \begin{bmatrix} A & | & b \end{bmatrix}$$

Proof. $[\Rightarrow]$

Let x^* be a solution to 8.1. Then, b is a linear combination, via the solution x^* of the columns of A. Then, from Proposition 218,

$$\text{rank } \begin{bmatrix} A & | & b \end{bmatrix} = \text{rank } \begin{bmatrix} A & | & 0 \end{bmatrix} = \text{rank } \begin{bmatrix} A \end{bmatrix}.$$

$[\Leftarrow]$

1st proof.

We want to show that

$$\exists x \in \mathbb{R}^n \text{ such that } Ax^* = b, \text{ i.e., } b = \sum_{j=1}^{n} x_j^* \cdot C^j(A).$$

By assumption, rank A = rank $\begin{bmatrix} A & | & b \end{bmatrix} := r$. Since rank $A = r$, there are r linearly independent column vectors of A, say $C^{j_1}(A), ..., C^{j_r}(A)$. Since rank $\begin{bmatrix} A & | & b \end{bmatrix} = r$, $C^{j_1}(A), ..., C^{j_r}(A), b$ are linearly dependent and from Lemma 196, b is a linear combinations of the vectors $C^{j_1}(A), ..., C^{j_r}(A)$, i.e., $\exists (x_j)_{j\in R}$ such that $b = \sum_{j\in R} x_j \cdot C^j(A)$ and

$$b = \sum_{j\in R} x_j \cdot C^j(A) + \sum_{j'\in\{1,...,n\}\setminus R} 0 \cdot C_{j'}(A).$$

Then, $x^* = \left(x_j^*\right)_{j=1}^{n}$ such that

$$x_j^* = \begin{cases} x_j & if \quad j \in R \\ \\ 0 & if \quad j' \in \{1,...,n\}\setminus R \end{cases}$$

is a solution to $Ax = b$.

Second proof.

By assumption , rank$[A]$ = rank$\begin{bmatrix} A & | & b \end{bmatrix} := r$. Then, from Proposition ..., there exist r linearly independent row vectors of A. Perform row interchanging operations on $\begin{bmatrix} A & | & b \end{bmatrix}$ to get the r linearly independent rows of A as the first r rows, i.e., the following matrix

$$\begin{bmatrix} A' & | & b' \\ A'' & | & b'' \end{bmatrix}.$$

Observe that $\operatorname{rank} A' = \operatorname{rank}\begin{bmatrix} A' & | & b' \end{bmatrix} = r$. Interchange columns of $\begin{bmatrix} A' \\ A'' \end{bmatrix}$ to have r linearly independent columns as the first r columns. From Remark 299, reordering columns of A and rows of $\begin{bmatrix} A & | & b \end{bmatrix}$ does not change the rank of A, does not change the rank of $\begin{bmatrix} A & | & b \end{bmatrix}$ and does not change the set of solutions. Therefore, we constructed the following system

$$\begin{bmatrix} A^* & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} b' \\ b'' \end{bmatrix},$$

where $A_{12} \in \mathbb{M}(r, n-r)$, $A_{21} \in \mathbb{M}(m-r, r)$, $A_{22} \in \mathbb{M}(m-r, n-r)$, $x^1 \in \mathbb{R}^r$, $x^2 \in \mathbb{R}^{n-r}$, $b' \in \mathbb{R}^r$, $b'' \in \mathbb{R}^{m-r}$,

(x^1, x^2) has been obtained from x performing on it the same permutations performed on the columns of A,

(b', b'') has been obtained from b performing on it the same permutations performed on the rows of $\begin{bmatrix} A & | & b \end{bmatrix}$.

Since

$$\operatorname{rank}\begin{bmatrix} A^* & A_{12} & b' \end{bmatrix} = \operatorname{rank} A^* = r,$$

then the r rows of $\begin{bmatrix} A^* & A_{12} & b' \end{bmatrix}$ are linearly independent and since

$$\operatorname{rank}\begin{bmatrix} A^* & A_{12} & b' \\ A_{21} & A_{22} & b'' \end{bmatrix} = r,$$

then the r rows of $\begin{bmatrix} A^* & A_{12} & b' \end{bmatrix}$ are a basis of the span of the rows of $\begin{bmatrix} A^* & A_{12} & b' \\ A_{21} & A_{22} & b'' \end{bmatrix}$.Then the last $m-r$ rows are linear combinations of the first r rows. Therefore, using again Remark 299, we have that $Ax = b$ is equivalent to

$$\begin{bmatrix} A^* & A_{12} \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = b'$$

or, using Remark 74.2,

$$A^* x^1 + A_{12} x_2 = b'$$

and

$$x^1 = (A^*)^{-1}\left(b' - A_{12}x^2\right) \in \mathbb{R}^r$$

while x_2 can be chosen arbitrarily; more precisely

$$\left\{\left(x^1, x^2\right) \in \mathbb{R}^n : x_1 = (A^*)^{-1}\left(b' - A_{12}x^2\right) \in \mathbb{R}^r \text{ and } x_2 \in \mathbb{R}^{n-r}\right\}$$

is the nonempty set of solution to the system $Ax = b$. ■

Exercise 303 *Apply the procedure described in the proof of Roche - Capelli theorem to solve the system* $Ax = b$, *where*

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}.$$

The main step in the exercise amount to rewrite A *and* b *as follows.*

$$\begin{bmatrix} A^* & A_{12} & b' \\ A_{21} & A_{22} & b'' \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 1 & 4 \\ 2 & 1 & 2 & 2 & 5 \\ 3 & 3 & 3 & 3 & 9 \end{bmatrix}$$

and therefore $x^1 = (x_3, x_4)$, $x^2 = (x_1, x_2)$, $b' = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $b'' = (9)$. *Then,*

$$x^1 = (A^*)^{-1}\left(b' - A_{12}x^2\right) \in \mathbb{R}^r$$

specializes in

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}\left(\begin{bmatrix} 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) =$$

$$= \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 - x_1 + x_2 \\ 5 - 2x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 + 2 \\ 1 \end{bmatrix}.$$

Theorem 304 *(Cramer) A system with n equations and n unknowns*

$$A_{n\times n}x = b$$

with $\det A \neq 0$*, has a unique solution* $x = (x_1, ..., x_i, ..., x_n)$ *where for* $i \in \{1, ..., n\}$,

$$x_i = \frac{\det A_i}{\det A}$$

and A_i is the matrix obtained from A substituting the column vector b in the place of the $i-th$ column.

Proof. since $\det A \neq 0$, A^{-1} exists and it is unique. Moreover, from $Ax = b$, we get $A^{-1}Ax = A^{-1}b$ and

$$x = A^{-1}b$$

Moreover

$$A^{-1}b = \frac{1}{\det A} Adj\ A \cdot b$$

It is then enough to verify that

$$Adj\ A \cdot b = \begin{bmatrix} \det A_1 \\ ... \\ \det A_i \\ ... \\ \det A_n \end{bmatrix}$$

which we omit (see Exercise 7.34, page 268, in Lipschutz (1991). ■

The combinations of Rouche-Capelli and Cramer's Theorem allow to give a method to solve any linear system - apart from computational difficulties.

Remark 305 ***Rouche'-Capelli and Cramer's Theorem based method.***

Let the following system with m equations and n unknowns be given:

$$A_{m\times n}x = b.$$

0. Simplify the system using elementary row operations on $\begin{bmatrix} A & | & b \end{bmatrix}$ *and elementary column operations on A. Those operation do not change* rank A, rank B, *set of solution to system $Ax = b$.*

1. Compute rank A *and* rank $\begin{bmatrix} A & | & b \end{bmatrix}$.

i. If

$$\text{rank } A \neq \text{rank } \begin{bmatrix} A & | & b \end{bmatrix},$$

then the system has no solution.

ii. If

$$\text{rank } A = \text{rank } \begin{bmatrix} A & | & b \end{bmatrix} := r,$$

then the system has solutions which can be computed as follows.

2. Extract a square r-dimensional invertible submatrix A_r from A.

i. Discard the equations, if any, whose corresponding rows are not part of A_r.

ii. In the remaining equations, bring on the right hand side the terms containing unknowns whose coefficients are not part of the matrix A_r, if any.

iii. You then get a system to which Cramer's Theorem can be applied, treating as constant the expressions on the right hand side and which contain $n - r$ unknowns. Those unknowns can be chosen arbitrarily. Sometimes it is said that then the system has "∞^{n-r}" solutions or that the system admits $n - r$ degrees of freedom. More formally, we can say what follows.

Definition 306 *Given $S, T \subseteq \mathbb{R}^n$, we define the sum of the sets S and T, denoted by $S + T$, as follows*

$$\{x \in \mathbb{R}^n : \exists s \in S, \exists t \in T \text{ such that } x = s + t\}.$$

Proposition 307 *Assume that the set S of solutions to the system $Ax = b$ is nonempty and let $x^* \in S$. Then*

$$S = \{x^*\} + \ker l_A := \left\{x \in \mathbb{R}^n : \exists x^0 \in \ker l_A \text{ such that } x = x^* + x^0\right\}$$

Proof. [$\subseteq$]

Take $x \in S$. We want to find $x^0 \in \ker A$ such that $x = x^* + x^0$. Take $x^0 = x - x^*$. Clearly $x = x^* + (x - x^*)$. Moreover,

$$Ax_0 = A(x - x^*) \overset{(1)}{=} b - b = 0 \tag{8.2}$$

where (1) follows from the fact that $x, x^* \in S$.

[$\supseteq$]

Take $x = x^* + x^0$ with $x^* \in S$ and $x^0 \in \ker A$. Then

$$Ax = Ax^* + Ax_0 = b + 0 = b.$$

■

Remark 308 *The above proposition implies that a linear system either has no solutions, or has a unique solution, or has infinite solutions.*

Definition 309 *V is an affine subspace of $\mathbb{R}^n$ if there exists a vector subspace W of $\mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$ such that*

$$V = \{x\} + W$$

We say that the[1] *dimension of the affine subspace V is* $\dim W$.

Remark 310 *Let $a \in \mathbb{R}$ be given. First of all observe that*

$$\left\{(x, y) \in \mathbb{R}^2 : y = ax\right\} = \ker l,$$

where $l \in \mathcal{L}\left(\mathbb{R}^2, \mathbb{R}\right)$ and $l(x, y) = ax - y$. Let's present a geometric description of Proposition 307. We want to verify that the following two sets are equal.

$$S := \{(x_0, y_0)\} + \left\{(x, y) \in \mathbb{R}^2 : y = ax\right\},$$

$$T := \left\{(x, y) \in \mathbb{R}^2 : y = a(x - x_0) + y_0\right\}.$$

In words, we want to verify that the affine space "$\{(x_0, y_0)\}$ plus $\ker l$" is nothing but the set of points belonging to the line with slope a and going through the point (x_0, y_0).

$S \subseteq T$. Take $(x', y') \in S$; then $\exists x'' \in \mathbb{R}$ such that $x' = x_0 + x''$ and $y' = y_0 + ax''$. We have tho check that $y' = a(x' - x_0) + y_0$.Indeed, $a(x' - x_0) + y_0 = a(x_0 + x'' - x_0 + y) = ax'' + y_0$.

$T \subseteq S$. Take (x', y') such that $y' = a(x' - x_0) + y_0$. Take $x'' = x' + x_0$.Then $x' = x_0 + x''$ and $y' = y_0 + ax''$, as desired.

Remark 311 *Since $\dim \ker l_A = n - \operatorname{rank} A$, the above Proposition and Definition say that if a nonhomogeneous systems has solutions, then the set of solutions is an affine space of dimension $n - \operatorname{rank} A$.*

Example 312 *Consider the system (in one equation and two unknowns):*

$$x_1 + x_2 - 1 = 0. \tag{8.3}$$

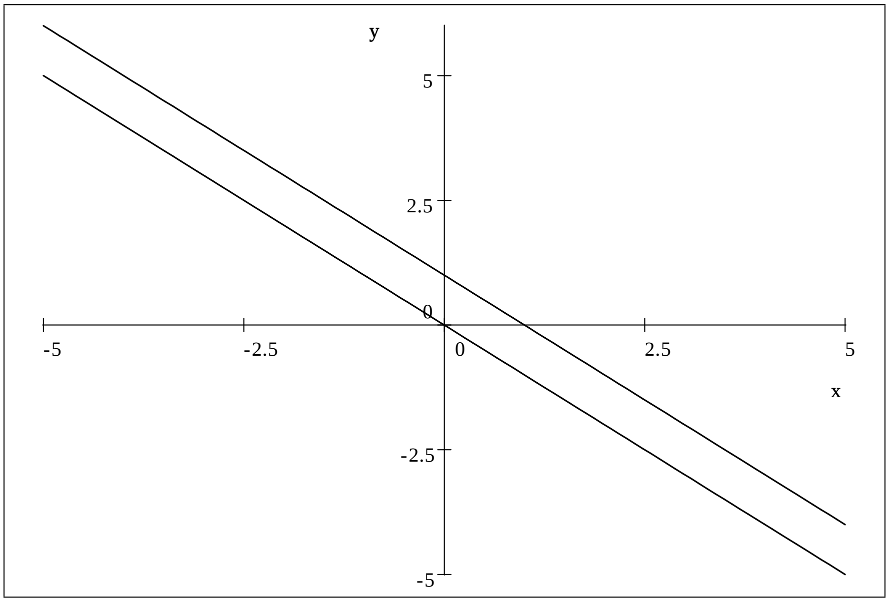
Then $x^* := (1, 0)$ is a solution to the above system. Moreover, defined $l : \mathbb{R}^2 \to \mathbb{R}$, $l(x_1, x_2) = x_1 + x_2$

$$\ker l := \{l(x) = 0\} = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = -x_1\right\}.$$

The set of solutions to (8.3) *is* $\{(0, 1)\} + \ker l$. *Observe that* $\{(0, 1)\} + \ker l = \{(1, 0)\} + \ker l = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\} + \ker l$.

$-x$

[1] If W' and W'' are vector subspaces of $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $V := \{x\} + W' = \{x\} + W''$, then $W' = W''$.
Take $w \in W'$, then $x + w \in V = \{x\} + W''$. Then there exists $x \in \{x\}$ and $\widehat{w} \in W''$ such that $x + w = x + \widehat{w}$. Then $w = \widehat{w} \in W''$, and $W' \subseteq W''$. Similar proof applies for the opposite inclusion.



Remark 313 ***Exercise 314*** *Apply the algorithm described in Remark 305 to solve the following linear system.*

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 4x_1 + 5x_2 + 6x_3 = 2 \\ 5x_1 + 7x_2 + 9x_3 = 3 \end{cases}$$

The associated matrix $\begin{bmatrix} A & | & b \end{bmatrix}$ *is*

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 \\ 4 & 5 & 6 & | & 2 \\ 5 & 7 & 9 & | & 3 \end{bmatrix}$$

1. Since the third row of the matrix $\begin{bmatrix} A & | & b \end{bmatrix}$ *is equal to the sum of the first two rows, and since*

$$\det \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = 5 - 8 = -3,$$

we have that

$$\text{rank } A = \text{rank } \begin{bmatrix} A & | & b \end{bmatrix} = 2,$$

and the system has solutions.

2. Define

$$A_2 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

i. Discarding the equations, whose corresponding rows are not part of A_2 *and, in the remaining equations, bringing on the right hand side the terms containing unknowns whose coefficients are not part of the matrix* A_2*, we get*

$$\begin{cases} x_1 + 2x_2 = 1 - 3x_3 \\ 4x_1 + 5x_2 = 2 - 6x_3 \\ 5x_1 + 7x_2 = 3 - 9x_3 \end{cases}$$

iii. Then, using Cramer's Theorem, recalling that $\det A_2 = -3$*, we get*

$$x_1 = \frac{\det \begin{bmatrix} 1 - 3x_3 & 2 \\ 2 - 6x_3 & 5 \end{bmatrix}}{-3} = x_3 - \frac{1}{3},$$

$$x_2 = \frac{\det \begin{bmatrix} 1 & 1 - 3x_3 \\ 4 & 2 - 6x_3 \end{bmatrix}}{-3} = \frac{2}{3} - 2x_3,$$

and the solution set is

$$\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1 = x_3-\frac{1}{3}, x_2=\frac{2}{3}-2x_3\right\}.$$

Remark 315 ***How to find a basis of*** $\ker$.

Let $A\in\mathbb{M}(m,n)$ *be given and* $rank A = r\le\min\{m,n\}$. *Then, from the second proof of Rouche'-Capelli's Theorem, we have that system*

$$Ax=0$$

admits the following set of solutions

$$\left\{\left(x^1,x^2\right)\in\mathbb{R}^r\times\mathbb{R}^{n-r} : x^1=(A^*)^{-1}\left(-A_{12}\cdot x^2\right)\right\} \tag{8.4}$$

Observe that $\dim\ker l_A = n-r := p$. *Then, a basis of* $\ker l_A$ *is*

$$\mathcal{B}=\left\{\begin{bmatrix} -[A^*]^{-1}A_{12}\cdot e_p^1 \\ e_p^1\end{bmatrix},...,\begin{bmatrix} -[A^*]^{-1}A_{12}\cdot e_p^p \\ e_p^p\end{bmatrix}\right\}.$$

To check the above statement, we check that 1. $\mathcal{B}\subseteq\ker l_A$, *and 2.* $\mathcal{B}$ *is linearly independent*[2].

1. *It follows from* (8.4);
2. *It follows from the fact that* $\det\left[e_p^1,...,e_p^p\right]=\det I = 1$.

Example 316

$$\begin{cases} x_1+x_2+x_3+2x_4=0\\ x_1-x_2+x_3+2x_4=0\end{cases}$$

Defined

$$A^*=\begin{bmatrix}1&1\\1&-1\end{bmatrix},\qquad A_{12}=\begin{bmatrix}1&2\\1&2\end{bmatrix}$$

the starting system can be rewritten as

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=-\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}\begin{bmatrix}1&2\\1&2\end{bmatrix}\begin{bmatrix}x_3\\x_4\end{bmatrix}.$$

Then a basis of $\ker$ *is*

$$\mathcal{B}=\left\{\begin{bmatrix}-\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}\begin{bmatrix}1&2\\1&2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ \begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix},\begin{bmatrix}-\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}\begin{bmatrix}1&2\\1&2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ \begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix}\right\}=$$

$$=\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}\right\}$$

Example 317 *Discuss the following system (i.e., say if admits solutions).*

$$\begin{cases} x_1 &+& x_2 &+& x_3 &=& 4\\ x_1 &+& x_2 &+& 2x_3 &=& 8\\ 2x_1 &+& 2x_2 &+& 3x_3 &=& 12\end{cases}$$

The augmented matrix $[A|b]$ *is:*

$$\begin{bmatrix}1&1&1&|&4\\1&1&2&|&8\\2&2&3&|&12\end{bmatrix}$$

[2] Observe that $\mathcal{B}$ is made up by p vectors and $\dim\ker l_A = p$.

$$\text{rank}\,[A|b] = \text{rank} \begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 1 & 1 & 2 & | & 8 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} = 2 = \text{rank } A$$

From Step 2 of Rouche'-Capelli and Cramer's method, we can consider the system

$$\begin{cases} x_2 & + & x_3 & = & 4 - x_1 \\ x_2 & + & 2x_3 & = & 8 - x_1 \end{cases}$$

Therefore, x_1 *can be chosen arbitrarily and since* $\det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1$,

$$x_2 = \det \begin{bmatrix} 4 - x_1 & 1 \\ 8 - x_1 & 2 \end{bmatrix} = -x_1$$

$$x_3 = \det \begin{bmatrix} 1 & 4 - x_1 \\ 1 & 8 - x_1 \end{bmatrix} = 4$$

Therefore, the set of solution is

$$\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = -x_1, x_3 = 4\right\}$$

Example 318 *Discuss the following system*

$$\begin{cases} x_1 & + & x_2 & = & 2 \\ x_1 & - & x_2 & = & 0 \end{cases}$$

The augmented matrix $[A|b]$ *is:*

$$\begin{bmatrix} 1 & 1 & | & 2 \\ 1 & -1 & | & 0 \end{bmatrix}$$

Since $\det A = -1 - 1 = -2 \neq 0$,

$$\text{rank}\,[A|b] = \text{rank}A = 2$$

and the system has a unique solution:

$$x_1 = \frac{\det \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}}{-2} = \frac{-2}{-2} = 1$$

$$x_2 = \frac{\det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}}{-2} = \frac{-2}{-2} = 1$$

Therefore, the set of solution is

$$\{(1, 1)\}$$

Example 319 *Discuss the following system*

$$\begin{cases} x_1 & + & x_2 & = & 2 \\ x_1 & + & x_2 & = & 0 \end{cases}$$

The augmented matrix $[A|b]$ *is:*

$$\begin{cases} 1 & 1 & | & 2 \\ 1 & 1 & | & 0 \end{cases}$$

Since $\det A = 1 - 1 = 0$, *and* $\det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = -2 \neq 0$, *we have that*

$$\text{rank}\,[A|b] = 2 \neq 1 = \text{rank}A$$

and the system has no solutions. Therefore, the set of solution is $\varnothing$.

Example 320 *Discuss the following system*

$$\begin{cases} x_1 & + & x_2 & = & 2 \\ 2x_1 & + & 2x_2 & = & 4 \end{cases}$$

The augmented matrix $[A|b]$ *is:*

$$\begin{bmatrix} 1 & 1 & | & 2 \\ 2 & 2 & | & 4 \end{bmatrix}$$

From rank properties,

$$\operatorname{rank}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 1$$

$$\operatorname{rank}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = 1$$

Recall that elementary operations of ***rows*** *on the augmented matrix do not change the rank of either the augmented or coefficient matrices.*

Therefore

$$\operatorname{rank}[A|b] = 1 = \operatorname{rank} A$$

and the system has infinite solutions. More precisely, the set of solutions is

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 2 - x_2\right\}$$

Example 321 *Say for which value of the parameter* $k \in \mathbb{R}$ *the following system has one, infinite or no solutions:*

$$\begin{cases} (k-1)\,x + (k+2)\,y = 1 \\ -x + ky = 1 \\ x - 2y = 1 \end{cases}$$

$$[A|b] = \begin{bmatrix} k-1 & k+2 & | & 1 \\ -1 & k & | & 1 \\ 1 & -2 & | & 1 \end{bmatrix}$$

$$\det[A|b] = \det\begin{bmatrix} k-1 & k+2 & 1 \\ -1 & k & 1 \\ 1 & -2 & 1 \end{bmatrix} = \det\begin{bmatrix} -1 & k \\ 1 & -2 \end{bmatrix} - \det\begin{bmatrix} k-1 & k+2 \\ 1 & -2 \end{bmatrix} + \det\begin{bmatrix} k-1 & k+2 \\ -1 & k \end{bmatrix} =$$

$$(2-k) - (-2k+2-k-2) + \left(k^2 - k + k + 2\right) = 2 - k + 2k + k + k^2 + 2 = 2k + k^2 + 4$$

$\Delta = -1 - 17 < 0$. *Therefore, the determinant is never equal to zero and* $\operatorname{rank}\ [A|b] = 3$. *Since* $\operatorname{rank} A_{3\times 2} \le 2$, *the solution set of the system is empty of each value of* k.

Remark 322 *To solve a parametric linear system* $Ax = b$, *where* $A \in \mathbb{M}(m, n) \setminus \{0\}$, *it is convenient to proceed as follows.*

1. *Perform "easy" row operations on* $[A|b]$*;*

2. *Compute* $\min\{m, n+1\} := k$ *and consider the* $k \times k$ *submatrices of the matrix* $[A|b]$.

   There are two possibilities.

   Case 1. There exists a $k \times k$ *submatrix whose determinant is different from zero for some values of the parameters;*

   Case 2. All $k \times k$ *submatrices have zero determinant for each value of the parameters.*

   If Case 2 occurs, then at least one row of $[A|b]$ *is a linear combinations of other rows; therefore you can eliminate it. We can therefore assume that we are in Case 1. In that case, proceed as described below.*

3. *among those $k \times k$ submatrices, choose a matrix A^* which is a submatrix of A, if possible; if you have more than one matrix to choose among, choose "the easiest one" from a computational viewpoint, i.e., that one with highest number of zeros, the lowest number of times a parameters appear, ... ;*

4. *compute $\det A^*$, a function of the parameter;*

5. *analyze the cases $\det A^* \neq 0$ and possibly $\det A^* = 0$.*

Example 323 *Say for which value of the parameter $a \in \mathbb{R}$ the following system has one, infinite or no solutions:*

$$\begin{cases} ax_1 + x_2 + x_3 & = & 2 \\ x_1 - ax_2 & = & 0 \end{cases}$$

Example 324 *Say for which value of the parameter $a \in \mathbb{R}$ the following system has one, infinite or no solutions:*

$$\begin{cases} ax_1 + x_2 + x_3 & = & 2 \\ x_1 - ax_2 & = & 0 \\ 2ax_1 + ax_2 & = & 4 \end{cases}$$

$$[A|b] = \begin{bmatrix} a & 1 & 1 & 2 \\ 1 & -a & 0 & 0 \\ 2a & a & 0 & 4 \end{bmatrix}$$

$$\det \begin{bmatrix} a & 1 & 1 \\ 1 & -a & 0 \\ 2a & a & 0 \end{bmatrix} = a + 2a^2 = 0 \qquad \text{if} \qquad a = 0, -\frac{1}{2}.$$

Therefore, if $a \neq 0, -\frac{1}{2}$,

$$rank\,[A|b] = rangoA = 3$$

and the system has a unique solution.

If $a = 0$,

$$[A|b] = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Since $\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$, *rank* $A = 2$. *On the other hand,*

$$\det \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = -4 \neq 0$$

and therefore rank $[A|b] = 3$, *and*

$$rank\,[A|b] = 3 \neq 2 = rangoA$$

and the system has no solutions.

If $a = -\frac{1}{2}$,

$$[A|b] = \begin{bmatrix} -\frac{1}{2} & 1 & 1 & 2 \\ 1 & \frac{1}{2} & 0 & 0 \\ -1 & -\frac{1}{2} & 0 & 4 \end{bmatrix}$$

Since

$$\det \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} = -\frac{1}{2},$$

$$rangoA = 2.$$

Since

$$\det \begin{bmatrix} -\frac{1}{2} & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 4 \end{bmatrix} = -\det \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = -4$$

and therefore

$$rank\,[A|b] = 3,$$

$$rank\,[A|b] = 3 \neq 2 = rangoA$$

and the system has no solutions.

Example 325 *Say for which value of the parameter $a \in \mathbb{R}$ the following system has one, infinite or no solutions:*

$$\begin{cases} (a+1)\,x_1 & + & (-2)\,x_2 & + & 2ax_3 & = & a \\ ax_1 & + & (-a)\,x_2 & + & x_3 & = & -2 \\ 2 & + & (2a-4)\,x_2 & + & (4a-2)\,x_3 & = & 2a+4 \end{cases}$$

Then,

$$[A|b] = \left[\begin{array}{ccc|c} a+1 & -2 & 2a & a \\ a & -a & 1 & -2 \\ 2 & 2a-4 & 4a-2 & 2a+4 \end{array}\right].$$

It is easy to see that we are in Case 2 described in Remark 322: all 3×3 submatrices of $[A|b]$ have determinant equal to zero - indeed, the last row is equal to 2 times the first row plus (-2) times the second row. We can then erase the third equation/row to get the following system and matrix.

$$\begin{cases} (a+1)\,x_1 & + & (-2)\,x_2 & + & 2ax_3 & = & a \\ ax_1 & + & (-a)\,x_2 & + & x_3 & = & -2 \end{cases}$$

$$[A|b] = \left[\begin{array}{ccc|c} a+1 & -2 & 2a & a \\ a & -a & 1 & -2 \end{array}\right]$$

$$\det \begin{bmatrix} -2 & 2a \\ -a & 1 \end{bmatrix} = 2a^2 - 2 = 0,$$

whose solutions are $-1, 1$. Therefore, if $a \in \mathbb{R}\backslash\{-1,1\}$,

$$rank\,[A|b] = 2 = rangoA$$

and the system has infinite solutions. Let's study the system for $a \in \{-1, 1\}$.
If $a = -1$, we get

$$[A|b] = \left[\begin{array}{ccc|c} 0 & -2 & -2 & -1 \\ -1 & 1 & 1 & -2 \end{array}\right]$$

and since

$$\det \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix} = 2 \neq 0$$

we have again

$$rank\,[A|b] = 2 = rangoA$$

If $a = 1$,we have

$$[A|b] = \left[\begin{array}{ccc|c} 2 & -2 & 2 & 1 \\ 1 & -1 & 1 & -2 \end{array}\right]$$

and

$$rank\,[A|b] = 2 > 1 = rangoA$$

and the system has no solution..

Example 326 *Say for which value of the parameter $a \in \mathbb{R}$ the following system has one, infinite or no solutions:*

$$\begin{cases} ax_1 & + & x_2 & = & 1 \\ x_1 & + & x_2 & = & a \\ 2x_1 & + & x_2 & = & 3a \\ 3x_1 & + & 2x_2 & = & a \end{cases}$$

$$[A|b] = \begin{bmatrix} a & 1 & | & 1 \\ 1 & 1 & | & a \\ 2 & 1 & | & 3a \\ 3 & 2 & | & a \end{bmatrix}$$

Observe that

$$rank \ A_{4\times 2} \leq 2.$$

$$\det \begin{bmatrix} 1 & 1 & a \\ 2 & 1 & 3a \\ 3 & 2 & a \end{bmatrix} = 3a = 0 \qquad if \qquad a = 0.$$

Therefore, if $a \in \mathbb{R}\setminus\{0\}$,

$$rank\,[A|b] = 3 > 2 \geq rank \ A_{4\times 2}$$

If $a = 0$,

$$[A|b] = \begin{bmatrix} 0 & 1 & | & 1 \\ 1 & 1 & | & 0 \\ 2 & 1 & | & 0 \\ 3 & 2 & | & 0 \end{bmatrix}$$

and since

$$\det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} = -1$$

the system has no solution for $a = 0$.

Summarizing, $\forall a \in \mathbb{R}$, *the system has no solutions.*

8.3 Exercises

Problem sets: 20,23,24.

From Lipschutz (1991), page 179: 5.56; starting from page 263, 7.17 → 7.20.

Part II

Some topology in metric spaces

Chapter 9

Metric spaces

9.1 Definitions and examples

Definition 327 *Let X be a nonempty set. A metric or distance on X is a function $d : X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$*

1. (a.) $d(x,y) \geq 0$, and (b.) $d(x,y) = 0 \Leftrightarrow x = y$,
2. $d(x,y) = d(y,x)$,
3. $d(x,z) \leq d(x,y) + d(y,z)$ (Triangle inequality).
(X,d) is called a metric space.

Remark 328 *Observe that the definition requires that $\forall x, y \in X$, it must be the case that $d(x,y) \in \mathbb{R}$.*

Example 329 *n-dimensional Euclidean space with Euclidean metric.*
Given $n \in \mathbb{N}$, take $X = \mathbb{R}^n$, and

$$d_{2,n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x,y) \mapsto \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}.$$

$(X, d_{2,n})$ was shown to be a metric space in Proposition 58, Section 2.3. $d_{2,n}$ is called the Euclidean distance in $\mathbb{R}^n$. In what follows, unless needed, we write simply d_2 in the place of $d_{2,n}$.

Proposition 330 *(Discrete metric space) Given a nonempty set X and the function*

$$d : X^2 \to \mathbb{R}, \quad d(x,y) = \begin{cases} 0 & if \quad x = y \\ 1 & if \quad x \neq y, \end{cases}$$

(X,d) is a metric space, called discrete metric space.

Proof. 1a. $0, 1 \geq 0$.
1b. From the definition, $d(x,y) = 0 \Leftrightarrow x = y$.
2. It follows from the fact that $x = y \Leftrightarrow y = x$ and $x \neq y \Leftrightarrow y \neq x$.
3. If $x = z$, the result follows. If $x \neq z$, then it cannot be $x = y$ and $y = z$, and again the result follows.
■

Proposition 331 *Given $n \in \mathbb{N}, p \in [1, +\infty)$, $X = \mathbb{R}^n$,*

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x,y) \mapsto \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}},$$

(X,d) is a metric space.

Proof. 1a. It follows from the definition of absolute value.
1b. [⇐]Obvious.
[⇒] $\left(\sum_{i=1}^{n}|x_i - y_i|^p\right)^{\frac{1}{p}} = 0 \Leftrightarrow \sum_{i=1}^{n}|x_i - y_i|^p = 0 \Rightarrow$ for any i , $|x_i - y_i| = 0 \Rightarrow$for any i , $x_i - y_i = 0$.
2. It follows from the fact $|x_i - y_i| = |y_i - x_i|$.
3. First of all observe that

$$d(x,z) = \left(\sum_{i=1}^{n}|(x_i - y_i) + (y_i - z_i)|^p\right)^{\frac{1}{p}}.$$

Then, it is enough to show that

$$\left(\sum_{i=1}^{n}|(x_i - y_i) + (y_i - z_i)|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n}|x_i - y_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|(y_i - z_i)|^p\right)^{\frac{1}{p}}$$

which is a consequence of Proposition 332 below. ■

Proposition 332 *Taken* $n \in \mathbb{N}, p \in [1,+\infty), X = \mathbb{R}^n, a, b \in \mathbb{R}^n$

$$\left(\sum_{i=1}^{n}|a_i + b_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|b_i|^p\right)^{\frac{1}{p}}$$

Proof. It follows from the proof of the Proposition 335 below. ■

Definition 333 *Let* $\mathbb{R}^\infty$ *be the set of sequences in* $\mathbb{R}$.

Definition 334 *For any* $p \in [1,+\infty)$, *define*[1]

$$l^p = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty : \sum_{n=1}^{+\infty}|x_n|^p < +\infty\right\},$$

i.e., roughly speaking, l^p *is the set of sequences whose associated series are absolutely convergent.*

Proposition 335 *(Minkowski inequality).* $\forall (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in l^p, \forall p \in [1,+\infty)$,

$$\left(\sum_{n=1}^{+\infty}|x_n + y_n|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{+\infty}|x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{+\infty}|y_n|^p\right)^{\frac{1}{p}}. \tag{9.1}$$

Proof. If either $(x_n)_{n\in\mathbb{N}}$ or $(y_n)_{n\in\mathbb{N}}$ are such that $\forall n \in \mathbb{N}$, $x_n = 0$ or $\forall n \in \mathbb{N}$, $y_n = 0$, i.e., if either sequence is the constant sequence of zeros, then (9.1) is trivially true.

Then, we can consider the case in which

$$\exists \alpha, \beta \in \mathbb{R}_{++} \text{ such that } \left(\sum_{n=1}^{+\infty}|x_n|^p\right)^{\frac{1}{p}} = \alpha \text{ and } \left(\sum_{n=1}^{+\infty}|y_n|^p\right)^{\frac{1}{p}} = \beta. \tag{9.2}$$

Define

$$\forall n \in \mathbb{N}, \quad \widehat{x}_n = \left(\frac{|x_n|}{\alpha}\right)^p \text{ and } \quad \widehat{y}_n = \left(\frac{|y_n|}{\beta}\right)^p. \tag{9.3}$$

Then

$$\sum_{n=1}^{+\infty}\widehat{x}_n = \sum_{n=1}^{+\infty}\widehat{y}_n = 1. \tag{9.4}$$

For any $n \in \mathbb{N}$, from the triangle inequality for the absolute value, we have

$$|x_n + y_n| \le |x_n| + |y_n|;$$

[1] For basic results on series, see, for example, Section 10.5 in Apostol (1967).

since $\forall p \in [1,+\infty)$, $f_p : \mathbb{R}_+ \to \mathbb{R}$, $f_p(t) = t^p$ is an increasing function, we have

$$|x_n + y_n|^p \le (|x_n| + |y_n|)^p . \tag{9.5}$$

Moreover, from (9.3),

$$(|x_n| + |y_n|)^p = \left(\alpha \left|\widehat{x}_n\right|^{\frac{1}{p}} + \beta \left|\widehat{y}_n\right|^{\frac{1}{p}}\right)^p = (\alpha+\beta)^p \left(\left(\frac{\alpha}{\alpha+\beta}\left|\widehat{x}_n\right|^{\frac{1}{p}} + \frac{\beta}{\alpha+\beta}\left|\widehat{y}_n\right|^{\frac{1}{p}}\right)^p\right). \tag{9.6}$$

Since $\forall p \in [1,+\infty)$, f_p is convex (just observe that $f_p''(t) = p(p-1)t^{p-2} \ge 0$), we get

$$\left(\frac{\alpha}{\alpha+\beta}\left|\widehat{x}_n\right| + \frac{\beta}{\alpha+\beta}\left|\widehat{y}_n\right|\right)^p \le \frac{\alpha}{\alpha+\beta}\left|\widehat{x}_n\right| + \frac{\beta}{\alpha+\beta}\left|\widehat{y}_n\right| \tag{9.7}$$

From $(9.5), (9.6)$ and (9.7), we get

$$|x_n + y_n|^p \le (\alpha+\beta) \cdot \left(\frac{\alpha}{\alpha+\beta}\left|\widehat{x}_n\right|^p + \frac{\beta}{\alpha+\beta}\left|\widehat{y}_n\right|\right).$$

From the above inequalities and basic properties of the series, we then get

$$\sum_{n=1}^{+\infty}|x_n + y_n|^p \le (\alpha+\beta)^p\left(\frac{\alpha}{\alpha+\beta}\sum_{n=1}^{+\infty}\left|\widehat{x}_n\right| + \frac{\beta}{\alpha+\beta}\sum_{n=1}^{+\infty}\left|\widehat{y}_n\right|\right) \overset{(9.4)}{=} (\alpha+\beta)^p\left(\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}\right) = (\alpha+\beta)^p .$$

Therefore, using (9.2), we get

$$\sum_{n=1}^{+\infty}|x_n + y_n|^p \le \left(\left(\sum_{n=1}^{+\infty}|x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{+\infty}|y_n|^p\right)^{\frac{1}{p}}\right)^p,$$

and therefore the desired result. ■

Proposition 336 (l^p, d_p) *with*

$$d_p : l^p \times l^p \to \mathbb{R}, \quad d_p\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\right) = \left(\sum_{n=1}^{+\infty}|x_n - y_n|^p\right)^{\frac{1}{p}}.$$

is a metric space.

Proof. We first of all have to check that $d_p(x,y) \in \mathbb{R}$, i.e., that $\left(\sum_{n=1}^{+\infty}|x_n - y_n|^p\right)^{\frac{1}{p}}$ converges.

$$\left(\sum_{n=1}^{+\infty}|x_n - y_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{+\infty}|x_n + (-y_n)|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{+\infty}|x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{+\infty}|y_n|^p\right)^{\frac{1}{p}} < +\infty,$$

where the first inequality follows from Minkowski inequality and the second inequality from the assumption that we are considering sequences in l^p.

Properties 1 and 2 of the distance follow easily from the definition. Property 3 is again a consequence of Minkowski inequality:

$$d_p(x,z) = \left(\sum_{n=1}^{+\infty}|(x_n - y_n) + (y_n - z_n)|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{+\infty}|x_n - y_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{+\infty}|(y_n - z_n)|^p\right)^{\frac{1}{p}} := d_p(x,y) + d_p(y,z).$$

■

Definition 337 *Let T be a non empty set. $\mathcal{B}(T)$ is the set of all bounded real functions defined on T, i.e.,*

$$\mathcal{B}(T) := \{f : T \to \mathbb{R} \quad : \quad \sup\{|f(x)| : x \in T\} < +\infty\},$$

and[2]

$$d_\infty : \mathcal{B}(T) \times \mathcal{B}(T) \to \mathbb{R}, \quad d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in T\}$$

Definition 339

$$l^\infty = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty : \sup\{|x_n| : n \in \mathbb{N}\} < +\infty\right\}$$

is called the set of bounded real sequences, and, still using the symbol of the previous definition,

$$d_\infty : l^\infty \times l^\infty \to \mathbb{R}, \quad d_\infty\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\right) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}$$

Proposition 340 *$(\mathcal{B}(T), d_\infty)$ and (l^∞, d_∞) are metric spaces, and d_∞ is called the sup metric.*

Proof. We show that $(\mathcal{B}(T), d_\infty)$ is a metric space. As usual, the difficult part is to show property 3 of d_∞, which is done below.

$\forall f, g, h \in \mathcal{B}(T), \forall x \in T,$

$$|f(x) - g(x)| \le |f(x) - h(x)| + |h(x) - g(x)| \le$$

$$\le \sup\{|f(x) - h(x)| : x \in T\} + \sup\{|h(x) - g(x)| : x \in T\} =$$

$$= d_\infty(f,h) + d_\infty(h,g).$$

Then, $\forall x \in T$,

$$d_\infty(f,g) := \sup |f(x) - g(x)| \le d_\infty(f,g) + d_\infty(h,g).$$

■

Exercise 341 *If (X,d) is a metric space, then*

$$\left(X, \frac{d}{1+d}\right)$$

is a metric space.

Proposition 342 *Given a metric space (X,d) and a set Y such that $\varnothing \ne Y \subseteq X$, then $\left(Y, d_{|Y\times Y}\right)$ is a metric space.*

Proof. By definition. ■

Definition 343 *Given a metric space (X,d) and a set Y such that $\varnothing \ne Y \subseteq X$, then $\left(Y, d_{|Y\times Y}\right)$, or simply, (Y,d) is called a metric subspace of X.*

Example 344 *1. Given $\mathbb{R}$ with the (Euclidean) distance $d_{2,1}$, $([0,1), d_{2,1})$ is a metric subspace of $(\mathbb{R}, d_{2,1})$.*
2. Given $\mathbb{R}^2$ with the (Euclidean) distance $d_{2,2}$, $(\{0\} \times \mathbb{R}, d_{2,2})$ is a metric subspace of $\left(\mathbb{R}^2, d_{2,2}\right)$.

Exercise 345 *Let $C([0,1])$ be the set of continuous functions from $[0,1]$ to $\mathbb{R}$. Show that a metric on that set is defined by*

$$d(f,g) = \int_0^1 |f(x) - g(x)| \, dx,$$

where $f, g \in C([0,1])$.

---

[2]

Definition 338 *Observe that $d_\infty(f,g) \in \mathbb{R}$:*

$$d_\infty(f,g) := \sup\{|f(x) - g(x)| : x \in T\} \le \sup\{|f(x)| : x \in T\} + \sup\{|g(x)| : x \in T\} < +\infty.$$

Example 346 *Let X be the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and consider $d(f,g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)|$. (X,d) is **not** a metric space because d is not a function from X^2 to $\mathbb{R}$: it can be $\sup_{x\in\mathbb{R}} |f(x) - g(x)| = +\infty$.*

Example 347 *Let $X = \{a,b,c\}$ and $d : X^2 \to \mathbb{R}$ such that*

$$d(a,b) = d(b,a) = 2$$

$$d(a,c) = d(c,a) = 0$$

$$d(b,c) = d(c,b) = 1.$$

*Since $d(a,b) = 2 > 0 + 1 = d(a,c) + d(b,c)$, then (X,d) is **not** a metric space.*

Example 348 *Given $n \in \mathbb{N}, p \in (0,1)$, $X = \mathbb{R}^2$, define*

$$d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \left(\sum_{i=1}^{2} |x_i - y_i|^p\right)^{\frac{1}{p}},$$

*(X,d) is **not** a metric space, as shown below. Take $x = (0,1), y = (1,0)$ and $z = (0,0)$. Then*

$$d(x,y) = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}},$$

$$d(x,z) = (0^p + 1^p)^{\frac{1}{p}} = 1$$

$$d(z,y) = 1.$$

Then, $d(x,y) - (d(x,z) + d(z,y)) = 2^{\frac{1}{p}} - 2 > 0$.

9.2 Open and closed sets

Definition 349 *Let (X,d) be a metric space. $\forall x_0 \in X$ and $\forall r \in \mathbb{R}_{++}$, the open r-ball of x_0 in (X,d) is the set*

$$B_{(X,d)}(x_0,r) = \{x \in X : d(x,x_0) < r\}.$$

If there is no ambiguity about the metric space (X,d) we are considering, we use the lighter notation $B(x_0,r)$ in the place of $B_{(X,d)}(x_0,r)$.

Example 350 *1.*

$$B_{(\mathbb{R},d_2)}(x_0,r) = (x_0 - r, x_0 + r)$$

is the open interval of radius r centered in x_0.

2.

$$B_{(\mathbb{R}^2,d_2)}(x_0,r) = \left\{(x_1,x_2) \in \mathbb{R}^2 : \sqrt{(x_1 - x_{01})^2 + (x_2 - x_{02})^2} < r\right\}$$

is the open disk of radius r centered in x_0.

3. In $\mathbb{R}^2$ with the metric d given by

$$d((x_1,x_2),(y_1,y_2)) = \max\{|x_1 - y_1, |x_2 - y_2||\}$$

the open ball $B(0,1)$ can be pictured as done below:

`a square around zero.`

Definition 351 *Let (X,d) be a metric space. x is an interior point of $S \subseteq X$ if there exists an open ball centered in x and contained in S, i.e., $\exists\, r \in \mathbb{R}_{++}$ such that $B(x,r) \subseteq S$.*

Definition 352 *The set of all interior points of S is called the Interior of S and it is denoted by $\mathrm{Int}_{(X,d)}\, S$ or simply by $\mathrm{Int}\, S$.*

Remark 353 Int $(S) \subseteq S$, *simply because* $x \in \text{Int }(S) \Rightarrow x \in B(x,r) \subseteq S$, *where the first inclusion follows from the definition of open ball and the second one from the definition of Interior. In other words, to find interior points of* S, *we can limit our search to points belonging to* S.

It is not true that $\forall S \subseteq X,\ S \subseteq \text{Int }(S)$, *as shown below. We want to prove that*

$$\neg(\forall S \subseteq X, \forall x \in S, \quad x \in S \Rightarrow x \in IntS),$$

i.e.,

$$(\exists S \subseteq X \text{ and } x \in S \text{ such that } x \notin \text{Int } S).$$

Take $(X,d) = (\mathbb{R}, d_2)$, $S = \{1\}$ *and* $x = 1$. *Then, clearly* $1 \in \{1\}$, *but* $1 \notin \text{Int } S : \forall r \in \mathbb{R}_{++}$, $(1-r, 1+r) \nsubseteq \{1\}$.

Remark 354 *To understand the following example, recall that* $\forall a, b \in \mathbb{R}$ *such that* $a < b$, $\exists c \in \mathbb{Q}$ *and* $d \in \mathbb{R}\backslash\mathbb{Q}$ *such that* $c, d \in (a,b)$ *- see, for example, Apostol (1967).*

Example 355 *Let* $(\mathbb{R}, d_2)$ *be given.*

1. $\text{Int } \mathbb{N} = \text{Int } \mathbb{Q} = \varnothing$.
2. $\forall a, b \in \mathbb{R}$, $a < b$, $\text{Int }[a,b] = \text{Int }[a,b) = \text{Int }(a,b] = \text{Int }(a,b) = (a,b)$.
3. $\text{Int } \mathbb{R} = \mathbb{R}$.
4. $\text{Int } \varnothing = \varnothing$.

Definition 356 *Let* (X,d) *be a metric space. A set* $S \subseteq X$ *is open in* (X,d), *or* (X,d)*-open, or open with respect to the metric space* (X,d), *if* $S \subseteq \text{Int } S$, *i.e.,* $S = \text{Int } S$, *i.e.,*

$$\forall x \in S, \exists r \in \mathbb{R}_{++} \text{ such that } B_{(X,d)}(x,r) := \{y \in X : d(y,x) < r\} \subseteq S.$$

Remark 357 *Let* $(\mathbb{R}, d_2)$ *be given. From Example 355, it follows that*

$\mathbb{N}, \mathbb{Q}, [a,b], [a,b), (a,b]$ *are not open sets, and* $(a,b), \mathbb{R}$ *and* $\varnothing$ *are open sets. In particular, open interval are open sets, but there are open sets which are not open interval. Take for example* $S = (0,1) \cup (2,3)$.

Exercise 358 $\forall n \in \mathbb{N}, \forall i \in \{1, ..., n\}, \forall a_i, b_i \in \mathbb{R}$ *with* $a_i < b_i$,

$$\times_{i=1}^{n} (a_i, b_i)$$

is $(\mathbb{R}^n, d_2)$ *open.*

Proposition 359 *Let* (X,d) *be a metric space. An open ball is an open set.*

Proof. Take $y \in B(x_0, r)$. Define

$$\delta = r - d(x_0, y). \tag{9.8}$$

First of all, observe that, since $y \in B(x_0, r)$, $d(x_0, y) < r$ and then $\delta \in \mathbb{R}_{++}$. It is then enough to show that $B(y, \delta) \subseteq B(x_0, r)$, i.e., we assume that

$$d(z, y) < \delta \tag{9.9}$$

and we want to show that $d(z, x_0) < r$. From the triangle inequality

$$d(z, x_0) \leq d(z, y) + d(y, x_0) \overset{(9.9),(9.8)}{<} \delta + (r - \delta) = r,$$

as desired. ■

Example 360 *In a discrete metric space* (X,d), $\forall x \in X, \forall r \in (0,1]$, $B(x,r) := \{y \in X : d(x,y) < r\} = \{x\}$ *and* $\forall r > 1$, $B(x,r) := \{y \in X : d(x,y) < r\} = X$. *Then, it is easy to show that any subset of a discrete metric space is open, as verified below. Let* (X,d) *be a discrete metric space and* $S \subseteq X$. *For any* $x \in S$, *take* $\varepsilon = \frac{1}{2}$; *then* $B\left(x, \frac{1}{2}\right) = \{x\} \subseteq S$.

Definition 361 *Let a metric space* (X,d) *be given. A set* $T \subseteq X$ *is* (X,d) *closed or closed in* (X,d) *if its complement in* X, *i.e.,* $X\backslash T$ *is open in* (X,d).

If no ambiguity arises, we simply say that T is closed in X, or even, T is closed; we also write T^C in the place of $X\backslash T$.

Remark 362 *S is open $\Leftrightarrow S^C$ is closed, simply because S^C closed $\Leftrightarrow \left(S^C\right)^C = S$ is open.*

Example 363 *The following sets are closed in $(\mathbb{R}, d_2)$: $\mathbb{R}; \mathbb{N}; \varnothing; \forall a, b \in \mathbb{R}$, $a < b$, $\{a\}$ and $[a, b]$.*

Remark 364 ***It is false that:***

$$S \text{ is not open} \Rightarrow S \text{ is closed}$$

(and therefore that S is not closed $\Rightarrow S$ is open), i.e., there exist sets which are not open and not closed, for example $(0, 1]$ in $(\mathbb{R}, d_2)$. There are also two sets which are both open and closed: $\varnothing$ and $\mathbb{R}^n$ in $(\mathbb{R}^n, d_2)$.

Proposition 365 *Let a metric space (X, d) be given.*
1. *$\varnothing$ and X are open sets.*
2. *The union of any (finite or infinite) collection of open sets is an open set.*
3. *The intersection of any finite collection of open sets is an open set.*

Proof. 1.
$\forall x \in X$, $\forall r \in \mathbb{R}_{++}$, $B(x, r) \subseteq X$. $\varnothing$ is open because it contains no elements.
2.
Let $\mathcal{I}$ be a collection of open sets and $S = \cup_{A \in \mathcal{I}} A$. Assume that $x \in S$. Then there exists $A \in \mathcal{I}$ such that $x \in A$. Then, for some $r \in \mathbb{R}_{++}$

$$x \in B(x, r) \subseteq A \subseteq S$$

where the first inclusion follows from fact that A is open and the second one from the definition of S.
3.
Let $\mathcal{F}$ be a collection of open sets, i.e., $\mathcal{F} = \{A_n\}_{n \in N}$, where $N \subseteq \mathbb{N}$, $\#N$ is finite and $\forall n \in N$, A_n is an open set. Take $S = \cap_{n \in N} A_n$. If $S = \varnothing$, we are done. Assume that $S \neq \varnothing$ and that $x \in S$. Then from the fact that each set A is open and from the definition of S as the intersection of sets

$$\forall n \in N, \exists r_n \in \mathbb{R}_{++} \text{ such that } x \in B(x, r_n) \subseteq A_n$$

Since N is a finite set, there exists a positive $r^* = \min\{r_n : n \in N\} > 0$. Then

$$\forall n \in N, x \in B(x, r^*) \subseteq B(x, r_n) \subseteq A_n$$

and from the very definition of intersections

$$x \in B(x, r^*) \subseteq \cap_{n \in N} B(x, r_n) \subseteq \cap_{n \in N} A_n = S.$$

■

Remark 366 *The assumption that $\#N$ is finite cannot be dispensed with:*

$$\cap_{n=1}^{+\infty} B\left(0, \frac{1}{n}\right) = \cap_{n=1}^{+\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not open.

Remark 367 *A generalization of metric spaces is the concept of topological spaces. In fact, we have the following definition which " assumes the previous Proposition".*

Let X be a nonempty set. A collection $\mathcal{T}$ of subsets of X is said to be a topology on X if
1. *$\varnothing$ and X belong to $\mathcal{T}$,*
2. *The union of any (finite or infinite) collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$,*
3. *The intersection of any finite collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.*

$(X, \mathcal{T})$ is called a topological space.
The members of $\mathcal{T}$ are said to be open set with respect to the topology $\mathcal{T}$, or $(X, \mathcal{T})$ open.

Proposition 368 *Let a metric space (X, d) be given.*

1. $\varnothing$ and X are closed sets.

2. The intersection of any (finite or infinite) collection of closed sets is a closed set.

3. The union of any finite collection of closed sets is a closed set.

Proof. 1

It follows from the definition of closed set, the fact that $\varnothing^C = X$, $X^C = \varnothing$ and Proposition 365.

2.

Let $\mathcal{I}$ be a collection of closed sets and $S = \cap_{B\in\mathcal{I}} B$. Then, from de Morgan's laws,

$$S^C = (\cap_{B\in\mathcal{I}} B)^C = \cup_{B\in\mathcal{I}} B^C$$

Then from Remark 362, $\forall B \in \mathcal{I}$, B^C is open and from Proposition 365.1, $\cup_{B\in\mathcal{I}} B^C$ is open as well.

2.

Let $\mathcal{F}$ be a collection of closed sets, i.e., $\mathcal{F} = \{B_n\}_{n\in N}$, where $N \subseteq \mathbb{N}$, $\#N$ is finite and $\forall n \in N$, B_n is an open set. Take $S = \cup_{n\in N} B_n$. Then, from de Morgan's laws,

$$S^C = (\cup_{n\in N} B_n)^C = \cap_{n\in N} B_n^C$$

Then from Remark 362, $\forall n \in N$, B^C is open and from Proposition 365.2, $\cap_{B\in\mathcal{I}} B^C$ is open as well. ■

Remark 369 *The assumption that $\#N$ is finite cannot be dispensed with:*

$$\left(\cap_{n=1}^{+\infty} B\left(0, \frac{1}{n}\right)\right)^C = \cup_{n=1}^{+\infty} B\left(0, \frac{1}{n}\right)^C = \cup_{n=1}^{+\infty}\left(\left(-\infty, -\frac{1}{n},\right] \cup \left[\frac{1}{n}, +\infty\right)\right) = \mathbb{R}\setminus\{0\}.$$

is not closed.

Definition 370 *If S is both closed and open in (X, d), S is called clopen in (X, d).*

Remark 371 *In any metric space (X, d), X and $\varnothing$ are clopen.*

Proposition 372 *In any metric space (X, d), $\{x\}$ is closed.*

Proof. We want to show that $X\setminus\{x\}$ is open. If $X = \{x\}$, then $X\setminus\{x\} = \varnothing$, and we are done. If $X \neq \{x\}$, take $y \in X$, where $y \neq x$. Taken

$$r = d(y, x) \tag{9.10}$$

with $r > 0$, because $x \neq y$. We are left with showing that $B(y, r) \subseteq X\setminus\{x\}$, which is true because of the following argument. Suppose otherwise; then $x \in B(y, r)$, i.e., $r \overset{(9.10)}{=} d(y, x) < r$, a contradiction. ■

Remark 373 *From Example 360, any set in any discrete metric space is open. Therefore, the complement of each set is open, and therefore each set is then clopen.*

Definition 374 *Let a metric space (X, d) and a set $S \subseteq X$ be given. x is an boundary point of S if any open ball centered in x intersects both S and its complement in X, i.e.,*

$$\forall r \in \mathbb{R}_{++}, \quad B(x, r) \cap S \neq \varnothing \qquad \wedge \qquad B(x, r) \cap S^C \neq \varnothing.$$

Definition 375 *The set of all boundary points of S is called the Boundary of S and it is denoted by $\mathcal{F}(S)$.*

Exercise 376 $\mathcal{F}(S) = \mathcal{F}\left(S^C\right)$.

Exercise 377 *$\mathcal{F}(S)$ is a closed set.*

Definition 378 *The closure of S, denoted by* Cl (S) *is the intersection of all closed sets containing S, i.e.,* Cl $(S) = \cap_{S'\in\mathcal{S}} S'$ *where* $\mathcal{S} := \{S' \subseteq X : S'$ *is closed and* $S' \supseteq S\}$.

Proposition 379 *1.* Cl (S)*is a closed set;*

2. S is closed $\Leftrightarrow S =$ Cl (S).

Proof. 1.
It follows from the definition and Proposition 368.
2.
[$\Leftarrow$]
It follows from 1. above.
[$\Rightarrow$]
Since S is closed, then $S \in \mathcal{S}$. Therefore, Cl $(S) = S \cap (\cap_{S' \in \mathcal{S}} S') = S$. ■

Definition 380 *$x \in X$ is an accumulation point for $S \subseteq X$ if any open ball centered at x contains points of S different from x, i.e., if*

$$\forall r \in \mathbb{R}_{++}, \quad (S \backslash \{x\}) \cap B(x,r) \neq \varnothing$$

The set of accumulation points of S is denoted by $D(S)$ and it is called the Derived set of S.

Definition 381 *$x \in X$ is an isolated point for $S \subseteq X$ if $x \in S$ and it is not an accumulation point for S, i.e.,*

$$x \in S \text{ and } \exists r \in \mathbb{R}_{++} \text{ such that } \quad (S \backslash \{x\}) \cap B(x,r) = \varnothing,$$

or

$$\exists r \in \mathbb{R}_{++} \text{ such that } \quad S \cap B(x,r) = \{x\}.$$

The set of isolated points of S is denoted by $Is(S)$.

Proposition 382 $D(S) = \{x \in \mathbb{R}^n : \ \forall r \in \mathbb{R}_{++}, \quad S \cap B(x,r)$ *has an infinite cardinality*$\}$.

Proof. [$\subseteq$]
Suppose otherwise, i.e., x is an accumulation point of S and $\exists r \in \mathbb{R}_{++}$ such that $S \cap B(x,r) = \{x_1, ..., x_n\}$.Then defined $\delta := \min\{d(x, x_i) : i \in \{1, ..., n\}\}$, $(S \backslash \{x\}) \cap B\left(x, \frac{\delta}{2}\right) = \varnothing$, a contradiction.
[$\supseteq$]
Since $S \cap B(x,r)$ has an infinite cardinality, then $(S \backslash \{x\}) \cap B(x,r) \neq \varnothing$. ■

9.2.1 Sets which are open or closed in metric subspaces.

Remark 383 *1.* $[0,1)$ *is* $([0,1), d_2)$ *open.*
2. $[0,1)$ *is not* $(\mathbb{R}, d_2)$ *open. We want to show*

$$\neg \left\langle \forall x_0 \in [0,1), \exists r \in \mathbb{R}_{++} \text{ such that } B_{(\mathbb{R},d_2)}(x_0, r) \right\rangle = (x_0 - r, x_0 + r) \subseteq [0,1), \tag{9.11}$$

i.e.,

$$\exists x_0 \in [0,1) \text{ such that } \forall r \in \mathbb{R}_{++}, \ \exists x' \in \mathbb{R} \text{ such that } x' \in (x_0 - r, x_0 + r) \text{ and } x' \notin [0,1).$$

It is enough to take $x_0 = 0$ and $x' = -\frac{r}{2}$.
3. Let $([0,+\infty), d_2)$ be given. $[0,1)$ is $([0,+\infty), d_2)-$open, as shown below. By definition of open set, - go back to Definition 356 and read it again - we have that, given the metric space $((0,+\infty), d_2)$, $[0,1)$ is open if

$$\forall x_0 \in [0,1), \ \exists r \in \mathbb{R}_{++} \text{ such that } B_{([0,+\infty),d_2)}(x_0, r) := \left\{ \underline{x \in [0,+\infty)} : d(x_0, x) < r \right\} = [0, r) \subseteq [0,1).$$

If $x_0 \in (0,1)$, then take $r = \min\{x_0, 1 - x_0\} > 0$.
If $x_0 = 0$, then take $r = \frac{1}{2}$. Therefore, we have $B_{(\mathbb{R}_+,d_2)}\left(0, \frac{1}{2}\right) = \left\{x \in \mathbb{R}_+ : |x - 0| < \frac{1}{2}\right\} = \left[0, \frac{1}{2}\right) \subseteq [0,1)$.

Remark 384 *1.* $(0,1)$ *is* $((0,1), d_2)$ *closed.*
2. $(0,1]$ *is* $((0,+\infty), d_2)$ *closed, simply because* $(1,+\infty)$ *is open.*

Proposition 385 *Let a metric space (X,d), a metric subspace (Y,d) of (X,d) and a set $S \subseteq Y$ be given.*

$$S \text{ is open in } (Y,d) \Leftrightarrow \text{there exists a set } O \text{ open in } (X,d) \text{ such that } S = Y \cap O.$$

Proof. Preliminary remark.

$$\forall x_0 \in Y, \forall r \in \mathbb{R}_{++},$$
$$B_{(Y,d)}(x_0,r) := \{x \in Y : d(x_0,x) < r\} = Y \cap \{x \in X : d(x_0,x) < r\} = Y \cap B_{(X,d)}(x_0,r). \tag{9.12}$$

[⇒]
Taken $x_0 \in S$, by assumption $\exists r_{x_0} \in \mathbb{R}_{++}$ such that $B_{(Y,d)}(x_0,r) \subseteq S \subseteq Y$. Then

$$S = \cup_{x_0 \in S} B_{(Y,d)}(x_0,r) \overset{(9.12)}{=} \cup_{x_0 \in S}\left(Y \cap B_{(X,d)}(x_0,r)\right) \overset{\text{distributive laws}}{=} Y \cap \left(\cup_{x_0 \in S} B_{(X,d)}(x_0,r)\right),$$

and the it is enough to take $O = \cup_{x_0 \in S} B_{(X,d)}(x_0,r)$ to get the desired result.

[⇐]
Take $x_0 \in S$. then, $x_0 \in O$, and, since, by assumption, O is open in (X,d), $\exists r \in \mathbb{R}_{++}$ such that $B_{(X,d)}(x_0,r) \subseteq O$. Then

$$B_{(Y,d)}(x_0,r) \overset{(9.12)}{=} Y \cap B_{(X,d)}(x_0,r) \subseteq O \cap Y = S,$$

where the last equality follows from the assumption. Summarizing, $\forall x_0 \in S, \exists r \in \mathbb{R}_{++}$ such that $B_{(Y,d)}(x_0,r) \subseteq S$, as desired. ■

Corollary 386 *Let a metric space (X,d), a metric subspace (Y,d) of (X,d) and a set $S \subseteq Y$ be given.*

1.
$$\langle S \text{ closed in } (Y,d)\rangle \Leftrightarrow \langle \text{there exists a set } C \text{ closed in } (X,d) \text{ such that } S = Y \cap C.\rangle.$$

2.
$$\langle S \text{ open (respectively, closed) in } (X,d)\rangle \underset{\not\Leftarrow}{\Rightarrow} \langle S \text{ open (respectively, closed) in } (Y,d)\rangle.$$

3. If Y is open (respectively, closed) in X,

$$\langle S \text{ open (respectively, closed) in } (X,d)\rangle \Leftarrow \langle S \text{ open (respectively, closed) in } (Y,d)\rangle.$$

i.e., "the implication ⇐ in the above statement 2. does hold true".

Proof. 1.
$\langle S \text{ closed in } (Y,d)\rangle \overset{\text{def.}}{\Leftrightarrow} \langle Y \backslash S \text{ open in } (Y,d)\rangle \overset{\text{Prop. 385}}{\Leftrightarrow}$
$\Leftrightarrow \langle \text{there exists an open set } S'' \text{ in } (X,d) \text{ such that } Y\backslash S = S'' \cap Y\rangle \Leftrightarrow$
$\Leftrightarrow \langle \text{there exists a closed set } S' \text{ in } (X,d) \text{ such that } S = S' \cap Y\rangle,$
where the last equivalence is proved below;
[⇐]
Take $S'' = X\backslash S'$, open in (X,d) by definition. We want to show that
if $S'' = X\backslash S$, $S = S' \cap Y$ and $Y \subseteq X$, then $Y\backslash S = S'' \cap Y$:

$$\begin{array}{lll} x \in Y\backslash S & iff & x \in Y \;\wedge\; x \notin S \\ & & x \in Y \;\wedge\; (x \notin S' \cap Y) \\ & & x \in Y \;\wedge (\neg\; (x \in S' \cap Y)) \\ & & x \in Y \;\wedge (\neg\; (x \in S' \wedge x \in Y)) \\ & & x \in Y \;\wedge ((x \notin S' \vee x \notin Y)) \\ & & (x \in Y \;\wedge x \notin S') \;\vee ((x \in Y \wedge x \notin Y)) \\ & & x \in Y \;\wedge x \notin S' \end{array}$$

$$\begin{array}{lll} x \in S'' \cap Y & iff & x \in Y \;\wedge\; x \in S'' \\ & & x \in Y \;\wedge\; (x \in X \wedge x \notin S') \\ & & (x \in Y \;\wedge\; x \in X) \wedge x \notin S' \\ & & x \in Y \wedge x \notin S' \end{array}$$

[⇒]
Take $S' = X\backslash S$.Then S' is closed in (X,d). We want to show that
if $T' = X\backslash S''$, $Y\backslash S = S'' \cap Y, Y \subseteq X$, then $S = S' \cap Y$.

Observe that we want to show that $Y\backslash S = Y\backslash\left(S' \cap Y\right)$,or from the assumptions, we want to show that

$$S'' \cap Y = Y\backslash\left(\left(X\backslash S''\right) \cap Y\right).$$

$$\begin{array}{lll} x \in Y\backslash\left(\left(X\backslash S''\right) \cap Y\right) & iff & x \in Y \ \ \wedge \ \left(\neg\left(x \in X\backslash S'' \wedge x \in Y\right)\right) \\ & & x \in Y \ \ \wedge \ \left(x \notin X\backslash S'' \vee x \notin Y\right) \\ & & x \in Y \ \ \wedge\left(x \in S'' \vee x \notin Y\right) \\ & & \left(x \in Y \wedge x \in S''\right) \ \ \vee\left(x \in Y \wedge x \notin Y\right) \\ & & x \in Y \wedge x \in S'' \\ & & x \in S'' \cap Y \end{array}$$

2. and 3.
Exercises. ∎

9.3 Sequences

Unless otherwise specified, up to the end of the chapter, we assume that

$$X \text{ is a metric space with metric } d,$$

and

$$\mathbb{R}^n \text{ is the metric space with Euclidean metric.}$$

Definition 387 *A sequence in X is a function $x : \mathbb{N} \to X$.*

Usually, for any $n \in \mathbb{N}$, the value $x(n)$ is denoted by x_n,which is called the n-th term of the sequence; the sequence is denoted by $(x_n)_{n\in\mathbb{N}}$.

Definition 388 *Given a nonempty set X, X^∞ is the set of sequences $(x_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$, $x_n \in X$.*

Definition 389 *A strictly increasing sequence of natural numbers is a sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such*

$$1 \le k_1 < k_2 < \ \ldots < k_n < \ \ldots$$

Definition 390 *A subsequence of a sequence $(x_n)_{n\in\mathbb{N}}$ is a sequence $(y_n)_{n\in\mathbb{N}}$ such that there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers such that $\forall n \in \mathbb{N}$, $y_n = x_{k_n}$.*

Definition 391 *A sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ is said to be (X,d) convergent to $x_0 \in X$ (or convergent to $x_0 \in X$ with respect to the metric space (X,d)) if*

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, \quad d(x_n, x_0) < \varepsilon \tag{9.13}$$

x_0 is called the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ and we write

$$\lim_{n\to+\infty} x_n = x_0, \text{ or } \ x_n \overset{n}{\to} x_0. \tag{9.14}$$

$(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) is convergent if there exist $x_0 \in X$ such that (9.13) holds. In that case, we say that the sequence converges to x_0 and x_0 is the limit of the sequence.[3]

Remark 392 *A more precise, and heavy, notation for (9.14) would be*

$$\lim_{\substack{n\to+\infty \\ (X,d)}} x_n = x_0 \quad \text{or} \quad x_n \underset{(X,d)}{\overset{n}{\to}} x_0$$

Remark 393 *Observe that $\left(\frac{1}{n}\right)_{n\in\mathbb{N}_+}$ converges with respect to $(\mathbb{R}, d_2)$ and it does not converge with respect to $(\mathbb{R}_{++}, d_2)$.*

Proposition 394 $\lim_{n\to+\infty} x_n = x_0 \Leftrightarrow \lim_{n\to+\infty} d(x_n, x_0) = 0.$

[3] For the last sentence in the Definition, see, for example, Morris (2007), page 121.

Proof. Observe that we can define the sequence $(d(x_n,x_0))_{n\in\mathbb{N}}$ in $\mathbb{R}$. Then from definition 391, we have that $\lim_{n\to+\infty} d(x_n,x_0)=0$ means that

$$\forall \varepsilon>0, \exists n_0\in\mathbb{N} \text{ such that } \forall n>n_0, \quad |d(x_n,x_0)-0|<\varepsilon.$$

■

Remark 395 *Since $(d(x_n,x_0))_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$, all well known results hold for that sequence. Some of those results are listed below.*

Proposition 396 *(Some properties of sequences in $\mathbb{R}$).*
All the following statements concern sequences in $\mathbb{R}$.
1. Every convergent sequence is bounded.
2. Every increasing (decreasing) sequence that is bounded above (below) converges to its sup *(*inf*).*
3. Every sequence has a monotone subsequence.
4. (Bolzano-Weierstrass 1) Every bounded sequence has a convergent subsequence.
5. (Bolzano-Weierstrass 2) Every sequence contained in a closed and bounded set has a convergent subsequence in the set.
Moreover, suppose that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}$ and $\lim_{n\to\infty} x_n=x_0$ and $\lim_{n\to+\infty} y_n=y_0$. Then
6. $\lim_{n\to+\infty}(x_n+y_n)=x_0+y_0$;
7. $\lim_{n\to+\infty} x_n\cdot y_n=x_0\cdot y_0$;
8. if $\forall n\in\mathbb{N}$, $x_n\neq 0$ and $x_0\neq 0$, $\lim_{n\to+\infty}\frac{1}{x_n}=\frac{1}{x_0}$;
9. if $\forall n\in\mathbb{N}$, $x_n\leq y_n$, then $x_0\leq y_0$;
10. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence such that $\forall n\in\mathbb{N}$, $x_n\leq z_n\leq y_n$, and assume that $x_0=y_0$. Then $\lim_{n\to+\infty} z_n=x_0$.

Proof. See Villanacci, (in progress), Basic Facts on sequences, series and integrals in $\mathbb{R}$, mimeo. ■

Proposition 397 *If $(x_n)_{n\in\mathbb{N}}$ converges to x_0 and $(y_n)_{n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$, then $(y_n)_{n\in\mathbb{N}}$ converges to x_0.*

Proof. By definition of subsequence, there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers, i.e., $1<k_1<k_2<\ ...<k_n<\ ...$, such that $\forall n\in\mathbb{N}$, $y_n=x_{k_n}$.
If $n\to+\infty$, then $k_n\to+\infty$. Moreover, $\forall n$, $\exists k_n$ such that

$$d(x_0,x_{k_n})=d(x_0,y_n)$$

Taking limits of both sides for $n\to+\infty$, we get the desired result. ■

Proposition 398 *A sequence in (X,d) converges at most to one element in X.*

Proof. Assume that $x_n\xrightarrow{n}p$ and $x_n\xrightarrow{n}q$; we want to show that $p=q$. From the Triangle inequality,

$$\forall n\in\mathbb{N}, \qquad 0\leq d(p,q)\leq d(p,x_n)+d(x_n,q) \tag{9.15}$$

Since $d(p,x_n)\to 0$ and $d(x_n,q)\to 0$, Proposition 396.10 and (9.15) imply that $d(p,q)=0$ and therefore $p=q$. ■

Proposition 399 *Given a sequence $(x_n)_{n\in\mathbb{N}}=\left(\left(x_n^i\right)_{i=1}^k\right)_{n\in\mathbb{N}}$ in $\mathbb{R}^k$,*

$$\left\langle (x_n)_{n\in\mathbb{N}}\ \mathbb{R}^k \textit{ converges to } x\right\rangle \Leftrightarrow \left\langle \forall i\in\{1,...,k\}, \left(x_n^i\right)_{n\in\mathbb{N}}\ \mathbb{R} \textit{ converges to } x^i\right\rangle,$$

and

$$\lim_{n\to+\infty} x_n=\left(\lim_{n\to+\infty} x_n^i\right)_{i=1}^n.$$

Proof. [⇒]
Observe that

$$\left|x_n^i - x^i\right| = \sqrt{\left(x_n^i - x^i\right)^2} \leq d\left(x_n x\right).$$

Then, the result follows.
[⇐]
By assumption, $\forall \varepsilon > 0$ and $\forall i \in \{1, ..., k\}$, there exists n_0 such that $\forall n > n_0$, we have $\left|x_n^i - x^i\right| < \frac{\varepsilon}{\sqrt{k}}$. Then $\forall n > n_0$,

$$d\left(x_n x\right) = \left(\sum_{i=1}^{k} \left|x_n^i - x^i\right|^2\right)^{\frac{1}{2}} < \left(\sum_{i=1}^{k} \left|\frac{\varepsilon}{\sqrt{k}}\right|^2\right)^{\frac{1}{2}} = \left(\varepsilon^2 \sum_{i=1}^{k} \frac{1}{k}\right)^{\frac{1}{2}} = \varepsilon.$$

■

Proposition 400 *Suppose that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}^k$ and $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to+\infty} y_n = y_0$. Then*

1. $\lim_{n\to+\infty} (x_n + y_n) = x_0 + y_0$;
2. $\forall c \in \mathbb{R}$, $\lim_{n\to+\infty} c \cdot x_n = c \cdot x_0$;
3. $\lim_{n\to+\infty} x_n \cdot y_n = x_0 \cdot y_0$.

Proof. It follows from Propositions 396 and 399. ■

Example 401 *In Proposition 340 , we have seen that $(\mathcal{B}([0,1]), d_\infty)$ is a metric space. Observe that defined $\forall n \in \mathbb{N}$,*

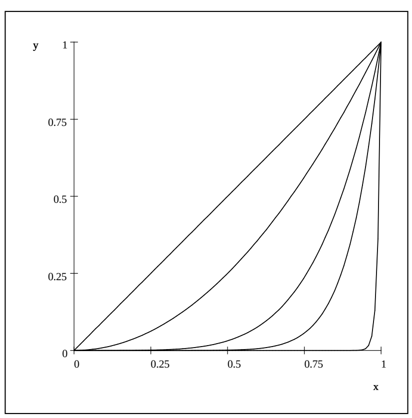
$$f_n : [0,1] \to \mathbb{R}, \qquad t \mapsto t^n,$$

we have that $(f_n)_n \in \mathcal{B}([0,1])^\infty$. Moreover, $\forall \bar{t} \in [0,1]$, $\left(f_n\left(\bar{t}\right)\right)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ and it converges in $(\mathbb{R}, d_2)$. In fact,

$$\lim_{n\to+\infty} \bar{t}^n = \begin{cases} 0 & if \quad \bar{t} \in [0,1) \\ 1 & if \quad \bar{t} = 1. \end{cases}$$

Define

$$f : [0,1] \to \mathbb{R}, \qquad t \mapsto \begin{cases} 0 & if \quad t \in [0,1) \\ 1 & if \quad t = 1. \end{cases}.$$



We want to check that ***it is false that***

$$f_m \underset{(\mathcal{B}([0,1]), d_\infty)}{\rightarrow} f,$$

i.e., it is false that $d_\infty(f_m, f) \underset{m}{\rightarrow} 0$. Then, we have to check

$$\neg \left\langle \ \forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \text{ such that } \forall n > N_\varepsilon,\ d_\infty(f_n, f) < \varepsilon \ \right\rangle,$$

i.e.,

$$\exists \varepsilon > 0 \quad \text{such that } \forall N_\varepsilon \in \mathbb{N},\ \exists n > N_\varepsilon \text{ such that } d_\infty(f_n, f) \geq \varepsilon \ .$$

Then, taken $\varepsilon = \frac{1}{4}$, *it suffice to show that*

$$\left\langle\ \forall m \in \mathbb{N},\ \exists \bar{t} \in (0,1) \text{ such that } \left|f_m(\bar{t}) - f(\bar{t})\right| \geq \tfrac{1}{4}\ \right\rangle.$$

It is then enough to take $\bar{t} = \left(\frac{1}{2}\right)^m$.

Exercise 402 *For any metric space* (X,d) *and* $(x_n)_{n\in\mathbb{N}} \in X^\infty$,

$$\left\langle x_n \underset{(X,d)}{\to} x \right\rangle \Leftrightarrow \left\langle x_n \underset{\left(X,\frac{1}{1+d}\right)}{\to} x \right\rangle.$$

9.4 Sequential characterization of closed sets

Proposition 403 *Let* (X,d) *be a metric space and* $S \subseteq X$.[4]

$$\langle\ S \text{ is closed}\ \rangle \quad \Leftrightarrow \quad \langle\ \text{any } (X,d) \text{ convergent sequence } (x_n)_{n\in\mathbb{N}} \in S^\infty \text{ converges to an element of } S\ \rangle.$$

Proof. We want to show that

$$S \text{ is closed} \Leftrightarrow$$

$$\Leftrightarrow \left\langle \left\langle\ (x_n)_{n\in\mathbb{N}} \text{ is such that } \begin{array}{l} 1.\ \forall n \in \mathbb{N}, x_n \in S, \quad \text{and} \\ 2.\ x_n \to x_0 \end{array} \right\rangle \Rightarrow 3.\ x_0 \in S \right\rangle.$$

[⇒]

We are going to show the contrapositive of the desired statement.[5] Therefore, we assume that there exists a sequence $(x_n)_{n\in\mathbb{N}} \in S^\infty$ such that $x_n \to x_0$ and $x_0 \notin S$. We want to show that S is not closed, i.e., $X \setminus S$ is not open, i.e., there exists $\overline{x} \in X \setminus S$ such that for any $r > 0$, we do have $B(\overline{x}, r) \cap S \neq \varnothing$. Indeed, take $\overline{x} = x_0$. Since $x_n \to x_0$, then for any $r > 0$ there exists $N_r \in \mathbb{N}$ such that for any $n > N_r$, we have $x_n \in B(x_0, r)$. Since for any $n \in \mathbb{N}$, $x_n \in S$, we have that for any $n > N_r$, $x_n \in B(x_0, r) \cap S$, as desired.

[⇐]

Suppose otherwise, i.e., S is not closed. Then, $X \backslash S$ is not open. Then, $\exists\ \overline{x} \in X\backslash S$ such that $\forall n \in \mathbb{N}$, $\exists x_n \in X$ such that $x_n \in B\left(\overline{x}, \frac{1}{n}\right) \cap S$, i.e.,

i. $\overline{x} \in X\backslash S$

ii. $\forall n \in \mathbb{N}$, $x_n \in S$,

iii. $d(x_n, \overline{x}) < \frac{1}{n}$, and therefore $x_n \to \overline{x}$,

and i., ii. and iii. contradict the assumption. ■

Remark 404 *The Appendix to this chapter contains some other characterizations of closed sets and summarizes all the presented characterizations of open and closed sets.*

9.5 Compactness

Definition 405 *Let* (X,d) *be a metric space,* S *a subset of* X, *and* Γ *be a set of arbitrary cardinality. A family* $\mathcal{S} = \{S_\gamma\}_{\gamma\in\Gamma}$ *such that* $\forall \gamma \in \Gamma$, S_γ *is* (X,d) *open, is said to be an open cover of* S *if* $S \subseteq \cup_{\gamma\in\Gamma} S_\gamma$.

A subfamily $\mathcal{S}'$ *of* $\mathcal{S}$ *is called a subcover of* S *if* $S \subseteq \cup_{S'\in\mathcal{S}'} S'$.

Definition 406 *A metric space* (X,d) *is compact if every open cover of* X *has a finite subcover.*

A set $S \subseteq X$ *is compact in* X *if every open cover of* S *has a finite subcover of* S.

Example 407 *Any finite set in any metric space is compact.*

Take $S = \{x_i\}_{i=1}^n$ *in* (X,d) *and an open cover* $\mathcal{S}$ *of* S. *For any* $i \in \{1,...,n\}$, *take an open set in* $\mathcal{S}$ *which contains* x_i; *call it* S_i. *Then* $\mathcal{S}' = \{S_i : i \in \{1,...,n\}\}$ *is the desired open subcover of* $\mathcal{S}$.

[4] Proposition 467 in Appendix 9.8.1 presents a different proof of the result below.

[5] Recall that the contrapositive of the statement "if p, then q" is "if not q, then not p."

Example 408 *1.* $(0,1)$ *is not compact in* $(\mathbb{R}, d_2)$.
We want to show that the following statement is true:

$$\neg \langle \forall \mathcal{S} \text{ such that } \cup_{S\in\mathcal{S}} S \supseteq (0,1)\,,\ \exists \mathcal{S}' \subseteq \mathcal{S} \text{ such that } \#\mathcal{S}' \text{ is finite and } \cup_{S\in\mathcal{S}'} S \supseteq (0,1)\rangle\,,$$

i.e.,

$$\exists \mathcal{S} \text{ such that } \cup_{S\in\mathcal{S}} S \supseteq (0,1) \text{ and } \forall \mathcal{S}' \subseteq \mathcal{S} \text{ either } \#\mathcal{S}' \text{ is infinite or } \cup_{S\in\mathcal{S}'} S \not\supseteq (0,1)\,.$$

Take $\mathcal{S} = \left(\left(\frac{1}{n},1\right)\right)_{n\in\mathbb{N}\backslash\{0,1\}}$ *and* $\mathcal{S}'$ *any finite subcover of* $\mathcal{S}$*.Then there exists a finite set* N *such that* $\mathcal{S}' = \left(\left(\frac{1}{n},1\right)\right)_{n\in N}$*. Take* $n^* = \max\{n\in N\}$*. Then,* $\cup_{S\in\mathcal{S}'} S = \cup_{n\in N}\left(\frac{1}{n},1\right) = \left(\frac{1}{n^*},1\right)$ *and* $\left(\frac{1}{n^*},1\right) \not\supseteq (0,1)$.

2. $(0,1]$ *is not compact in* $((0,+\infty), d_2)$*. Take* $\mathcal{S} = \left(\left(\frac{1}{n},2\right)\right)_{n\in\mathbb{N}}$ *and* $\mathcal{S}'$ *any finite subcover of* $\mathcal{S}$*.Then there exists a finite set* N *such that* $\mathcal{S}' = \left(\left(\frac{1}{n},2\right)\right)_{n\in N}$*. Take* $n^* = \max\{n\in N\}$*. Then,* $\cup_{S\in\mathcal{S}'} S = \cup_{n\in N}\left(\frac{1}{n},2\right) = \left(\frac{1}{n^*},2\right)$ *and* $\left(\frac{1}{n^*},2\right) \not\supseteq (0,1]$.

Proposition 409 *Let* (X,d) *be a metric space.*

$$X \text{ compact and } C \subseteq X \text{ closed } \Rightarrow \langle C \text{ compact}\rangle\,.$$

Proof. Take an open cover $\mathcal{S}$ of C. Then $\mathcal{S} \cup (X\backslash C)$ is an open cover of X. Since X is compact, then there exists an open covers $\mathcal{S}'$of $\mathcal{S} \cup (X\backslash C)$ which cover X. Then $\mathcal{S}'\backslash \{X\backslash C\}$ is a finite subcover of $\mathcal{S}$ which covers C. ■

9.5.1 Compactness and bounded, closed sets

Definition 410 *Let* (X,d) *be a metric space and a nonempty subset* S *of* X*.* S *is bounded in* (X,d) *if* $\exists r \in \mathbb{R}_{++}$ *such that* $\forall x, y \in S$, $d(x,y) < r$.

Proposition 411 *Given a metric space* (X,d) *and a nonempty subset* S *of* X*, then*

$$S \text{ is bounded} \Leftrightarrow \exists r^* \in \mathbb{R}_{++} \text{ and } \exists \overline{z} \in X \text{ such that } S \subseteq B(\overline{z}, r^*).$$

Proof. [$\Rightarrow$] Take $r^* = r$ and an arbitrary point $\overline{z}$ in $S \subseteq X$.
[$\Rightarrow$] Take $x, y \in S$. Then

$$d(x,y) \le d(x,\overline{z}) + d(\overline{z},y) < 2r.$$

Then it is enough to take $r^* = 2r$. ■

Exercise 412 *Show that*
1.given $S \subseteq (X,d)$*, then*

$$S \text{ is bounded} \Leftrightarrow \forall z \in X \ \ \exists r_z \in \mathbb{R}_{++} \text{ such that } S \subseteq B(z, r_z);$$

2. given, $S \subseteq \mathbb{R}^n$*, then*

$$S \text{ is bounded} \Leftrightarrow \exists \underline{x}, \overline{x} \in \mathbb{R}^n \text{ such that for any } x \in S \text{ and for any } i \in \{1,...,n\},\ \underline{x}_i < x_i < \overline{x}_i.$$

Proposition 413 *The finite union of bounded set is bounded.*

Proof. Take $n \in \mathbb{N}$ and $\{S_i\}_{i=1}^n$ such that $\forall i \in \{1,...,n\}$, S_i is bounded. Then, $\forall i \in \{1,...,n\}$, $\exists r \in \mathbb{R}_{++}$ such that $S_i \subseteq B(\overline{z}, r_i)$. Take $r = \max\{r_i\}_{i=1}^n$. Then $\cup_{i=1}^n S_i \subseteq \cup_{i=1}^n B(\overline{z}, r_i) \subseteq B(\overline{z}, r)$. ■

Proposition 414 *Let* (X,d) *be a metric space and* S *a subset of* X*.*

$$S \text{ compact} \ \Rightarrow \ S \text{ bounded.}$$

Proof. If $S = \varnothing$, we are done. Assume then that $S \ne \varnothing$, and take $\overline{x} \in S$ and $\mathcal{B} = \{B(\overline{x}, n)\}_{n\in\mathbb{N}}$. $\mathcal{B}$ is an open cover of X and therefore of S. Then, there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that

$$\mathcal{B}' = \{B(\overline{x}, n_i)\}_{i\in N}\,,$$

where N is a finite set and $\mathcal{B}'$ covers S.
Then takes $n^* = \max_{i\in N} n_i$, we get $S \subseteq B(\overline{x}, n^*)$ as desired. ■

Proposition 415 *Let (X, d) be a metric space and S a subset of X.*

$$S \text{ compact } \Rightarrow S \text{ closed.}$$

Proof. If $S = X$, we are done by Proposition 368. Assume that $S \neq X$: we want to show that $X \backslash S$ is open. Take $y \in S$ and $x \in X \backslash S$. Then, taken $r_y \in \mathbb{R}$ such that

$$0 < r_y < \frac{1}{2} d(x, y),$$

we have

$$B(y, r_y) \cap B(x, r_y) = \varnothing.$$

Now, $\mathcal{S} = \{B(y, r_y) : y \in S\}$ is an open cover of S, and since S is compact, there exists a finite subcover $\mathcal{S}'$ of $\mathcal{S}$ which covers S, say

$$\mathcal{S}' = \{B(y_n, r_n)\}_{n \in N},$$

such that N is a finite set. Take

$$r^* = \min_{n \in N} r_n,$$

and therefore $r^* > 0$. Then $\forall n \in N$,

$$B(y_n, r_n) \cap B(x, r_n) = \varnothing,$$

$$B(y_n, r_n) \cap B(x, r^*) = \varnothing,$$

and

$$(\cup_{n \in \mathbb{N}} B(y_n, r_n)) \cap B(x, r^*) = \varnothing.$$

Since $\{B(y_n, r_n)\}_{n \in N}$ covers S, we then have

$$S \cap B(x, r^*) = \varnothing,$$

or

$$B(x, r^*) \subseteq X \backslash S.$$

Therefore, we have shown that

$$\forall x \in X \backslash S, \exists r^* \in \mathbb{R}_{++} \text{ such that } B(x, r^*) \subseteq X \backslash S,$$

i.e., $X \backslash S$ is open and S is closed. ■

Remark 416 *Summarizing, we have seen that in any metric space*

$$S \text{ compact } \Rightarrow S \text{ bounded and closed.}$$

The opposite implication is false. In fact, the following sets are bounded, closed and not compact.

1. Let the metric space $((0, +\infty), d_2)$. $(0, 1]$ is closed from Remark 384, it is clearly bounded and it is not compact from Example 408.2 .

2. (X, d) where X is an infinite set and d is the discrete metric.

X is closed, from Remark 373 .

X is bounded: take $x \in X$ and $r = 2$.

X is not compact. Take $\mathcal{S} = \{B(x, 1)\}_{x \in X}$. Then $\forall x \in X$ there exists a unique element S_x in $\mathcal{S}$ such that $x \in S_x$. [6]

Remark 417 *In next section we are going to show that if (X, d) is an Euclidean space with the Euclidean distance and $S \subseteq X$, then*

$$S \text{ compact } \Leftarrow S \text{ bounded and closed.}$$

[6] For other examples, see among others, page 155, Ok (2007).

9.5.2 Sequential compactness

Definition 418 *Let a metric space (X,d) be given. $S \subseteq X$ is sequentially compact if every sequence of elements of S has a subsequence which converges to an element of S, i.e.,*

$$\langle (x_n)_{n\in\mathbb{N}} \text{ is a sequence in } S \rangle \Rightarrow \langle \exists \text{ a subsequence } (y_n)_{n\in\mathbb{N}} \text{ of } (x_n)_{n\in\mathbb{N}} \text{ such that } y_n \to x \in S \rangle .$$

In what follows, we want to prove that in metric spaces, compactness is equivalent to sequential compactness. To do that requires some work and the introduction of some, useful in itself, concepts.

Proposition 419 *(Nested intervals) For every $n \in \mathbb{N}$, define $I_n = [a_n, b_n] \subseteq \mathbb{R}$ such that $I_{n+1} \subseteq I_n$. Then $\cap_{n\in\mathbb{N}} I_n \neq \varnothing$.*

Proof. By assumption,

$$a_1 \le a_2 \le ... \le a_n \le ... \tag{9.16}$$

and

$$...b_n \le b_{n-1} \le ... \le b_1 \tag{9.17}$$

Then,

$$\forall m, n \in \mathbb{N}, \quad a_m < b_n$$

simply because, if $m > n$, then $a_m < b_m \le b_n$,where the first inequality follows from the definition of interval I_m and the second one from (9.17), and if $m \le n$, then $a_m \le a_n \le b_n$,where the first inequality follows from (9.16) and the second one from the definition of interval I_n.

Then $A := \{a_n : n \in \mathbb{N}\}$ is nonempty and bounded above by b_n for any n.Then $\sup A := s$ exists.

Since $\forall n \in \mathbb{N}$, b_n is an upper bound for A,

$$\forall n \in \mathbb{N}, \quad s \le b_n$$

and from the definition of sup,

$$\forall n \in \mathbb{N}, \quad a_n \le s$$

Then

$$\forall n \in \mathbb{N}, \quad a_n \le s \le b_n$$

and

$$\forall n \in \mathbb{N}, \quad I_n \neq \varnothing.$$

■

Remark 420 *The statement in the above Proposition is false if instead of taking closed bounded intervals we take either open or unbounded intervals. To see that consider $I_n = \left(0, \frac{1}{n}\right)$ and $I_n = [n, +\infty]$.*

Proposition 421 *(Bolzano- Weirstrass) If $S \subseteq \mathbb{R}^n$ has infinite cardinality and is bounded, then S admits at least an accumulation point, i.e., $D(S) \neq \varnothing$.*

Proof. Step 1. $n = 1$.

Since S is bounded, $\exists a_0, b_0 \in \mathbb{R}$ such that $S \subseteq [a_0, b_0] := B_0$.Divide B_0 in two subinterval of equal length:

$$\left[a_0, \frac{a_0 + b_0}{2}\right] \text{ and } \left[\frac{a_0 + b_0}{2}, b_0\right]$$

Choose an interval which contains an infinite number of points in S. Call $B_1 = [a_1, b_1]$ that interval. Proceed as above for B_1. We therefore obtain a family of intervals

$$B_0 \supseteq B_1 \supseteq .... \supseteq B_n \supseteq ...$$

Observe that *lenght* $B_0 := b_0 - a_0$ and

$$\forall n \in \mathbb{N}, lenght\ B_n = \frac{b_0 - a_0}{2^n}.$$

Therefore, $\forall \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall n > N$, *lenght* $B_n < \varepsilon$.

From Proposition 419, it follows that

$$\exists x \in \cap_{n=0}^{+\infty} B_n$$

We are now left with showing that x is an accumulation point for S

$$\forall r \in \mathbb{R}_{++}, B(x,r) \text{ contains an infinite number of points.}$$

By construction, $\forall n \in \mathbb{N}$, B_n contains an infinite number of points; it is therefore enough to show that

$$\forall r \in \mathbb{R}_{++}, \exists n \in \mathbb{N} \text{ such that } B(x,r) \supseteq B_n.$$

Observe that

$$B(x,r) \supseteq B_n \Leftrightarrow (x-r, x+r) \supseteq [a_n, b_n] \Leftrightarrow x-r < a_n < b_n < x+r \Leftrightarrow \max\{x-a_n, b_n-x\} < r$$

Moreover, since $x \in [a_n, b_n]$,

$$\max\{x-a_n, b_n-x\} < b_n - a_n = \mathit{lenght}\ B_n = \frac{b_0 - a_0}{2^n}$$

Therefore, it suffices to show that

$$\forall r \in \mathbb{R}_{++}, \exists n \in \mathbb{N} \text{ such that } \frac{b_0 - a_0}{2^n} < r$$

i.e., $n \in \mathbb{N}$ and $n > \log_2(b_0 - a_0)$.
Step 2. Omitted (See Ok (2007)). ■

Remark 422 *The above Proposition does not say that there exists an accumulation point which belongs to S. To see that, consider $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.*

Proposition 423 *Let a metric space (X,d) be given and consider the following statements.*

1. *S is compact set;*
2. *Every infinite subset of S has an accumulation point which belongs to S, i.e.,*

$$\langle T \subseteq S \ \wedge \#T \text{ is infinite}\rangle \Rightarrow \langle D(T) \cap S \neq \varnothing\rangle,$$

3. *S is sequentially compact*
4. *S is closed and bounded.*

Then

$$1. \Leftrightarrow 2. \Leftrightarrow 3 \Rightarrow 4.$$

If $X = \mathbb{R}^n$, $d = d_2$, then we also have that

$$3 \Leftarrow 4.$$

More precisely, S is $(\mathbb{R}^n, d_2)$ compact $\Leftrightarrow$ S is $(\mathbb{R}^n, d_2)$ closed and bounded.

Proof. $(1) \Rightarrow (2)$[7]

Take an infinite subset $T \subseteq S$ and suppose otherwise. Then, no point in S is an accumulation point of T, i.e., $\forall x \in S\ \exists r_x > 0$ such that

$$B(x, r_x) \cap T \setminus \{x\} = \varnothing.$$

Then[8]

$$B(x, r_x) \cap T \subseteq \{x\}. \tag{9.19}$$

[7]Proofs of $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are taken from Aliprantis and Burkinshaw (1990), pages 38-39.

[8]In general,

$$A \backslash B = C \ \Rightarrow A \subseteq C \cup B, \tag{9.18}$$

as shown below.

Since

$$S \subseteq \cup_{x \in S} B(x, r_x)$$

and S is compact, $\exists x_1, ..., x_n$ such that

$$S \subseteq \cup_{i=1}^{n} B(x_i, r_i)$$

Then, since $T \subseteq S$,

$$T = S \cap T \subseteq \left(\cup_{i=1}^{n} B(x_i, r_i)\right) \cap T = \cup_{i=1}^{n} \left(B(x_i, r_i) \cap T\right) \subseteq \{x_1, ..., x_n\}$$

where the last inclusion follows from (9.19). But then $\#T \leq n$, a contradiction.

(2) $\Rightarrow$ (3)

Take a sequence $(x_n)_{n \in \mathbb{N}}$ of elements in S.

If $\# \{x_n : n \in \mathbb{N}\}$ is finite, then $\exists x_{n^*}$ such that $x_j = x_{n^*}$ for j in an infinite subset of $\mathbb{N}$, and $(x_{n^*}, ..., x_{n^*}, ...)$ is the required convergent subsequence - converging to $x_{n^*} \in S$.

If $\# \{x_n : n \in \mathbb{N}\}$ is infinite, then there exists a subsequence $(y_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ with an infinite amount distinct values, i.e., such that $\forall n, m \in \mathbb{N}, n \neq m$, we have $y_n \neq y_m$. To construct the subsequence $(y_n)_{n \in \mathbb{N}}$, proceed as follows.

$y_1 = x_1 := x_{k_1}$,
$y_2 = x_{k_2} \notin \{x_{k_1}\}$,
$y_3 = x_{k_3} \notin \{x_{k_1}, x_{k_2}\}$,
...
$y_n = x_{k_n} \notin \left\{x_{k_1}, x_{k_2}, ...x_{k_{n-1}}\right\}$,
...

Since $T := \{y_n : n \in \mathbb{N}\}$ is an infinite subset of S, by assumption it does have an accumulation point x in S; moreover, we can redefine $(y_n)_{n \in \mathbb{N}}$ in order to have $\forall n \in \mathbb{N}$, $y_n \neq x$ [9], as follows. If $\exists k$ such that $y_k = x$, take the (sub)sequence $(y_{k+1}, y_{k+2}, ...) = (y_{k+n})_{n \in \mathbb{N}}$. With some abuse of notation, call still $(y_n)_{n \in \mathbb{N}}$ the sequence so obtained. Now take a further subsequence as follows, using the fact that x is an accumulation point of $\{y_n : n \in \mathbb{N}\} := T$,

$y_{m_1} \in T$ such that $d(y_{m1}, x) < \frac{1}{1}$,
$y_{m_2} \in T$ such that $d(y_{m2}, x) < \min \left\{\frac{1}{2}, (d(y_m, x))_{m \leq m_1}\right\}$,
$y_{m_3} \in T$ such that $d(y_{m3}, x) < \min \left\{\frac{1}{3}, (d(y_m, x))_{m \leq m_2}\right\}$,
...
$y_{m_n} \in T$ such that $d(y_{m_n}, x) < \min \left\{\frac{1}{n}, (d(y_m, x))_{m \leq m_{n-1}}\right\}$,

Observe that since $\forall n, d(y_{m_n}, x) < \min \left\{(d(y_m, x))_{m \leq m_{n-1}}\right\}$, we have that $\forall n$, $m_n > m_{n-1}$ and therefore $(y_{m_n})_{n \in \mathbb{N}}$ is a subsequence of $(y_n)_{n \in \mathbb{N}}$ and therefore of $(x_n)_{n \in \mathbb{N}}$. Finally, since

$$\lim_{n \to +\infty} d(y_{m_n}, x) < \lim_{n \to +\infty} \frac{1}{n} = 0$$

we also have that

$$\lim_{n \to +\infty} y_{m_n} = x$$

Since $A \backslash B = A \cap B^C$, by assumption, we have

$$\left(A \cap B^C\right) \cup B = C \cup B$$

Moreover,

$$\left(A \cap B^C\right) \cup B = (A \cup B) \cap \left(B^C \cup B\right) = A \cup B \supseteq A$$

Observe that the inclusion in (9.18) can be strict, i.e., it can be

$$A \backslash B = C \ \wedge A \subset C \cup B;$$

just take $A = \{1\}, B = \{2\}$ and $C = \{1\}$:

$$A \backslash B = \{1\} = C \ \wedge A = \{1\} \subset C \cup B = \{1, 2\} .;$$

[9] Below we need to have $d(y_n, x) > 0$.

as desired.
(3) $\Rightarrow$ (1)
It is the content of Proposition 432 below.
(1) $\Rightarrow$ (4)
It is the content of Remark 416.
If $X = \mathbb{R}^n$, (4) $\Rightarrow$ (2)
Take an infinite subset $T \subseteq S$. Since S is bounded T is bounded as well. Then from Bolzano-Weirestrass theorem, i.e., Proposition 421, $D(T) \neq \varnothing$. Since $T \subseteq S$, from Proposition 458, $D(T) \subseteq D(S)$ and since S is closed, $D(S) \subseteq S$. Then, summarizing $\varnothing \neq D(T) \subseteq S$ and therefore $D(T) \cap S = D(T) \neq \varnothing$, as desired. ■

To complete the proof of the above Theorem it suffices to show sequential compactness implies compactness, which is done below, and it requires some preliminary results.

Definition 424 *Let (X, d) be a metric space and S a subset of X. S is totally bounded if $\forall \varepsilon > 0, \exists$ a finite set $T \subseteq S$ such that $S \subseteq \cup_{x \in T} B(x, \varepsilon)$.*

Proposition 425 *Let (X, d) be a metric space and S a subset of X.*

$$S \text{ totally bounded} \quad \Rightarrow \quad S \text{ bounded.}$$

Proof. It follows from the definition of totally bounded sets and from Proposition 413. ■

Remark 426 *In the previous Proposition, the opposite implication does not hold true.*

Example 427 *Take (X, d) where X is an infinite set and d is the discrete metric. Then, if $\varepsilon = \frac{1}{2}$, a ball is needed to "take care of each element in X" . Similar situation arises in the following probably more interesting example.*

Example 428 *Consider the metric space $\left(l^2, d_2\right)$ - see Proposition 336. Recall that*

$$l^2 = \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^\infty : \sum_{n=1}^{+\infty} |x_n|^2 < +\infty \right\}$$

and

$$d_2 : l^2 \times l^2 \to \mathbb{R}_+, \qquad \left((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}\right) \mapsto \left(\sum_{n=1}^{+\infty} |x_n - y_n|^2 \right)^{\frac{1}{2}}.$$

Define $e_m = (e_{m,n})_{n \in \mathbb{N}}$ such that

$$e_{m,n} = \begin{cases} 1 & if \quad n = m \\ 0 & if \quad n \neq m, \end{cases}$$

and $S = \{e_m : m \in \mathbb{N}\}$. In other words, $S =$

$$\{(1, 0, 0, ..., 0, ...), (0, 1, 0, ..., 0, ...), (0, 0, 1, ..., 0, ...), ...\}.$$

Observe that $\forall m \in \mathbb{N}$, $\sum_{n=1}^{+\infty} |e_{m,n}|^2 = 1$ and therefore $S \subseteq l^2$. We now want to check that S is bounded, but not totally bounded. The main ingredient of the argument below is that

$$\forall m, p \in \mathbb{N} \text{ such that } m \neq p, \quad d(e_m, e_p) = \sqrt{2}. \tag{9.20}$$

1. S is bounded. For any $m \in \mathbb{N}$, $d(e_1, e_m) = \left(\sum_{n=1}^{+\infty} |e_{1,n}, e_{m,n}|^2 \right)^{\frac{1}{2}} = 2^{\frac{1}{2}}$.

2. S is not totally bounded. We want to show that $\exists \varepsilon > 0$ such that for any finite subset T of S there exists $x \in S$ such that $x \notin \cup_{x \in T} B(x, \varepsilon)$. Take $\varepsilon = 1$and let $T = \{e_k : k \in N\}$ with N arbitrary finite subset of $\mathbb{N}$. Then, for $k' \in \mathbb{N} \backslash N$, $e_{k'} \in S$ and from (9.20), for any $k' \in \mathbb{N} \backslash N$ and $k \in N$, $d(e_k, e_{k'}) = \sqrt{2} > 1$. Therefore, for $k' \in \mathbb{N} \backslash N$, $e_{k'} \notin \cup_{k \in N} B(e_k, 1)$.

Remark 429 *In $(\mathbb{R}^n, d_2)$, S bounded $\Rightarrow$ S totally bounded.*

Lemma 430 *Let (X,d) be a metric space and S a subset of X.*

$$S \text{ sequentially compact} \quad \Rightarrow \quad S \text{ totally bounded.}$$

Proof. Suppose otherwise, i.e., $\exists \varepsilon > 0$ such that for any finite set $T \subseteq S$, $S \nsubseteq \cup_{x\in T} B(x,\varepsilon)$. We are now going to construct a sequence in S which does not admit any convergent subsequence, contradicting sequential compactness.

Take an arbitrary

$$x_1 \in S.$$

Then, by assumption $S \nsubseteq B(x_1,\varepsilon)$. Then take $x_2 \in S \backslash B(x_1,\varepsilon)$, i.e.,

$$x_2 \in S \text{ and } d(x_1,x_2) > \varepsilon.$$

By assumption, $S \nsubseteq B(x_1,\varepsilon) \cup B(x_2,\varepsilon)$. Then, take $x_3 \in S \backslash (B(x_1,\varepsilon) \cup B(x_2,\varepsilon))$, i.e.,

$$x_3 \in S \text{ and for } i \in \{1,2\}, \quad d(x_3,x_i) > \varepsilon.$$

By the axiom of choice, we get that

$$\forall n \in \mathbb{N}, \ \ x_n \in S \text{ and for } i \in \{1,...,n-1\}, \quad d(x_n,x_i) > \varepsilon.$$

Therefore, we have constructed a sequence $(x_n)_{n\in\mathbb{N}} \in S^\infty$ such that

$$\forall i,j \in \mathbb{N}, \text{ if } i \neq j, \text{ then } d(x_i,x_j) > \varepsilon. \tag{9.21}$$

But, then it is easy to check that $(x_n)_{n\in\mathbb{N}}$ does not have any convergent subsequence in S, as verified below. Suppose otherwise, then $(x_n)_{n\in\mathbb{N}}$ would admit a subsequence $(x_m)_{m\in\mathbb{N}} \in S^\infty$ such that $x_m \to x \in S$. But, by definition of convergence, $\exists N \in \mathbb{N}$ such that $\forall m > N$, $d(x_m,x) < \frac{\varepsilon}{2}$, and therefore

$$d(x_m,x_{m+1}) \leq d(x_m,x) + d(x_{m+1},x) < \varepsilon,$$

contradicting (9.21). ■

Lemma 431 *Let (X,d) be a metric space and S a subset of X.*

$$\left\langle \begin{array}{l} S \textit{ sequentially compact} \\ \mathcal{S} \textit{ is an open cover of } S \end{array} \right\rangle \quad \Rightarrow \quad \left\langle \ \exists \varepsilon > 0 \textit{ such that } \forall x \in S, \exists O_x \in \mathcal{S} \textit{ such that } B(x,\varepsilon) \subseteq O_x \ \right\rangle.$$

Proof. Suppose otherwise; then

$$\forall n \in \mathbb{N}_+, \ \exists x_n \in S \text{ such that } \forall O \in \mathcal{S}, \quad B\left(x_n, \frac{1}{n}\right) \nsubseteq O. \tag{9.22}$$

By sequential compactness, the sequence $(x_n)_{n\in\mathbb{N}} \in S^\infty$ admits a subsequence, without loss of generality the sequence itself, $(x_n)_{n\in\mathbb{N}} \in S^\infty$ such that $x_n \to x \in S$. Since $\mathcal{S}$ is an open cover of S, $\exists\ O \in \mathcal{S}$ such that $x \in O$ and, since O is open, $\exists \varepsilon > 0$ such that

$$B(x,\varepsilon) \subseteq O. \tag{9.23}$$

Since $x_n \to x$, $\exists M \in \mathbb{N}$ such that$\{x_{M+i}, i \in \mathbb{N}\} \subseteq B\left(x, \frac{\varepsilon}{2}\right)$. Now, take $n > \max\left\{M, \frac{2}{\varepsilon}\right\}$. Then,

$$B\left(x_n, \frac{1}{n}\right) \subseteq B(x,\varepsilon). \tag{9.24}$$

i.e., $d(y,x_n) < \frac{1}{n} \Rightarrow d(y,x) < \varepsilon$, as shown below.

$$d(y,x) \leq d(y,x_n) + d(x_n,x) < \frac{1}{n} + \frac{\varepsilon}{2} < \varepsilon.$$

From (9.23) and (9.24), we get $B\left(x_n, \frac{1}{n}\right) \subseteq O \in \mathcal{S}$, contradicting (9.22). ■

Proposition 432 *Let (X,d) be a metric space and S a subset of X.*

$$S \text{ sequentially compact} \quad \Rightarrow \quad S \text{ compact.}$$

Proof. Take an open cover $\mathcal{S}$ of S. Since S is sequentially compact, from Lemma 431,

$$\exists \overline{\varepsilon} > 0 \text{ such that } \forall x \in S \ \exists O_x \in \mathcal{S} \text{ such that } B(x, \overline{\varepsilon}) \subseteq O_x.$$

Moreover, from Lemma 430 and the definition of total boundedness, there exists a finite set $T \subseteq S$ such that $S \subseteq \cup_{x\in T} B(x, \overline{\varepsilon}) \subseteq \cup_{x\in T} O_x$. But then $\{O_x : x \in T\}$ is the required subcover of $\mathcal{S}$ which covers S. ■

We conclude our discussion on compactness with some results we hope will clarify the concept of compactness in $\mathbb{R}^n$.

Proposition 433 *Let X be a proper subset of $\mathbb{R}^n$, and C a subset of X.*

C is bounded and $(\mathbb{R}^n, d_2)$ closed
$(\Updownarrow 1)$
C is $(\mathbb{R}^n, d_2)$ compact
$(\Updownarrow 2)$
C is (X, d_2) compact
$\Downarrow$ (*not* $\Uparrow$) 3
C is bounded and (X, d_2) closed

Proof. [1 $\Updownarrow$]
It is the content of (Propositions 414, 415 and last part of) Proposition 423.
To show the other result, observe preliminarily that

$$(X \cap S_\alpha) \cup (X \cap S_\beta) = X \cap (S_\alpha \cup S_\beta)$$

[2 $\Downarrow$]
Take $\mathcal{T} := \{T_\alpha\}_{\alpha\in A}$ such that $\forall \alpha \in A$, T_α is (X,d) open and $C \subseteq \cup_{\alpha\in A} T_\alpha$. From Proposition 385,

$$\forall \alpha \in A, \ \ \exists \ S_\alpha \text{ such that } S_\alpha \text{ is } (\mathbb{R}^n, d_2) \text{ open and } T_\alpha = X \cap S_\alpha.$$

Then

$$C \subseteq \cup_{\alpha\in A} T_\alpha \subseteq \cup_{\alpha\in A} (X \cap S_\alpha) = X \cap (\cup_{\alpha\in A} S_\alpha).$$

We then have that

$$C \subseteq \cup_{\alpha\in A} S_\alpha,$$

i.e., $\mathcal{S} := \{S_\alpha\}_{\alpha\in A}$ is a $(\mathbb{R}^n, d_2)$ open cover of C and since C is $(\mathbb{R}^n, d_2)$ compact, then there exists a finite subcover $\{S_i\}_{i\in N}$ of $\mathcal{S}$ such that

$$C \subseteq \cup_{i\in N} S_i.$$

Since $C \subseteq X$, we then have

$$C \subseteq (\cup_{i\in N} S_i) \cap X = \cup_{i\in N} (S_i \cap X) = \cup_{i\in N} T_i,$$

i.e., $\{T_i\}_{i\in N}$ is a (X,d) open subcover of $\{T_\alpha\}_{\alpha\in A}$ which covers C, as required.
[2 $\Uparrow$]
Take $\mathcal{S} := \{S_\alpha\}_{\alpha\in A}$ such that $\forall \alpha \in A$, S_α is $(\mathbb{R}^n, d_2)$ open and $C \subseteq \cup_{\alpha\in A} S_\alpha$. From Proposition 385,

$$\forall \alpha \in A, \ \ T_\alpha := X \cap S_\alpha \text{ is } (X,d) \text{ open.}$$

Since $C \subseteq X$, we then have

$$C \subseteq (\cup_{\alpha\in A} S_\alpha) \cap X = \cup_{\alpha\in A} (S_\alpha \cap X) = \cup_{\alpha\in A} T_\alpha.$$

Then, by assumption, there exists $\{T_i\}_{i\in N}$ is an open subcover of $\{T_\alpha\}_{\alpha\in A}$ which covers C, and therefore there exists a set N with finite cardinality such that

$$C \subseteq \cup_{i\in N} T_i = \cup_{i\in N} (S_i \cap X) = (\cup_{i\in N} S_i) \cap X \subseteq (\cup_{i\in N} S_i),$$

i.e., $\{S_i\}_{i\in N}$ is a $(\mathbb{R}^n, d_2)$ open subcover of $\{S_\alpha\}_{\alpha\in A}$ which covers C, as required.
[3 $\Downarrow$]
It is the content of Propositions 414, 415.
[3 not $\Uparrow$]
See Remark 416.1. ■

Remark 434 *The proof of part [2 $\Updownarrow$] above can be used to show the following result.*
Given a metric space (X,d), a metric subspace (Y,d) a set $C \subseteq Y$, then

$$\begin{array}{c} C \text{ is } (Y,d) \text{ compact} \\ \Updownarrow \\ C \text{ is } (X,d) \text{ compact} \end{array}$$

In other words, (X',d) compactness of $C \subseteq X' \subseteq X$ is an intrinsic property of C: it does not depend by the subspace X'you are considering. On the other hand, as we have seen, closedness and openness are ***not*** *an intrinsic property of the set.*

Remark 435 *Observe also that to define "anyway" compact sets as closed and bounded sets would not be a good choice. The conclusion of the extreme value theorem (see Theorem 522) would not hold in that case. That theorem basically says that a continuous real valued function on a compact set admits a global maximum. It is not the case that a continuous real valued function on a closed and bounded set admits a global maximum: consider the continuous function*

$$f:(0,1] \to \mathbb{R}, \qquad f(x) = \frac{1}{x}.$$

The set $(0,1]$ is bounded and closed (in $((0,+\infty), d_2)$ and f has no maximum on $(0,1]$.

9.6 Completeness

9.6.1 Cauchy sequences

Definition 436 *Let (X,d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ is a Cauchy sequence if*

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \text{ such that } \forall l,m > N, \quad d(x_l, x_m) < \varepsilon.$$

Proposition 437 *Let a metric space (X,d) and a sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ be given.*

1. *$(x_n)_{n\in\mathbb{N}}$ is convergent $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is Cauchy, but not vice-versa;*
2. *$(x_n)_{n\in\mathbb{N}}$ is Cauchy $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is bounded;*
3. *$(x_n)_{n\in\mathbb{N}}$ is Cauchy and it has a subsequence converging to $x \in X$ $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is convergent to $x \in X$.*

Proof. 1.
[$\Rightarrow$] Since $(x_n)_{n\in\mathbb{N}}$ is convergent, by definition, $\exists x \in X$ such that $x_n \to x$, $\exists N \in \mathbb{N}$ such that $\forall l,m > N$, $d(x,x_l) < \frac{\varepsilon}{2}$ and $d(x,x_m) < \frac{\varepsilon}{2}$. But then $d(x_l,x_m) \le d(x_l,x) + d(x,x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
[$\not\Leftarrow$]
Take $X = (0,1)$, $d =$ absolute value, $(x_n)_{n\in\mathbb{N}} \in (0,1)^\infty$ such that $\forall n \in \mathbb{N}$, $x_n = \frac{1}{n}$.
$(x_n)_{n\in\mathbb{N}}$ is Cauchy:

$$\forall \varepsilon > 0, \quad d\left(\frac{1}{l}, \frac{1}{m}\right) = \left|\frac{1}{l} - \frac{1}{m}\right| < \left|\frac{1}{l}\right| + \left|\frac{1}{m}\right| = \frac{1}{l} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where the last inequality is true if $\frac{1}{l} < \frac{\varepsilon}{2}$ and $\frac{1}{m} < \frac{\varepsilon}{2}$, i.e., if $l > \frac{2}{\varepsilon}$ and $m > \frac{2}{\varepsilon}$. Then, it is enough to take $N > \frac{2}{\varepsilon}$ and $N \in \mathbb{N}$, to get the desired result.
$(x_n)_{n\in\mathbb{N}}$ is not convergent to any point in $(0,1)$:
take any $\overline{x} \in (0,1)$. We want to show that

$$\exists \varepsilon > 0 \text{ such that } \forall N \in \mathbb{N}\ \exists n > N \text{ such that } d(x_n, \overline{x}) > \varepsilon.$$

Take $\varepsilon = \frac{\overline{x}}{2} > 0$ and $\forall N \in \mathbb{N}$, take $n^* \in \mathbb{N}$ such that $\frac{1}{n^*} < \min\left\{\frac{1}{N}, \frac{\overline{x}}{2}\right\}$. Then, $n^* > N$, and

$$\left|\frac{1}{n^*} - \overline{x}\right| = \overline{x} - \frac{1}{n^*} > \overline{x} - \frac{\overline{x}}{2} = \frac{\overline{x}}{2} = \varepsilon.$$

2.

Take $\varepsilon = 1$. Then $\exists N \in \mathbb{N}$ such that $\forall l, m > N, \quad d(x_l, x_m) < 1$. If $N = 1$, we are done. If $N \geq 1$, define

$$r = \max\{1, d(x_1, x_N), ..., d(x_{N-1}, x_N)\}.$$

Then

$$\{x_n : n \in \mathbb{N}\} \subseteq B(x_N, r).$$

3.
Let $(x_{n_k})_{k\in\mathbb{N}}$ be a convergent subsequence to $x \in X$. Then,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x).$$

Since $d(x_n, x_{n_k}) \to 0$, because the sequence is Cauchy, and $d(x_{n_k n}, x) \to 0$, because the subsequence is convergent, the desired result follows. ■

9.6.2 Complete metric spaces

Definition 438 *A metric space (X, d) is complete if every Cauchy sequence is a convergent sequence.*

Remark 439 *If a metric space is complete, to show convergence you do not need to guess the limit of the sequence: it is enough to show that the sequence is Cauchy.*

Example 440 $((0,1),$ *absolute value) is not a complete metric space; it is enough to consider* $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$.

Example 441 *Let (X, d) be a discrete metric space. Then, it is complete. Take a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$. Then, we claim that $\exists N \in \mathbb{N}$ and $\overline{x} \in X$ such that $\forall n > N, \quad x_n = \overline{x}$. Suppose otherwise:*

$$\forall N \in \mathbb{N}, \exists m, m' > N \text{ such that } x_m \neq x_{m'},$$

but then $d(x_m, x_{m'}) = 1$, contradicting the fact that the sequence is Cauchy.

Example 442 *$(\mathbb{Q}, d_2)$ is not a complete metric space. Since $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall x \in \mathbb{R}\backslash\mathbb{Q}$, we can find $(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}^\infty$ such that $x_n \to x$.*

Proposition 443 $\left(\mathbb{R}^k, d_2\right)$ *is complete.*

Proof. 1. $(\mathbb{R}, d_2)$ is complete.
Take a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$. Then, from Proposition 437.2, it is bounded. Then from Bolzano-Weierstrass Theorem (i.e., Proposition 396.4), $(x_n)_{n\in\mathbb{N}}$ does have a convergent subsequence - i.e., $\exists (y_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ which is a subsequence of $(x_n)_{n\in\mathbb{N}}$ and such that $y_n \to a \in \mathbb{R}$. Then from Proposition 437.3.
2. For any $k \geq 2$, $\left(\mathbb{R}^k, d_2\right)$ is complete.
Take a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \in \left(\mathbb{R}^k\right)^\infty$ For $i \in \{1, ..., k\}$, consider $\left(x_n^i\right)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$. Then, for any $n, m \in \mathbb{N}$,

$$\left|x_n^i - x_m^i\right| < \|x_n - x_m\|.$$

Then, $\forall i \in \{1, ..., k\}$, $\left(x_n^i\right)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ is Cauchy and therefore from 1. above, $\forall i \in \{1, ..., k\}$, $\left(x_n^i\right)_{n\in\mathbb{N}}$ is convergent. Finally, from Proposition 399, the desired result follows. ■

Example 444 *For any nonempty set T, $(\mathcal{B}(T), d_\infty)$ is a complete metric space.*
Let $(f_n)_{n\in\mathbb{N}} \in (\mathcal{B}(T))^\infty$ be a Cauchy sequence. For any $\overline{x} \in T$, $(f_n(\overline{x}))_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ is a Cauchy sequence, and since $\mathbb{R}$ is complete, it has a convergent subsequence, without loss of generality, $(f_n(\overline{x}))_{n\in\mathbb{N}}$ itself converging say to $f_{\overline{x}} \in \mathbb{R}$. Define

$$f : T \to \mathbb{R}, \qquad : \overline{x} \mapsto f_{\overline{x}}.$$

We are going to show that (i). $f \in \mathcal{B}(T)$, and (ii) $f_n \to f$.
(i). Since $(f_n)_n$ is Cauchy,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \text{ such that } \forall l, m > N, \quad d_\infty(f_l, f_m) := \sup_{x\in T} |f_l(x) - f_m(x)| < \varepsilon.$$

Then,

$$\forall x \in T, \quad |f_l(x) - f_m(x)| \leq \sup_{x \in T} |f_l(x) - f_m(x)| = d_\infty(f_l, f_m) < \varepsilon. \tag{9.25}$$

Taking limits of both sides of (9.25) *for* $l \to +\infty$, *and using the continuity of the absolute value function, we have that*

$$\forall x \in T, \quad \lim_{l \to +\infty} |f_l(x) - f_m(x)| = |f(x) - f_m(x)| < \varepsilon. \tag{9.26}$$

Since[10]

$$\forall x \in T, \big|\ |f(x)| - |f_m(x)|\ \big| \leq \ |\ f(x) - f_m(x)\ | < \varepsilon,$$

and therefore,

$$\forall x \in T, |f(x)| \leq f_m(x) + \varepsilon.$$

Since $f_l \in \mathcal{B}(T)$, $f \in \mathcal{B}(T)$ *as well.*
(ii) From (9.26), *we also have that*

$$\forall x \in T, |f(x) - f_m(x)| < \varepsilon,$$

and by definition of sup

$$d_\infty(f_m, f) := \sup_{x \in T} |f_m(x) - f(x)| < \varepsilon,$$

i.e., $d_\infty(f_m, f) \to 0$.

For future use, we also show the following result.

Proposition 445

$$\mathcal{BC}(X) := \{f : X \to \mathbb{R} : f \textit{ is bounded and continuous}\}$$

endowed with the metric $d(f, g) = \sup_{x \in X} |f(x) - g(x)|$ *is a complete metric space.*

Proof. See Stokey and Lucas (1989), page 47. ■

9.6.3 Completeness and closedness

Proposition 446 *Let a metric space* (X, d) *and a metric subspace* (Y, d) *of* (X, d) *be given.*

1. Y *complete* $\Rightarrow Y$ *closed;*

2. Y *complete* $\Leftarrow Y$ *closed* <u>*and*</u> X *complete.*

Proof. 1.
Take $(x_n)_{n \in \mathbb{N}} \in Y^\infty$ such that $x_n \to x$. From Proposition 403, it is enough to show that $x \in Y$. Since $(x_n)_{n \in \mathbb{N}}$ is convergent in X, then it is Cauchy. Since Y is complete, by definition, $x_n \to x \in Y$.
2.
Take a Cauchy sequence $(x_n)_{n \in \mathbb{N}} \in Y^\infty$. We want to show that $x_n \to x \in Y$. Since $Y \subseteq X$, $(x_n)_{n \in \mathbb{N}}$ is Cauchy in X, and since X is complete, $x_n \to x \in X$. But since Y is closed, $x \in Y$. ■

Remark 447 *An example of a metric subspace* (Y, d) *of* (X, d) *which is closed and not complete is the following one.* $(X, d) = (\mathbb{R}_{++}, d_2)$, $(Y, d) = ((0, 1], d_2)$ *and* $(x_n)_{n \in \mathbb{N}} = \left(\frac{1}{n}\right)_{n \in \mathbb{N}}$.

Corollary 448 *Let a complete metric space* (X, d) *and a metric subspace* (Y, d) *of* (X, d) *be given. Then,*

$$Y \textit{ complete} \Leftrightarrow Y \textit{ closed.}$$

[10] See, for example, page 37 in Ok (2007).

9.7 Fixed point theorem: contractions

Definition 449 *Let (X,d) be a metric space. A function $\varphi : X \to X$ is said to be a contraction if*

$$\exists k \in (0,1) \text{ such that } \forall x,y \in X, \quad d(\phi(x), \phi(y)) \le k \cdot d(x,y).$$

The inf *of the set of k satisfying the above condition is called contraction coefficient of ϕ.*

Example 450 *1. Given $(\mathbb{R}, d_2)$,*

$$f_\alpha : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \alpha x$$

is a contraction iff $|\alpha| < 1$; in that case $|\alpha|$ is the contraction coefficient of f_α.
2. Let S be a nonempty open subset of $\mathbb{R}$ and $f : S \to S$ a differentiable function. If

$$\sup_{x \in S} |f'(x)| < 1,$$

then f is a contraction.

Definition 451 *For any $f,g \in X \subseteq \mathcal{B}(T)$, we say that $f \le g$ if $\forall x \in T$, $f(x) \le g(x)$.*

Proposition 452 *(Blackwell) Let the following objects be given:*
1. a nonempty set T;
2. X is a nonempty subset of the set $\mathcal{B}(T)$ such that $\forall f \in X$, $\forall \alpha \in \mathbb{R}_+$, $f + \alpha \in X$;
3. $\phi : X \to X$ is increasing, i.e., $f \le g \Rightarrow \phi(f) \le \phi(g)$;
4. $\exists \delta \in (0,1)$ such that $\forall f \in X, \forall \alpha \in \mathbb{R}_+, \phi(f+\alpha) \le \phi(f) + \delta\alpha$.
Then ϕ is a contraction with contraction coefficient δ.

Proof. $\forall f,g \in X$, $\forall x \in T$

$$f(x) - g(x) \le |f(x) - g(x)| \le \sup_{x \in T} |f(x) - g(x)| = d_\infty(f,g).$$

Therefore, $f \le g + d_\infty(f,g)$, and from Assumption 3,

$$\phi(f) \le \phi(g + d_\infty(f,g)).$$

Then, from Assumption 4,

$$\exists \delta \in (0,1) \text{ such that } \phi(g + d_\infty(f,g)) \le \phi(g) + \delta d_\infty(f,g),$$

and therefore

$$\phi(f) \le \phi(g) + \delta d_\infty(f,g). \tag{9.27}$$

Since the argument above is symmetric with respect to f and g, we also have

$$\phi(g) \le \phi(f) + \delta d_\infty(f,g). \tag{9.28}$$

From (9.27) and (9.28) and the definition of absolute value, we have

$$|\phi(f) - \phi(g)| \le \delta d_\infty(f,g),$$

as desired. ■

Proposition 453 *(Banach fixed point theorem) Let (X,d) be a complete metric space. If $\phi : X \to X$ is a contraction with coefficient k, then*

$$\exists!\ x^* \in X \text{ such that } x^* = \phi(x^*). \tag{9.29}$$

and

$$\forall x_0 \in X \text{ and } \forall n \in \mathbb{N}, \qquad d(\phi^n(x_0), x^*) \le k^n \cdot d(x_0, x^*), \tag{9.30}$$

where $\phi^n := (\overset{1}{\phi} \circ \overset{2}{\phi} \circ .... \circ \overset{n}{\phi})$.
Proof. *(9.29) holds true.*

Take any $x_0 \in X$ and define the sequence

$$(x_n)_{n\in\mathbb{N}} \in X^\infty, \text{ with } \forall n \in \mathbb{N}, x_{n+1} = \phi(x_n).$$

We want to show that 1. that $(x_n)_{n\in\mathbb{N}}$ is Cauchy, 2. its limit is a fixed point for ϕ, and 3. that fixed point is unique.

1. First of all observe that

$$\forall n \in \mathbb{N}, d(x_{n+1}, x_n) \leq k^n d(x_1, x_0), \tag{9.31}$$

where k is the contraction coefficient of ϕ, as shown by induction below.

Step 1: $\mathcal{P}(1)$ is true:

$$d(x_2, x_1) = d(\phi(x_1), \phi(x_0)) \leq kd(x_1, x_0)$$

from the definition of the chosen sequence and the assumption that ϕ is a contraction.

Step 2. $\mathcal{P}(n-1) \Rightarrow \mathcal{P}(n)$:

$$d(x_{n+1}, x_n) = d(\phi(x_n), \phi(x_{n-1})) \leq kd(x_n, x_{n-1}) \leq k^n d(x_1, x_0)$$

from the definition of the chosen sequence, the assumption that ϕ is a contraction and the assumption of the induction step.

Now, for any $m, l \in \mathbb{N}$ with $m > l$,

$$d(x_m, x_l) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{l+1}, x_l) \leq$$

$$\leq \left(k^{m-1} + k^{m-2} + ... + k^l\right) d(x_1, x_0) \leq k^l \tfrac{1-k^{m-l}}{1-k} d(x_1, x_0),$$

where the first inequality follows from the triangle inequality, the third one from the following computation[11]*:*

$$k^{m-1} + k^{m-2} + ... + k^l = k^l\left(1 + k + ... + k^{m-l+1}\right) = k^l \frac{1-k^{m-l}}{1-k}.$$

Finally, since $k \in (0,1)$,we get

$$d(x_m, x_l) \leq \frac{k^l}{1-k} d(x_1, x_0). \tag{9.32}$$

If $x_1 = x_0$, then for any $m, l \in \mathbb{N}$ with $m > l, d(x_m, x_l) = 0$ and $\forall n \in \mathbb{N}$, $x_n = x_0$ and the sequence is converging and therefore it is Cauchy. Therefore, consider the case $x_1 \neq x_0$.From (9.32) it follows that $(x_n)_{n\in\mathbb{N}} \in X^\infty$ is Cauchy: $\forall \varepsilon > 0$ choose $N \in \mathbb{N}$ such that $\frac{k^N}{1-k} d(x_1, x_0) < \varepsilon$, i.e., $k^N < \frac{\varepsilon(1-k)}{d(x_1,x_0)}$ and $N > \frac{\log \frac{\varepsilon(1-k)}{d(x_1,x_0)}}{\log \kappa}$.

2. Since (X, d) is a complete metric space, $(x_n)_{n\in\mathbb{N}} \in X^\infty$ does converge say to $x^* \in X$, and, in fact, we want to show that $\phi(x^*) = x^*$. Then, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n > N$,

$$d(\phi(x^*), x^*) \leq d(\phi(x^*), x_{n+1}) + d\left(x^{n+1}, x^*\right) \leq$$

$$\leq d(\phi(x^*), \phi(x_n)) + d\left(x^{n+1}, x^*\right) \leq kd(x^*, x_n) + d\left(x^{n+1}, x^*\right) \leq$$

$$\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

---

[11] We are also using the basic facat used to study geometrical series. Define

$$s_n \equiv 1 + a + a^2 + ... + a^n;$$

:

Multiply both sides of the above equality by$(1-a)$:

$$(1-a)s_n \equiv (1-a)\left(1 + a + a^2 + ... + a^n\right)$$

$$(1-a)s_n \equiv \left(1 + a + a^2 + ... + a^n\right) - \left(a + a^2 + ... + a^{n+1}\right) = 1 - a^{n+1}$$

Divide both sides by $(1-a)$:

$$s_n \equiv \left(1 + a + a^2 + ... + a^n\right) - \left(a + a^2 + ... + a^{n+1}\right) = \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a} - \frac{a^{n+1}}{1-a}$$

where the first equality comes from the triangle inequality, the second one from the construction of the sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$*, the third one from the assumption that* ϕ *is a contraction and the last one from the fact that* $(x_n)_{n\in\mathbb{N}}$ *converges to* x^**. Since* ε *is arbitrary,* $d(\phi(x^*), x^*) = 0$*, as desired.*

3. Suppose that $\widehat{x}$ *is another fixed point for* ϕ *- beside* x^**. Then,*

$$d(\widehat{x}, x^*) = d(\phi(\widehat{x}), \phi(x^*)) \leq kd(\widehat{x}, x^*)$$

and assuming $\widehat{x} \neq x^*$ *would imply* $1 \leq k$*, a contradiction of the fact that* ϕ *is a contraction with contraction coefficient* k*.*

(9.30) *hods true.*

We show the claim by induction on $n \in \mathbb{N}$*.*

$\mathcal{P}(1)$ *is true.*

$$d(\phi(x_0), x^*) = d(\phi(x_0), \phi(x^*)) \leq k \cdot d(x_0, x^*),$$

where the equality follows from the fact that x^* *is a fixed point for* ϕ*, and the inequality by the fact that* ϕ *is a contraction.*

$\mathcal{P}(n-1)$ *is true implies that* $\mathcal{P}(n)$ *is true.*

$$d(\phi^n(x_0), x^*) = d(\phi^n(x_0), \phi(x^*)) = d\left(\phi\left(\phi^{n-1}(x_0)\right), \phi(x^*)\right) \leq$$

$$\leq k \cdot d\left(\phi^{n-1}(x_0), x^*\right) \leq k \cdot k^{n-1} \cdot d(x_0, x^*) = k^n \cdot d(x_0, x^*).$$

■

9.8 Appendices.

9.8.1 Some characterizations of open and closed sets

Remark 454 *From basic set theory, we have* $A^C \cap B = \varnothing \Leftrightarrow B \subseteq A$*, as verified below.*

$$\neg\left\langle \exists x : x \in A^C \wedge x \in B \right\rangle = \langle \forall x : x \in A \vee \neg(x \in B)\rangle =$$

$$= \langle \forall x : \neg(x \in B) \vee x \in A\rangle \stackrel{(*)}{=} \langle \forall x : x \in B \Rightarrow x \in A\rangle,$$

where $(*)$ *follows from the fact that* $\langle p \Rightarrow q\rangle = \langle(\neg p) \vee q\rangle$*.*

Proposition 455 S *is open* $\Leftrightarrow S \cap \mathcal{F}(S) = \varnothing$*.*

Proof. [⇒]

Suppose otherwise, i.e., $\exists x \in S \cap \mathcal{F}(S)$. Since $x \in \mathcal{F}(S)$, $\forall r \in \mathbb{R}_{++}$, $B(x,r) \cap S^C \neq \varnothing$. Then, from Remark 454, $\forall r \in \mathbb{R}_{++}$, it is false that $B(x,r) \subseteq S$, contradicting the assumption that S is open.

[⇐]

Suppose otherwise, i.e., $\exists x \in S$ such that

$$\forall r \in \mathbb{R}_{++}, B(x,r) \cap S^C \neq \varnothing \tag{9.33}$$

Moreover

$$x \in B(x,r) \cap S \neq \varnothing \tag{9.34}$$

But (9.33) and (9.34) imply $x \in \mathcal{F}(S)$. Since $x \in S$, we would have $S \cap \mathcal{F}(S) \neq \varnothing$, contradicting the assumption. ■

Proposition 456 S *is closed* $\Leftrightarrow \mathcal{F}(S) \subseteq S$*.*

Proof.

$$S \text{ closed} \Leftrightarrow S^C \text{ open} \stackrel{(1)}{\Leftrightarrow} S^C \cap \mathcal{F}\left(S^C\right) = \varnothing \stackrel{(2)}{\Leftrightarrow} S^C \cap \mathcal{F}(S) = \varnothing \stackrel{(3)}{\Leftrightarrow} \mathcal{F}(S) \subseteq S$$

where

(1) follows from Proposition 455;

(2) follows from Remark 376

(3) follows Remark 454. ■

Proposition 457 *S is closed* $\Leftrightarrow D(S) \subseteq S$.

Proof. We are going to use Proposition 456, i.e., S is closed $\Leftrightarrow \mathcal{F}(S) \subseteq S$.
[$\Rightarrow$]
Suppose otherwise, i.e.,

$$\exists x \notin S \text{ such that } \forall r \in \mathbb{R}_{++},\ (S \setminus \{x\}) \cap B(x,r) \neq \varnothing$$

and since $x \notin S$, it is also true that

$$\forall r \in \mathbb{R}_{++},\ S \cap B(x,r) \neq \varnothing \tag{9.35}$$

and

$$\forall r \in \mathbb{R}_{++},\ S^C \cap B(x,r) \neq \varnothing \tag{9.36}$$

From (9.35) and (9.36), it follows that $x \in \mathcal{F}(S)$, while $x \notin S$, which from Proposition 456 contradicts the assumption that S is closed.
[$\Leftarrow$]
Suppose otherwise, i.e., using Proposition 456,

$$\exists x \in \mathcal{F}(S) \text{ such that } x \notin S$$

Then, by definition of $\mathcal{F}(S)$,

$$\forall r \in \mathbb{R}_{++}, \quad B(x,r) \cap S \neq \varnothing.$$

Since $x \notin S$, we also have

$$\forall r \in \mathbb{R}_{++}, \quad B(x,r) \cap (S \setminus \{x\}) \neq \varnothing,$$

i.e., $x \in D(S)$ and $x \notin S$, a contradiction. ■

Proposition 458 $\forall S, T \subseteq X,\ S \subseteq T \Rightarrow D(S) \subseteq D(T)$.

Proof. Take $x \in D(S)$. Then

$$\forall r \in \mathbb{R}_{++}, \quad (S \setminus \{x\}) \cap B(x,r) \neq \varnothing. \tag{9.37}$$

Since $S \subseteq T$, we also have

$$(T \setminus \{x\}) \cap B(x,r) \supseteq (S \setminus \{x\}) \cap B(x,r). \tag{9.38}$$

From (9.37) and (9.38), we get $x \in D(T)$. ■

Proposition 459 $S \cup D(S)$ *is a closed set.*

Proof. Take $x \in (S \cup D(S))^C$ i.e., $x \notin S$ and $x \notin D(S)$. We want to show that

$$\exists r \in \mathbb{R}_{++} \text{ such that } B(x,r) \cap (S \cup D(S)) = \varnothing,$$

i.e.,

$$\exists r \in \mathbb{R}_{++} \text{ such that } (B(x,r) \cap S) \cup (B(x,r) \cap D(S)) = \varnothing,$$

Since $x \notin D(S)$, $\exists r \in \mathbb{R}_{++}$ such that $B(x,r) \cap (S \setminus \{x\}) = \varnothing$. Since $x \notin S$, we also have that

$$\exists r \in \mathbb{R}_{++} \text{ such that } S \cap B(x,r) = \varnothing. \tag{9.39}$$

We are then left with showing that $B(x,r) \cap D(S) = \varnothing$. If $y \in B(x,r)$, then from (9.39), $y \notin S$ and $B(x,r) \cap S \setminus \{y\} = \varnothing$, i.e., $y \notin D(S)$,i.e., $B(x,r) \cap D(S) = \varnothing$, as desired. ■

Proposition 460 $\mathrm{Cl}\ (S) = S \cup D(S)$.

Proof. [⊇]
Since

$$S \subseteq \text{Cl }(S) \tag{9.40}$$

from Proposition 458,

$$D(S) \subseteq D(\text{Cl }(S)). \tag{9.41}$$

Since Cl (S) is closed, from Proposition 457,

$$D(\text{Cl }(S)) \subseteq \text{Cl }(S) \tag{9.42}$$

From (9.40) , and (9.41) , (9.42), we get

$$S \cup D(S) \subseteq \text{Cl }(S)$$

[⊆]
Since, from Proposition 459, $S \cup D(S)$ is closed and contains S, then by definition of Cl (S),

$$\text{Cl }(S) \subseteq S \cup D(S).$$

■

To proceed in our analysis, we need the following result.

Lemma 461 *For any metric space (X, d) and any $S \subseteq X$, we have that*
1. $X = \text{Int } S \cup \mathcal{F}(S) \cup \text{Int } S^C$, *and*
2. $(\text{Int } S \cup \mathcal{F}(S))^C = \text{Int } S^C$.

Proof. If either $S = \varnothing$ or $S = X$, the results are trivial. Otherwise, observe that either $x \in S$ or $x \in X \setminus S$.
1. If $x \in S$,then
either $\exists r \in \mathbb{R}_{++}$ such that $B(x, r) \subseteq S$ and then $x \in \text{Int } S$,
or $\forall r \in \mathbb{R}_{++}$, $B(x, r) \cap S^C \neq \varnothing$ and then $x \in \mathcal{F}(S)$.
Similarly, if $x \in X \setminus S$,then
either $\exists r' \in \mathbb{R}_{++}$ such that $B(x, r') \subseteq X \setminus S$ and then $x \in \text{Int } (X \setminus S)$,
or $\forall r' \in \mathbb{R}_{++}$, $B(x, r') \cap S \neq \varnothing$ and then $x \in \mathcal{F}(S)$.
2. By definition of Interior and Boundary of a set, $(\text{Int } S \cup \mathcal{F}(S)) \cap \text{Int } S^C = \varnothing$.
Now, for arbitrary sets $A, B \subseteq X$ such that $A \cup B = X$ and $A \cap B = \varnothing$, we have what follow:
$A \cup B = X \Leftrightarrow (A \cup B)^C = X^C \Leftrightarrow A^C \cap B^C = \varnothing$, and from Remark 454, $B^C \subseteq A$;
$A \cap B = \varnothing \Leftrightarrow A \cap \left(B^C\right)^C = \varnothing \Rightarrow A \subseteq B^C$.
Therefore we can the desired result. ■

Proposition 462 Cl $(S) = \text{Int } S \cup \mathcal{F}(S)$.

Proof. From Lemma 461, it is enough to show that

$$(\text{Cl }(S))^C = \text{Int } S^C.$$

[⊇]
Take $x \in \text{Int } S^C$. Then, $\exists r \in \mathbb{R}_{++}$ such that $B(X, r) \subseteq S^C$ and therefore $B(x, r) \cap S = \varnothing$ and, since $x \notin S$,

$$B(x, r) \cap (S \setminus \{x\}) = \varnothing.$$

Then $x \notin S$ and $x \notin D(S)$, i.e.,

$$x \notin S \cup D(S) = \text{Cl }(S)$$

where last equality follows from Proposition 460. In other words, $x \in (\text{Cl }(S))^C$.
[⊆]
Take $x \in (\text{Cl }(S))^C = (D(S) \cup S)^C$. Since $x \notin D(S)$,

$$\exists r \in \mathbb{R}_{++} \text{ such that } \quad (S \setminus \{x\}) \cap B(x, r) = \varnothing \tag{9.43}$$

Since $x \notin S$,

$$\exists r \in \mathbb{R}_{++} \text{ such that } \; S \cap B(x,r) = \varnothing \tag{9.44}$$

i.e.,

$$\exists r \in \mathbb{R}_{++} \text{ such that } B(x,r) \subseteq S^C \tag{9.45}$$

and $x \in \text{Int } S^C$. ■

Definition 463 *$x \in X$ is an adherent point for S if $\forall r \in \mathbb{R}_{++}, B(x,r) \cap S \neq \varnothing$ and*

$$Ad(S) := \{x \in X : \forall r \in \mathbb{R}_{++}, B(x,r) \cap S \neq \varnothing\}$$

Corollary 464 *1.* Cl $(S) = Ad(S)$*.*
2. A set S is closed $\Leftrightarrow Ad(S) = S$.

Proof. 1.
[$\subseteq$]
$x \in \text{Cl }(S) \Rightarrow \langle x \in IntS$ or $\mathcal{F}(S)\rangle$ and in both cases the desired conclusion is insured.
[$\supseteq$]
If $x \in S$, then, by definition of closure, $x \in \text{Cl }(S)$. If $x \notin S$,then $S = S \backslash \{x\}$ and, from the assumption, $\forall r \in \mathbb{R}_{++}, B(x,r) \cap (S \backslash \{x\}) \neq \varnothing$, i.e., $x \in D(S)$ which is contained in Cl $(S)$from Proposition 460.
2. It follows from 1. above and Proposition 379.2. ■

Proposition 465 *Let $S \subseteq \mathbb{R}^n$ be given. Then, $\mathcal{F}(S) = \text{Cl}(S) \setminus Int(S)$.*

Proof. We want to show

$$\langle \; \forall r \in \mathbb{R}_{++}, B(x,r) \cap S \neq \varnothing \text{ and } B(x,r) \cap S^c \neq \varnothing \; \rangle \Leftrightarrow \langle \; x \in \text{Cl}(S) \backslash IntS \; \rangle \, .$$

From Definition **??** and **??**, $x \in \text{Cl}(S) \backslash \text{Int}(S)$ iff $\forall r \in \mathbb{R}_{++}, B(x,r) \cap S \neq \varnothing$ and $\neg(\exists \delta > 0$ such that $B(x,\delta) \subseteq S)$, i.e., $\forall \delta > 0$,

$$B(x,\delta) \cap (\mathbb{R}^n \backslash S) = B(x,\delta) \cap \mathbb{R}^n \cap S^C = B(x,\delta) \cap S^c \neq \varnothing.$$

Therefore, $x \in \text{Cl}(S) \backslash \text{Int}(S)$ iff $\forall r \in \mathbb{R}_{++}, B(x,r) \cap S \neq \varnothing$ and $B(x,\delta) \cap S^C \neq \varnothing$. ■

Proposition 466 *$x \in$ Cl $(S) \Leftrightarrow \exists (x_n)_{n \in \mathbb{N}}$ in S converging to x.*

Proof. [$\Rightarrow$]
From Corollary 464, if $x \in \text{Cl }(S)$ then $\forall n \in \mathbb{N}$, we can take $x_n \in B\left(x, \frac{1}{n}\right) \cap S$. Then $d(x, x_n) < \frac{1}{n}$ and $\lim_{n \to +\infty} d(x, x_n) = 0$.
[$\Leftarrow$]
By definition of convergence,

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ such that } \forall n > n_\varepsilon, \;\; d(x_n, x) < \varepsilon \text{ or } x_n \in B(x, \varepsilon)$$

or

$$\forall \varepsilon > 0, \;\, B(x,\varepsilon) \cap S \supseteq \{x_n : n > n_\varepsilon\}$$

and

$$\forall \varepsilon > 0, \;\, B(x,\varepsilon) \cap S \neq \varnothing$$

i.e., $x \in Ad(S)$, and from the Corollary 464.1, the desired result follows. ■

Proposition 467 *S is closed $\Leftrightarrow$ any convergent sequence $(x_n)_{n \in \mathbb{N}}$ with elements in S converges to an element of S.*

Proof. We are going to show that S is closed using Proposition 457, i.e., S is closed $\Leftrightarrow D(S) \subseteq S$. We want to show that

$$\langle D(S) \subseteq S \rangle \Leftrightarrow \left\langle \left\langle \; (x_n)_{n \in \mathbb{N}} \text{ is such that } \begin{array}{l} 1. \; \forall n \in \mathbb{N}, x_n \in S, \quad \text{and} \\ 2. \; x_n \to x_0 \end{array} \right\rangle \Rightarrow x_0 \in S \right\rangle,$$

[$\Rightarrow$]

Suppose otherwise, i.e., there exists $(x_n)_{n\in\mathbb{N}}$ such that 1. $\forall n \in \mathbb{N}, x_n \in S$. and 2.$x_n \to x_0$, but $x_0 \notin S$.
By definition of convergent sequence, we have

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, \quad d(x_n, x_0) < \varepsilon$$

and, since $\forall n \in \mathbb{N}, x_n \in S$,

$$\{x_n : n > n_0\} \subseteq B(x_0, \varepsilon) \cap (S \setminus \{x_0\})$$

Then,

$$\forall \varepsilon > 0, B(x_0, \varepsilon) \cap (S \setminus \{x_0\}) \neq \varnothing$$

and therefore $x_0 \in D(S)$ while $x_0 \notin S$, contradicting the fact that S is closed.
[$\Leftarrow$]
Suppose otherwise, i.e., $\exists\ x_0 \in D(S)$ and $x_0 \notin S$. We are going to construct a convergent sequence $(x_n)_{n\in\mathbb{N}}$ with elements in S which converges to x_0 (a point not belonging to S).
From the definition of accumulation point,

$$\forall n \in \mathbb{N}, \quad (S \setminus \{x_0\}) \cap B\left(x, \frac{1}{n}\right) \neq \varnothing.$$

Then, we can take $x_n \in (S \setminus \{x\}) \cap B\left(x, \frac{1}{n}\right)$, and since $d(x_n, x_0) < \frac{1}{n}$, we have that $d(x_n, x_0) \to 0$. ■
Summarizing, the following statements are equivalent:

1. S is open (i.e., $S \subseteq \text{Int } S$)
2. S^C is closed,
3. $S \cap \mathcal{F}(S) = \varnothing$,

and the following statements are equivalent:

1. S is closed,
2. S^C is open,
3. $\mathcal{F}(S) \subseteq S$,
4. $S = \text{Cl}(S)$.
5. $D(S) \subseteq S$,
6. $Ad(S) = S$,
7. any convergent sequence $(x_n)_{n\in\mathbb{N}}$ with elements in S converges to an element of S.

9.8.2 Norms and metrics

In these notes, a field K can be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 468 *A norm on a vector space E on a field $K \in \{\mathbb{R}, \mathbb{C}\}$ is a function*

$$||\cdot|| : E \to \mathbb{R}, \quad x \mapsto ||x||$$

which satisfies the following properties: $\forall x, y \in E, \forall \lambda \in K$,

1. $||x|| \geq 0$ *(non negativity),*
2. $||x + y|| \leq ||x|| + ||y||$ *(triangle inequality),*
3. $||\lambda x|| = |\lambda| \cdot ||x||$ *(homogeneity),* [12]
4. $||x|| = 0 \Rightarrow x = 0$ *(separation).*

[12] If $\lambda \in \mathbb{R}$, $|\lambda|$ is the absolute value of λ; if $\lambda = (a, b) \in \mathbb{C}$, then $|\lambda| = \sqrt{a^2 + b^2}$.

Proposition 469 *Given a norm* $||\cdot||$ *on* E, $\forall x, y \in E$

1. $x = 0 \Rightarrow ||x|| = 0$
2. $||x - y|| = ||y - x||$
3. $|(||x|| - ||y||)| \leq ||x - y||$.

Proof.

1. Since E is a vector space, $\forall x \in E$, $0x = 0$. Then
$$||0|| = ||0x|| \stackrel{(a)}{=} |0| \cdot ||x|| \stackrel{(b)}{=} 0||x|| = 0$$
where (a) follows from property 3 of norm and (b) from the definition of absolute value.

2. $||x-y|| = ||-y+x|| = {}^{13}||-y-(-x)|| \stackrel{(c)}{=} ||(-1)y+(-1)(-x)|| = ||(-1)(y-x)|| = |-1|\cdot||y-x|| = ||y-x||$ where (c) follows from Proposition 136.

3. From the definition of absolute value, we want to show that
$$-||x - y|| \leq ||x|| - ||y|| \leq ||x - y||.$$
Indeed,
$$||x|| = ||x - y + y|| \leq ||x - y|| + ||y||,$$
i.e., $||x - y|| \geq ||x|| - ||y||$, and
$$||y|| = ||y - x + x|| \leq ||y - x|| + ||x|| = ||x - y|| + ||x||,$$
i.e., $-||x - y|| \leq ||x|| - ||y||$, as desired.

■

Proposition 470 *If properties 2. and 3. in Definition 468 hold true, then property 1. in the same definition holds true.*

Proof. We want to show that

$$\langle ||x + y|| \leq ||x|| + ||y|| \wedge ||\lambda x|| = |\lambda| \cdot ||x|| \rangle$$
$$\Rightarrow \langle \forall x \in E, ||x|| \geq 0 \rangle$$

Observe that if $x = 0$, from Proposition 469.2 (which uses only property 3 of Definition 468) we have $||x|| = 0$. Then

$$\begin{aligned} ||0|| &\leq ||x - x|| \leq ||x|| + ||-x|| \text{ and therefore} \\ ||x|| &\geq -||x|| = -|-1| \cdot ||x|| = -||x||. \end{aligned}$$

Now, if $||x|| < 0$, we would have a negative number strictly larger than a positive number, which is a contradiction. ■

Definition 471 *The pair* $(E, ||\cdot||)$, *where* E *is a vector space and* $||\cdot||$ *is a norm, is called a normed vector space.*

Remark 472 *Normed spaces are, by definition, vector space.*

Definition 473 *A seminorm is a function satisfying properties 1, 2 and 3 in Definition 468.*

Definition 474 *Given a non-empty set* X, *a function* $d : X \times X \to \mathbb{R}$ *is called a metric or a distance on* X *if* $\forall x, y, z \in X$,

[13] $\forall v \in V$ $(-1)v = -v$ and $-(-v) = (-1)((-1)v) = ((-1)(-1))v = 1 \cdot v = v$.

1. $d(x, y) \geq 0$ *(non negativity),*
2. $d(x, y) = 0 \Leftrightarrow x = y$ *(coincidence),*
3. $d(x, y) = d(y, x)$ *(symmetry),*
4. $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality),*

(X, d) *is called a metric space.*

Definition 475 *Given a normed vector space* $(E, ||\cdot||)$*, the metric*

$$d : E^2 \to \mathbb{R}, (x, y) \mapsto ||x - y||$$

is called the metric induced by the norm $||\cdot||$*.*

Proposition 476 *Given a normed vector space* $(E, ||\cdot||)$*,*

$$d : E \times E \to \mathbb{R}, \quad (x, y) \mapsto ||x - y||$$

is a metric and (E, d) *is a metric space.*

Proof.

1. It follows from the fact that $x, y \in E \Rightarrow x - y \in E$ and property 1 of the norm.
2. It follows from property 1 of the norm and Proposition 469.1.
3. It follows from Proposition 469.2.
4. $d(x, z) = ||x - z|| = ||(x - y) + (y - z)|| \leq ||x - y|| + ||y - z|| = d(x, y) + d(y, z).$

■

Proposition 477 *If* $||\cdot||$ *is a norm on a vector space* E *and*

$$d : E \times E \to \mathbb{R}, \quad (x, y) \mapsto ||x - y||$$

then $\forall x, y, z \in E, \forall \lambda \in K$

a. $d(x, 0) = ||x||$
b. $d(x + z, y + z) = d(x, y)$ *(translation invariance)*
c. $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ *(homogeneity).*

Proof.

a. $d(x, 0) = ||x - 0|| = ||x||$
b. $d(x + z, y + z) = ||(x + z) - (y + z)|| = ||x - y|| = d(x, y)$
c. $d(\lambda x, \lambda y) = ||\lambda x - \lambda y|| = ||\lambda(x - y)|| = |\lambda| \cdot ||x - y|| = |\lambda| d(x, y).$

■

Proposition 478 *Let* (E, d) *be a metric space such that* d *satisfies translation invariance and homogeneity. Then*

$$n : E \to \mathbb{R}, \quad x \mapsto d(x, 0)$$

is a norm and $\forall x, y \in E$*,* $n(y - x) = d(x, y)$*.*

Proof.

1. $n(x) = d(x, 0) \geq 0$, where the inequality follows from property 1 in Definition 4,

2.

$$n(x+y) = d(x+y,0) \overset{(a)}{=} d(x+y-y,0-y) = d(x,-y) \overset{(b)}{\leq} d(x,0)+d(0,-y) =$$
$$\overset{(c)}{=} d(x,0)+d(-y,0) \overset{(d)}{=} d(0,y)+d(0,x) = n(y)+n(x),$$

where (a) follows from translation invariance, (b) from triangle inequality in Definition 474, (c) from symmetry in Definition 474 and (d) from homogeneity.

3.

$$n(\lambda x) = d(\lambda x, 0) = |\lambda| d(x,0) = |\lambda| n(x),$$

4.

$$n(x) = 0 \Rightarrow d(x,0) = 0 \Rightarrow x = 0.$$

It follows that

$$n(y-x) = d(y-x,0) = d(y-x+x, 0+x) = d(y,x) = d(x,y).$$

■

Remark 479 *The above Proposition suggests that the following statement is* ***false:***

Given a metric space (E,d), then $n_d : E \to \mathbb{R}, : x \mapsto d(x,0)$ is a norm on E.

The fact that the above statement is false is verified below. Take an arbitrary vector space E with the discrete metric d,

$$d : E \times E \to \mathbb{R}, \quad d(x,y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

First of all, let's verify that d does not satisfy (translation invariance and homogeneity), otherwise from Proposition 478, we would contradict the desired result. Indeed homogeneity fails.

Take $x \neq y$ and $\lambda = 2$ then

$$d(x,y) = 1 \quad \text{and} \quad d(\lambda x, \lambda y) = 1$$
$$|\lambda| d(x,y) = 2 \neq 1.$$

Let's now show that in the case of the discrete metric

$$n : E \to \mathbb{R}, \quad x \mapsto d(x,0)$$

is not a norm. Take $x \neq 0$ and $\lambda = 2$ then

$$||\lambda x|| = d(\lambda x, 0) = 1$$
$$|\lambda| d(x,0) = 2.$$

9.8.3 Distance between sets, diameters and "topological separation".

Definition 480 *Let (X,d) be a metric space. The distance between a point $x \in X$ and a non-empty subset A of X is denoted and defined by*

$$d(x,A) = \inf\{d(x,a) : a \in A\}$$

The distance between two non-empty subsets A and B of X is denoted and defined by

$$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$$

Definition 481 *The* diameter *of a non-empty subset A of X is denoted and defined by*

$$d(A) = \sup\{d(a_1,a_2) : a_1, a_2 \in A\}$$

If the diameter of A is finite, then A is said to be bounded, otherwise A is said to be unbounded.

Proposition 482 *Let A and B be non-empty subsets of a metric space (X,d) and let $x \in X$. Then*

1. *$d(x,A)$ and $d(A,B)$ are non-negative real numbers;*
2. *If $x \in A$, then $d(x,A) = 0$;*
3. *If $A \cap B \neq \varnothing$, then $d(A,B) = 0$;*
4. *If A is finite, then $d(A) < \infty$, i.e., A is bounded.*

Proof. 1.It follows directly from Definition 327.
2. Since $x \in A$, we have that

$$0 \overset{\text{Def 327}}{=} d(x,x) \overset{\text{Def 480}}{\in} \{d(x,a) : a \in A\} \subseteq \mathbb{R}_+$$

3.

$$A \cap B \neq \varnothing \Rightarrow \exists : x \text{ such that } x \in A \text{ and } x \in B$$

But, $d(x,x) = 0$, therefore we can proceed like we did in point 2 above.
4. Since A is finite, $d(A) = sup\{d(a_1,a_2) : a_1, a_2 \in A\}$ is the max of a finite set and therefore it is finite.
■

Remark 483 *The converses of the statements 2, 3 and 4 in Proposition 482 do not hold true, as verified below.*

1. *$d(x,A) = 0 \nRightarrow x \in A$:*
Consider the euclidean topology on $\mathbb{R}$ and take $A = (0,1)$. We have that $0 \notin A$, while $d(0,A) = 0$.

2. *$d(A,B) = 0 \nRightarrow A \cap B \neq \varnothing$:*
Consider the euclidean topology on $\mathbb{R}$ and take $A = (0,1)$ and $B = [1,2)$. We have that $A \cap B = \varnothing$, while $d(A,B) = 0$.

3. *$d(A) < \infty \nRightarrow A$ is finite:*
Consider the euclidean topology on $\mathbb{R}$ and take $A = [0,1]$. A is not finite, but we have that $d(A) = 1$.

Remark 484 *For the empty set, the following conventions are adopted:*

1. *$d(x,\varnothing) = +\infty$;*
2. *$d(A,\varnothing) = d(\varnothing,A) = +\infty$;*
3. *$d(\varnothing) = -\infty$.*

Proposition 485 *Let $x \in X$ with (X,d) metric space. The closure of a subset A of X is equal to the set of points whose distance from A is zero, i.e.,*

$$\text{Cl}(A) = \{x : d(x,A) = 0\}$$

Proof. From Proposition 368 we know that $\text{Cl}(A) = A \cup D(A)$. We need to show that the points of A and $D(A)$ have distance zero from A. First, from Proposition 482 point 2, we have that $\forall x \in A, d(x,A) = 0$. Now consider $D(A)$

$$x \in D(A) \overset{\text{Def accumulation point}}{\Leftrightarrow} \forall\ G \text{ open, such that } x \in G, (G \setminus \{x\}) \cap A \neq \varnothing \Rightarrow$$

$$\overset{(X,d)\text{ metric space}}{\Rightarrow} \forall r > 0, (B_{(X,d)}(x,r) \setminus \{x\}) \cap A \neq \varnothing$$

Now assume that $\exists\ x_0 \in D(A)$ such that $d(x_0,A) = \varepsilon \neq 0$. But then,

$$B_{(X,d)}\left(x_0, \frac{\varepsilon}{2}\right) \cap A = \varnothing.$$

Therefore x_0 is not an accumulation point for A. It follows that $x \in D(A) \Rightarrow d(x,A) = 0$. In conclusion, since all the points of A and $D(A)$ have distance zero from A, we have that the points of $\text{Cl}(A) = A \cup D(A)$ have distance 0 from A. ■

Remark 486 *From the previous Proposition, in a metric space (X,d), $A \subseteq X$ is closed if $A = \{x \in X : d(x,A) = 0\}$.*

Corollary 487 *In a metric space (X,d), all finite sets are closed.*

Proof. From the definition of distance - see Definition 327 - since $d(x,y)=0 \Leftrightarrow x=y$, we have that the only point with zero distance from a singleton set $\{x\} \subseteq X$ is the point x itself. Hence by the above proposition we have that, in a metric space, singleton sets are closed. But since by Corollary **??** any finite union of closed sets is closed, we have that finite sets are closed as well. ■

Corollary 488 *Let (X,d) be a metric space. Let A be an (X,d)-closed set, then $x \notin A \Rightarrow d(x,a) > 0$.*

Proof.

$$d(x,A) \neq 0 \Rightarrow x \notin \mathrm{Cl}(A) \overset{\mathrm{Cl}(A)=A}{\Rightarrow} x \notin A.$$

■

Proposition 489 *(A topological separation property) Let (X,d) be a metric space and A,B closed disjoint subsets of X. Then there exist open disjoint sets G and H such that $A \subseteq G$ and $B \subseteq H$.*

Proof. If either A or B is empty, we can take G and H equal to $\varnothing$ and X. Assume now that A and B are non-empty. Take $a \in A$ and $b \in B$. Since A and B are disjoint, i.e., $A \cap B = \varnothing$,

$$\langle a \in A \text{ and } a \notin B \rangle \quad \text{and} \quad \langle b \notin A \text{ and } b \in B \rangle.$$

From the Corollary 488, $\exists\, \delta_a > 0$ and $\delta_b > 0$ such that $d(a,B) = \delta_a$ and $d(b,A) = \delta_b$. For any $a \in A$ and any $b \in B$, define,

$$S_a = B\left(a, \frac{1}{3}\delta_a\right) \quad \text{and} \quad S_b = B\left(b, \frac{1}{3}\delta_b\right).$$

We want to show that our desired sets G and H are indeed

$$G = \bigcup_{a \in A} S_a \quad \text{and} \quad H = \bigcup_{b \in B} S_b.$$

1. G and H are clearly open
2. Clearly, $A \subseteq G$ and $B \subseteq H$
3. We are left with showing that $G \cap H = \varnothing$. Suppose otherwise, i.e., $\exists x \in G \cap H$. Then by definition of G and H,

$$\exists a_x \in A \text{ and } b_x \in B \text{ such that } x \in B\left(a_x, \frac{1}{3}\delta_a\right) \cap B\left(b_x, \frac{1}{3}\delta_b\right). \tag{9.46}$$

Since $a_x \in A$ and $b_x \in B$ and A and B are disjoint we have that $d(a_x, b_x) = \varepsilon > 0$. Then, by definition of distance between a point and a set,

$$d(a_x, B) := \delta_a \leq \varepsilon \text{ and } d(b_x, A) := \delta_b \leq \varepsilon. \tag{9.47}$$

Moreover, from (9.46),

$$d(a_x, x) < \frac{1}{3}\delta_a, \qquad d(b_x, x) < \frac{1}{3}\delta_b \tag{9.48}$$

Then

$$\varepsilon := d(a_x, b_x) \leq d(a_x, x) + d(b_x, x) \overset{(9.48)}{<} \frac{1}{3}\delta_a + \frac{1}{3}\delta_b \overset{(9.47)}{\leq} \frac{2}{3}\varepsilon.$$

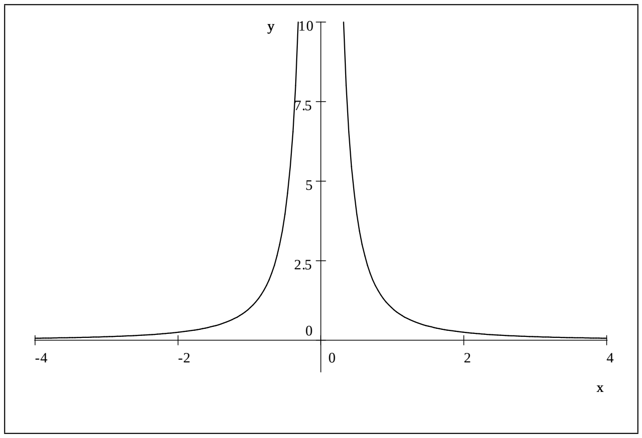
i.e., $0 < \varepsilon < \frac{2}{3}\varepsilon$, a contradiction.

■

Example 490 *Consider the metric space $(\mathbb{R}^2, d)$ and the following two subsets of $\mathbb{R}^2$:*

$$A = \{(x,y) : xy \geq -1,\ x < 0\}, \qquad B = \{(x,y) : xy \geq 1,\ x > 0\}.$$

A and B are both closed and disjoint. However, $d(A,B) = 0$.



Lemma 491 *Let (X, d) be a metric space and $S \subseteq X$. Then*

$$d(S) = d(\mathrm{Cl}(S)).$$

Proof. By applying the definition of diameter, we want to prove that , defined

$$A := \{d(s_1, s_2) : s_1, s_2 \in S\} \text{ and } B := \{d(s_1, s_2) : s_1, s_2 \in Cl(S)\} := \sup B,$$

$$\sup A = \sup B.$$

[$\leq$] It follows from the fact that, since $S \subseteq \mathrm{Cl}(S)$, $A \subseteq B$.
[$\geq$] From Proposition 485 we have that $\mathrm{Cl}(S) = \{x : d(x, S) = 0\}$. Therefore $\forall\, x \in \mathcal{F}(S)$, $\exists\, s_* \in S$ such that $d(x, s_*) = 0$[14]. Taken $s_1, s_2 \in$Cl(S), since $\mathrm{Cl}(S) = \mathcal{F}(S) \cup (S)$, we have three possible cases.

- Case 1: $s_1, s_2 \in S$. In this case $d(s_1, s_2) \in A$.
- Case 2: $s_1 \in \mathcal{F}(S)$ and $s_2 \in S$. From Proposition 485 mentioned above, it is possible to find $s_3 \in S$ such that $d(s_1, s_3) = 0$. Hence, from the triangle inequality, $d(s_1, s_2) \leq d(s_1, s_3) + d(s_3, s_2) = d(s_3, s_2) \in A$.
- Case 3: $s_1, s_2 \in \mathcal{F}(S)$. Just like we did for Case 2, we can find $s_3, s_4 \in S$ such that $d(s_1, s_3) = 0$ and $d(s_2, s_4) = 0$. Therefore $d(s_1, s_2) \leq d(s_1, s_3) + d(s_3, s_2) \leq d(s_3, s_4) + d(s_4, s_2) = d(s_3, s_4) \in A$.

In all cases we have found that for every element in B it is possible to find an element in A which is greater or equal, therefore $\sup A \geq \sup B$, as desired. ■

Definition 492 *Let A be a nonempty set in $\mathbb{R}^n$. Then, a function $f : A \to \mathbb{R}^m$ is said to satisfy a Lipschitz condition on A, or simply to be Lipschitz continuous on A, if*

$$\exists s \in \mathbb{R} \text{ such that } \forall x, y \in A, \quad \|f(x) - f(y)\| \leq s\|x - y\|.$$

Example 493 *The norm function satisfies a Lipschitz condition. In fact,*

$$\|\|x\| - \|y\|\| = |\|x\| - \|y\|| \overset{(1)}{\leq} \|x - y\|,$$

*where (1) follows from Proposition **??**.3: indeed, the norm function satisfies the Lipschitz condition with $s = 1$.*

Proposition 494 *Let A be a nonempty set in $\mathbb{R}^n$, and let the function $f : A \to \mathbb{R}^m$ satisfy a Lipschitz condition on A. Then, f is continuous on A.*

[14] Remember that, from Proposition **??**, $\mathrm{Cl}(S) = Int(S) \cup \mathcal{F}(S)$. Since $Int(S) \subseteq S \subseteq$Cl$(S)$,

$$\mathcal{F}(S) \cup S = \mathrm{Cl}(S).$$

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points of A convergent to $a \in A$, i.e., such that

$$\lim_{n\to+\infty} \|a_n - a\| = 0. \tag{9.49}$$

Since $\exists s \in \mathbb{R}$ such that $\|f(a_n) - f(a)\| \leq s\|a_n - a\|$, from (9.49),

$$\lim_{n\to+\infty} \|f(a_n) - f(a)\| = 0,$$

whence the sequence $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(a)$. Since this is true $\forall a \in A$, f is continuous on A. ■

Proposition 495 *The distance function is Lipschitz continuous and therefore it is continuous on $\mathbb{R}^n$.*

Proof. 1st proof. Let $x, y \in \mathbb{R}^n$ and A be a nonempty set in $\mathbb{R}^n$. Then, by definition of distance as inf, $\forall \varepsilon > 0\ \exists a \in A$ such that

$$\|y - a\| < d_A(y) + \varepsilon. \tag{9.50}$$

Now,

$$d_A(x) \overset{\text{Def. ??}}{\leq} \|x - a\| = \|x - y + y - a\| \leq \|x - y\| + \|y - a\| \overset{(9.50)}{<} \|x - y\| + d_A(y) + \varepsilon,$$

whence, since $\varepsilon > 0$ is arbitrary,

$$d_A(x) \leq \|x - y\| + d_A(y),$$

or

$$d_A(x) - d_A(y) \leq \|x - y\|.$$

Since the above argument is perfectly symmetric with respect to x and y, we can repeat it interchanging x with y in order to get

$$d_A(y) - d_A(x) \leq \|y - x\|,$$

and then, as desired,

$$|d_A(x) - d_A(y)| \leq \|x - y\|.$$

2. (Nguyen (2016)).
Take $x, y \in A$. Then, for any $u \in A$, we have

$$d(x, A) \leq \|x - u\| \leq \|x - y\| + \|y - u\|.$$

Taking inf of both sides of the above inequalities with respect to u, we get

$$d(x, A) \leq \|x - u\| \leq \|x - y\| + \inf\{\|y - u\| : u \in A\} \overset{\text{Def. } d(\ldots,A)}{=} \|x - y\| + d(y, A),$$

or

$$d(x, A) - d(y, A) \leq \|x - y\|. \tag{9.51}$$

Repeating the above argument interchanging x with y, we get

$$d(y, A) - d(x, A) \leq \|x - y\|,$$

and therefore

$$d(x, A) - d(y, A) \geq -\|x - y\|. \tag{9.52}$$

(9.51) and (9.52) are the desired result. ■

9.9 Exercises

Problem sets: 3,4,5,6.

From Lipschutz (1965), starting from page 54: 1, 18, 19, 20, 23, 28 (observe that Lipschutz uses the word "range" in the place of "image");

starting from page 120: 1, 3, 6, 7, 25, 29.

Chapter 10

Functions

10.1 Limits of functions

In what follows we take for given metric spaces (X, d) and (X', d') and sets $S \subseteq X$ and $T \subseteq X'$.

Definition 496 *Given $x_0 \in D(S)$, i.e., given an accumulation point x_0 for S, and $f : S \to T$, we write*

$$\lim_{x \to x_0} f(x) = l \in X'$$

if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in \left(B_{(X,d)}(x_0, \delta) \cap S\right) \setminus \{x_0\} \Rightarrow f(x) \in B_{(X',d')}(l, \varepsilon)$$

or

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that} \langle\, x \in S \quad \wedge \quad 0 < d(x, x_0) < \delta \rangle \Rightarrow d'(f(x), l) < \varepsilon$$

Proposition 497 *Given $x_0 \in D(S)$ and $f : S \to T$,*

$$\langle \lim_{x \to x_0} f(x) = l \rangle$$
$$\Leftrightarrow$$
$$\left\langle \begin{array}{l} \text{for any sequence } (x_n)_{n\in\mathbb{N}} \text{ in } S \text{ such that } \forall n \in \mathbb{N},\ \ x_n \neq x_0 \text{ and } \lim_{n\to+\infty} x_n = x_0, \\ \lim_{n\to+\infty} f(x_n) = l. \end{array} \right\rangle$$

Proof. `for the following proof see also Proposition 6.2.4, page 123 in Morris.`
[⇒]
Take

a sequence $(x_n)_{n\in\mathbb{N}}$ in S such that $\forall n \in \mathbb{N}$, $x_n \neq x_0$ and $\lim_{n\to+\infty} x_n = x_0$.

We want to show that $\lim_{n\to+\infty} f(x_n) = l$, i.e.,

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, \quad d(f(x_n), l) < \varepsilon.$$

Since $\lim_{x\to x_0} f(x) = l$,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in S \quad \wedge \quad 0 < d(x, x_0) < \delta \Rightarrow d(f(x), l) < \varepsilon.$$

Since $\lim_{n\to+\infty} x_n = x$

$$\forall \delta > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, \quad 0 \overset{(*)}{<} d(x_n, x_0) < \delta,$$

where $(*)$ follows from the fact that $\forall n \in \mathbb{N}$, $x_n \neq x_0$.
Therefore, combining the above results, we get

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0, d(f(x_n), l) < \varepsilon$$

as desired.

[$\Leftarrow$]
Suppose otherwise, then

$$\begin{aligned} &\exists \varepsilon > 0 \text{ such that } \forall \delta_n = \tfrac{1}{n}, \text{ i.e., } \forall\ n \in \mathbb{N},\ \exists x_n \in S \text{ such that} \\ &x_n \in S\ \wedge 0 < d(x_n, x_0) < \tfrac{1}{n} \text{and } d(f(x), l) \geq \varepsilon. \end{aligned} \tag{10.1}$$

Consider $(x_n)_{n \in \mathbb{N}}$; then, from the above and from Proposition 394, $x_n \to x_0$, and from the above (specifically the fact that $0 < d(x_n, x_0)$), we also have that $\forall n \in \mathbb{N},\quad x_n \neq x_0$. Then by assumption, $\lim_{n \to +\infty} f(x_n) = l$, i.e., by definition of limit,

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that if } n > N, \text{ then } |f(x_n - l)| < \varepsilon,$$

contradicting (10.1). ■

Proposition 498 *(uniqueness) Given $x_0 \in D(S)$ and $f : S \to T$,*

$$\left\langle \lim_{x \to x_0} f(x) = l_1 \ \textit{and}\ \lim_{x \to x_0} f(x) = l_2 \right\rangle \Rightarrow \langle l_1 = l_2 \rangle$$

Proof. It follows from Proposition 497 and Proposition 398. ■

Proposition 499 *Given $S \subseteq X$, $x_0 \in D(S)$ and $f, g : S \to \mathbb{R}$, , and*

$$\lim_{x \to x_0} f(x) = l \ \textit{and}\ \lim_{x \to x_0} g(x) = m$$

1. $\lim_{x \to x_0} f(x) + g(x) = l + m$;

2. $\lim_{x \to x_0} f(x) \cdot g(x) = l \cdot m$;

3. if $m \neq 0$ and $\forall x \in S,\ g(x) \neq 0$, $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{l}{m}$.

Proof. It follows from Proposition 497 and Proposition 396. ■

Remark 500 *Given $f : \mathbb{R}^2 \to \mathbb{R}$, it is false that*
if $\lim_{x \to x_0} f(x, y_0) = f(x_0, y_0)$ and $\lim_{y \to y_0} f(x_0, y) = f(x_0, y_0)$, then $\lim_{(x,y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$.
A desired counterexample is presented below. Let the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \textit{if} \quad (x, y) \neq 0, \\ 0 & \textit{if} \quad (x, y) = 0, \end{cases}$$

be given. Then,
1.

$$\lim_{x \to 0} f(x, 0) = \lim_{x \to 0} 0 = 0 = f(0, 0), \qquad \lim_{y \to 0} f(0, y) = \lim_{y \to 0} 0 = 0 = f(0, 0);$$

2. We want to show
$\neg(\forall \varepsilon > 0, \exists \delta > 0$ *such that* $(x, y) \in B(0, \delta) \Rightarrow |f(x, y)| < \varepsilon)$, *i.e.,*
$\exists\ \varepsilon > 0$ *such that* $\forall \delta > 0\ \exists (\overline{x}, \overline{y}) \in \mathbb{R}^2$ *such that* $\|(\overline{x}, \overline{y})\| < \delta$ *and* $f(\overline{x}, \overline{y}) \geq \varepsilon$.
Take $\varepsilon = \frac{1}{4}$ *and* $(\overline{x}, \overline{y}) = (\overline{x}, \overline{x}) \in B(0, \delta) \ \setminus\ \{0\}$. *Then* $f(\overline{x}, \overline{x}) = \frac{(\overline{x})^2}{(\overline{x})^2 + (\overline{x})^2} = \frac{1}{2} > \frac{1}{4}$, *as desired.*

Proposition 501 *Let $f : S \subseteq X \to \mathbb{R}^m$, $x_0 \in D(S)$ and for any $j \in \{1, ..., m\}$, $f_j : S \to \mathbb{R}$ be such that $\forall x \in S$,*

$$f(x) = (f_j(x))_{j=1}^m$$

Then, if either $\lim_{x \to x_0} f(x) = l \in \mathbb{R}^m$ or for any $j \in \{1, ..., m\}$, $\lim_{x \to x_0} f_j(x) = l_j \in \mathbb{R}$, then

$$\lim_{x \to x_0} f(x) = \left(\lim_{x \to x_0} f_j(x) \right)_{j=1}^m.$$

Proof. It follows from Propositions 497 and 399. ■

10.2 Continuous Functions

Definition 502 *Given a metric space (X,d) and a set $V \subseteq X$, an open neighborhood of V is an open set containing V.*

Remark 503 *Sometimes, an open neighborhood is simply called a neighborhood.*

Definition 504 *Take $S \subseteq (X,d), T \subseteq (Y,d')$, $x_0 \in S$ and $f : S \to T$. Then, f is $(X,d)-(Y,d')$ continuous at x_0 if*

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in \left(B_{(X.d)}(x_0,\delta) \cap S\right) \Rightarrow f(x) \in B_{(X',d')}(f(x_0), \varepsilon),$$

i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in S \ \wedge d(x,x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \varepsilon,$$

i.e.,

$$\forall \varepsilon > 0, \exists \delta > 0, \ f\left(B_{(X,d)}(x_0,\delta) \cap S\right) \subseteq B_{(X',d')}(f(x_0), \varepsilon),$$

i.e.,

for any open neighborhood V of $f(x_0)$,
there exists an open neighborhood U of x_0 such that $f(U \cap S) \subseteq V$.

If f is continuous at x_0 for every x_0 in S, f is continuous on S.

Remark 505 *If x_0 is an isolated point of S, f is continuos at x_0. If x_0 is an accumulation point for S, f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.*

Proposition 506 *Suppose that $Z \subseteq X''$, where (X''', d'') is a metric space and*

$$f : S \to T, \ g : W \supseteq f(S) \to Z$$

$$h : S \to Z, \ h(x) = g(f(x))$$

If f is continuous at $x_0 \in S$ and g is continuous at $f(x_0)$, then h is continuous at x_0.

Proof. Exercise (see Apostol (1974), page 79) or Ok, page 206. ■

Proposition 507 *Take $f, g : S \subseteq \mathbb{R}^n \to \mathbb{R}$. If f and g are continuous, then*

1. *$f + g$ is continuous;*
2. *$f \cdot g$ is continuous;*
3. *if $\forall x \in S$, $g(x) \neq 0$, $\frac{f}{g}$ is continuous.*

Proof. If x_0 is an isolated point of S, from Remark 505, we are done. If x_0 is an accumulation point for S, the result follows from Remark 505 and Proposition 499. ■

Proposition 508 *Let $f : S \subseteq X \to \mathbb{R}^m$, and for any $j \in \{1, ..., m\}$ $f_j : S \to \mathbb{R}$ be such that $\forall x \in S$,*

$$f(x) = (f_j(x))_{j=1}^m$$

Then,

$$\langle f \text{ is continuous} \rangle \Leftrightarrow \langle \forall j \in \{1, ..., m\}, \ f_j \text{ is continuous} \rangle$$

Proof. The proof follows the strategy used in Proposition 399. ■

Definition 509 *Given for any $i \in \{1, ..., n\}$, $S_i \subseteq \mathbb{R}$, $f : \times_{i=1}^n S_i \to \mathbb{R}$ is continuous in each variable separately if $\forall i \in \{1, ..., n\}$ and $\forall x_i^0 \in S_i$,*

$$f_{x_i^0} : \times_{k \neq i} S_k \to \mathbb{R}, \qquad f_{x_i^0}\left((x_k)_{k \neq i}\right) = f\left(x_1, .., x_{i-1}, x_i^0, x_{i+1}, ..., x_n\right)$$

is continuous.

Proposition 510 *Given for any* $i \in \{1, ..., n\}$,

$$f : \times_{i=1}^{n} S_i \to \mathbb{R} \text{ is continuous} \Rightarrow f \text{ is continuous in each variable separately}$$

Proof. Exercise. ■

Remark 511 *It is false that*

$$f \text{ is continuous in each variable separately} \Rightarrow f \text{ is continuous}$$

To see that consider $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & if \quad (x,y) \neq 0 \\ \\ 0 & if \quad (x,y) = 0 \end{cases}$$

The following Proposition is useful to show continuity of functions using the results about continuity of functions from $\mathbb{R}$ to $\mathbb{R}$.

Proposition 512 *For any* $k \in \{1, ..., n\}$, *take* $S_k \subseteq X$, *and define* $S := \times_{k=1}^{n} S_k \subseteq X^n$. *Moreover, take* $i \in \{1, ..., n\}$ *and let*

$$g : S_i \to Y, \qquad : x_i \mapsto g(x_i)$$

be a continuous function and

$$f : S \to Y, \qquad (x_k)_{k=1}^{n} \mapsto g(x_i).$$

Then f *is continuous.*

Example 513 *An example of the objects described in the above Proposition is the following one.*

$$g : [0, \pi] \to \mathbb{R}, \quad g(x) = \sin x,$$

$$f : [0, \pi] \times [-\pi, 0] \to \mathbb{R}, \quad f(x,y) = \sin x.$$

Proof. of Proposition 512. We want to show that

$$\forall x_0 \in S, \forall \varepsilon > 0 \;\; \exists \delta > 0 \text{ such that } d(x, x_0) < \delta \wedge x \in S \Rightarrow d(f(x), f(x_0)) < \varepsilon$$

We know that

$$\forall x_{i0} \in S_i, \forall \varepsilon > 0 \;\; \exists \delta' > 0 \text{ such that } d(x_i, x_{i0}) < \delta' \wedge x_i \in S \Rightarrow d(g(x_i), g(x_{i0})) < \varepsilon$$

Take $\delta = \delta'$. Then $d(x, x_0) < \delta \wedge x \in S \Rightarrow d(x_i, x_{i0}) < \delta' \wedge x_i \in S$ and $\varepsilon > d(g(x_i), g(x_{i0})) = d(f(x), f(x_0))$, as desired. ■

Exercise 514 *Show that the following function is continuous.*

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x_1 x_2) = \begin{pmatrix} e^{x_1} + \cos(x_1 \cdot x_2) \\ \frac{\sin^2 x_1}{e^{x_2}} \\ x_1 + x_2 \end{pmatrix}$$

From Proposition 508, it suffices to show that each component function is continuous. We are going to show that $f_1 : \mathbb{R}^2 \to \mathbb{R}$,

$$f_1(x_1 x_2) = e^{x_1} + \cos(x_1 \cdot x_2)$$

is continuous, leaving the proof of the continuity of the other component functions to the reader.

1. $f_{11} : \mathbb{R}^2 \to \mathbb{R}$, $f_{11}(x_1, x_2) = e^{x_1}$ *is continuous from Proposition 512 and "Calculus 1";*

2. $h_1 : \mathbb{R}^2 \to \mathbb{R}$, $h_1(x_1, x_2) = x_1$ *is continuous from Proposition 512 and "Calculus 1",*

$h_2 : \mathbb{R}^2 \to \mathbb{R}$, $h_2(x_1, x_2) = x_2$ *is continuous from Proposition 512 and "Calculus 1",*

$g : \mathbb{R}^2 \to \mathbb{R}$, $g(x_1, x_2) = h_1(x_1, x_2) \cdot h_2(x_1, x_2) = x_1 \cdot x_2$ *is continuous from Proposition 507.2,*

$\phi : \mathbb{R} \to \mathbb{R}$, $\phi(x) = \cos x$ *is continuous from "Calculus 1",*

$f_{12} : \mathbb{R}^2 \to \mathbb{R}$, $f_{12}(x_1, x_2) = (\phi \circ g)(x_1, x_2) = \cos(x_1 \cdot x_2)$ *is continuous from Proposition 506 (continuity of composition).*

3. $f_1 = f_{11} + f_{12}$ *is continuous from Proposition 507.1.*

The following Proposition is useful in the proofs of several results.

Proposition 515 *Let S, T be arbitrary sets, $f : S \to T$, $\{A_i\}_{i=1}^n$ a family of subsets of S and $\{B_i\}_{i=1}^n$ a family of subsets of T. Then*

1. " inverse image preserves inclusions, unions, intersections and set differences", i.e.,

   a. $B_1 \subseteq B_2 \Rightarrow f^{-1}(B)_1 \subseteq f\left({}^{-1}B_2\right)$,
   b. $f^{-1}\left(\cup_{i=1}^n B_i\right) = \cup_{i=1}^n f^{-1}(B_i)$,
   c. $f^{-1}\left(\cap_{i=1}^n B_i\right) = \cap_{i=1}^n f^{-1}(B_i)$,
   d. $f^{-1}(B_1 \backslash B_2) = f^{-1}(B_1) \backslash f^{-1}(B_2)$,

2. " image preserves inclusions, unions, only", i.e.,

   e. $A_1 \subseteq A_2 \Rightarrow f(A)_1 \subseteq f(A_2)$,
   f. $f\left(\cup_{i=1}^n A_i\right) = \cup_{i=1}^n f(A_i)$,
   g. $f\left(\cap_{i=1}^n A_i\right) \subseteq \cap_{i=1}^n f(A_i)$, and
   if f is one-to-one, then $f\left(\cap_{i=1}^n A_i\right) = \cap_{i=1}^n f(A_i)$,
   h. $f(A_1 \backslash A_2) \supseteq f(A_1) \backslash f(A_2)$,and
   if f is one-to-one and onto, then $f(A_1 \backslash A_2) = f(A_1) \backslash f(A_2)$,

3. "relationship between image and inverse image"

   i. $A_1 \subseteq f^{-1}(f(A_1))$, and
   if f is one-to-one, then $A_1 = f^{-1}(f(A_i))$,
   l. $B_1 \supseteq f\left(f^{-1}(B_1)\right)$, and
   if f is onto, then $B_1 = f\left(f^{-1}(B_1)\right)$.

Proof.
...
g.
(i) . $y \in f(A_1 \cap A_2) \Leftrightarrow \exists x \in A_1 \cap A_2$ such that $f(x) = y$;
(ii) . $y \in f(A_1) \cap f(A_2) \Leftrightarrow y \in f(A_1) \wedge y \in f(A_2) \Leftrightarrow (\exists x_1 \in A_1$ such that $f(x_1) = y) \wedge (\exists x_2 \in A_2$ such that $f(x_2) = y)$
To show that $(i) \Rightarrow (ii)$ it is enough to take $x_1 = x$ and $x_2 = x$.
... ■

Proposition 516 *$f : X \to Y$ is continuous $\Leftrightarrow$*

$$V \subseteq Y \text{ is open } \Rightarrow \quad f^{-1}(V) \subseteq X \text{ is open.}$$

Proof. $[\Rightarrow]$
Take a point $x_0 \in f^{-1}(V)$. We want to show that

$$\exists r > 0 \text{ such that } B(x_0, r) \subseteq f^{-1}(V)$$

Define $y_0 = f(x_0) \in V$. Since $V \subseteq Y$ is open,

$$\exists \varepsilon > 0 \text{ such that } B(y_0, \varepsilon) \subseteq V \tag{10.2}$$

Since f is continuous,

$$\forall \varepsilon > 0, \exists \delta > 0, f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon) = B(y_0, \varepsilon) \tag{10.3}$$

Then, taken $r = \delta$, we have

$$B(x_0, r) = \ B(x_0, \delta) \overset{(1)}{\subseteq} f^{-1}(f(B(x_0, \delta))) \overset{(2)}{\subseteq} f^{-1}(B(y_0, \varepsilon)) \overset{(3)}{\subseteq} f^{-1}(V)$$

where (1) follows from 3.i in Proposition 515,
(2) follows from 1.a in Proposition 515 and (10.3)
(3) follows from 1.a in Proposition 515 and (10.2).
[$\Leftarrow$]
Take $x_0 \in X$ and define $y_0 = f(x_0)$; we want to show that f is continuous at x_0.
Take $\varepsilon > 0$, then $B(y_0, \varepsilon)$ is open and, by assumption,

$$f^{-1}(B(y_0, \varepsilon)) \subseteq X \text{ is open.} \tag{10.4}$$

Moreover, by definition of y_0,

$$x_0 \in f^{-1}(B(y_0, \varepsilon)) \tag{10.5}$$

(10.4) and (10.5) imply that

$$\exists \delta > 0 \text{ such that } B(x_0, \delta) \subseteq f^{-1}(B(y_0, \varepsilon)) \tag{10.6}$$

Then

$$f(B(x_0, \delta)) \overset{(1)}{\subseteq} f\left(f^{-1}(B(y_0, \varepsilon))\right) \overset{(2)}{\subseteq} (B(y_0, \varepsilon))$$

where
(1) follows from 2.e in Proposition 515 and (10.6),
(2) follows from 2.l in Proposition 515 ■

Proposition 517 *$f : X \to Y$ is continuous $\Leftrightarrow$*

$$V \subseteq Y \text{ closed } \Rightarrow \quad f^{-1}(V) \subseteq X \text{ closed.}$$

Proof. [$\Rightarrow$]
V closed in $Y \Rightarrow Y \backslash V$ open. Then

$$f^{-1}(Y \backslash V) = f^{-1}(Y) \ \backslash \ f^{-1}(V) = X \backslash f^{-1}(V) \tag{10.7}$$

where the first equality follows from 1.d in Proposition 515.
Since f is continuous and $Y \backslash V$ open,then from (10.7) $X \backslash f^{-1}(V) \subseteq X$ is open and therefore $f^{-1}(V)$ is closed.
[$\Leftarrow$]
We want to show that for every open set V in Y, $f^{-1}(V)$ is open.

$$V \text{ open} \Rightarrow Y \backslash V \text{ closed} \Rightarrow f^{-1}(Y \backslash V) \text{ closed} \Leftrightarrow X \backslash f^{-1}(V) \text{ closed} \Leftrightarrow f^{-1}(V) \text{ open.}$$

■

Definition 518 *A function $f : X \to Y$ is open if*

$$S \subseteq X \text{ open } \Rightarrow f(S) \text{ open;}$$

it is closed if

$$S \subseteq X \text{ closed } \Rightarrow f(S) \text{ closed.}$$

Exercise 519 *Through simple examples show the relationship between open,closed and continuous functions.*

We can summarize our discussion on continuous function in the following Proposition.

Proposition 520 *Let f be a function between metric spaces (X, d) and (Y, d'). Then the following statements are equivalent:*

1. *f is continuous;*
2. *$V \subseteq Y$ is open $\Rightarrow \quad f^{-1}(V) \subseteq X$ is open;*
3. *$V \subseteq Y$ closed $\Rightarrow \quad f^{-1}(V) \subseteq X$ closed;*
4. *$\forall x_0 \in X$, $\forall (x_n)_{n \in \mathbb{N}} \in X^{\infty}$ such that $\lim_{n \to +\infty} x_n = x_0$, $\quad \lim_{n \to +\infty} f(x_n) = f(x_0)$.*

10.3 Continuous functions on compact sets

Proposition 521 *Given $f : X \to Y$, if S is a compact subset of X and f is continuous, then $f(S)$ is a compact subset (of Y).*

Proof. Let $\mathcal{F}$ be an open covering of $f(S)$, so that

$$f(S) \subseteq \cup_{A \in \mathcal{F}} A. \tag{10.8}$$

We want to show that $\mathcal{F}$ admits an open subcover which covers $f(S)$. Since f is continuous,

$$\forall A \in \mathcal{F}, \quad f^{-1}(A) \text{ is open in } X$$

Moreover,

$$S \overset{(1)}{\subseteq} f^{-1}(f(S)) \overset{(2)}{\subseteq} f^{-1}(\cup_{A \in \mathcal{F}} A) \overset{(3)}{=} \cup_{A \in \mathcal{F}} f^{-1}(A)$$

where
(1) follows from 3.i in Proposition 515,
(2) follows from 1.a in Proposition 515 and (10.8),
(3) follows from 1.b in Proposition 515.
In other words $\left\{ f^{-1}(A) \right\}_{A \in \mathcal{F}}$ is an open cover of S. Since S is compact there exists $A_1, .., A_n \in F$ such that

$$S \subseteq \cup_{i=1}^{n} f^{-1}(A_i).$$

Then

$$f(S) \overset{(1)}{\subseteq} f\left(\cup_{i=1}^{n} \left(f^{-1}(A)\right)\right) \overset{(2)}{=} \cup_{i=1}^{n} f\left(f^{-1}(A_i)\right) \overset{(3)}{\subseteq} \cup_{i=1}^{n} A_i$$

where
(1) follows from 1.a in Proposition 515,
(2) follows from 2.f in Proposition 515,
(3) follows from 3.l in Proposition 515. ■

Proposition 522 ***(Extreme Value Theorem)*** *If S a nonempty, compact subset of X and $f : S \to \mathbb{R}$ is continuous, then f admits global maximum and minimum on S, i.e.,*

$$\exists x_{\min}, x_{\max} \in S \text{ such that } \forall x \in S, \quad f(x_{\min}) \leq f(x) \leq f(x_{\max}).$$

Proof. From the previous Proposition $f(S)$ is closed and bounded. Therefore, since $f(S)$ is bounded, there exists $M = \sup f(S)$. By definition of sup,

$$\forall \varepsilon > 0, \ B(M, \varepsilon) \cap f(S) \neq \varnothing$$

Then[1], $\forall n \in \mathbb{N}$,take

$$\alpha_n \in B\left(M, \frac{1}{n}\right) \cap f(S).$$

Then, $(\alpha_n)_{n \in \mathbb{N}}$ is such that $\forall n \in \mathbb{N}$, $\alpha_n \in f(S)$ and $0 < d(\alpha_n, M) < \frac{1}{n}$. Therefore, $\alpha_n \to M$,and since $f(S)$ is closed, $M \in f(S)$. But $M \in f(S)$ means that $\exists x_{\max} \in S$ such that $f(x_{\max}) = M$ and the fact that $M = \sup f(S)$ implies that $\forall x \in S, \quad f(x) \leq f(x_{\max})$. Similar reasoning holds for $x_{\min}$. ■

We conclude the section showing a result useful in itself and needed to show the inverse function theorem - see Section 15.3.

Proposition 523 *Let $f : X \to Y$ be a function from a metric space (X, d) to another metric space (Y, d'). Assume that f is one-to-one and onto. If X is compact and f is continuous, then the inverse function f^{-1} is continuous.*

[1] The fact that $M \in f(S)$ can be also proved as follows: from Proposition 464 , $M \in \text{Cl } f(S) = f(S)$, where the last equality follows from the fact that $f(S)$ is closed.

Proof. Exercise. ■

We are going to use the above result to show that a " well behaved" consumer problem does have a solution.

Let the following objects be given.

Price vector $p \in \mathbb{R}^n_{++}$, consumption vector $x \in \mathbb{R}^n$, consumer's wealth $w \in \mathbb{R}_{++}$, continuous utility function $u : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto u(x)$. The consumer solves the following problem. For given, $p \in \mathbb{R}^n_{++}$,$w \in \mathbb{R}_{++}$, find x which gives the maximum value to the utility function u under the constraint $x \in C(p,w)$ defined as

$$\{x = (x_i)_{i=1}^n \in \mathbb{R}^n : \forall i \in \{1,...,n\}, x_i \geq 0 \text{ and } px \leq w\}.$$

As an application of Propositions 522 and 423, we have to show that for any $p \in \mathbb{R}^n_{++}$,$w \in \mathbb{R}_{++}$,

1. $C(p,w) \neq \varnothing$,
2. $C(p,w)$ is bounded, and
3. $C(p,w)$ is closed,

1. $0 \in C(p,w)$.
2. Clearly if $S \subseteq \mathbb{R}^n$,then S is bounded iff

S is bounded below, i.e., $\exists \underline{x} = (\underline{x}_i)_{i=1}^n \in \mathbb{R}^n$such that $\forall x = (x_i)_{i=1}^n \in S$, we have that $\forall i \in \{1,...,n\}$, $x_i \geq \underline{x}_i$, and

S is bounded above, i.e., $\exists \overline{x} = (\overline{x}_i)_{i=1}^n \in \mathbb{R}^n$such that $\forall x = (x_i)_{i=1}^n \in S$, we have that $\forall i \in \{1,...,n\}$, $x_i \leq \overline{x}_i$.

$C(p,w)$ is bounded below by zero, i.e., we can take $\underline{x} = 0$. $C(p,w)$ is bounded above because for every $i \in \{1,...,n\}$,

$$x_i \leq \frac{w - \sum_{i' \neq i} p_{i'} x_{i'}}{p_i} \leq \frac{w}{p_i},$$

where the first inequality comes from the fact that $px \leq w$, and the second inequality from the fact that $p \in \mathbb{R}^n_{++}$ and $x \in \mathbb{R}^n_+$.Then we can take $\overline{x} = (m, m, ..., m)$, where $m = \max \left\{ \frac{w}{p_i} \right\}_{i=1}^n$.

3. Define

$$\text{for } i \in \{1,...,n\},\ g_i : \mathbb{R}^n \to \mathbb{R}, \quad x = (x_i)_{i=1}^n \mapsto x_i,$$

and

$$h : \mathbb{R}^n \to \mathbb{R}, \quad x = (x_i)_{i=1}^n \mapsto w - px.$$

All the above functions are continuous and clearly,

$$C(p,w) = \{x = (x_i)_{i=1}^n \in \mathbb{R}^n : \forall i \in \{1,...,n\},\ g_i(x) \geq 0 \text{ and } h(x) \geq 0\}.$$

Moreover,

$$C(p,w) = \{x = (x_i)_{i=1}^n \in \mathbb{R}^n : \forall i \in \{1,...,n\},\ g_i(x) \in [0,+\infty) \text{ and } h(x) \in [0,+\infty)\} =$$

$$= \cap_{i=1}^n g_i^{-1}([0,+\infty)) \cap h^{-1}([0,+\infty)).$$

$[0,+\infty)$ is a closed set and since for any g_i is continuous and h is continuous, from Proposition 520.3,the following sets are closed

$$\forall i \in \{1,...,n\},\ g_i^{-1}([0,+\infty)) \text{ and } h^{-1}([0,+\infty)).$$

Then the desired result follows from the fact that intersection of closed set is closed.

10.4 Exercises

From Lipschutz (1965), starting from page 61: 30, 32, 34; starting from page 106:19,20.

Chapter 11

Correspondence, maximum theorem and a fixed point theorem

11.1 Continuous Correspondences

Definition 524 *Let[1] two metric spaces (X, d_X) and (Y, d_Y) be given. A correspondence, or set-valued function, φ from X to Y is a rule which associates a subset of Y with each element of X, and it is described by the notation*

$$\varphi : X \to\to Y, \ \varphi : x \mapsto\mapsto \varphi(x).$$

Remark 525 *In other words, a correspondence $\varphi : X \to\to Y$ can be identified with a function from X to 2^Y (the set of all subsets of Y).*

Moreover, if we identify x with $\{x\}$, a function from X to Y can be thought as a particular correspondence.

Remark 526 *Some authors make part of the Definition of correspondence the fact that φ is not empty valued, i.e., that $\forall x \in X$, $\varphi(x) \neq \varnothing$.*

In what follows, unless otherwise stated, (X, d_X) and (Y, d_Y) are assumed to be metric spaces and are denoted by X and Y, respectively.

Definition 527 *Given $U \subseteq X$, $\varphi(U) = \cup_{x \in U} \varphi(x) = \{y \in Y : \exists x \in U \text{ such that } y \in \varphi(x)\}$.*

Definition 528 *The graph of $\varphi : X \to\to Y$ is*

$$\text{graph } \varphi := \{(x, y) \in X \times Y : y \in \varphi(x)\}.$$

Definition 529 *Consider $\varphi : X \to\to Y$. φ is Upper Hemi-Continuous (UHC) at $x \in X$ if*

$\varphi(x) \neq \varnothing$ and

for every open neighborhood V of $\varphi(x)$, there exists an open neighborhood U of x such that for every $x' \in U$, $\varphi(x') \subseteq V$ (or $\varphi(U) \subseteq V$).

φ is UHC if it is UHC at every $x \in X$.

Definition 530 *Consider $\varphi : X \to\to Y$. φ is Lower Hemi-Continuous (LHC) at $x \in X$ if*

$\varphi(x) \neq \varnothing$ and

for any open set V in Y such that $\varphi(x) \cap V \neq \varnothing$, there exists an open neighborhood U of x such that for every $x' \in U$, $\varphi(x') \cap V \neq \varnothing$.

φ is LHC if it is LHC at every $x \in X$.

Example 531 *Consider $X = \mathbb{R}_+$ and $Y = [0, 1]$, and*

$$\varphi_1(x) = \begin{cases} [0,1] & if \quad x = 0 \\ \{0\} & if \quad x > 0. \end{cases}$$

[1] This chapter is based mainly on McLean (1985), Hildebrand (1974), HIldebrand and Kirman (1974) and Ok (2007).

$$\varphi_2(x) = \begin{cases} \{0\} & if \quad x = 0 \\ [0,1] & if \quad x > 0. \end{cases}$$

φ_1 is UHC and not LHC; φ_2 is LHC and not UHC.

Some (partial) intuition about the above definitions can be given as follows.

Upper Hemi-Continuity does not allow "explosions". In other words, φ *is not UHC at* x if there exists a small enough open neighborhood of x such that φ does "explode", i.e., it becomes much bigger in that neighborhood.

Lower Hemi-Continuity does not allow "implosions". In other words, φ *is not LHC at* x if there exists a small enough open neighborhood of x such that φ does "implode", i.e., it becomes much smaller in that neighborhood.

In other words, "UHC $\Rightarrow$ no explosion" and "LHC $\Rightarrow$ no implosion"(or "explosion $\Rightarrow$ not UHC" and "implosion $\Rightarrow$ not LHC)". On the other hand, opposite implications are false, i.e.,

it is false that "explosion $\Leftarrow$ not UHC" and "implosion $\Leftarrow$ not LHC", or, in an equivalent manner,

it is false that "no explosion $\Rightarrow$ UHC" and "no implosion $\Rightarrow$ LHC".

An example of a correspondence which neither explodes nor explodes and which is not UHC and not LHC is presented below.

$$\varphi : \mathbb{R}_+ \to\to \mathbb{R}, \qquad \varphi : x \mapsto\mapsto \begin{cases} [1,2] & if \quad x \in [0,1) \\ [3,4] & if \quad x \in [1,+\infty) \end{cases}$$

φ does not implode or explode if you move away from 1 (in a small open neighborhood of 1): on the right of 1, φ does not change; on the left, it changes completely. Clearly, φ is neither UHC nor LHC (in 1).

The following correspondence is both UHC and LHC:

$$\varphi : \mathbb{R}_+ \to\to \mathbb{R}, \qquad \varphi : x \mapsto\mapsto [x, x+1]$$

A, maybe disturbing, example is the following one

$$\varphi : \mathbb{R}_+ \to\to \mathbb{R}, \qquad \varphi : x \mapsto\mapsto (x, x+1).$$

Observe that the graph of the correspondence under consideration "does not implode, does not explode, does not jump". In fact, the above correspondence is LHC, but it is not UHC in any $x \in \mathbb{R}_+$, as verified below. We want to show that

$$not \left\langle \begin{array}{l} \text{for every neighborhood } V \text{ of } \varphi(x), \\ \\ \text{there exists a neighborhood } U \text{ of } x \text{ such that for every } x' \in U,\ \varphi(x') \subseteq V \end{array} \right\rangle$$

i.e.,

$$\left\langle \begin{array}{l} \text{there exists a neighborhood } V^* \text{ of } \varphi(x) \text{ such that.} \\ \\ \text{for every neighborhood } U \text{ of } x \text{ there exists } x' \in U \text{ such that } \varphi(x') \nsubseteq V \end{array} \right\rangle$$

Just take $V = \varphi(x) = (x, x+1)$; then for any open neighborhood U of x and, in fact, $\forall x' \in U \setminus \{x\}$, $\varphi(x') \nsubseteq V$.

Example 532 *The correspondence below*

$$\varphi : \mathbb{R}_+ \to\to \mathbb{R}, \qquad \varphi : x \mapsto\mapsto \begin{cases} [1,2] & if \quad x \in [0,1] \\ [3,4] & if \quad x \in [1,+\infty) \end{cases}$$

is UHC, but not LHC.

Definition 533 $\varphi : X \to\to Y$ *is a continuous correspondence if it is both UHC and LHC.*

Remark 534 *Summarizing the above results, we can maybe say that a correspondence which is both UHC and LHC, in fact a continuous correspondence, is a correspondence which agrees with our intuition of a graph without explosions, implosions or jumps.*

Proposition 535 *1. If $\varphi : X \to\to Y$ is either UHC or LHC and it is a function, then it is a continuous function.*

2. If $\varphi : X \to\to Y$ is a continuous function, then it is a UHC and LHC correspondence.

Proof.
1.
Case 1. φ is UHC.
First proof. Use the fourth characterization of continuous function in Definition 504.
Second proof. Recall that a function $f : X \to Y$ is continuous iff $[V \text{ open in } Y] \Rightarrow \left[f^{-1}(V) \text{ open in } X\right]$. Take V open in Y. Consider $x \in f^{-1}(V)$, i.e., x such that $f(x) \in V$. By assumption f is UHC and therefore $\exists$ an open neighborhood U of x such that $f(U) \subseteq V$. Then, $U \subseteq f^{-1} \circ f(U) \subseteq f^{-1}(V)$. Then, for any $x \in f^{-1}(V)$, we have found an open set U which contains x and is contained in $f^{-1}(V)$, i.e., $f^{-1}(V)$ is open.
Case 2. φ is LHC.
See Remark 542 below.
2.
The results follows from the definitions and again from Remark 542 below.
■

Remark 536 *Propositions below partially answer the following question: are Upper and Lower Hemi-Continuity preserved under set operations?*

Proposition 537 *Let $\varphi_1, \varphi_2 : X \to\to Y$ be given. If φ_1 and φ_2 are UHC, then*

$$\varphi : X \to\to Y, \quad x \mapsto \varphi_1(x) \cup \varphi_2(x)$$

is UHC.

Proof. By assumption, for $k = 1, 2$, φ_k is UHC at x, for any $x \in X$. i.e.,
i. $\varphi_k(x) \neq \varnothing$, and
ii. for any open neighborhood V_k of $\varphi_k(x)$, there exists an open neighborhood U_k of x such that $\forall\, x'_k \in U_k$, $\varphi(x'_k) \subseteq V_k$.
We want to show that conditions described in i) and ii) above do hold true for $\varphi := \varphi_1 \cup \varphi_1$.
i'. $\varphi(x) := \varphi_1(x) \cup \varphi_2(x) \neq \varnothing$ simply because, by assumption, $\varphi_1(x) \neq \varnothing$ and $\varphi_2(x) \neq \varnothing$.
ii'. We want to show that, taken an open neighborhood V of $\varphi_1(x) \cup \varphi_2(x)$, then there exists an open neighborhood U of x such that $\forall x' \in U$, $\varphi_1(x') \cup \varphi_2(x') \subseteq V$.
Since V is open and $V \supseteq \varphi_1(x) \cup \varphi_2(x)$, then

$$V \text{ is an open neighborhood of } \varphi_1(x) \text{ and of } \varphi_2(x). \tag{11.1}$$

Then, from (11.1) and assumption ii.,

$$\text{there exists an open neighborhood } U_1 \text{ of } x \text{, such that, } \forall x'_1 \in U_1,\ \varphi_1(x'_1) \subseteq V, \tag{11.2}$$

and

$$\text{there exists an open neighborhood } U_2 \text{ of } x \text{, such that, } \forall x'_2 \in U_2,\ \varphi_2(x'_2) \subseteq V. \tag{11.3}$$

Taken $U = U_1 \cap U_2$, we want to show that

$$\forall x' \in U,\ \varphi_1(x') \cup \varphi_2(x') \subseteq V.$$

Indeed, $x' \in U \overset{\text{def. intersection}}{\Rightarrow} x' \in U_1 \overset{(11.2)}{\Rightarrow} \varphi_1(x') \subseteq V$ and $x' \in U \overset{\text{def. intersection}}{\Rightarrow} x' \in U_2 \overset{(14.36)}{\Rightarrow} \varphi_2(x') \subseteq V$. Then $x' \in U \Rightarrow \varphi(x') := \varphi_1(x') \cup \varphi_2(x') \subseteq V$, as desired. ■

Proposition 538 *Let $\varphi_1, \varphi_2 : X \to\to Y$ be given. If φ_1 and φ_2 are LHC, then*

$$\varphi : X \to\to Y, \quad x \mapsto \varphi_1(x) \cup \varphi_2(x)$$

is LHC.

Proof. By assumption, for $k = 1, 2$, φ_k is LHC at x, for any $x \in X$, i.e.,

i. $\varphi_k(x) \neq \varnothing$ and

ii. for any open set V_k in Y such that $\varphi_k(x) \cap V_k \neq \varnothing$, there exists an open neighborhood U_k of x such that for every $x'_k \in U_k$, $\varphi_k(x'_k) \cap V_k \neq \varnothing$.

We want to show i) and ii) for $\varphi_1 \cup \varphi_2$.

i'. Obvious.

ii'. We want to show that taken an open set V in Y such that $\varphi_k(x) \cap V \neq \varnothing$, i.e., such that

$$(\varphi_1(x) \cup \varphi_2(x)) \cap V \neq \varnothing, \tag{11.4}$$

then there exists an open neighborhood U of x such that, for every $x' \in U$, $(\varphi_1(x') \cup \varphi_2(x')) \cap V \neq \varnothing$.

From (11.4), we have that $(\varphi_1(x) \cap V) \cup (\varphi_2(x) \cap V) \neq \varnothing$. Then, either $\varphi_1(x) \cap V \neq \varnothing$ or $\varphi_2(x) \cap V \neq \varnothing$. Without loss of generality, assume that $\varphi_1(x) \cap V \neq \varnothing$. Then, from assumption ii., we have that there exists an open neighborhood U_1 of x such that for every $x'_1 \in U_1$, $\varphi_1(x'_1) \cap V \neq \varnothing$. Then,

$$(\varphi_1(x') \cup \varphi_2(x')) \cap V = (\varphi_1(x') \cap V) \cup (\varphi_2(x') \cap V) \neq \varnothing,$$

as desired. ■

Proposition 539 *Let the correspondence $\varphi : X \to\to Y$ be given. φ is UHC $\underset{\not\Leftarrow}{\Rightarrow}$ $Cl\varphi$ is UHC.*

Proof. [⇒]

Let φ be UHC at $x \in X$. Then, by assumption, $\varphi(x) \neq \varnothing$ and

$$\begin{array}{l} \text{for every open neighborhood } V \text{ of } \varphi(x), \\ \text{there exists an open neighborhood } U \text{ of } x \text{ such that for every } x' \in U,\ \varphi(x') \subseteq V. \end{array} \tag{11.5}$$

We want to show that $\mathrm{Cl}\varphi(x) \neq \varnothing$, which follows from the definition of Closure, and

for every open neighborhood V_1 of $\mathrm{Cl}\varphi(x)$,

there exists an open neighborhood U_1 of x such that for every $x_1 \in U_1$, $\mathrm{Cl}\varphi(x_1) \subseteq V_1$.

Since $\mathrm{Cl}(\varphi(x)) \subseteq V_1$, then $\mathrm{Cl}(\varphi(x))$ and V_1^C are closed disjoint sets. Then from Proposition 489 there exists open sets H_1 and H_2 such that

$$\text{a. } H_1 \cap H_2 = \varnothing, \quad \text{b. } \mathrm{Cl}(\varphi(x)) \subseteq H_1, \text{ and} \quad \text{c. } V_1^C \subseteq H_2. \tag{11.6}$$

It then suffices to show that for any $x \in U$, where U is defined in (11.5), we have that $\mathrm{Cl}\varphi(x) \subseteq V_1$. Since, from (11.6.*b.*), we have $\mathrm{Cl}(\varphi(x)) \subseteq H_1$, it suffices to show that $H_1 \subseteq V_1$. Suppose otherwise, i.e., there exists $y \in H_1 \cap V_1^C \overset{(11.6.c.)}{\subseteq} H_2$, contradicting (11.6.*a.*).

[⇍]

It is enough to consider $\varphi : (0,1) \to\to \mathbb{R}$, $x \mapsto\mapsto (0, x)$ (where $(0, x) := \{z \in \mathbb{R} : 0 < z < x\}$. ■

Very often, checking if a correspondence is UHC or LHC is not easy. We present some *related* concepts which are more convenient to use.

Definition 540 *$\varphi : X \to\to Y$ is "sequentially LHC" at $x \in X$ if $\varphi(x) \neq \varnothing$ and*
for every sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $x_n \to x$, and for every $y \in \varphi(x)$,
there exists a sequence $(y_n)_{n\in\mathbb{N}} \in Y^\infty$ such that $\forall\, n \in \mathbb{N}$, $y_n \in \varphi(x_n)$ and $y_n \to y$.
φ is "sequentially LHC" if it is "sequentially LHC" at every $x \in X$.

Proposition 541 [2] *Let $\varphi : X \to\to Y$ be given. φ is LHC at $x \in X \Leftrightarrow \varphi$ is LHC in terms of sequences at $x \in X$.*

[2] See Proposition 4 page 229 in Ok (2007).

Proof. [⇒]

Take

$$(x_n)_{n\in\mathbb{N}} \in X^{\infty} \text{ such that } x_n \to x \text{ and } y \in \varphi(x). \tag{11.7}$$

We want to find $(y_n) \in Y^{\infty}$ such that $\forall n \in \mathbb{N}$, $y_n \in \varphi(x_n)$ and $y_n \to y$.

Since, from (11.7), $y \in \varphi(x)$, then for any $k \in \mathbb{N}$, we have $B\left(y, \frac{1}{k}\right) \cap \varphi(x) \neq \varnothing$. Since, by assumption, φ is LHC , then

$$\forall k \in \mathbb{N},\ \exists \delta_k > 0 \text{ such that } \forall x_k \in B(x, \delta_k)\text{, we have } B\left(y, \frac{1}{k}\right) \cap \varphi(x_k) \neq \varnothing. \tag{11.8}$$

We now construct a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers as follows. Since, from (11.7), $x_n \to x$, we have that

for $k = 1$, there exists $n_1 \in \mathbb{N}$ such that $\forall n \geq n_1 \geq 1$, we have $x_n \in B(x, \delta_1)$ and $n_1 \geq 1$.

For $k = 2$, there exists $n_2' \in \mathbb{N}$ such that $\forall n \geq n_2'$, we have $x_n \in B(x, \delta_2)$; we can then choose $n_2 \in \{n_1 + 1, n_1 + 2, ...\}$, $n_2 \geq n_2'$ and $n_2 \geq n_1 + 1 \geq 2$.

Then, it is easy to show by an induction argument that:

For arbitrary $k \in \mathbb{N}$, we then have that $\exists n_k \in \{n_{k-1} + 1, n_{k+1} + 2, ...\} \subseteq \mathbb{N}$ such that $\forall n \geq n_k$, we have $x_n \in B(x, \delta_k)$ and $n_k \geq n_{k-1} + 1 \geq k$.

Then, we have constructed a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that

$$\forall k \in \mathbb{N},\ \exists n_k \in \mathbb{N} \text{ such that } \forall n \in \{n_k, n_k + 1, ..., n_{k+1} - 1\},\ x_n \in B(x, \delta_k). \tag{11.9}$$

Then, from (11.9) and (11.8), we have that

$$\forall k \in \mathbb{N}\ \exists n_k \in \mathbb{N} \text{ such that } \forall n \in \{n_k, n_k + 1, ..., n_{k+1} - 1\},\ B\left(y, \frac{1}{k}\right) \cap \varphi(x_n) \neq \varnothing.$$

Then

$$\forall k \in \mathbb{N}\ \exists n_k \in \mathbb{N} \text{ such that } \forall n \in \{n_k, n_k + 1, ..., n_{k+1} - 1\},\text{ we can take } y_n \in B\left(y, \frac{1}{k}\right) \cap \varphi(x_n). \tag{11.10}$$

We have then constructed a sequence $(y_n)_{n\in\mathbb{N}} \in Y^{\infty}$ such that $\forall n \in \mathbb{N}$, $y_n \in \varphi(x_n)$ - observe that, by construction, $\cup_{k\in\mathbb{N}} \{n_k, n_k + 1, ..., n_{k+1} + 1\} = \mathbb{N}$ because, as we have shown above, for any $k \in \mathbb{N}$, we do have $n_k \geq k$.

To conclude the proof we are left with showing that

$$y_n \to y. \tag{11.11}$$

i.e., that $\forall \varepsilon > 0\ \exists k^* \in \mathbb{N}$ such that $\forall n \geq k^*$, $y_n \in B(y, \varepsilon)$.

Observe that from (11.10), we have that

if $k = 1$, then for any $n \in \{n_1 \geq 1, ..., n_2 - 1\}$, we have that $y_n \in B\left(y, \frac{1}{1}\right)$;

if $k = 2$, then for any $n \in \{n_2 \geq 1, ..., n_3 - 1\}$, we have that $y_n \in B\left(y, \frac{1}{2}\right) \subseteq B\left(y, \frac{1}{1}\right)$,

and

for arbitrary $k \in \mathbb{N}$, for any $n \in \{n_k \geq 1, ..., n_{k+1} - 1\}$, we have that $y_n \in B\left(y, \frac{1}{k}\right) \subseteq B\left(y, \frac{1}{k-1}\right) \subseteq B\left(y, \frac{1}{1}\right)$.

Therefore,

$$\forall k \in \mathbb{N}\ \ \exists n_k \in \mathbb{N} \text{ such that } \forall n \geq n_k\text{, we have } y_n \in B\left(y, \frac{1}{k}\right).$$

Then, taken $k^* \in \mathbb{N}$ and $k^* > \frac{1}{\varepsilon}$, we have that there exists $n_{k^*} \in \mathbb{N}$ such that $\forall n \geq n_{k^*}$, we have $y_n \in B\left(y, \frac{1}{k^*}\right) \subseteq B(y, \varepsilon)$, as desired.

[⇐]

Assume otherwise, i.e., there exists an open set V such that

$$\varphi(x) \cap V \neq \varnothing \tag{11.12}$$

and such that for any open neighborhood U of x, there exists $x_U \in U$ such that $\varphi(x_U) \cap V = \varnothing$.

Consider the following family of open neighborhood of x

$$\left\{B\left(x,\frac{1}{n}\right):n\in\mathbb{N}\right\}.$$

Then $\forall n\in\mathbb{N}$, $\exists\, x_n\in B\left(x,\frac{1}{n}\right)$, and therefore $x_n\to x$, such that

$$\varphi\left(x_n\right)\cap V=\varnothing. \tag{11.13}$$

From (11.12), we can take $y\in\varphi\left(x\right)\cap V$. By assumption, we know that there exists a sequence $\left(y_n\right)_{n\in\mathbb{N}}\in Y^{\infty}$ such that $\forall n\in\mathbb{N}$, $y_n\in\varphi\left(x_n\right)$ and $y_n\to y$. Since V is open and $y\in V$, $\exists\,\overline{n}$ such that $n>\overline{n}$ implies that $y_n\in V$. Therefore, for any $n>\overline{n}$,

$$y\in\varphi\left(x_n\right)\cap V. \tag{11.14}$$

But (11.14) contradicts (11.13). ■

Thanks to the above Proposition from now on we talk simply of Lower Hemi-Continuous correspondences.

Remark 542 *If $\varphi:X\to\to Y$ is LHC and it is a function, then it is a continuous function. The result follows from the characterization of Lower Hemi-Continuity in terms of sequences and from the characterization of continuous functions presented in Proposition 520.*

Proposition 543 *For any $i\in\{1,..,n\}$, let $\varphi_i:X\to\to Y$ be LHC at $x\in X$. Then*

1) *the correspondence $x\mapsto\mapsto\sum_{i=1}^{n}\varphi_i\left(x\right):X\to\to Y$ is LHC at x.*

2) *the correspondence $x\mapsto\mapsto\times_{i=1}^{n}\varphi_i\left(x\right):X\to\to Y_1\times...\times Y_n$ is LHC at x.*

Proof. See HIldebrand and Kirman (1976), pages 197-198. ■

Definition 544 *Given a set $A\subseteq\mathbb{R}^n$, the convex hull of A is denoted by convA and it is defined as follows.*[3]

$$\left\{\sum_{i=1}^{m}\lambda_i a^i\in\mathbb{R}^n : m\in\mathbb{N}\text{ and, for } i\in\{1,...,m\}\,, a^i\in A, \lambda_i\geq 0, \sum_{i=1}^{n}\lambda_i=1\right\}.$$

Proposition 545 *Let the correspondence $\varphi:X\to\to\mathbb{R}^m$ be LHC at x. Then the correspondence convex hull of φ, i.e., $x\mapsto\mapsto\text{conv}\varphi\left(x\right):X\to\to\mathbb{R}^m$, is LHC at x.*

Proof. See Hildenbrand and Kirman (1976), pages 197-198. ■

Definition 546 *$\varphi:X\to\to Y$ is closed, or "sequentially UHC", at $x\in X$ if $\varphi\left(x\right)\neq\varnothing$ and*

for every sequence $\left(x_n\right)_{n\in\mathbb{N}}\in X^{\infty}$ such that $x_n\to x$, and for every sequence $\left(y_n\right)_{n\in\mathbb{N}}\in Y^{\infty}$ such that $y_n\in\varphi\left(x_n\right)$ and $y_n\to y$,

it is the case that $y\in\varphi\left(x\right)$.

φ is closed if it is closed at every $x\in X$.

Proposition 547 *Let $\varphi_i:X\to\to Y$, $x\mapsto\mapsto\varphi_i\left(x\right)$ for $i\in\{1,...,k\}$ be given. If φ_i is LHC at $x\in X$, then the following correspondences are LHC as well.*

1.

$$\pi:X\to\to Y^k,\quad x\mapsto\mapsto\times_{i=1}^{k}\varphi_i\left(x\right).$$

2. Assume that Y is a vector space.

$$\Sigma:X\to\to Y,\quad x\mapsto\mapsto\sum_{i=1}^{k}\varphi_i\left(x\right),$$

3.

$$\text{conv}\left(\varphi\right):X\to\to\mathbb{R}^m,\quad x\mapsto\text{conv}\left(\varphi\left(x\right)\right).$$

[3] For a discussion of the concept of convex hull and related concepts see Villanacci, A., (in progress), Basic Convex Analysis, mimeo, Università degli Studi di Firenze, and the references listed there.

Proof. See Proposition 8, page 27, Proposition 9, page 28 and Proposition 10, page 28, respectively, in Hildenbrand (1974). ■

Proposition 548 *φ is closed* $\Leftrightarrow$ graph φ *is a closed set in* $X \times Y$. [4]

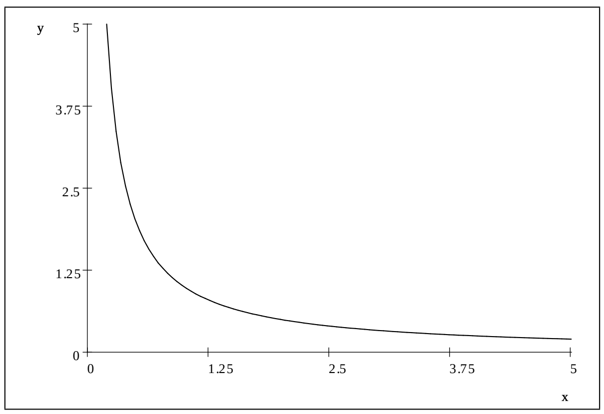
Proof. An equivalent way of stating the Definition of closed correspondence is the following one: for every sequence $(x_n, y_n)_{n\in\mathbb{N}} \in (X \times Y)^\infty$ such that $\forall n \in \mathbb{N}$, $(x_n, y_n) \in$ graph φ and $(x_n, y_n) \to (x, y)$, it is the case that $(x, y) \in$ graph φ. Then, from the characterization of closed sets in terms of sequences, i.e., Proposition 403, the desired result follows. ■

Remark 549 *Because of the above result, many author use the expression "φ has closed graph" in the place of "φ is closed".*

Remark 550 *The definition of closed correspondence does NOT reduce to continuity in the case of functions, as the following example shows.*

$$\varphi_3 : \mathbb{R}_+ \to\to \mathbb{R},\ \varphi_3(x) = \begin{cases} \{0\} & if \quad x = 0 \\ \left\{\frac{1}{x}\right\} & if \quad x > 0. \end{cases}$$

φ is a closed correspondence, but it is not a continuous function.



Definition 551 *A set $C \subseteq \mathbb{R}^n$ is convex if $\forall x_1,\ x_2 \in C$ and $\forall \lambda \in [0,1]$, $(1-\lambda)x_1 + \lambda x_2 \in C$.*

Definition 552 *$\varphi : X \to\to Y$ is closed (non-empty, convex, compact ...) valued if for every $x \in X$, $\varphi(x)$ is a closed (non-empty, convex, compact ...) set.*

Proposition 553 *Let $\varphi : X \to\to Y$ be given. φ closed* $\overset{\Rightarrow}{\not\Leftarrow}$ *φ closed valued.*

Proof. [$\Rightarrow$]

We want to show that given $x \in X$, if $(y_n)_{n\in\mathbb{N}}$ is a sequence of elements in $\varphi(x)$ which converges to y, then $y \in \varphi(x)$. Setting for any $n \in \mathbb{N}$, $x_n = x$, we get $x_n \to x, \quad y_n \in \varphi(x_n), \quad y_n \to y$. Then, since φ is closed, $y \in \varphi(x)$, as desired.

[$\not\Leftarrow$]

φ_2 in Example 531 is closed valued, but not closed. ■

Remark 554 *Let $\varphi : X \to\to Y$ be given. φ is UHC* $\overset{\not\Rightarrow}{\not\Leftarrow}$ *φ is closed.*

[$\not\Rightarrow$]

$$\varphi_4 : \mathbb{R}_+ \to\to \mathbb{R},\ \varphi_4(x) = [0,1)$$

is UHC and not closed.

[$\not\Leftarrow$]

φ_3 in Remark 550 is closed and not UHC, simply because it is not a continuous "function".

[4] $((X \times Y), d^*)$ with $d^*((x,x'),(y,y')) := \max\{d(x,x'), d'(y,y')\}$ is a metric space.

Proposition 555 *Let a closed correspondence $\varphi : X \rightarrow\rightarrow Y$ and a compact set $K \subseteq X$ be given. Then $\varphi(K)$ is closed.*

Proof. Given $(y_n)_{n\in\mathbb{N}} \in (\varphi(K))^\infty$ such that $y_n \to y \in Y$, we want to show that $y \in \varphi(K)$.

Since for any $n \in \mathbb{N}$, we have $y_n \in \varphi(K)$, then there exists $x_n \in K$ such that $y_n \in \varphi(x_n)$. Since K is compact by assumption, the sequence $(x_n)_{n\in\mathbb{N}} \in K^\infty$ admits a subsequence $(x_v)_{v\in\mathbb{N}}$ such that $x_v \to x \in K$. Then, we have that $x_v \in K \subseteq X$, $y_v \to y$, $x_v \to x$. Then since φ is closed $y \in \varphi(x) \subseteq K$, as desired. ■

Proposition 556 *Let $\varphi : X \rightarrow\rightarrow Y$ be given. If φ is UHC (at x) and closed valued (at x), then φ is closed (at x).*

Proof. Take an arbitrary $x \in X$. We want to show that φ is closed at x, i.e., given $(x_n)_{n\in\mathbb{N}} \in X^\infty$, if $x_n \to x$, $\quad y_n \in \varphi(x_n)$ and $y_n \to y$, then $y \in \varphi(x)$. Since $\varphi(x)$ is a closed set, it suffices to show that $y \in \text{Cl}(\phi(x))$, i.e.,[5] $\forall \varepsilon > 0,\ B(y,\varepsilon) \cap \varphi(x) \neq \varnothing$.

Consider $\left\{B\left(z, \frac{\varepsilon}{2}\right) : z \in \varphi(x)\right\}$. Then, $\cup_{z\in\varphi(x)} B\left(z, \frac{\varepsilon}{2}\right) := V$ is open and contains $\varphi(x)$. Since φ is UHC at x, then there exists an open neighborhood U of x such that

$$\varphi(U) \subseteq V. \tag{11.15}$$

Since $x_n \to x \in U$, $\exists \widehat{n} \in \mathbb{N}$ such that $\forall n > \widehat{n}, \quad x_n \in U$, and, from (11.15), $\varphi(x_n) \subseteq V$. Since $y_n \in \varphi(x_n)$,

$$\forall n > \widehat{n}, \qquad y_n \in V := \cup_{z\in\varphi(x)} B\left(z, \frac{\varepsilon}{2}\right). \tag{11.16}$$

From (11.16), $\forall n > \widehat{n}$, $\exists z_n^* \in \varphi(x)$ such that $y_n \in B\left(z_n^*, \frac{\varepsilon}{2}\right)$ and then

$$d(y_n, z_n^*) < \frac{\varepsilon}{2}. \tag{11.17}$$

Since $y_n \to y$, $\exists n^*$ such that $\forall n > n^*$,

$$d(y_n, y) < \frac{\varepsilon}{2}. \tag{11.18}$$

From (11.17) and (11.18), $\forall n > \max\{\widehat{n}, n^*\}$, $z_n^* \in \varphi(x)$ and $d(y, z^*) \leq d(y, y_n) + d(y_n, z^*) < \varepsilon$, i.e., $z_n^* \in B(y,\varepsilon) \cap \varphi(x)$ and then for any $\varepsilon > 0$, $B(y,\varepsilon) \cap \varphi(x) \neq \varnothing$, as desired. ■

Proposition 557 *Let $\varphi : X \rightarrow\rightarrow Y$ be given. If φ is closed and there exists a compact set $K \subseteq Y$ such that $\varphi(X) \subseteq K$, then φ is UHC.*

Therefore, in simpler terms, if φ is closed (at x)and Y is compact, then φ is UHC (at x).

Proof. Assume that there exists $x \in X$ such that φ is not UHC at $x \in X$, i.e., there exist an open neighborhood V of $\varphi(x)$ such that for every open neighborhood U_x of x, $\ \varphi(U_x) \cap V^C \neq \varnothing$. In particular, $\forall n \in \mathbb{N}, \quad \varphi\left(B\left(x, \frac{1}{n}\right)\right) \cap V^C \neq \varnothing$. Therefore, we can construct a sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $x_n \to x$ and for any $n \in \mathbb{N}$, $\varphi(x_n) \cap V^C \neq \varnothing$. Now, take $y_n \in \varphi(x_n) \cap V^C$. Since $y_n \in \varphi(X) \subseteq K$ and K is compact, and therefore sequentially compact, up to a subsequence, $y_n \to y \in K$. Moreover, since $\forall n \in \mathbb{N}$, $y_n \in V^C$ and V^C is closed,

$$y \in V^C. \tag{11.19}$$

Since φ is closed and $x_n \to x, \quad y_n \in \varphi(x_n), \quad y_n \to y$, we have that $y \in \varphi(x)$. Since, by assumption, $\varphi(x) \subseteq V$, we have that

$$y \in V. \tag{11.20}$$

But (11.20) contradicts (11.19). ■

None of the Assumptions of the above Proposition can be dispensed of. All the examples below show correspondences which are not UHC.

Example 558 *1.*

$$\varphi : \mathbb{R}_+ \rightarrow\rightarrow \mathbb{R},\ \varphi(x) = \begin{cases} \left\{\frac{1}{2}\right\} & if \quad x \in [0,2] \\ \{1\} & if \quad x > 2. \end{cases}$$

$Y = [0,1]$, *but φ is not closed.*

[5] See Corollary 464.

2.

$$\varphi : \mathbb{R}_+ \to\to \mathbb{R},\ \varphi(x) = \begin{cases} \{0\} & if \quad x = 0 \\ \left\{\frac{1}{x}\right\} & if \quad x > 0. \end{cases}$$

φ is closed, but $\varphi(X) = \mathbb{R}_+$, which is closed, but not bounded.

3.

$$\varphi : [0,1] \to\to [0,1),\ \varphi(x) = \begin{cases} \{0\} & if \quad x \in [0,1) \\ \left\{\frac{1}{2}\right\} & if \quad x = 1. \end{cases}$$

φ is closed (in Y), but $Y = [0,1)$ is not compact. Observe that if you consider

$$\varphi : [0,1] \to\to [0,\overset{\downarrow}{1}],\ \varphi(x) = \begin{cases} \{0\} & if \quad x \in [0,1) \\ \left\{\frac{1}{2}\right\} & if \quad x = 1. \end{cases},$$

then φ is not closed.

Proposition 559 *For any $i \in \{1,...,n\}$, let the correspondences $\varphi_i : X \to\to Y$ be compact valued and UHC at $x \in X$. Then*

1. The correspondence $: X \to\to Y^n,\ x \mapsto\mapsto \times_{i=1}^n \varphi_i(x)$:s compact valued and UHC at x.

2. If Y is a vector space, then the correspondence $: X \to\to Y,\ : x \mapsto\mapsto \sum_{i=1}^n \varphi_i(x)$ is compact valued and UHC at x.

Proof. See Propositions 4 and 5, page 25 in Hildenbrand (1974). ■

Proposition 560 *Let the correspondence $\varphi : X \to\to \mathbb{R}^m$ be compact valued and UHC at $x \in X$. Then the convex hull of φ, $x \mapsto\mapsto conv\varphi(x) : X \to\to \mathbb{R}^m$, is compact valued and UHC at x.*

Proof. See Hildenbrand (1974), page 26. ■

Proposition 561 *Let $\varphi_1, \varphi_2 : X \to\to Y$ be given. Assume that $\varphi_1(x) \cap \varphi_2(x) \neq \varnothing$. If φ_1 is compact-valued and UHC and φ_2 is closed then*

$$\varphi : X \to\to Y,\ such\ that,\ x \mapsto \varphi_1(x) \cap \varphi_2(x),\ is\ UHC\ and\ compact\text{-}valued.$$

Proof. See Hildenbrand and Kirman (1976), page 195. ■

In Proposition 570 below, we present a characterization of UHC correspondences in the case of compact valued correspondences (in the proof of that result, Proposition 568 below is needed).

Definition 562 *Consider $\varphi : X \to\to Y$, $V \subseteq Y$.*

The strong inverse image of V via φ is

$${}^s\varphi^{-1}(V) := \{x \in X : \varphi(x) \subseteq V\};$$

.

The weak inverse image of V via φ is

$${}^w\varphi^{-1}(V) := \{x \in X : \varphi(x) \cap V \neq \varnothing\}.$$

Remark 563 *1. $\forall\, V \subseteq Y,\ {}^s\varphi^{-1}(V) \subseteq {}^w\varphi^{-1}(V)$.*

2. If φ is a function, the usual definition of inverse image coincides with both above definitions.

Proposition 564 *Consider $\varphi : X \to\to Y$.*

1.1. φ is UHC $\Leftrightarrow$ for every open set V in Y, ${}^s\varphi^{-1}(V)$ is open in X;

1.2. φ is UHC $\Leftrightarrow$ for every closed set V in Y, ${}^w\varphi^{-1}(V)$ is closed in X;

2.1. φ is LHC $\Leftrightarrow$ for every open set V in Y, ${}^w\varphi^{-1}(V)$ is open in X;

2.2. φ is LHC $\Leftrightarrow$ for every closed set V in Y, ${}^s\varphi^{-1}(V)$ is closed in X.[6]

[6] Part 2.2 of the Proposition will be used in the proof of the Maximum Theorem.

Proof.

[1.1., $\Rightarrow$] Consider V open in Y. Take $x_0 \in {}^s\varphi^{-1}(V)$; by definition of ${}^s\varphi^{-1}$, $\varphi(x_0) \subseteq V$. By definition of UHC correspondence, $\exists$ an open neighborhood U of x_0 such that $\forall x \in U$, $\varphi(x) \in V$. Then $x_0 \in U \subseteq {}^s\varphi^{-1}(V)$.

[1.1., $\Leftarrow$] Take an arbitrary $x_0 \in X$ and an open neighborhood V of $\varphi(x_0)$. Then $x_0 \in {}^s\varphi^{-1}(V)$ and ${}^s\varphi^{-1}(V)$ is open by assumption. Therefore (just identifying U with ${}^s\varphi^{-1}(V)$), we have proved that φ is UHC.

To show 1.2, preliminarily, observe that

$$\left({}^w\varphi^{-1}(V)\right)^C = {}^s\varphi^{-1}\left(V^C\right). \tag{11.21}$$

(To see that, simply observe that $\left({}^w\varphi^{-1}(V)\right)^C := \{x \in X : \varphi(x) \cap V = \varnothing\}$ and ${}^s\varphi^{-1}\left(V^C\right) := \left\{x \in X : \varphi(x) \subseteq V^C\right\}$)

[1.2., $\Rightarrow$] V closed $\Leftrightarrow V^C$ open $\overset{Assum., (1.1)}{\Leftrightarrow} {}^s\varphi^{-1}\left(V^C\right) \overset{(11.21)}{=} \left({}^w\varphi^{-1}(V)\right)^C$ open $\Leftrightarrow {}^w\varphi^{-1}(V)$ closed.

[1.2., $\Leftarrow$]

From $(1.1.)$, it suffices to show that $\forall$ open set V in Y, ${}^s\varphi^{-1}(V)$ is open in X. Then,

V open $\Leftrightarrow V^C$ closed $\overset{Assum.}{\Leftrightarrow} {}^w\varphi^{-1}\left(V^C\right)$ closed $\Leftrightarrow \left({}^w\varphi^{-1}\left(V^C\right)\right)^C \overset{(11.21)}{=} {}^s\varphi^{-1}(V)$ open.

[2.1 $\Rightarrow$]

Let V be an open set in Y, we want to show that ${}^w\varphi^{-1}(V)$ is open in X. If ${}^w\varphi^{-1}(V) = \varnothing$ we are done. Assume then that ${}^w\varphi^{-1}(V) \neq \varnothing$ and take $x_0 \in {}^w\varphi^{-1}(V)$.Then by definition of ${}^w\varphi^{-1}(V)$ we have that $\varphi(x_0) \cap V \neq \varnothing$ and from the assumption that φ is LHC, we have that

$$\textit{there exists an open neighborhood } U_{x_0} \textit{ of } x_0 \textit{ such that } \forall x \in U_{x_0},\ \varphi(x) \cap V \neq \varnothing; \tag{11.22}$$

Summarizing, we have shown that for any $x_0 \in {}^w\varphi^{-1}(V)$, we have *there exists an open neighborhood U_{x_0} of x_0 such that $U_{x_0} \subseteq {}^w\varphi^{-1}(V)$*, as desired.

[2.1 $\Leftarrow$]

We want to show that $\forall x \in X$ and any open set V such that $\varphi(x) \cap V \neq \varnothing$, there exists an open neighborhood U_x of x such that $\forall x' \in U_x$, $\varphi(x') \cap V \neq \varnothing$.

Since $\varphi(x) \cap V \neq \varnothing$, then $x \in {}^w\varphi^{-1}(V)$. By assumption, ${}^w\varphi^{-1}(V)$ is open; then there exists an open neighborhood U_x of x, such that $U_x \subseteq {}^w\varphi^{-1}(V)$, i.e., $\forall x' \in U_x$, $\varphi(x') \cap V \neq \varnothing$ as desired.

[2.2 $\Rightarrow$]

V closed $\Leftrightarrow V^C$ open $\overset{assum.,(2.1)}{\Leftrightarrow} {}^w\varphi^{-1}\left(V^C\right)$ is open $\Leftrightarrow \left({}^w\varphi^{-1}\left(V^C\right)\right)^C \overset{(11.21)}{=} {}^s\varphi^{-1}(V)$ is closed.

[2.2 $\Leftarrow$]

From 2.1 it suffices to show that for any open set V in Y, ${}^w\varphi^{-1}(V)$ is open in X.

V open $\Leftrightarrow V^C$ closed $\overset{assum.}{\Rightarrow} {}^s\varphi^{-1}\left(V^C\right)$ closed $\Leftrightarrow \left({}^s\varphi^{-1}\left(V^C\right)\right)^C \overset{(11.21)}{=} {}^w\varphi^{-1}(V)$ open.

■

Remark 565 *Observe that φ is UHC $\begin{matrix}\not\Rightarrow \\ \not\Leftarrow\end{matrix}$ for every closed set V in Y, ${}^s\varphi^{-1}(V)$ is closed in X.*

[$\not\Rightarrow$]

Consider

$$\varphi : \mathbb{R}_+ \rightarrow\rightarrow \mathbb{R}, \quad \varphi(x) = \begin{cases} [0,2] & if \quad x \in [0,1] \\ [0,1] & if \quad x > 1. \end{cases}$$

φ is UHC and $[0,1]$ is closed, but ${}^s\varphi^{-1}([0,1]) := \{x \in \mathbb{R}_+ : \varphi(x) \subseteq [0,1]\} = (1, +\infty)$ is not closed.

[$\not\Leftarrow$]

Consider

$$\varphi : \mathbb{R}_+ \rightarrow\rightarrow \mathbb{R}_+, \quad \varphi(x) = \begin{cases} \left[0, \frac{1}{2}\right] \cup \{1\} & if \quad x = 0 \\ [0,1] & if \quad x > 0. \end{cases}$$

For any closed set in $Y := \mathbb{R}_+$, ${}^s\varphi^{-1}(V)$ can be only one of the following set, and each of them is closed: $\{0\}, \mathbb{R}_+, \varnothing$. On the other hand, φ is not UHC in 0.

Definition 566 *Let the vector spaces* (X, d_X), (Y, d_Y) *and* (Z, d_Z) *and the correspondences* $\varphi : X \to\to Y, \psi : Y \to\to Z$ *be given. The composition of* φ *with* ψ *is*
$\psi \circ \varphi : X \to\to Z,$

$$(\psi \circ \varphi)(x) := \cup_{y \in \varphi(x)} \psi(y) = \{z \in Z : \exists x \in X \text{ such that } z \in \psi(\varphi(x))\}$$

.

Proposition 567 *Consider* $\varphi : X \to\to Y, \psi : Y \to\to Z$. *If* φ *and* ψ *are UHC, then* $\psi \circ \varphi$ *is UHC.*

Proof.
Step 1. ${}^s(\psi \circ \varphi)^{-1}(V) = {}^s\varphi^{-1}\left({}^s\psi^{-1}(V)\right)$.

$${}^s(\psi \circ \varphi)^{-1}(V) = \{x \in X : \psi(\varphi(x)) \subseteq V\} = \{x \in X : \forall y \in \varphi(x), \psi(y) \subseteq V\} =$$

$$= \left\{x \in X : \forall y \in \varphi(x), y \in {}^s\psi^{-1}(V)\right\} = \left\{x \in X : \varphi(x) \subseteq {}^s\psi^{-1}(V)\right\} = {}^s\varphi^{-1}\left({}^s\psi^{-1}(V)\right).$$

Step 2. Desired result.
Take V open in Z. From Theorem 564, we want to show that ${}^s(\psi \circ \varphi)^{-1}(V)$ is open in X. From step 1, we have that ${}^s(\varphi \circ \psi)^{-1}(V) = {}^s\varphi^{-1}\left({}^s\psi^{-1}(V)\right)$. Now, ${}^s\psi^{-1}(V)$ is open because ψ is UHC, and ${}^s\varphi^{-1}\left({}^s\psi^{-1}(V)\right)$ is open because φ is UHC.
■

Proposition 568 *Consider* $\varphi : X \to\to Y$. *If* φ *is UHC and compact valued, and* $A \subseteq X$ *is a compact set, then* $\varphi(A)$ *is compact.*

Proof.
Consider an arbitrary open cover $\{C_\alpha\}_{\alpha \in I}$ for $\varphi(A)$. Since $\varphi(A) := \cup_{x \in A} \varphi(x)$ and φ is compact valued, there exists a finite set $N_x \subseteq I$ such that

$$\varphi(x) \subseteq \cup_{\alpha \in N_x} C_\alpha := G_x. \tag{11.23}$$

Since for every $\alpha \in N_x$, C_α is open, then G_x is open. Since φ is UHC, ${}^s\varphi^{-1}(G_x)$ is open. Moreover, $x \in {}^s\varphi^{-1}(G_x)$: this is the case because, by definition, $x \in {}^s\varphi^{-1}(G_x)$ iff $\varphi(x) \subseteq G_x$, which is just (11.23). Therefore, $\left\{{}^s\varphi^{-1}(G_x)\right\}_{x \in A}$ is an open cover of A. Since, by assumption, A is compact, there exists a finite set $\{x_i\}_{i=1}^m \subseteq A$ such that $A \subseteq \cup_{i=1}^m \left({}^s\varphi^{-1}(G_{x_i})\right)$. Finally,

$$\varphi(A) \subseteq \varphi\left(\cup_{i=1}^m \left({}^s\varphi^{-1}(G_{x_i})\right)\right) \overset{(1)}{\subseteq} \cup_{i=1}^m \varphi\left({}^s\varphi^{-1}(G_{x_i})\right) \overset{(2)}{\subseteq} \cup_{i=1}^m G_{x_i} = \cup_{i=1}^m \cup_{\alpha \in N_{x_i}} C_\alpha,$$

and $\left\{\{C_\alpha\}_{\alpha \in N_{x_i}}\right\}_{i=1}^m$ is a finite subcover of $\{C_\alpha\}_{\alpha \in I}$. We are left with showing (1) and (2) above.
(1). In general, it is the case that $\varphi(\cup_{i=1}^m S_i) \subseteq \cup_{i=1}^m \varphi(S_i)$.
$y \in \varphi(\cup_{i=1}^m S_i) \Leftrightarrow \exists x \in \cup_{i=1}^m S_i$ such that $y \in \varphi(x) \Rightarrow \exists i$ such that $y \in \varphi(x) \subseteq \varphi(S_i) \Rightarrow y \in \cup_{i=1}^m \varphi(S_i)$.
(2). In general, it is the case that $\varphi\left({}^s\varphi^{-1}(A)\right) \subseteq A$.
$y \in \varphi\left({}^s\varphi^{-1}(A)\right) \Rightarrow \exists x \in {}^s\varphi^{-1}(A)$ such that $y \in \varphi(x)$. But, by definition of ${}^s\varphi^{-1}(A)$, and since $x \in {}^s\varphi^{-1}(A)$, it follows that $\varphi(x) \subseteq A$ and therefore $y \in A$.
■

Remark 569 *Observe that the assumptions in the above Proposition cannot be dispensed of, as verified below.*

Consider $\varphi : \mathbb{R}_+ \to\to \mathbb{R}, \varphi(x) = [0,1)$. *Observe that* φ *is* UHC *and bounded valued but not closed valued, and* $\varphi([0,1]) = [0,1)$ *is not compact.*

Consider $\varphi : \mathbb{R}_+ \to\to \mathbb{R}, \varphi(x) = \mathbb{R}_+$. *Observe that* φ *is* UHC *and closed valued, but not bounded valued, and* $\varphi([0,1]) = \mathbb{R}_+$ *is not compact.*

Consider $\varphi : \mathbb{R}_+ \to\to \mathbb{R}_+$, $\varphi(x) = \begin{cases} \{x\} & if \quad x \neq 1 \\ \{0\} & if \quad x = 1. \end{cases}$ *Observe that* φ *is not* UHC *and* $\varphi([0,1]) = [0,1)$ *is not compact.*

Proposition 570 *Let a compact valued correspondence $\varphi : X \rightarrow\rightarrow Y$ be given. Then*
φ it is UHC at x
$\Leftrightarrow$
i. $\varphi(x) \neq \varnothing$, and
ii. for every $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $x_n \to x$ and for every $(y_n)_{n\in\mathbb{N}} \in Y^\infty$ such that $y_n \in \varphi(x_n)$, there exists a subsequence $(y_\nu)_{\nu\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$, such that $y_\nu \to y \in \varphi(x)$.

Proof. [$\Rightarrow$]
Since $x_n \to x$, then $K := \{x_n : n \in \mathbb{N}\}$ is a compact set (just use the definition of compact sets in terms of open sets or the characterization of compactness in terms of sequential compactness). Since φ is UHC and compact valued by assumption and since K is compact, then from Proposition 568, we have that $\varphi(K)$ is compact. Since for any $n \in \mathbb{N}$, $y_n \in \varphi(x_n) \subseteq \varphi(K)$, then $\{y_n : n \in \mathbb{N}\} \subseteq \varphi(K)$ and therefore it admits a subsequence $(y_\nu)_{\nu\in\mathbb{N}}$ such that $y_\nu \to y \in \varphi(K)$. We are left with showing that $y \in \varphi(x)$. Since φ is compact valued, then it is closed valued and therefore since φ is UHC, then from Proposition 556, φ is closed at x.Then since $x_v \to x$, $y_\nu \to y$ and for any $v \in \mathbb{N}$, $y_\nu \in \varphi(x_\nu)$, we have that $y \in \varphi(x)$, as desired.
[$\Leftarrow$]
Suppose otherwise, i.e., φ is not UHC at $x \in X$, i.e.,

$$\begin{array}{l}\text{there exists an open neighborhood } V \text{ of } \varphi(x) \text{ such that for any open neighborhood } U \text{ of } x, \\ \text{there exists } x' \in U \text{ such that } \varphi(x') \cap V^C \neq \varnothing.\end{array} \tag{11.24}$$

Then, for any $n \in \mathbb{N}$, there exists $x'_n \in B\left(x, \frac{1}{n}\right)$ such that $\varphi(x'_n) \cap V^C \neq \varnothing$. Therefore, $x'_n \to x$ and

$$\text{there exists a sequence } (y_n)_{n\in\mathbb{N}} \text{ such that for any } n \in \mathbb{N},\ y_n \in \varphi(x'_n) \cap V^C. \tag{11.25}$$

By assumption, there exists a subsequence $(y_\nu)_{\nu\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$, such that

$$y_\nu \to y \in \varphi(x) \overset{(11.24)}{\subseteq} V. \tag{11.26}$$

Then, from (11.24), for any $\nu \in \mathbb{N}$, we have $y_\nu \in V^C$, a closed set and, therefore, since $y_\nu \to y$, then we do have $y \in V^C$, contradicting (11.26). ■

Remark 571 *Below, we summarize some facts we showed in the present Section, in a somehow informal manner.*

$$\langle\text{(if } \varphi \text{ is a fcn., it is cnt.}\rangle \Leftrightarrow \langle\varphi \text{ is UHC}\rangle \underset{\not\Leftarrow}{\overset{\not\Rightarrow}{}} \langle\varphi \text{ is sequentially UHC,i.e., closed}\rangle \underset{\Leftarrow}{\overset{\not\Rightarrow}{}} \langle\text{(if } \varphi \text{ is a fcn., it is cnt.}\rangle$$

$$\langle\text{(if } \varphi \text{ is a fcn, it is continuous}\rangle \Leftrightarrow \langle\varphi \text{ is LHC}\rangle \Leftrightarrow \langle\varphi \text{ is sequentially LHC}\rangle$$

$$\langle\varphi \text{ UHC and closed valued at } x\rangle \Rightarrow \langle\varphi \text{ is closed at } x\rangle$$
$$\langle\varphi \text{ UHC at } x\ \rangle \Leftarrow \langle\varphi \text{ is closed at } x \text{ and } \operatorname{Im}\varphi \text{ contained in a compact set}\rangle$$

11.2 The Maximum Theorem

Theorem 572 *(Maximum Theorem) Let the metric spaces (Π, d_Π), (X, d_X), the correspondence $\beta : \Pi \rightarrow\rightarrow X$ and a function $u : X \times \Pi \to \mathbb{R}$ be given.*[7] *Define*

$$\begin{array}{l}\xi : \Pi \rightarrow\rightarrow X, \\ \xi(\pi) = \{z \in \beta(\pi) : \forall x \in \beta(\pi), u(z,\pi) \geq u(x,\pi)\} = \arg\max_{x\in\beta(\pi)} u(x,\pi),\end{array}$$

Assume that
β is non-empty valued, compact valued and continuous,

[7] Obviously, β stands for "budget correspondence" and u for "utility function".

u is continuous.
Then
1. ξ is non-empty valued, compact valued, UHC and closed,and
2.

$$v : \Pi \to \mathbb{R}, \qquad v : \pi \mapsto \max_{x \in \beta(\pi)} u(x, \pi).$$

is continuous.

Proof.

ξ is non-empty valued.

It is a consequence of the fact that β is non-empty valued and compact valued and of the Extreme Value Theorem - see Proposition 522.

ξ is compact valued.

We are going to show that for any $\pi \in \Pi$, $\xi(\pi)$ is a sequentially compact set. Consider a sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $\{x_n : n \in \mathbb{N}\} \subseteq \xi(\pi)$. Since $\xi(\pi) \subseteq \beta(\pi)$ and $\beta(\pi)$ is compact by assumption, without loss of generality, up to a subsequence, $x_n \to x_0 \in \beta(\pi)$. We are left with showing that $x_0 \in \xi(\pi)$. Take an arbitrary $z \in \beta(\pi)$. Since $\{x_n : n \in \mathbb{N}\} \subseteq \xi(\pi)$, we have that $u(x_n, \pi) \geq u(z, \pi)$. By continuity of u, taking limits with respect to n of both sides, we get $u(x_0, \pi) \geq u(z, \pi)$, i.e., $x_0 \in \xi(\pi)$, as desired.

ξ is UHC.

From Proposition 564, it suffices to show that given an arbitrary closed set V in X, ${}^w\xi^{-1}(V) := \{\pi \in \Pi : \xi(\pi) \cap V \neq \varnothing\}$ is closed in Π. Consider an arbitrary sequence $(\pi_n)_{n\in\mathbb{N}}$ such that $\{\pi_n : n \in \mathbb{N}\} \subseteq {}^w\xi^{-1}(V)$ and such that $\pi_n \to \pi_0$. We have to show that $\pi_0 \in {}^w\xi^{-1}(V)$.

Take a sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that for every n, $x_n \in \xi(\pi_n) \cap V \neq \varnothing$. Since $\xi(\pi_n) \subseteq \beta(\pi_n)$, it follows that $x_n \in \beta(\pi_n)$. We can now show the following

Claim. There exists a subsequence $(x_{n_k})_{n_k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \to x_0$ and $x_0 \in \beta(\pi_0)$.

Proof of the Claim.

Since $\{\pi_n : n \in \mathbb{N}\} \cup \{\pi_0\}$ is a compact set (Show it), and since, by assumption, β is UHC and compact valued, from Proposition 568, $\beta(\{\pi_n : n \in \mathbb{N}\} \cup \{\pi_0\})$ is compact. Since $\{x_n\}_n \subseteq \beta(\{\pi_n\} \cup \{\pi_0\})$, there exists a subsequence $(x_{n_k})_{n_k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which converges to some x_0. Since β is compact valued, it is closed valued, too. Then, β is UHC and closed valued and from Proposition 556, β is closed. Since

$$\pi_{n_k} \to \pi_0, \quad x_{n_k} \in \beta(\pi_{n_k}), \quad x_{n_k} \to x_0,$$

the fact that β is closed implies that $x_0 \in \beta(\pi_0)$.

End of the Proof of the Claim.

Choose an arbitrary element z_0 such that $z_0 \in \beta(\pi_0)$. Since we assumed that $\pi_n \to \pi_0$ and since β is LHC, there exists a sequence $(z_n)_{n\in\mathbb{N}} \in X^\infty$ such that $z_n \in \beta(\pi_n)$ and $z_n \to z_0$.

Summarizing, and taking the subsequences of $(\pi_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ corresponding to $(x_{n_k})_{n_k\in\mathbb{N}}$, we have for any n_k,

$$\begin{array}{l} \pi_{n_k} \to \pi_0, \\ x_{n_k} \to x_0, \ \ x_{n_k} \in \xi(\pi_{n_k}), \quad x_0 \in \beta(\pi_0), \\ z_{n_k} \to z_0, \ \ z_{n_k} \in \beta(\pi_{n_k}), \quad z_0 \in \beta(\pi_0). \end{array}$$

Then for any n_k, we have that $u(x_{n_k}, \pi_{n_k}) \geq u(z_{n_k}, \pi_{n_k})$. Since u is continuous, taking limits, we get that $u(x_0, \pi_0) \geq u(z_0, \pi_0)$ .Since the choice of z_0 in $\beta(\pi_0)$ was arbitrary, we have then $x_0 \in \xi(\pi_0)$.

Finally, since $(x_{n_k})_{n_k\in\mathbb{N}} \in V^\infty$, $x_{n_k} \to x_0$ and V is closed, $x_0 \in V$. Then $x_0 \in \xi(\pi_0) \cap V$ and $\pi_0 \in \{\pi \in \Pi : \xi(\pi) \cap V \neq \varnothing\} := {}^w\xi^{-1}(V)$, which was the desired result.

ξ is closed.

ξ is UHC and compact valued, and therefore closed valued. Then, from Proposition 556, it is closed, too.

v is a continuous function.

The basic idea of the proof is that v is a function and "it is equal to" the composition of UHC correspondences; therefore, it is a continuous function. A precise argument goes as follows.

Let the following correspondences be given:

$$(\xi, id) : \Pi \to\to X \times \Pi, \qquad \pi \mapsto \xi(\pi) \times \{\pi\},$$

$$\beta : X \times \Pi \to\to \mathbb{R}, \qquad (x, \pi) \mapsto \{u(x, \pi)\}.$$

Then, from Definition 566,

$$(\beta \circ (\xi, id))(\pi) = \cup_{(x,\pi)\in\xi(\pi)\times\{\pi\}} \{u(x, \pi)\}.$$

By definition of ξ,

$$\forall \pi \in \Pi, \forall \overline{x} \in \xi(\pi), \quad \cup_{(x,\pi)\in\xi(\pi)\times\{\pi\}} \{u(x, \pi)\} = \{u(\overline{x}, \pi)\},$$

and

$$\forall \pi \in \Pi, \quad (\beta \circ (\xi, id))(\pi) = \{u(\overline{x}, \pi)\} = \{v(\pi)\}. \tag{11.27}$$

Now, (ξ, id) is UHC, and since u is a continuous function, β is UHC as well. From Proposition 567, $\beta \circ (\xi, id)$ is UHC and, from 11.27, v is a continuous function.

■

A sometimes more useful version of the Maximum Theorem is one which does not use the fact that β is UHC.

Theorem 573 *(Maximum Theorem) Consider the correspondence $\beta : \Pi \to\to X$ and the function $u : X \times \Pi \to \mathbb{R}$ defined in Theorem 572 and Π, X Euclidean spaces.*

Assume that

β is non-empty valued, compact valued, convex valued, closed and LHC.

u is continuous.

Then

1. ξ is a non-empty valued, compact valued, closed and UHC correspondence;

2. v is a continuous function.

Proof.

The desired result follows from next Proposition.

■

Proposition 574 *Assume that $\varphi : X \to\to Y$ is non-empty valued, compact valued, convex valued, closed and LHC. Then φ is UHC.*

Proof.

Since[8] by assumption, φ is compact valued, we can apply Proposition 570, and therefore it suffices to show that for every $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $x_n \to x$ and for every $(y_n)_{n\in\mathbb{N}} \in Y^\infty$ such that $y_n \in \varphi(x_n)$, there exists a subsequence $(y_\nu)_{\nu\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$, such that $y_\nu \to y \in \varphi(x)$.

Indeed, it suffices to show that $(y_\nu)_{\nu\in\mathbb{N}}$ is bounded: if that is the case, then it admits a convergent subsequence, i.e., there exists a subsequence $(y_{\nu'})_{\nu'\in\mathbb{N}}$ of $(y_\nu)_{\nu\in\mathbb{N}}$ such that $y_{\nu'} \to y$. Moreover, for any $\nu' \in \mathbb{N}$, $y_{\nu'} \in \varphi(x_{\nu'})$ and $x_{\nu'} \to x$. Therefore, since φ is closed by assumption, we do have that $y \in \varphi(x)$ and our proof is complete. Therefore, below we prove that $(y_\nu)_{\nu\in\mathbb{N}}$ is bounded.

Since, by assumption, φ is LHC, then, from Proposition 541, for every sequence $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $x_n \to x$, and for every $z \in \varphi(x)$,

there exists a sequence $(z_n)_{n\in\mathbb{N}} \in Y^\infty$ such that $\forall\, n \in \mathbb{N}$, $z_n \in \varphi(x_n)$ and $z_n \to z$. Since for any $n \in \mathbb{N}$, $y_n \in \varphi(x_n)$ and since φ is convex valued by assumption, we have that for any $n \in \mathbb{N}$, $[y_n, z_n] \subseteq \varphi(x_n)$, where $[y_n, z_n] := \{(1-\lambda)\, y_n + \lambda x_n : \lambda \in [0, 1]\}$ is the segment from y_n to z_n.

[8] The proof is taken from Hildenbrand (1974), Lemma 1, page 33. I added some details which should be carefully checked.

Now, suppose our claim is false, i.e., $(y_n)_{n\in\mathbb{N}}$ is unbounded.

Claim 1. For any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for any $n > N_\varepsilon$, there exists $y'_n \in [y_n, z_n]$ such that $d(y'_n, \varphi(x)) = \varepsilon$.

Proof of Claim 1.

Define $B(\varphi(x), \varepsilon) = \{w \in Y : d(w, \varphi(x)) < \varepsilon\}$ which is a bounded and convex set containing the (bounded, convex) set $\varphi(x)$. Since, by assumption, $(y_n)_{n\in\mathbb{N}}$ is unbounded and $z_n \to z \in \varphi(x)$, then $\exists N_\varepsilon \in \mathbb{N}$ such that $\forall n > N_\varepsilon$, $y_n \notin B(\varphi(x), \varepsilon)$ and $z_n \in B(\varphi(x), \varepsilon)$. Then, (exercise)[9] there exists $y'_n \in [y_n, z_n] \cap \mathcal{F}(B(\varphi(x), \varepsilon))$, where $\mathcal{F}(S)$ is the boundary or frontier of the set S. We are left with showing that $\forall n > N_\varepsilon$, $d(y'_n, \varphi(x)) = \varepsilon$,which is done below.

Suppose $d(y'_n, \varphi(x)) < \varepsilon$. Since $d(, \varphi(x)) : X \to \mathbb{R}$ is a continuous function - from Proposition 495 - then $B(\varphi(x), \varepsilon)$ is open and then $y'_n \in B(\varphi(x), \varepsilon) = \text{Int}(B(\varphi(x), \varepsilon))$, contradicting the fact that $y'_n \in \mathcal{F}(B(\varphi(x), \varepsilon))$.

Suppose that $d(y'_n, \varphi(x)) > \varepsilon$. Since $d(, \varphi(x)) : X \to \mathbb{R}$ is a continuous function, then there exists $r > 0$ such that $\forall w \in B(y'_n, r)$, $d(w, \varphi(x)) > \varepsilon$, i.e., $y'_n \in \text{Int}(\{u \in Y : d(u, \varphi(x)) > \varepsilon\}) = \text{Int}\left(\{u \in Y : d(u, \varphi(x)) \leq \varepsilon\}^C\right) \subseteq \text{Int}\left(\{u \in Y : d(u, \varphi(x)) < \varepsilon\}^C\right) = \text{Int}\left(B(\varphi(x), \varepsilon)^C\right)$, again contradicting the fact that $y'_n \in \mathcal{F}(B(\varphi(x), \varepsilon))$.

End of the proof of Claim 1.

Claim 2. $(y'_n)_{n\in\mathbb{N}}$ is bounded.

Proof of Claim 2.

We want to show that there exists $M' \in \mathbb{R}_{++}$ such that $\forall n, m \in \mathbb{N}$, $d(y'_n, y'_m) < M'$.

From Proposition 351 in villanacci, Convex Analysis, mimeo, or Proposition 1.9.1 in Webster (1994) , we have the following result.

Let A be a nonempty closed set in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Then $\exists a^0 \in A$ such that $d(x, A) = \|x - a^0\|$.

Then, since $\varphi(x)$ is compact and therefore closed, we have that for any $n, m \in \mathbb{N}$, there exists $z_n, z_m \in \varphi(x)$ such that

$$\varepsilon \overset{\text{Claim 1}}{=} d(y'_n, \varphi(x)) = d(y'_n, z_n) \text{ and } \varepsilon \overset{\text{Claim 1}}{=} d(y'_m, \varphi(x)) = d(y'_m, z_m).$$

Then,

$$d(y'_n, y'_m) \leq d(y'_n, z_n) + d(z_n, z_m) + d(z_m, y'_m) = \varepsilon + M + \varepsilon,$$

as desired.

End of the proof of Claim 2.

We are now ready to get the desired contradiction. From Claim 2, up to a subsequence, we have that $y'_n \to \widetilde{y}$, and since from Claim 1, for any $n > N_\varepsilon$, $d(y'_n, \varphi(x)) = \varepsilon$, and $d(, \varphi(x)) : X \to \mathbb{R}$ is a continuous function, we do have $d(\widetilde{y}, \varphi(x)) = \varepsilon > 0$ and therefore

$$\widetilde{y} \notin \varphi(x). \tag{11.28}$$

Observe also that we have found

$(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $x_n \to x$ and $(y'_n)_{n\in\mathbb{N}} \in Y^\infty$ such that $y'_n \in [y_n, z_n] \subseteq \varphi(x_n)$ and $y'_n \to \widetilde{y}$. Since, by assumption, φ is closed, then

$$\widetilde{y} \in \varphi(x),$$

contradicting (11.28), as desired.

■

The following result allows to substitute the requirement "β is LHC" with the easier to check requirement "$Cl\beta$ is LHC".

Proposition 575 *Consider the correspondence $\varphi : \Pi \to\to X$. φ is LHC $\Leftrightarrow$ $Cl\phi$ is LHC.*

Proof.

Preliminary Claim.

V open set, ,$Cl\phi(\pi) \cap V \neq \varnothing \Rightarrow \varphi(\pi) \cap V \neq \varnothing$.

Proof of the Preliminary Claim.

Take $z \in Cl\phi(\pi) \cap V \neq \varnothing$. Since V is open, $\exists \varepsilon > 0$ such that $B(z, \varepsilon) \subseteq V$. Since $z \in Cl\phi(\pi)$, $\exists \{z_n\} \subseteq \varphi(\pi)$ such that $z_n \to z$. But then $\exists n_\varepsilon$ such that $n > n_\varepsilon \Rightarrow z_n \in B(z, \varepsilon) \subseteq V$. But $z_n \in V$ and $z_n \in \varphi(\pi)$ implies that $\varphi(\pi) \cap V \neq \varnothing$.

[9] That result is used to show that a bounded set is Peano measurable iff its boundary is Peano measurable.

End of the Proof of the Preliminary Claim.

[⇒]

Take an open set V such that $Cl\phi(\pi) \cap V \neq \varnothing$. We want to show that there exists an open set U^* such that $\pi \in U^*$ and $\forall \xi \in U^*$, $Cl\phi(\xi) \cap V \neq \varnothing$. From the preliminary remark, it must be the case that $\varphi(\pi) \cap V \neq \varnothing$. Then, since φ is LHC, there exists an open set U such that $\pi \in U$ and $\forall \xi \in U^*$, $\varphi(\pi) \cap V \neq \varnothing$. Since $Cl\phi(\pi) \supseteq \varphi(\pi)$, we also have $Cl\phi(\pi) \cap V \neq \varnothing$. Choosing $U^* = U$, we are done.

[⇐]

Since $\varphi(\pi) \cap V \neq \varnothing$, then $Cl\phi(\pi) \cap V \neq \varnothing$, and, by assumption, $\exists$ open set U' such that $\pi \in U'$ and $\forall \xi \in U'$, $Cl\phi(\xi) \cap V \neq \varnothing$. Then, from the preliminary remark, it must be the case that

$\varphi(\pi) \cap V \neq \varnothing$.

■

Remark 576 *In some economic models, a convenient strategy to show that a correspondence β is LHC is the following one. Introduce a correspondence $\widehat{\beta}$; show that $\widehat{\beta}$ is LHC; show that* $\mathrm{Cl}\,\widehat{\beta} = \beta$. *Then from the above Proposition 575, the desired result follows - see, for example, point 5 the proof of Proposition 591 below.*

We can summarize what said above in the following result.

Theorem 577 *(Maximum Theorem: summary) Let the metric spaces (Π, d_Π), (X, d_X), the correspondence $\beta : \Pi \rightarrow\rightarrow X$ and a function $u : X \times \Pi \rightarrow \mathbb{R}$ be given.*[10] *Define*

$$\xi : \Pi \rightarrow\rightarrow X,$$
$$\xi(\pi) = \{z \in \beta(\pi) : \forall x \in \beta(\pi), u(z, \pi) \geq u(x, \pi)\} = \arg\max_{x \in \beta(\pi)} u(x, \pi),$$

Assume that

1. β is non-empty valued, compact valued and either

a. continuous,

b. convex valued, closed and LHC, or

c. convex valued, closed and such that $\mathrm{Cl}(\varphi)$ *is LHC,*

2. u is continuous.

Then

1. ξ is non-empty valued, compact valued, usc and closed,and

2.

$$v : \Pi \rightarrow \mathbb{R}, \qquad v : \pi \mapsto \max_{x \in \beta(\pi)} u(x, \pi).$$

is continuous.

Proof. We list precise references for the proof of cases a., b. and c of Assumption 1.

Assumption 1a.

Theorem 572.

Assumption 1b.

Theorem 573.

Assumption 1c.

It follows from Proposition 573. ■

Remark 578 *There are other version of the maximum theorems; in a footnote on page 306, Ok (2007) points out the existence of two more general versions of the theorem by Walker (1979) and by Leininger (1984).*

11.3 Fixed point theorems

A thorough analysis of the many versions of fixed point theorems existing in the literature is outside the scope of this notes. Below, we present a useful relatively general version of fixed point theorems both in the case of functions and correspondences.

[10] Obviously, β stands for "budget correspondence" and u for "utility function".

Theorem 579 ***(The Brouwer Fixed Point Theorem)***
For any $n \in \mathbb{N}$, let S be a nonempty, compact, convex subset of $\mathbb{R}^n$. If $f : S \to S$ is a continuous function, then $\exists x \in S$ such that $f(x) = x$.

Proof. For a (not self-contained) proof, see Ok (2007), page 279. ■
Just to try to avoid having a Section without a proof, let's show the following extremely simple version of that theorem.

Proposition 580 *If $f : [0,1] \to [0,1]$ is a continuous function, then $\exists x \in [0,1]$ such that $f(x) = x$.*

Proof. If $f(0) = 0$ or $f(1) = 1$, the result is true. Then suppose otherwise, i.e., $f(0) \neq 0$ and $f(1) \neq 1$, i.e., since the domain of f is $[0,1]$, suppose that $f(0) > 0$ and $f(1) < 1$. Define

$$g : [0,1] \to \mathbb{R}, \qquad : x \mapsto x - f(x).$$

Clearly, g is continuous, $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$. Then, from the intermediate value for continuous functions, $\exists x \in [0,1]$ such that $g(x) = x - f(x) = 0$, i.e., $x = f(x)$, as desired. ■

Theorem 581 ***(Kakutani's Fixed Point Theorem)*** *For any $n \in \mathbb{N}$, if*
1. X is a nonempty, compact, convex subset of $\mathbb{R}^n$, and
2. if $\varphi : X \to\to X$ is a nonempty and convex valued,and
a. closed correspondence, or
b. closed valued and usc, or
c. closed valued and lsc,
then $\exists x \in X$ such that $\varphi(x) \ni x$.

Proof. We list precise references for the proof of cases a., b. and c of Assumption 2.
Assumption 2a.
See Ok (2007), page 331, or Hildebrand (1974) page 39)
Assumption 2b.
From Proposition 556, or Proposition 3b, page 295, Ok (2007),

$$\langle \varphi \text{ usc and closed valued at } x \rangle \Rightarrow \langle \varphi \text{ is closed at } x \rangle$$

Assumption 2c.
See Ok (2007), Corollary 1, page 337. ■

11.4 Application of the maximum theorem to the consumer problem

Definition 582 *(Mas Colell (1996), page 17) Commodities are goods and services available for purchases in the market.*

We assume the number of commodities is finite and equal to C. Commodities are indexed by superscript $c = 1, ..., C$.

Definition 583 *A commodity vector is an element of the commodity space $\mathbb{R}^C$.*

Definition 584 *`(almost Mas Colell(1996), page 18)` A consumption set is a subset of the commodity space $\mathbb{R}^C$. It is denoted by X. Its elements are the vector of commodities the individual can conceivably consume given the physical or institutional constraints imposed by the environment.*

Example 585 *See `Mas colell pages 18, 19.`*

Common assumptions on X are that it is convex,bounded below and unbounded. Unless otherwise stated, we make the following stronger

Assumption 1 $X = \mathbb{R}^C_+ := \left\{x \in \mathbb{R}^C : x \geq 0\right\}$.

Definition 586 *$p \in \mathbb{R}^C$ is the vector of commodity prices.*

Households' choices are limited also by an economic constraint: they cannot buy goods whose value is bigger than their wealth, i.e., it must be the case that $px \leq w$, where w is household's wealth.

Remark 587 *w can take different specifications. For example, we can have $w = pe$, where $e \in \mathbb{R}^C$ is the vector of goods owned by the household, i.e., her endowments.*

Assumption 2 All commodities are traded in markets at publicly observable prices, expressed in monetary unit terms.

Assumption 3 All commodities are assumed to be strictly goods (and not "bad"), i.e., $p \in \mathbb{R}^C_{++}$.

Assumption 4 Households behave as if they cannot influence prices.

Definition 588 *The budget set is*

$$\beta(p, w) := \left\{x \in \mathbb{R}^C_+ : px \leq w\right\}.$$

With some abuse of notation we define the budget correspondence as

$$\beta : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \rightarrow\rightarrow \mathbb{R}^C,\ \beta(p, w) = \left\{x \in \mathbb{R}^C_+ : px \leq w\right\}.$$

Definition 589 *The utility function is*

$$u : X \rightarrow \mathbb{R}, \qquad x \mapsto u(x)$$

Definition 590 *The Utility Maximization Problem (UMP) is*

$$\max_{x \in \mathbb{R}^C_+} \quad u(x) \qquad s.t. \quad px \leq w, \ \text{or } x \in \beta(p, w).$$

$\xi : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \rightarrow\rightarrow \mathbb{R}^C$, $\xi(p, w) = \arg\max(UMP)$ *is the demand correspondence.*

Theorem 591 *ξ is a non-empty valued, compact valued, closed and UHC correspondence and*

$$v : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R},\ v : (p, w) \mapsto \max(UMP),$$

i.e., the indirect utility function, is a continuous function.

Proof.

As an application of the (second version of) the Maximum Theorem, i.e., Theorem 573, we have to show that β is non-empty valued, compact valued, convex valued, closed and LHC.

1. β is non-empty valued.

$x = \left(\frac{w}{Cp^c}\right)_{c=1}^C \in \beta(p, w)$ (or, simpler, $0 \in \beta(p, w)$).

2. β is compact valued.

$\beta(p, w)$ is closed because is the intersection of the inverse image of closed sets via continuous functions.

$\beta(p, w)$ is bounded below by zero.

$\beta(p, w)$ is bounded above because for every c, $x^c \leq \frac{w - \sum_{c' \neq c} p^{c'} x^{c'}}{p^c} \leq \frac{w}{p^c}$, where the first inequality comes from the fact that $px \leq w$, and the second inequality from the fact that $p \in \mathbb{R}^C_{++}$ and $x \in \mathbb{R}^C_+$.

3. β is convex valued.

To see that, simply, observe that $(1 - \lambda) px' + \lambda px'' \leq (1 - \lambda) w + \lambda w = w$.

4. β is closed.

We want to show that for every sequence $\{(p_n, w_n)\}_n \subseteq \mathbb{R}^C_{++} \times \mathbb{R}_{++}$ such that

$(p_n, w_n) \rightarrow (p, w)$, $x_n \in \beta(p_n, w_n)$, $x_n \rightarrow x$,

it is the case that $x \in \beta(p, w)$.

Since $x_n \in \beta(p_n, w_n)$, we have that $p_n x_n \leq w_n$ and $x_n \geq 0$. Taking limits of both sides of both inequalities, we get $px \leq w$ and $x \geq 0$,i.e., $x \in \beta(p, w)$.

5. β is LHC.

We proceed as follows: a. Int β is LHC; b. *Cl* Int $\beta = \beta$. Then, from Proposition 575 the result follows.

a. Observe that Int $\beta(p,w) := \left\{x \in \mathbb{R}^C_+ : x \gg 0 \text{ and } px < w\right\}$ and that Int $\beta(p,w) \neq \varnothing$, since $x = \left(\frac{w}{2Cp^c}\right)_{c=1}^{C} \in \text{Int } \beta(p,w)$. We want to show that the following is true.

For every sequence $(p_n, w_n)_n \in \left(\mathbb{R}^C_{++} \times \mathbb{R}_{++}\right)^\infty$ such that $(p_n, w_n) \to (p,w)$ and for any $x \in Int\beta(p,w)$, there exists a sequence $\{x_n\}_n \subseteq \mathbb{R}^C_+$ such that $\forall n$, $x_n \in \text{Int } \beta(p_n, w_n)$ and $x_n \to x$.

$p_n x - w_n \to px - w < 0$ (where the strict inequality follows from the fact that $x \in \text{Int } \beta(p,w)$. Then, $\exists N$ such that $n \geq N \Rightarrow p_n x - w_n < 0$.

For $n \leq N$, choose an arbitrary $x_n \in \text{Int } \beta(p_n, w_n) \neq \varnothing$. Since $p_n x - w < 0$, for every $n > N$, there exists $\varepsilon_n > 0$ such that $z \in B(x, \varepsilon_n) \Rightarrow p_n z - w_n < 0$.

For any $n > N$, choose $x_n = x + \frac{1}{\sqrt{C}} \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\} \cdot \mathbf{1}$. Then,

$$d(x, x_n) = \left(C\left(\frac{1}{\sqrt{C}} \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\}\right)^2\right)^{\frac{1}{2}} = \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\} < \varepsilon_n,$$

i.e., $x_n \in B(x, \varepsilon_n)$ and therefore

$$p_n x_n - w_n < 0 \quad (1).$$

Since $x_n \gg x$, we also have

$$x_n \gg 0 \quad (2).$$

(1) and (2) imply that $x_n \in \text{Int } \beta(p_n, w_n)$. Moreover, since $x_n \gg x$, we have $0 \leqq \lim_{n\to+\infty}(x_n - x) = \lim_{n\to\infty} \frac{1}{\sqrt{C}} \cdot \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\} \cdot \mathbf{1} \leqq \lim_{n\to\infty} \frac{1}{n}\frac{1}{\sqrt{C}} \cdot \mathbf{1} = 0$, i.e., $\lim_{n\to+\infty} x_n = x$.[11]

b.

It follows from the fact that the budget correspondence is the intersection of the inverse images of half spaces via continuous functions.

2.

It follows from Proposition 592, part (4), and the Maximum Theorem.

■

Proposition 592 *For every* $(p,w) \in \mathbb{R}^C_{++} \times \mathbb{R}_{++}$,

(1) $\forall \alpha \in \mathbb{R}_{++}, \xi(\alpha p, \alpha w) = \xi(p,w)$;

(2) *if u is LNS,* $\forall x \in \mathbb{R}^C_+$, $x \in \xi(p,w) \Rightarrow px = w$;

(3) *if u is quasi-concave*[12], *ξ is convex valued;*

(4) *if u is strictly quasi-concave,*[13] *ξ is single valued, i.e., it is a function.*

Proof.

(1)

It simply follows from the fact that $\forall \alpha \in \mathbb{R}_{++}, \beta(\alpha p, \alpha w) = \beta(p,w)$.

(2)

Suppose otherwise, then $\exists x' \in \mathbb{R}^C_+$ such that $x' \in \xi(p,w)$ and $px' < w$. Therefore,$\exists \varepsilon' > 0$ such that $B(x', \varepsilon') \subseteq \beta(p,w)$ (take $\varepsilon' = d(x', H(p,w))$). Then, from the fact that u is LNS, there exists x^* such that $x^* \in B(x', \varepsilon') \subseteq \beta(p,w)$ and $u(x^*) > u(x')$, i.e., $x' \notin \xi(p,w)$, a contradiction.

(3)

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[11] Or simply

$$0 \leq \lim_{n\to\infty} d(x, x_n) \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

[12] A continuous function f is quasi-concave iff $\forall x', x'' \in X$, $\forall \lambda \in [0,1]$,

$$f((1-\lambda)x' + \lambda x'') \geq \min\left\{f(x'), f(x'')\right\}.$$

[13]

Definition 593 *f is strictly quasi-concave*
iff $\forall x', x'' \in X$, such that $x' \neq x''$, and $\forall \lambda \in (0,1)$, we have that

$$f((1-\lambda)x' + \lambda x'') > \min\left\{f(x'), f(x'')\right\}.$$

Assume there exist x', x'' such that $x', x'' \in \xi(p,w)$. We want to show that $\forall \lambda \in [0,1]$, $x^{\lambda} := (1-\lambda)x' + \lambda x'' \in \xi(p,w)$. Observe that $u(x') = u(x'') := u^*$. From the quasi-concavity of u, we have $u(x^{\lambda}) \geq u^*$. We are therefore left with showing that $x^{\lambda} \in \beta(p,w)$, i.e., β is convex valued. To see that, simply, observe that $px^{\lambda} = (1-\lambda)px' + \lambda px'' \leq (1-\lambda)w + \lambda w = w$.

(4) Assume otherwise. Following exactly the same argument as above we have $x', x'' \in \xi(p,w)$, and $px^{\lambda} \leq w$. Since u is strictly quasi concave, we also have that $u(x^{\lambda}) > u(x') = u(x'') := u^*$, which contradicts the fact that $x', x'' \in \xi(p,w)$.

■

Proposition 594 *If u is a continuous LNS utility function, then the indirect utility function has the following properties.*

For every $(p,w) \in \mathbb{R}^C_{++} \times \mathbb{R}_{++}$,

(1) $\forall \alpha \in \mathbb{R}_{++}, v(\alpha p, \alpha w) = v(p,w)$;

(2) *Strictly increasing in w and for every c, non increasing in p^c;*

(3) *for every $\overline{v} \in \mathbb{R}$, $\{(p,w) : v(p,w) \leq \overline{v}\}$ is convex.*

(4) *continuous.*

Proof.

(1) It follows from Proposition 592 (2).

(2)

If w increases, say by Δw, then, from Proposition 592 (2), $px(p,w) < w + \Delta w$. Define $x(p,w) := x'$. Then,$\exists \varepsilon' > 0$ such that $B(x', \varepsilon') \subseteq \beta(p, w+\Delta w)$ (take $\varepsilon' = d(x', H(p, w+\Delta w))$). Then, from the fact that u is LNS, there exists x^* such that $x^* \in B(x', \varepsilon') \subseteq \beta(p, w+\Delta w)$ and $u(x^*) > u(x')$. The result follows observing that $v(p, w+\Delta w) \geq u(x^*)$.

Similar proof applies to the case of a decrease in p. Assume $\Delta p^{c'} < 0$. Define $\Delta := (\Delta^c)_{c=1}^C \in \mathbb{R}^C$ with $\Delta^c = 0$ iff $c \neq c'$ and $\Delta^{c'} = \Delta p^{c'}$. Then,

$px(p,w) = w \Rightarrow (p+\Delta)x(p,w) = px(p,w) + \overset{(\leq 0)}{\Delta p^{c'} x^{c'}}(p,w) =$

$= w + \Delta p^{c'} x^{c'}(p,w) \leq w$. The remaining part of the proof is the same as in the case of an increase of w.

(3) Take $(p', w'), (p'', w'') \in \{(p,w) : v(p,w) \leq \underline{v}\} := S(\overline{v})$. We want to show that $\forall \lambda \in [0,1]$, $(p^{\lambda}, w^{\lambda}) := (1-\lambda)(p', w') + \lambda(p'', w'') \in S(\overline{v})$, i.e., $x \in \xi(p^{\lambda}, w^{\lambda}) \Rightarrow u(x) > \overline{v}$.

$x \in \xi(p^{\lambda}, w^{\lambda}) \Rightarrow p^{\lambda}x \leq w^{\lambda} \Leftrightarrow (1-\lambda)p' + \lambda p'' \leq (1-\lambda)w' + \lambda w''$.

Then, either $p'x \leq w'$ or $p''x \leq w''$. If $p'x \leq w'$, then $u(x) \leq v(p', w') \leq \overline{v}$. Similarly, if $p''x \leq w''$.

(4)

It was proved in Theorem 591.

■

Part III

Differential calculus in Euclidean spaces

Chapter 12

Partial derivatives and directional derivatives

12.1 Partial Derivatives

The[1] concept of partial derivative is not that different from the concept of "standard" derivative of a function from $\mathbb{R}$ to $\mathbb{R}$, in fact we are going to see that partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}$ are just standard derivatives of a naturally associated function from $\mathbb{R}$ to $\mathbb{R}$.

Recall that for any $k \in \{1, ..., n\}$, $e_n^k = (0, .., 1, ..., 0)$ is the $k - th$ vector in the canonical basis of $\mathbb{R}^n$.

Definition 595 *Let a set $S \subseteq \mathbb{R}^n$, a point $x_0 = (x_{0k})_{k=1}^n \in Int\ S$ and a function $f : S \to \mathbb{R}$ be given. If the following limit exists and it is finite*

$$\lim_{h \to 0} \frac{f\left(x_0 + h e_n^k\right) - f(x_0)}{h} \overset{(h := x_k - x_{0_k})}{=} \lim_{x_k \to x_{0k}} \frac{f\left(x_0 + (x_k - x_{0k}) e_n^k\right) - f(x_0)}{x_k - x_{0k}}, \tag{12.1}$$

then it is called the partial derivative of f with respect to the $k - th$ coordinate computed in x_0 and it is denoted by any of the following symbols

$$D_{x_k} f(x_0), \qquad D_k f(x_0), \qquad \frac{\partial f}{\partial x_k}(x_0), \qquad \frac{\partial f(x)}{\partial x_k}_{|x = x_0}.$$

Remark 596 *As said above, partial derivatives are not really a new concept. We are just treating f as a function of one variable at the time, keeping the other variables fixed. In other words, for simplicity taking $S = \mathbb{R}^n$ and using the notation of the above definition, we can define*

$$g_k : \mathbb{R} \to \mathbb{R}, \qquad g_k(x_k) = f\left(x_0 + (x_k - x_{0k}) e_n^k\right)$$

a function of only one variable, and

$$\begin{aligned} g_k'(x_{0k}) &\overset{(1)}{=} \lim_{x_k \to x_{0k}} \tfrac{g_k(x_k) - g_k\left(x_{0_k}\right)}{x_k - x_{0k}} \overset{(2)}{=} \\ &= \lim_{x_k \to x_{0k}} \tfrac{f\left(x_0 + \left(x_k - x_{0_k}\right) e_n^k\right) - f\left(x_0 + \left(x_{0_k} - x_{0_k}\right) e_n^k\right)}{x_k - x_{0k}} = \\ &= \lim_{x_k \to x_{0k}} \tfrac{f\left(x_0 + \left(x_k - x_{0_k}\right) e_n^k\right) - f(x_0)}{x_k - x_{0k}} \overset{(3)}{=} D_{x_k} f(x_0). \end{aligned} \tag{12.2}$$

where

(1) *follows from the definition of derivative of a function from $\mathbb{R}$ to $\mathbb{R}$,*

(2) *from the definition of the function g_k ,*

(3) *from the definition of partial derivative.*

[1] In this Part, I follow closely Section 5.14 and chapters 12 and 13 in Apostol (1974).

Example 597 *Given* $f : \mathbb{R}^3 \to \mathbb{R}$,

$$f(x_1, x_2, x_3) = e^{xy} \cos x + \sin yz$$

we have

$$\begin{pmatrix} D_{x_1} f(x) \\ D_{x_2} f(x) \\ D_{x_2} f(x) \end{pmatrix} = \begin{pmatrix} -(\sin x) e^{xy} + y(\cos x) e^{xy} \\ z \cos yz + x(\cos x) e^{xy} \\ y \cos yz \end{pmatrix}$$

Remark 598 *Loosely speaking, we can give the following geometrical interpretation of partial derivatives. Given* $f : \mathbb{R}^2 \to \mathbb{R}$ *admitting partial derivatives,* $\frac{\partial f(x_0)}{\partial x_1}$ *is the slope of the graph of the function obtained cutting the graph of* f *with a plane which is*

orthogonal to the $x_1 - x_2$ *plane, and*

going through the line parallel to the x_1 *axis and passing through the point* x_0*, line to which we have given the same orientation as the* x_1 *axis.*

Definition 599 *Given an open subset* S *in* $\mathbb{R}^n$ *and a function* $f : S \to \mathbb{R}$*, if* $\forall k \in \{1, ..., n\}$*, the limit in* (12.1) *exists, we call the gradient of* f *in* x_0 *the following vector*

$$(D_k f(x_0))_{k=1}^n$$

and we denote it by

$$Df(x_0)$$

Remark 600 *The existence of the gradient for* f *in* x_0 *does not imply continuity of the function in* x_0*, as the following example shows.*

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x_1, x_2) = \begin{cases} 0 \quad if & \begin{array}{l} either\ x_1 = 0\ or\ x_2 = 0 \\ i.e.,\ (x_1, x_2) \in (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \end{array} \\ 1 & otherwise \end{cases}$$

$$D_1 f(0) = \lim_{x_1 \to 0} \frac{f(x_1, 0) - f(0, 0)}{x_1 - 0} = \lim_{x_1 \to 0} \frac{0}{x_1 - 0} = 0$$

and similarly

$$D_2 f(0) = 0.$$

f *is not continuous in* 0*: we want to show that* $\exists \varepsilon > 0$ *such that* $\forall \delta > 0$ *there exists* $(x_1, x_2) \in \mathbb{R}^2$ *such that* $(x_1, x_2) \in B(0, \delta)$ *and* $|f(x_1, x_2) - f(0, 0)| \geq \varepsilon$*. Take* $\varepsilon = \frac{1}{2}$ *and any* $(x_1, x_2) \in B(0, \delta)$ *such that* $x_1 \neq 0$ *and* $x_2 \neq 0$*(for example,* $x_1 = x_2 = \frac{\delta}{2}$*; then* $\|(x_1, x_2)\| = \sqrt{\frac{\delta^2}{4} + \frac{\delta^2}{4}} = \sqrt{\frac{\delta^2}{2}} = \frac{1}{\sqrt{2}} \delta < \delta$*). Then* $|f(x_1, x_2) - f(0, 0)| = 1 > \varepsilon$*.*

12.2 Directional Derivatives

A first generalization of the concept of partial derivative of a function is presented in Definition 602 below.

Definition 601 *Given*

$$f : S \subseteq \mathbb{R}^n \to \mathbb{R}^m, \qquad x \mapsto f(x),$$

$\forall i \in \{1, .., m, \}$*, the function*

$$f_i : S \subseteq \mathbb{R}^n \to \mathbb{R}, \qquad x \mapsto\ i - th \text{ component of } f(x).$$

is called the $i - th$ *component function of* f*.*

Therefore,

$$\forall x \in S, \quad f(x) = (f_i(x))_{i=1}^m. \tag{12.3}$$

Definition 602 *Given $m, n \in \mathbb{N}$, a set $S \subseteq \mathbb{R}^n$, $x_0 \in$ Int S, $u \in \mathbb{R}^n$, $h \in \mathbb{R}$ such that $x_0 + hu \in S$, $f : S \to \mathbb{R}^m$, we call the directional derivative of f at x_0 in the direction u, denoted by the symbol*

$$f'(x_0; u),$$

the limit

$$\lim_{h\to 0} \frac{f(x_0 + hu) - f(x_0)}{h} \tag{12.4}$$

if it exists and it is componentwise finite.

Remark 603 *Assume that the limit in* (12.4) *exists and it is finite. Then, from* (12.3) *and using Proposition 501,*

$$f'(x_0; u) = \lim_{h\to 0} \frac{f(x_0 + hu) - f(x_0)}{h} = \left(\lim_{h\to 0} \frac{f_i(x_0 + hu) - f_i(x_0)}{h} \right)_{i=1}^m = \left(f_i'(x_0; u)\right)_{i=1}^m .$$

If $u = e_n^j$, the $j-th$ element of the canonical basis in $\mathbb{R}^n$, we then have

$$f'\left(x_0; e_n^j\right) = \left(\lim_{h\to 0} \frac{f_i\left(x_0 + he_n^j\right) - f_i(x_0)}{h} \right)_{i=1}^m = \left(f_i'\left(x_0; e_n^j\right)\right)_{i=1}^m \overset{(*)}{=} \left(D_{x_j} f_i(x_0)\right)_{i=1}^m := D_{x_j} f(x_0) \tag{12.5}$$

where equality $(*)$ follows from (12.2).

We can then construct a matrix whose n columns are the above vectors, a matrix which involves all partial derivative of all component functions of f. That matrix is formally defined below.

Definition 604 *Assume that $f = (f_i)_{i=1}^m : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ admits all partial derivatives in x_0. The Jacobian matrix of f at x_0 is denoted by $Df(x_0)$ and is the following $m \times n$ matrix:*

$$\begin{bmatrix} D_{x_1} f_1(x_0) & ... & D_{x_j} f_1(x_0) & ... & D_{x_n} f_1(x_0) \\ ... & & ... & & ... \\ D_{x_1} f_i(x_0) & ... & D_{x_j} f_i(x_0) & ... & D_{x_n} f_i(x_0) \\ ... & & ... & & ... \\ D_{x_1} f_m(x_0) & ... & D_{x_j} f_m(x_0) & ... & D_{x_n} f_m(x_0) \end{bmatrix} =$$

$$= \begin{bmatrix} D_{x_1} f(x_0) & ... & D_{x_j} f(x_0) & ... & D_{x_n} f(x_0) \end{bmatrix} =$$

$$= \begin{bmatrix} f'\left(x_0; e_n^1\right) & ... & f'\left(x_0; e_n^j\right) & ... & f'\left(x_0; e_n^n\right) \end{bmatrix}.$$

Remark 605 *(How to easily write the Jacobian matrix of a function.)*

To compute the Jacobian of f is convenient to construct a table as follows.

1. In the first column, write the m vector component functions $f_1, ..., f_i, ..., f_m$ of f.
2. In the first row, write the subvectors $x_1, ..., x_j, ..., x_n$ of x.
3. For each i and j, write the partial Jacobian matrix $D_{x_j} f_i(x)$ in the entry at the intersection of the $i-th$ row and $j-th$ column.

We then obtain the following table,

$$\begin{array}{c|ccccc|} & x_1 & ... & x_j & ... & x_n \\ f_1 & D_{x_1} f_1(x) & & D_{x_j} f_1(x) & & D_{x_n} f_1(x) \\ ... & & & & & \\ f_i & D_{x_1} f_i(x) & & D_{x_j} f_i(x) & & D_{x_n} f_i(x) \\ ... & & & & & \\ f_m & D_{x_1} f_m(x) & & D_{x_j} f_m(x) & & D_{x_n} f_m(x) \end{array}$$

where the Jacobian matrix is the part of the table between square brackets.

Example 606 *Given* $f : \mathbb{R}^4 \to \mathbb{R}^5$,

$$f(x, y, z, t) = \begin{pmatrix} \frac{xy}{x^2+1} \\ \frac{x+yz}{e^x} \\ \frac{xyzt}{e^t} \\ x + y + z + t \\ x^2 + t^2 \end{pmatrix}$$

its Jacobian matrix is

$$\begin{bmatrix} \frac{y}{x^2+1} - 2x^2 \frac{y}{(x^2+1)^2} & \frac{x}{x^2+1} & 0 & 0 \\ \frac{1}{e^x} - \frac{1}{e^x}(x+yz) & \frac{z}{e^x} & \frac{y}{e^x} & 0 \\ ty\frac{z}{e^t} & tx\frac{z}{e^t} & tx\frac{y}{e^t} & xy\frac{z}{e^t} - txy\frac{z}{e^t} \\ 1 & 1 & 1 & 1 \\ 2x & 0 & 0 & 2t \end{bmatrix}_{5\times 4}$$

Remark 607 *From Remark 603,*

$$\forall u \in \mathbb{R}^n,\ f'(x_0; u) \text{ exists } \Rightarrow Df(x_0) \text{ exists} \tag{12.6}$$

On the other hand, the opposite implication does not hold true. Consider the example in Remark 600. There, we have seen that

$$D_x f(0) = D_y f(0) = 0.$$

But if $u = (u_1, u_2)$ *with* $u_1 \neq 0$ *and* $u_2 \neq 0$*, we have*

$$\lim_{h\to 0} \frac{f(0 + hu) - f(0)}{h} = \lim_{h\to 0} \frac{1 - 0}{h}$$

does not exist .

Remark 608 *Again loosely speaking, we can give the following geometrical interpretation of directional derivatives. Take* $f : \mathbb{R}^2 \to \mathbb{R}$ *admitting directional derivatives.* $f(x_0; u)$ *with* $\|u\| = 1$ *is the slope the graph of the function obtained cutting the graph of* f *with a plane which is*

orthogonal to the $x_1 - x_2$ *plane, and*

going through the line going through the points x_0 *and* $x_0 + u$*, line to which we have given the same orientation as* u*.*

Example 609 *Take*

$$f : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto x \cdot x = \|x\|^2$$

Then, the existence of $f'(x_0; u)$ *can be checked computing the following limit.*

$$\lim_{h\to 0} \frac{f(x_0+hu)-f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+hu)(x_0+hu)-x_0x_0}{h} =$$

$$= \lim_{h\to 0} \frac{x_0x_0+hx_0u+hux_0+h^2uu-x_0x_0}{h} =$$

$$= \lim_{h\to 0} \frac{2hx_0u+h^2uu}{h} = \lim_{h\to 0} 2x_0u + huu = 2x_0u$$

Remark 610 *Given* $S \subseteq \mathbb{R}^n$*, a function* $f : S \to \mathbb{R}^m$ *and a point* $x_0 \in Int\ S$

$$f'(x_0; -u) = -f'(x_0; u).$$

Proof.

$$f'(x_0; -u) = \lim_{h\to 0} \frac{f(x_0+h(-u))-f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0-hu)-f(x_0)}{h} =$$

$$= -\lim_{k\to 0} \frac{f(x_0+(-h)u)-f(x_0)}{-h} \overset{k:=-h}{=} -\lim_{k\to 0} \frac{f(x_0+ku)-f(x_0)}{k} = -f'(x_0; u).$$

■

Remark 611 *It is **not** the case that*

$$\forall u \in \mathbb{R}^n,\ f'(x_0;u) \ \text{exists} \ \Rightarrow \ f \ \text{is continuous in} \ x_0 \tag{12.7}$$

as the following example shows. Consider

$$f:\mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & if \quad x \neq 0 \\ 0 & if \quad x=0, i.e., (x,y) \in \{0\} \times \mathbb{R} \end{cases}$$

Let's compute $f'(0;u)$. If $u_1 \neq 0$.

$$\lim_{h\to 0} \frac{f(0+hu)-f(0)}{h} = \lim_{h\to 0} \frac{hu_1 \cdot h^2u_2^2}{(h^2u_1^2+h^4u_2^4)\,h} = \lim_{h\to 0} \frac{u_1 \cdot u_2^2}{u_1^2+h^2u_2^4} = \frac{u_2^2}{u_1}$$

If $u_1 = 0$, we have

$$\lim_{h\to 0} \frac{f(0+hu)-f(0)}{h} = \lim_{h\to 0} \frac{f(0,hu_2)-f(0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

On the other hand, if $x = y^2$ and $x, y \neq 0$, i.e., along the graph of the parabola $x = y^2$ except the origin, we have

$$f(x,y) = f\left(y^2,y\right) = \frac{y^4}{y^4+y^4} = \frac{1}{2}$$

while

$$f(0,0) = 0.$$

To prove that f is not continuous in 0 we have to show that $\exists \varepsilon > 0$ such that $\forall \delta > 0$ there exists $(\bar{x},\bar{y}) \in \mathbb{R}^2$ such that $(\bar{x},\bar{y}) \in B(0,\delta)$ and $|f(\bar{x},\bar{y}) - f(0,0)| = \left|\frac{\bar{x}\bar{y}^2}{\bar{x}^2+\bar{y}^4}\right| \geq \varepsilon$. Take $\varepsilon = 1/4$ and consistently with the above argument $\bar{y}_n = \frac{\delta}{n}$ and $\bar{x}_n = \frac{\delta^2}{n^2}$. Then

$$\forall n \in \mathbb{N}, \quad |f(\bar{x}_n,\bar{y}_n)| = f\left(\bar{y}_n^2,\bar{x}_n\right) = \frac{1}{2} > \varepsilon = \frac{1}{4}.$$

We are left with showing that $\exists n \in \mathbb{N}$ such that

$$(\bar{x}_n,\bar{y}_n) \in B(0,\delta), i.e., \|(\bar{x}_n,\bar{y}_n)\| < \delta.$$

Indeed

$$\|(\bar{x}_n,\bar{y}_n)\| = \sqrt{\frac{\delta^4}{n^4}+\frac{\delta^2}{n^2}} \quad and \quad \lim_{n\to\infty} \sqrt{\frac{\delta^4}{n^4}+\frac{\delta^2}{n^2}} = 0$$

Then, by definition of limit, $\forall \delta > 0$, $\exists \bar{n} \in \mathbb{N}$ such that $\|(\bar{x}_n,\bar{y}_n)\| < \delta$, as desired.

Remark 612 *Roughly speaking, the existence of partial derivatives in a given point in all directions implies "continuity along straight lines" through that point; it does not imply " continuity along all possible curves through that point", as in the case of the parabola in the picture above.*

Remark 613 *We are now left with two problems:*

1. *Is there a definition of derivative whose existence implies continuity?*
2. *Is there any "easy" way to compute the directional derivative?*

Appendix `(to be corrected)`

There are other definitions of directional derivatives used in the literature.

Let the following objects be given: $m, n \in \mathbb{N}$, a set $S \subseteq \mathbb{R}^n$, $x_0 \in \text{Int}\, S$, $u \in \mathbb{R}^n$, $h \in \mathbb{R}$ such that $x_0 + hu \in S$, $f : S \to \mathbb{R}^m$,

Definition 614 *(our definition following* ***A****postol (1974)) We call the directional derivative of f at x_0 in the direction u according to Apostol, denoted by the symbol*

$$f'_A(x_0; u),$$

the limit

$$\lim_{h\to 0}\frac{f(x_0+hu)-f(x_0)}{h} \tag{12.8}$$

if it exists and it is finite.

Definition 615 *(****G****irsanov (1972)) We call the directional derivative of f at x_0 in the direction u according to Girsanov, denoted by the symbol*

$$f'_G(x_0; u),$$

the limit

$$\underline{\lim_{h\to 0^+}}\frac{f(x_0+hu)-f(x_0)}{h} \tag{12.9}$$

if it exists and it is finite.

Definition 616 *(****W****ikipedia) Take $u \in \mathbb{R}^n$ such that $\|u\| = 1$. We call the directional derivative of f at x_0 in the direction u according to Wikipedia, denoted by the symbol*

$$f'_W(x_0; u),$$

the limit

$$\underline{\lim_{h\to 0^+}}\frac{f(x_0+hu)-f(x_0)}{h} \tag{12.10}$$

if it exists and it is finite.

Fact 1. For given $x_0 \in S, u \in \mathbb{R}^n$

$$A \Rightarrow G \Rightarrow W,$$

while the opposite implications do not hold true. In particular, to see way $A \not\Leftarrow G$, just take $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x \geq 0 \end{cases},$$

and observe that while the right derivative in 0 is

$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{1-1}{h} = 0,$$

while the left derivative is

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{0-1}{h} = +\infty.$$

Fact 2. For given $x_0 \in S$,

$$f'_W(x,u) \text{ exists} \Rightarrow f'_G(x,v) \text{ exists for any } v = \alpha u \text{ and } \alpha \in \mathbb{R}_{++}.$$

Proof.

$$f'_G(x,v) = \lim_{h\to 0^+}\tfrac{f(x_0+hv)-f(x_0)}{h} = \alpha \lim_{h\to 0^+}\tfrac{f(x_0+h\alpha u)-f(x_0)}{\alpha h} \overset{k=\alpha h>0}{=}$$

$$= \alpha \lim_{h\to 0^+}\tfrac{f(x_0+ku)-f(x_0)}{k} = \alpha f'_W(x,u).$$

Fact 3. Assume that $u \neq 0$ and $x_0 \in \mathbb{R}^n$. Then the following implications are true:

$$\forall u \in \mathbb{R}^n,\ f'_A(x,u) \text{ exists} \quad \Leftrightarrow \quad \forall u \in \mathbb{R}^n,\ f'_G(x,u) \text{ exists} \quad \Leftrightarrow \quad \forall u \in \mathbb{R}^n \text{ such that } \|u\| = 1,\ f'_W(x,u) \text{ exists.}$$

Proof.
From Fact 1, we are left with showing just two implications.
$G \Rightarrow A$.
We want to show that

$$\forall u \in \mathbb{R}^n, \lim_{h \to 0^+} \frac{f(x_0 + hu) - f(x_0)}{h} \in \mathbb{R} \Rightarrow \forall v \in \mathbb{R}^n, \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h} \in \mathbb{R}.$$

Therefore, it suffices to show that $l := \lim_{h \to 0^-} \frac{f(x_0+hv)-f(x_0)}{h} \in \mathbb{R}$.Take $u = -v$. Then,

$$l = \lim_{h \to 0^-} \frac{f(x_0 - hu) - f(x_0)}{h} = - \lim_{h \to 0^-} \frac{f(x_0 - hu) - f(x_0)}{-h} \overset{k=-h}{=} - \lim_{k \to 0^+} \frac{f(x_0 + ku) - f(x_0)}{k} \in \mathbb{R}.$$

$W \Rightarrow G$.
The proof of this implication is basically the proof of Fact 2.We want to show that

$$\forall u \in \mathbb{R}^n \text{ such that } \|u\| = 1, \lim_{h \to 0^+} \frac{f(x_0 + hu) - f(x_0)}{h} \in \mathbb{R} \Rightarrow \forall v \in \mathbb{R}^n \backslash \{0\}, l := \lim_{h \to 0^+} \frac{f(x_0 + hv) - f(x_0)}{h} \in \mathbb{R}.$$

In fact,

$$l := \lim_{h \to 0^+} \frac{f\left(x_0 + h\|v\| \frac{v}{\|v\|}\right) - f(x_0)}{h} \in \mathbb{R},$$

simply because $\left\| \frac{v}{\|v\|} \right\| = 1$.

Remark 617 *We can give the following geometrical interpretation of directional derivatives. First of all observe that from Proposition 620,*

$$f'(x_0; u) := \lim_{h \to 0} \frac{f(x_0 + hu) - f(x_0)}{h} = df_{x_0}(u).$$

If $f : \mathbb{R} \to \mathbb{R}$, we then have

$$f'(x_0; u) = f'(x_0) \cdot u.$$

Therefore, if $u = 1$, we have

$$f'(x_0; u) = f'(x_0),$$

and if $u > 0$, we have

$$sign\ f'(x_0; u) = sign\ f'(x_0).$$

Take now $f : \mathbb{R}^2 \to \mathbb{R}$ admitting directional derivatives. Then,

$$f'(x_0; u) = Df(x_0) \cdot u \quad with \quad \|u\| = 1$$

is the slope the graph of the function obtained cutting the graph of f with a plane which is
orthogonal to the $x_1 - x_2$ plane, and
along the line going through the points x_0 and $x_0 + u$, in the direction from x_0 to $x_0 + u$.

Chapter 13

Differentiability

13.1 Total Derivative and Differentiability

If $f : \mathbb{R} \to \mathbb{R}$, we say that f is differentiable in x_0, if the following limit exists and it is finite

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and we write

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

or, in equivalent manner,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - f'(x_0) \cdot h}{h} = 0$$

and

$$f(x_0 + h) - (f(x_0) + f'(x_0) \cdot h) = r(h)$$

where

$$\lim_{h \to 0} \frac{r(h)}{h} = 0,$$

or

$$f(x_0 + h) = f(x_0) + f'(x_0) \cdot h + r(h),$$

or using what said in Section 6.1,

$$f(x_0 + h) = f(x_0) + l_{f'(x_0)}(h) + r(h)$$

where $l_{f'(x_0)} \in \mathcal{L}(\mathbb{R}, \mathbb{R})$

and $\lim_{h \to 0} \frac{r(h)}{h} = 0$

Definition 618 *Given a set $S \subseteq \mathbb{R}^n$, $x_0 \in Int\ S$, $f : S \to \mathbb{R}^m$, we say that f is differentiable at x_0 if there exists*

a linear function $df_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$

such that for any $u \in \mathbb{R}^n$, $u \neq 0$, such that $x_0 + u \in S$,

$$\lim_{u \to 0} \frac{f(x_0 + u) - f(x_0) - df_{x_0}(u)}{\|u\|} = 0 \tag{13.1}$$

In that case, the linear function df_{x_0} is called the total derivative or the differential or simply the derivative of f at x_0.

Remark 619 *Obviously, given the condition of the previous Definition, we can say that f is differentiable at x_0 if there exists a linear function $df_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\forall u \in \mathbb{R}^n$ such that $x_0 + u \in S$*

$$f(x_0 + u) = f(x_0) + df_{x_0}(u) + r(u), \qquad \textit{with} \qquad \lim_{u \to 0} \frac{r(u)}{\|u\|} = 0 \tag{13.2}$$

or

$$f(x_0 + u) = f(x_0) + df_{x_0}(u) + \|u\| \cdot E_{x_0}(u), \qquad \textit{with} \qquad \lim_{u \to 0} E_{x_0}(u) = 0 \tag{13.3}$$

The above equations are called the first-order Taylor formula (of f at x_0 in the direction u). Condition (13.3) *is the most convenient one to use in many instances.*

Proposition 620 *Assume that $f : S \to \mathbb{R}^m$ is differentiable at x_0 , then*

$$\forall u \in \mathbb{R}^n, \qquad f'(x_0; u) = df_{x_0}(u).$$

Proof.

$$f'(x_0; u) := \lim_{h \to 0} \frac{f(x_0 + hu) - f(x_0)}{h} \overset{(1)}{=}$$

$$= \lim_{h \to 0} \frac{f(x_0) + df_{x_0}(hu) + \|hu\| \cdot E_{x_0}(hu) - f(x_0)}{h} \overset{(2)}{=} \lim_{h \to 0} \frac{h df_{x_0}(u) + |h| \|u\| \cdot E_{x_0}(hu)}{h} \overset{(3)}{=}$$

$$= \lim_{h \to 0} df_{x_0}(u) + \lim_{h \to 0} sign\ (h) \cdot \|u\| \cdot E_{x_0}(hu) \overset{(4)}{=} df_{x_0}(u) + \|u\| \lim_{hu \to 0} sign\ (h) \cdot E_{x_0}(hu) \overset{(5)}{=} df_{x_0}(u),$$

where
(1) follows from (13.3) with hu in the place of u,
(2) from the fact that df_{x_0} is linear, and from a property of a norm,
(3) from the fact that $\frac{|h|}{h} = sign\ (h)$, [1]
(4) from the fact that $h \to 0$ implies that $hu \to 0$,
(5) from the assumption that f is differentiable in x_0. ■

Remark 621 *The above Proposition implies that if the differential exists, then it is unique - from the fact that the limit is unique, if it exists.*

Proposition 622 *If $f : S \to \mathbb{R}^m$ is differentiable at x_0, then f is continuous at x_0.*

Proof. We have to prove that

$$\lim_{u \to 0} f(x_0 + u) - f(x_0) = 0$$

i.e., from (13.3), it suffices to show that

$$\lim_{u \to 0} df_{x_0}(u) + \|u\| \cdot E_{x_0}(u) \overset{(1)}{=} df_{x_0}(0) + \lim_{u \to 0} \|u\| \cdot E_{x_0}(u) \overset{(2)}{=} 0$$

where
(1) follows from the fact that df_{x_0} is linear and therefore continuous, which is shown in Lemma 623 below.
(2) follows from the fact again that df_{x_0} is linear, and therefore $df_x(0) = 0$, and from (13.3). ■

[1] *sign* is the function defined as follows:

$$sign : \mathbb{R} \to \{-1, 0 + 1\}, \quad x \mapsto \begin{cases} -1 & if \quad x < 0 \\ 0 & if \quad x = 0 \\ +1 & if \quad x > 0. \end{cases}$$

Lemma 623 *If df_{x_0} is linear, then it is continuous.*

Proof. *1st proof.*

Since df_{x_0} is linear, then there exist $A \in \mathbb{M}(m,n)$ such that for any $u \in \mathbb{R}^n$, $df_{x_0}(u) = A \cdot u$. Then, the desired result follows from Remark 514.

2nd proof.

We have to show that $\forall x_0 \in \mathbb{R}^n$, $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\|x - x_0\| < \delta \Rightarrow \|l(x) - l(x_0)\| < \varepsilon$. Defined $[l] = A$, we have that

$$\|l(x) - l(x_0)\| = \|A \cdot x - x_0\| =$$

$$= \left\| \left(R^1(A) \cdot (x - x_0), ..., R^m(A) \cdot (x - x_0)\right) \right\| \overset{(1)}{\leq} \sum_{i=1}^m \left|R^i(A) \cdot (x - x_0)\right| \overset{(2)}{\leq} \tag{13.4}$$

$$\leq \sum_{i=1}^m \left\|R^i(A)\right\| \cdot \|x - x_0\| \leq m \cdot \left(\max_{i \in \{1,...,m\}} \left\{\left\|R^i(A)\right\|\right\}\right) \cdot \|x - x_0\|,$$

where (1) follows from Remark 56, and (2) from Proposition 53.4, i.e., Cauchy-Schwarz inequality. Take

$$\delta = \frac{\varepsilon}{m \cdot \left(\max_{i \in \{1,...,m\}} \left\{R^i(A)\right\}\right)}.$$

Then we have that $\|x - x_0\| < \delta$ implies that $m \cdot \left(\max_{i \in \{1,...,m\}} \left\{R^i(A)\right\}\right) \cdot \|x - x_0\| < \varepsilon$, and from (13.4), $\|l(x) - l(x_0)\| < \varepsilon$, as desired. ■

Remark 624 *The above Proposition is the answer to Question 1 in Remark 613. We still do not have a answer to Question 2 and another question naturally arises at this point:*

3. Is there an "easy" way of checking differentiability?

13.2 Total Derivatives in terms of Partial Derivatives.

In Remark 626 below, we will answer question 2 in Remark 613: Is there any "easy" way to compute the directional derivative?

Proposition 625 *Assume that $f = (f_j)_{j=1}^m : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable in x_0. The matrix associated with df_{x_0} with respect to the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is the Jacobian matrix $Df(x_0)$, i.e.,*

$$[df_{x_0}] = Df(x_0),$$

i.e.,

$$\forall x \in \mathbb{R}^n, \qquad df_{x_0}(x) = Df(x_0) \cdot x. \tag{13.5}$$

Proof. From Definitiion 269,

$$[df_{x_0}] = \begin{bmatrix} df_{x_0}(e_n^1) & ... & df_{x_0}(e_n^i) & ... & df_{x_0}(e_n^n) \end{bmatrix}_{m \times n}$$

From Proposition 620,

$$\forall i \in \{1, ..., n\}, \quad df_{x_0}(e^i) = f'(x_0; e^i),$$

and from (12.5)

$$f'(x_0; e^i) = (D_{x_i} f_j(x_0))_{j=1}^m.$$

Then

$$[df_{x_0}] = \begin{bmatrix} (D_{x_1} f_j(x_0))_{j=1}^m & ... & (D_{x_i} f_j(x_0))_{j=1}^m & ... & (D_{x_1} f_j(x_0))_{j=1}^m \end{bmatrix}_{m \times n},$$

as desired. ■

Remark 626 *From Proposition 620, part 1, and the above Proposition 625, we have that if f is differentiable in x_0, then $\forall u \in \mathbb{R}^m$*

$$\forall u \in \mathbb{R}^m, \qquad f'(x_0; u) = Df(x_0) \cdot u.$$

Remark 627 *From* (13.5)*, we get*

$$\|df_{x_0}(x)\| = \|[Df(x_0)]_{mn}\ x\| \overset{(1)}{\leq} \sum_{j=1}^{m} |Df_j(x_0) \cdot x| \overset{(2)}{\leq} \sum_{j=1}^{m} \|Df_j(x_0)\| \cdot \|x\|$$

where
(1) *follows from Remark 56, and*
(2) *from Cauchy-Schwarz inequality in Proposition 53.*
Therefore, defined $\alpha := \sum_{j=1}^{m} \|Df_j(x_0)\|$ *,we have that*

$$\|df_{x_0}(x)\| \leq \alpha \cdot \|x\|$$

and

$$\lim_{x \to 0} \|df_{x_0}(x)\| = 0$$

Remark 628 *We have seen that*

f differentiable in x_0	$\Rightarrow$	*f admits directional derivative in* x_0	$\underset{\not\Leftarrow}{\Rightarrow}$	$Df(x_0)$ *exists*
$\Downarrow$		(*not* $\Downarrow$)		(*not* $\Downarrow$)
f continuous in x_o		*f continuous in* x_o		*f continuous in* x_o

Therefore

$$f \text{ differentiable in } x_0 \quad \not\Leftarrow \quad Df(x_0) \text{ exists}$$

and

$$f \text{ differentiable in } x_0 \quad \not\Leftarrow \quad f \text{ admits directional derivative in } x_0$$

We still do not have an answer to question 3 in Remark 624: Is there an easy way of checking differentiability? We will provide an answer in Proposition 661.

Chapter 14

Some Theorems

We first introduce some needed definitions.

Definition 629 *Given an open $S \subseteq \mathbb{R}^n$,*

$$f : S \to \mathbb{R}^m, \qquad x := (x_j)_{j=1}^n \mapsto f(x) = (f_i(x))_{i=1}^m$$

$$I \subseteq \{1, ..., m\} \quad and \quad J \subseteq \{1, ..., n\},$$

the partial Jacobian of $(f_i)_{i \in I}$ with respect to $(x_j)_{j \in J}$ in $x_0 \in S$ is the following $(\#I) \times (\#J)$ submatrix of $Df(x_0)$

$$\left[\frac{\partial f_i(x_0)}{\partial x_j}\right]_{i \in I,\ j \in J},$$

and it is denoted by

$$D_{(x_j)_{j \in J}} (f_j)_{i \in I}(x_0)$$

Example 630 *Take:*
S an open subset of $\mathbb{R}^{n_1}$, with generic element $x' = (x_j)_{j=1}^{n_1}$,
T an open subset of $\mathbb{R}^{n_2}$, with generic element $x'' = (x_k)_{k=1}^{n_2}$ and

$$f : S \times T \to \mathbb{R}^m, \quad (x', x'') \mapsto f(x', x'')$$

Then, defined $n = n_1 + n_1$, we have

$$D_{x'} f(x_0) = \begin{bmatrix} D_{x_1} f_1(x_0) & \ldots & D_{x_{n_1}} f_1(x_0) \\ \ldots & & \ldots \\ D_{x_1} f_i(x_0) & \ldots & D_{x_{n_1}} f_i(x_0) \\ \ldots & & \ldots \\ D_{x_1} f_m(x_0) & \ldots & D_{x_{n_1}} f_m(x_0) \end{bmatrix}_{m \times n_1}$$

and, similarly,

$$f(x_0) := \begin{bmatrix} D_{x_{n_1+1}} f_1(x_0) & \ldots & D_{x_n} f_1(x_0) \\ \ldots & & \ldots \\ D_{x_{n_1+1}} f_i(x_0) & \ldots & D_{x_n} f_i(x_0) \\ \ldots & & \ldots \\ D_{x_{n_1+1}} f_m(x_0) & \ldots & D_{x_n} f_m(x_0) \end{bmatrix}_{m \times n_2}$$

and therefore

$$Df(x_0) := \begin{bmatrix} D_{x'} f(x_0) & D_{x''} f(x_0) \end{bmatrix}_{m \times n}$$

Definition 631 *Given an open set $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}$, assume that $\forall x \in S$, $Df(x) := \left(\frac{\partial f(x)}{\partial x_j}\right)_{j=1}^n$ exists. Then, $\forall j \in \{1, ..., n\}$, we define the $j-th$ partial derivative function as*

$$\frac{\partial f}{\partial x_j} : S \to \mathbb{R}, \qquad x \mapsto \frac{\partial f(x)}{\partial x_j}$$

Assuming that the above function has partial derivative with respect to x_k for $k \in \{1, ..., n\}$, we define it as the mixed second order partial derivative of f with respect to x_j and x_k and we write

$$\frac{\partial^2 f(x)}{\partial x_j \partial x_k} := \frac{\partial \frac{\partial f(x)}{\partial x_j}}{\partial x_k}$$

Definition 632 *Given $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, the Hessian matrix of f at x_0 is the $n \times n$ matrix*

$$D^2 f(x_0) := \left[\frac{\partial^2 f}{\partial x_j \partial x_k}(x_0) \right]_{j,k=1,...,n}$$

Remark 633 *$D^2 f(x_0)$ is the Jacobian matrix of the gradient function of f.*

Example 634 *Compute the Hessian function of $f : \mathbb{R}^3_{++} \to \mathbb{R}$,*

$$f(x, y, z) = \left(e^x \cos y + z^2 + x^2 \log y + \log x + \log z + 2t \log t\right)$$

We first compute the gradient:

$$\begin{pmatrix} 2x \ln y + (\cos y) e^x + \frac{1}{x} \\ -(\sin y) e^x + \frac{x^2}{y} \\ 2z + \frac{1}{z} \\ 2 \ln t + 2 \end{pmatrix}$$

and then the Hessian matrix

$$\begin{bmatrix} 2 \ln y + (\cos y) e^x - \frac{1}{x^2} & -(\sin y) e^x + 2\frac{x}{y} & 0 & 0 \\ -(\sin y) e^x + 2\frac{x}{y} & -(\cos y) e^x - \frac{x^2}{y^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{z^2} + 2 & 0 \\ 0 & 0 & 0 & \frac{2}{t} \end{bmatrix}$$

14.1 The chain rule

Proposition 635 *(Chain Rule) Given $S \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^m$,*

$$f : S \subseteq \mathbb{R}^n \to \mathbb{R}^m,$$

such that $Im\ f \subseteq T$

$$g : T \subseteq \mathbb{R}^m \to \mathbb{R}^p,$$

assume that f and g are differentiable in x_0 and $y_0 = f(x_0)$, respectively. Then

$$h : S \subseteq \mathbb{R}^n \to \mathbb{R}^p, \qquad h(x) = (g \circ f)(x)$$

is differentiable in x_0 and

$$dh_{x_0} = dg_{f(x_0)} \circ df_{x_0}.$$

Proof. We want to show that there exists a linear function $dh_{x_0} : \mathbb{R}^n \to \mathbb{R}^p$ such that

$$h(x_0 + u) = h(x_0) + dh_{x_0}(u) + \|u\| \cdot E^*_{x_0}(u), \qquad \text{with} \qquad \lim_{u \to 0} E^*_{x_0}(u) = 0,$$

and $dh_{x_0} = dg_{f(x_0)} \circ df_{x_0}$.
Taking u sufficiently small (in order to have $x_0 + u \in S$), we have

$$h(x_0 + u) - h(x_0) = g[f(x_0 + u)] - g[f(x_0)] = g[f(x_0 + u)] - g(y_0)$$

and defined

$$v = f(x_0 + u) - y_0$$

we get

$$h(x_0+u)-h(x_0)=g(y_0+v)-g(y_0).$$

Since f is differentiable in x_0, we get

$$v=df_{x_0}(u)+\|u\|\cdot E_{x_0}(u), \qquad \text{with} \qquad \lim_{u\to 0} E_{x_0}(u)=0. \tag{14.1}$$

Since g is differentiable in $y_0=f(x_0)$, we get

$$g(y_0+v)-g(y_0)=dg_{y_0}(v)+\|v\|\cdot E_{y_0}(v), \qquad \text{with} \qquad \lim_{v\to 0} E_{y_0}(v)=0. \tag{14.2}$$

Inserting (14.1) in (14.2), we get

$$g(y_0+v)-g(y_0)=dg_{y_0}(df_{x_0}(u)+\|u\|\cdot E_{x_0}(u))+\|v\|\cdot E_{y_0}(v)=dg_{y_0}(dg_{x_0}(u))+\|u\|\cdot dg_{y_0}(E_{x_0}(u))+\|v\|\cdot E_{y_0}(v)$$

Defined

$$E_{x_0}(u):=\begin{cases} 0 & \text{if} \quad u=0 \\ df_{y_0}(E_{x_0}(u))+\frac{\|v\|}{\|u\|}\cdot E_{y_0}(v) & \text{if} \quad u\neq 0 \end{cases},$$

we are left with showing that

$$\lim_{u\to 0} E_{x_0}(u)=0.$$

Observe that

$$\lim_{u\to 0} df_{y_0}(E_{x_0}(u))=0$$

since linear functions are continuous and from (14.1). Moreover, since $\lim_{u\to 0} v=\lim_{u\to 0}(f(x_0+u)-y_0)=0$, from (14.2), we get

$$\lim_{u\to 0} E_{y_0}(v)=0.$$

Finally, we have to show that $\lim_{u\to 0}\frac{\|v\|}{\|u\|}$ is bounded. Now, from the definition of u and from (627), defined $\alpha:=\sum_{j=1}^{m}\|Df_j(x_0)\|$,

$$\|v\|=\|df_{x_0}(u)+\|u\|\cdot E_{x_0}(u)\|\leq\|df_{x_0}(u)\|+\|u\|\,\|E_{x_0}(u)\|\leq(\alpha+\|E_{x_0}(u)\|)\cdot\|u\|$$

and

$$\lim_{u\to 0}\frac{\|v\|}{\|u\|}\leq\lim_{u\to 0}(\alpha+\|E_{x_0}(u)\|)=\alpha,$$

as desired. ■

Remark 636 *From Proposition 625 and Proposition 281, or simply* (7.10)*, we also have*

$$Dh(x_0)_{p\times n}=Dg(f(x_0))_{p\times m}\cdot Df(x_0)_{m\times n},$$

or

$$D_x h(x_0)=D_y g(y)_{|y=f(x_o)}\cdot D_x f(x_0),$$

Observe that $Dg(f(x_0))$ is obtained computing $Dg(y)$ and then substituting $f(x_0)$ in the place of y. We therefore also write $Dg(f(x_0))=Dg(y)_{|y=f(x_0)}$.

Example 637 *Take*

$$f:\mathbb{R}\to\mathbb{R}^2, \qquad x\mapsto\left(3x,4x+x^2\right) \tag{14.3}$$

and

$$g:\mathbb{R}^2\to\mathbb{R}, \qquad (x_1,x_2)\mapsto x_1+x_2.$$

Then

$$h:\mathbb{R}\to\mathbb{R} \qquad x\mapsto 3x+4x+x^2=7x+x^2$$

and

$$h'(x_0)=7+2x_0 \tag{14.4}$$

Therefore, from Remark 636*,*

$$Dh(x_0)_{1\times 1}=Dg(f(x_0))_{1\times 2}\cdot Df(x_0)_{2\times 1}=[1\ ,\ 1]\begin{bmatrix}3\\4+2x_0\end{bmatrix}=3+4+2x_0=7+2x_0.$$

Example 638 *Sometimes we have to solve (not very well formulated) problems as the following one. "Given the function*

$$u\left(g_1\left(l\right), ..., g_n\left(l\right), l\right),$$

what is the effect of a change in l on the value of u?"

Below, 1. we formulate the problem in a rigorous way;

2. applying the Chain Rule Proposition, we answer the question.

1. Let the following functions be given

$$\begin{array}{lll} u:\mathbb{R}^{n+1}\to\mathbb{R} & (x_1,...x_n,l)\mapsto u\left((x_1,...x_n,l)\right) & \\ g_i:\mathbb{R}\to\mathbb{R}, & l\mapsto g_i\left(l\right) & \text{for any } i\in\{1,..,n\}, \\ \varphi:\mathbb{R}\to\mathbb{R}^{n+1}, & l\mapsto\left(g_1\left(l\right),...,g_n\left(l\right),l\right) & \\ v:\mathbb{R}\to\mathbb{R}, & l\mapsto\left(u\circ\varphi\right)\left(l\right)=u\left(g_1\left(l\right),...,g_n\left(l\right),l\right) & \end{array}$$

Assume that any function defined above is differentiable.

2. First of all, we identify the symbols in the Proposition with the symbols in the case we are analyzing consistently with the following table.

Chain Rule Proposition		x_0		f		g		y		h		n		m		p
Example under analysis		l		φ		u		$(x_1,...x_n,l)$		v		1		$n+1$		1

Then, we have that

$$Dh\left(x_0\right)_{p\times n}=Dg\left(f\left(x_0\right)\right)_{p\times m}\cdot Df\left(x_0\right)_{m\times n}$$

or

$$D_xh\left(x_0\right)=D_yg\left(y\right)_{|y=f(x_o)}\cdot D_xf\left(x_0\right),$$

becomes

$$Dv\left(l\right)_{1\times 1}=Du\left(\varphi\left(l\right)\right)_{1\times(n+1)}\cdot D\varphi\left(l\right)_{(n+1)\times 1}$$

or

$$Dv\left(l\right)=D_{(x_1,...x_n,l)}u\left((x_1,...x_n,l)\right)_{|(x_1,...x_n,l)=\varphi(l)}\cdot D_l\varphi\left(l\right).$$

Since

$$D_l\varphi\left(l\right)=\left(g_1'\left(l\right),...,g_n'\left(l\right),1\right)$$

and

$$D_{(x_1,...x_n,l)}u\left(x_1,...x_n,l\right)_{|(x_1,...x_n,l)=\varphi(l)}=\left(D_{x_1}u\left(...\right),..,D_{x_n}u\left(...\right),D_lu\left(...\right)\right)_{|(x_1,...x_n,l)=\varphi(l)},$$

then,

$$Dv\left(l\right)_{1\times 1}=$$

$$=\left(D_{x_1}u\left(\varphi\left(l\right)\right),..,D_{x_n}u(\varphi\left(l\right),D_lu\left(\varphi\left(l\right)\right)\right)\cdot\left(g_1'\left(l\right),...,g_n'\left(l\right),1\right)=$$

$$=\left(\sum_{i=1}^n D_{x_i}u(\varphi\left(l\right)\cdot g_i'\left(l\right)\right)+D_lu(\varphi\left(l\right).$$

Definition 639 *Given* $f : S \subseteq \mathbb{R}^n \to \mathbb{R}^{m_1}$, $g : S \subseteq \mathbb{R}^n \to \mathbb{R}^{m_2}$,

$$(f,g) : \mathbb{R}^n \to \mathbb{R}^{m_1+m_2}, \qquad (f,g)(x) = (f(x), g(x))$$

Remark 640 *Clearly,*

$$D(f,g)(x_0) = \begin{bmatrix} Df(x_0) \\ Dg(x_0) \end{bmatrix}$$

Example 641 *Given*

$$f : \mathbb{R} \to \mathbb{R}^2, \qquad : x \mapsto (\sin x, \cos x)$$

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \qquad : (y_1, y_2) \mapsto (y_1 + y_2, y_1 \cdot y_2)$$

$$h = g \circ f : \mathbb{R} \to \mathbb{R}^2, \qquad : x \mapsto (\sin x + \cos x, \sin x \cdot \cos x)$$

verify the conclusion of the Chain Rule Proposition.

$$Df(x) = \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix}$$

$$Dg(y) = \begin{bmatrix} 1 & 1 \\ y_2 & y_1 \end{bmatrix}$$

$$Dg(f(x)) = \begin{bmatrix} 1 & 1 \\ \cos x & \sin x \end{bmatrix}$$

$$Dg(f(x)) \cdot Df(x) = \begin{bmatrix} 1 & 1 \\ \cos x & \sin x \end{bmatrix} \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix} = \begin{bmatrix} \cos x - \sin x \\ \cos^2 x - \sin^2 x \end{bmatrix} = Dh(x)$$

Example 642 *Take*

$$g : \mathbb{R}^k \to \mathbb{R}^n, \qquad : t \mapsto g(t)$$

$$f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m, \qquad : (x,t) \mapsto f(x,t)$$

$$h : \mathbb{R}^k \to \mathbb{R}^m, \qquad : t \mapsto f(g(t), t)$$

Then

$$\widetilde{g} := (g, id_{\mathbb{R}^k}) : \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k, \qquad t \mapsto (g(t), t)$$

and

$$h = f \circ \widetilde{g} = f \circ (g, id_{\mathbb{R}^k},)$$

Therefore, assuming that f, g, h *are differentiable,*

$$[Dh(t_0)]_{m\times k} = [Df(g(t_0), t_0)]_{m\times(n+k)} \cdot \begin{bmatrix} Dg(t_0) \\ I \end{bmatrix}_{(n+k)\times k} =$$

$$= \left[[D_x f(g(t_0), t_0)]_{m\times n} \mid [D_t f(g(t_0), t_0)]_{m\times k} \right] \cdot \begin{bmatrix} [Dg(t_0)]_{n\times k} \\ I_{k\times k} \end{bmatrix} =$$

$$[D_x f(g(t_0), t_0)]_{m\times n} \cdot [Dg(t_0)]_{n\times k} + [D_t f(g(t_0), t_0)]_{m\times k}$$

In the case $k = n = m = 1$, *the above expression*

$$\frac{df(x = g(t), t)}{dt} = \frac{\partial f(g(t), t)}{\partial x} \frac{dg(t)}{dt} + \frac{\partial f(g(t), t)}{\partial t}$$

or

$$\frac{df(g(t), t)}{dt} = \frac{\partial f(x,t)}{\partial x}_{|x=g(t)} \cdot \frac{dg(t)}{dt} + \frac{\partial f(x,t)}{\partial t}_{|x=g(t)}$$

14.2 Mean value theorem

Proposition 643 *(Mean Value Theorem) Let S be an open subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}^m$ a differentiable function. Let $x, y \in S$ be such that the line segment joining them is contained in S, i.e.,*

$$L(x, y) := \{z \in \mathbb{R}^n : \exists \lambda \in [0, 1] \text{ such that } z = (1 - \lambda)x + \lambda y\} \subseteq S.$$

Then

$$\forall a \in \mathbb{R}^m, \quad \exists\, z \in L(x, y) \quad \text{such that} \quad a \cdot \ [f(y) - f(x)] = a \cdot [Df(z) \cdot (y - x)]$$

Remark 644 *Under the assumptions of the above theorem, the following conclusion is* ***false:***

$$\exists\, z \in L(x, y) \text{ such that } f(y) - f(x) = Df(z) \cdot (y - x).$$

But if $f : S \to \mathbb{R}^{m=1}$, then setting $a \in \mathbb{R}^{m=1}$ equal to 1, we get that the above statement is indeed true.

Example 645 *Let $f : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$. Then*

$$Df(t) \cdot u = u(-\sin t, \cos t)$$

for every real u. The Mean-Value formula

$$f(y) - f(x) = Df(z) \cdot (y - x)$$

cannot hold when $x = 0$ and $y = 2\pi$, since the left member is zero and the right member is a vector of length 2π.

Proof. of Proposition 643

Define $u = y - x$. Since S is open and $L(x, y) \subseteq S$, $\exists \delta > 0$ such that $\forall t \in (-\delta, 1 + \delta)$ such that $x + tu = (1 - t)x + ty \in S$. Taken $a \in \mathbb{R}^m$, define

$$F : (-\delta, 1 + \delta) \to \mathbb{R}, \qquad : t \mapsto a \cdot f(x + tu) = \textstyle\sum_{j+1}^{m} a_j \cdot f_j(x + tu)$$

Then

$$F'(t) = \sum_{j+1}^{m} a_j \cdot [Df_j(x + tu)]_{1 \times n} \cdot u_{n \times 1} = a_{1 \times m} \cdot [Df(x + tu)]_{m \times n} \cdot u_{n \times 1}$$

and F is continuous on $[0, 1]$ and differentiable on $(0, 1)$; then, we can apply "Calculus 1" Mean Value Theorem and conclude that

$$\exists \theta \in (0, 1) \text{ such that } F(1) - F(0) = F'(\theta),$$

and by definition of F and u,

$$\exists \theta \in (0, 1) \text{ such that } f(y) - f(x) = a \cdot Df(x + \theta u) \cdot (y - x)$$

which choosing $z = x + \theta u$ gives the desired result. ∎

Remark 646 *Using the results we have seen on directional derivatives, the conclusion of the above theorem can be rewritten as follows.*

$$\exists\, z \in L(x, y) \text{ such that } f(y) - f(x) = f'(z; y - x)$$

As in the case of real functions of real variables, the Mean Value Theorem allows to give a simple relationship between the sign of the derivative and monotonicity of the function.

Definition 647 *A set $C \subseteq \mathbb{R}^n$ is convex if $\forall x_1,\ x_2 \in C$ and $\forall \lambda \in [0, 1]$, $(1 - \lambda)x_1 + \lambda x_2 \in C$.*

Proposition 648 *Let S be an open and convex subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}^m$ a differentiable function. If $\forall x \in S$, $df_x = 0$, then f is constant on S.*

Proof. Take arbitrary $x, y \in S$. Then since S is convex and f is differential, from the Mean Value Theorem, we have that

$$\forall a \in \mathbb{R}^m, \quad \exists\, z \in L(x,y) \quad \text{such that} \quad a\ [f(y) - f(x)] = a\,[Df(z)\cdot(y-x)] = 0.$$

Taken $a = f(y) - f(x)$, we get that

$$\|f(y) - f(x)\| = 0$$

and therefore

$$f(x) = f(y),$$

as desired. ■

Definition 649 *Given* $x := (x_i)_{i=1}^n, y := (y_i)_{i=1}^n \in \mathbb{R}^n$,

$$x \geq y \quad \textit{means} \quad \forall i \in \{1, ..., n\}, \quad x_i \geq y_i;$$

$$x > y \quad \textit{means} \quad x \geq y \quad \wedge \quad x \neq y;$$

$$x \gg y \quad \textit{means} \quad \forall i \in \{1, ..., n\}, \quad x_i > y_i.$$

Definition 650 $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ *is increasing if* $\forall x, y \in S$, $x > y \Rightarrow f(x) \geq f(y)$.
f is strictly increasing if $\forall x, y \in S$, $x > y \Rightarrow f(x) > f(y)$.

Proposition 651 *Take an open, convex subset S of $\mathbb{R}^n$, and $f : S \to \mathbb{R}$ differentiable.*
1. If $\forall x \in S$, $Df(x) \geq 0$, then f is increasing;
2. If $\forall x \in S$, $Df(x) >> 0$, then f is strictly increasing.

Proof. 1. Take $y \geq x$. Since S is convex, $L(x,y) \subseteq S$. Then from the Mean Value Theorem,

$$\exists\, z \in L(x,y) \text{ such that } f(y) - f(x) = Df(z)\cdot(y-x)$$

Since $y - x \geq 0$ and $Df(z) \geq 0$, the result follows.
2. Take $x > y$. Since S is convex, $L(x,y) \subseteq S$. Then from the Mean Value Theorem,

$$\exists\, z \in L(x,y) \text{ such that } f(y) - f(x) = Df(z)\cdot(y-x) \tag{14.5}$$

Since $y - x > 0$ and $Df(z) >> 0$, the result follows. ■

Remark 652 *The statement "If $\forall x \in S$, $Df(x) > 0$, then f is strictly increasing"is false as verified below.*
We want to find $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ such that $\forall x \in S$, $Df(x) > 0$ and f is not strictly increasing. Take

$$f : \mathbb{R}^n \to \mathbb{R}, \quad (x_1, x_2) \mapsto x_1. \tag{14.6}$$

Then

$$Df(x) = (1,0) > (0,0). \tag{14.7}$$

Now we want to to show that is false that f is strictly increasing, i.e., we want to show that we can have

$$x' := (x_1', x_2') > (x_1'', x_2'') := x'' \quad \textit{and} \quad f(x') \leq f(x'') \tag{14.8}$$

Take

$$x' = (0,2) \quad \textit{and} \quad x'' = (0,1) \tag{14.9}$$

Then

$$f(x') = 0 = f(x'') \tag{14.10}$$

as desired.

Corollary 653 *Take an open, convex subset S of $\mathbb{R}^n$, and $f \in C^1(S, \mathbb{R})$. If $\exists x_0 \in S$ and $u \in \mathbb{R}^n \setminus \{0\}$ such that $f'(x_0, u) > 0$, then $\exists\, \bar{t} \in \mathbb{R}_{++}$ such that $\forall t \in [0, \bar{t})$,*

$$f(x_0 + tu) \geq f(x_0).$$

Proof. Since f is C^1 and $f'(x_0, u) = Df(x_0) \cdot u > 0$, $\exists r > 0$ such that

$$\forall x \in B(x_0, r), \quad f'(x, u) > 0.$$

Then $\forall t \in (-r, r)$, $\left\| x_0 + \frac{1}{\|u\|} tu - x_0 \right\| = t < r$, and therefore

$$f'\left(x_0 + \frac{t}{\|u\|} u, u\right) > 0$$

Then, from the Mean Value Theorem, $\forall t \in \left[0, \frac{r}{2}\right]$,

$$f(x_0 + tu) - f(x_0) = f'(x_0 + tu, u) \geq 0.$$

■

Definition 654 *Given a function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$, $x_0 \in S$ is a point of local maximum for f if*

$$\exists \delta > 0 \text{ such that } \forall x \in B(x_0, \delta), \quad f(x_0) \geq f(x);$$

x_0 is a point of global maximum for f if

$$\forall x \in S, \quad f(x_0) \geq f(x).$$

$x_0 \in S$ is a point of strict local maximum for f if

$$\exists \delta > 0 \text{ such that } \forall x \in B(x_0, \delta), \quad f(x_0) > f(x);$$

x_0 is a point of strict global maximum for f if

$$\forall x \in S, \quad f(x_0) > f(x).$$

Local, global, strict minima are defined in obvious manner

Proposition 655 *If $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$ admits all partial derivatives in $x_0 \in Int\ S$ and x_0 is a point of local maximum or minimum, then $Df(x_0) = 0$.*

Proof. Since x_0 is a point of local maximum, $\exists \delta > 0$ such that $\forall x \in B(x_0, \delta)$, $f(x_0) \geq f(x)$. As in Remark 596, for any $k \in \{1, ..., n\}$, define

$$g_k : \mathbb{R} \to \mathbb{R}, \qquad g(x_k) = f\left(x_0 + (x_k - x_{0k})\, e_n^k\right).$$

Then g_k has a local maximum point at x_{0k}. Then from Calculus 1,

$$g_k'(x_{0k}) = 0$$

Since, again from Remark 596, we have

$$D_k f(x_0) = g_k'(x_0).$$

the result follows. ■

14.3 A sufficient condition for differentiability

`Proofs in the present and next section have to be carefully read (see also my handwritten notes).`

Definition 656 *$f = (f_i)_{i=1}^m : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable on $A \subseteq S$, or f is C^1 on S, or $f \in \mathcal{C}(A, \mathbb{R}^m)$ if $\forall i \in \{1, ..., m\}, j \in \{1, ..., n\}$,*

$$D_{x_j} f_i : A \to \mathbb{R}, \qquad x \mapsto D_{x_j} f_i(x) \qquad \text{is continuous.}$$

Definition 657 *$f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable on $A \subseteq S$, or f is C^2 on S, or $f \in \mathcal{C}^2(A, \mathbb{R}^m)$ if $\forall j, k \in \{1, ..., n\}$,*

$$\frac{\partial^2 f}{\partial x_j \partial x_k} : A \to \mathbb{R}, \qquad x \mapsto \frac{\partial^2 f}{\partial x_j \partial x_k}(x) \qquad \text{is continuous.}$$

The proof of the main result of the section, i.e., Proposition 661, requires some Lemmas.

Lemma 658 *If L is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$, then L is differentiable.*

Proof. *We want to show that there exists a linear function $T_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$ such that if $v \in B(x_0, \varepsilon) \subseteq S$, then*

$$L(x_0 + v) = L(x_0) + T_{x_0}(v) + \|v\| \cdot E_{x_0}(v) \quad \text{with} \quad \lim_{v \to 0} E_{x_0}(v) = 0 \tag{14.11}$$

Take

$$T_{x_0} = L. \tag{14.12}$$

Then by the definition of linear function

$$L(x_0 + v) = L(x_0) + L(v) = L(x_0) + T_{x_0}(v) \tag{14.13}$$

and taken

$$E_{x_0} = 0,$$

we get the desired result. ■

Lemma 659 *The projection function*

$$\pi^j : \mathbb{R}^m \to \mathbb{R}, \quad \pi^j : (y_1, ..., y_j, ..., y_n) \mapsto y_j \tag{14.14}$$

is differentiable.

Proof. *The claim is true because of Lemma 658 and because π^j is linear as shown below. Given x,y $\in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, then*

$$\pi^j(x + y) = x_j + y_j = \pi^j(x) + \pi^j(y) \quad \text{and} \quad \pi^j(\alpha x) = \alpha x_j = \alpha \pi^j(x) \tag{14.15}$$

■

Lemma 660 *Given $f : \mathbb{R}^n \to \mathbb{R}^m$, $f = (f_1, ..., f_j, ..., f_m)$, then f is differentiable if and only if f_j is differentiable for any $j \in \{1, ..., m\}$.*

Proof. *[⇒] Observe that $f_j = \pi^j \circ f$, where π^j is the projection function. Since π^j is differentiable from Lemma 659, and f is differentiable by assumption, from the Chain Rule, i.e. Proposition 635, f_j is differentiable.*

[⇐] By assumption

$$f_j(x_0 + v) = f_j(x_0) + Df_j(x_0) \cdot v + \|v\| \cdot E^j_{x_0}(v) \tag{14.16}$$

with

$$\lim_{v \to 0} E^j_{x_0}(v) = 0 \tag{14.17}$$

for any j. Therefore

$$f(x_0 + v)_{m \times n} = f(x_0)_{m \times 1} + Df(x_0)_{m \times n} \cdot v_{n \times 1} + \|v\| \begin{pmatrix} E^1_{x_0}(v) \\ . \\ . \\ . \\ E^m_{x_0}(v) \end{pmatrix}_{m \times n} \tag{14.18}$$

with

$$\lim_{v \to 0} \begin{pmatrix} E^1_{x_0}(v) \\ . \\ . \\ . \\ E^m_{x_0}(v) \end{pmatrix}_{m \times n} = 0, \tag{14.19}$$

as desired ■

Proposition 661 *If $f : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is C^1 in an open neighborhood of x_0, then it is differentiable in x_0.*

Proof. By assumption $\exists \delta > 0$ such that f is C^1 in $B(x_0, \delta)$. Since, by definition,

$$f \;\; is \;\; C^1 \Leftrightarrow for \;\; any \;\; j \in \{1, ..., m\}, \;\; f_j \;\; is \;\; C^1$$

and since, from Lemma 660,

$$f \;\; is \;\; differentiable \Leftrightarrow for \;\; any \;\; j \in \{1, ..., m\}, \;\; f_j \;\; is \;\; differentiable,$$

then it is enough to show that for any $j \in \{1, ..., m\}$, f_j is differentiable. For simplicity of notation- dropping the subscript j- we have to show that $f : S \to \mathbb{R}$ is differentiable at x_0.

Take an arbitrary $v \in \mathbb{R}^n \setminus \{0\}$ such that $x_0 + v \in B\left(x_0, \frac{\delta}{2}\right)$ and therefore

$$\|v\| < \frac{\delta}{2}. \tag{14.20}$$

Then recalling that $e_n^k := (0, ..., 1, ..., 0) \in \mathbb{R}^n$, define

$$\lambda = \|v\|, \qquad y = \tfrac{v}{\|v\|}, \qquad \text{and then } y = \lambda v, \tag{14.21}$$

and

$$v_0 = 0, \qquad\qquad v_1 = y_1 e_n^1 = (y_1, 0, ..., 0)$$

$$v_2 = y_1 e_n^1 + y_2 e_n^2 = v_1 + y_2 e_n^2 = (y_1, y_2, 0, ..., 0), \qquad ...$$

$$v_k = \textstyle\sum_{h=1}^k y_h e_n^h = v_{k-1} + y_k e_n^k = (y_1, y_2, ..., y_k, 0, ..., 0), \qquad ...$$

$$v_n = \textstyle\sum_{h=1}^n y_n e_n^h = y$$

Therefore,

$$\begin{aligned} &\textstyle\sum_{k=1}^n \left[f(x_0 + \lambda v_k) - f(x_0 + \lambda v_{k-1})\right] = \\ &= f(x_0 + \lambda v_1) - f(x_0 + \lambda v_0) + f(x_0 + \lambda v_2) - f(x_0 + \lambda v_1) + ... + f(x_0 + \lambda v_n) - f(x_0 + \lambda v_{n-1}) = \\ &= f(x_0 + \lambda v_n) - f(x_0 + \lambda v_0) = f(x_0 + \lambda y) - f(x_0 + 0) = f(x_0 + v) - f(x_0). \end{aligned} \tag{14.22}$$

For any $k \in \{1, ..., m\}$, define $b_k = x_0 + \lambda v_{k-1}$, i.e.,

$$b_k = x_0 + \lambda v_{k-1} = (x_{0_1} + \lambda y_1, ..., x_{0,k-1} + \lambda y_{k-1}, x_{0,k}, ..., x_{0,n})$$

and

$$b_{k+1} = b_k + \lambda y_k e_n^k = (x_{0_1} + \lambda y_1, ..., x_{0,k-1} + \lambda y_{k-1}, x_{0,k} + \lambda x_{0,k}, x_{0,k+1}, ..., x_{0,n}).$$

Therefore

$$f(x_0 + \lambda v_k) - f(x_0 + \lambda v_{k-1}) = f\left(x_0 + \lambda v_{k-1} + \lambda y_k e_n^k\right) - f(x_0 + \lambda v_{k-1}) = f\left(b_k + \lambda y_k e_n^k\right) - f(b_k) \tag{14.23}$$

From (14.22) and (14.23) we get

$$f(x_0 + v) - f(x_0) = \sum_{k=1}^n \left[f\left(b_k + \lambda y_k e_n^k\right) - f(b_k)\right]. \tag{14.24}$$

Observe that the $k - th$ term in the sum in (14.24) is

$$f\left(b_k + \lambda y_k e_n^k\right) - f(b_k) = f(b_k + (0, ..., \lambda y_k, ..., 0)) - f(b_k).$$

Moreover, if

$$|x_k| < 2\lambda |y_k|, \tag{14.25}$$

then

$$|x_k| < 2\lambda |y_k| < 2\|v\| \cdot \frac{\|v\|}{\|v\|} = 2\|v\| < \delta, \tag{14.26}$$

and also

$$b_k + x_k e_n^k \overset{\text{def. } b_k, v_k}{=} x_0 + \lambda v_{k-1} + x_k e_n^k = x_0 + (\lambda y_1, ..., \lambda y_{k-1}, x_k, 0, ..., 0)$$

is such that

$$\left\| b_k + x_k e_n^k - x_0 \right\| = \|\lambda y_1, ..., \lambda y_{k-1}, x_k, 0, ..., 0\| \leq \lambda \left(\sum_{i \neq k} (y_i)^2 + \left(\tfrac{x_k}{\lambda}\right)^2 \right)^{\frac{1}{2}} \overset{(14.25)}{<}$$

$$< \lambda \left(\sum_{i \neq k} (y_i)^2 + \left(\tfrac{2\lambda y_k}{\lambda} \right)^2 \right)^{\frac{1}{2}} = 2\lambda \|y\| \overset{(14.21)}{=} 2\|v\| \overset{(14.20)}{<} \delta.$$

Summarizing, if $|x_k| < 2\lambda |y_k|$, then

$$b_k + x_k e_n^k \in B(x_0, \delta). \tag{14.27}$$

Now, define for any $k \in \{1, ..., m\}$,

$$g_k : (-2\lambda |y_k|, +2\lambda |y_k|) \to \mathbb{R}, \quad x_k \mapsto g_k(x_k) := f\left(b_k + x_k e_n^k\right).$$

Then using the same trick used in Remark 596 we have that for any $x_k \in (-2\lambda |y_k|, +2\lambda |y_k|)$,

$$\lim_{h \to 0} \frac{g_k(x_k + h) - g_k(x_k)}{h} = \lim_{h \to 0} \frac{f\left(b_k + x_k e_n^k + h e_n^k\right) - f\left(b_k + x_k e_n^k\right)}{h} = D_{x_k} f\left(b_k + x_k e_n^k\right),$$

i.e., g_k is differentiable. Then we can use the Intermediate Value Theorem for differentiable functions from subsets of $\mathbb{R}$ to $\mathbb{R}$ applying it to $g_k : [0, \lambda y_k] \subseteq (-2\lambda |y_k|, +2\lambda |y_k|) \to \mathbb{R}$ or to $g_k : [\lambda y_k, 0] \subseteq (-2\lambda |y_k|, +2\lambda |y_k|) \to \mathbb{R}$ to conclude that there exists $\beta_k \in (0, \lambda y_k)$ such that $g_k(\lambda y_k) - g_k(0) = g'(\beta_k) \cdot (\lambda y_k)$, or, there exists $\gamma_k \in (\lambda y_k, 0)$ such that $g_k(0) - g_k(\lambda y_k) = g'(\gamma_k) \cdot (-\lambda y_k)$, i.e., $-g_k(0) + g_k(\lambda y_k) = g'(\gamma_k) \cdot (\lambda y_k)$.

Summarizing,

for any $k \in \{1, ..., m\}$ there exists $\alpha_k \in (0, \lambda y_k)$ or $\alpha_k \in (\lambda y_k, 0)$ such that

$$g_k(\lambda y_k) - g_k(0) = g'(\alpha_k) \cdot (\lambda y_k),$$

and

$$\begin{aligned} &\text{for any } k \in \{1, ..., m\} \text{ there exists } a_k := b_k + \alpha_k e_n^k \in L\left(b_k, b_k + \lambda y_k e_n^k\right) \text{ such that} \\ &f\left(b_k + \lambda y_k e_n^k\right) - f(b_k) = D_{x_k} f(a_k) \cdot \lambda y_k. \end{aligned} \tag{14.28}$$

Now, $\lim_{\lambda \to 0} a_k \overset{\text{def. } a_k}{=} \lim_{\lambda \to 0} \left(b_k + \alpha_k e_n^k\right) \overset{\text{def. } b_k}{=} \lim_{\lambda \to 0} \left(x_0 + \lambda v_{k-1} + \alpha_k e_n^k\right) \overset{\lambda \to 0 \Rightarrow \alpha_k \to 0}{=} x_0$. Then, since f is C^1, we do have $\lim_{\lambda \to 0} D_{x_k} f(a_k) = D_{x_k} f(x_0)$ and also

$$\text{for any } k \in \{1, ..., m\}, \qquad \lim_{\alpha_k \to x_0} D_k f(\alpha_k) = \lim_{\lambda \to 0} D_k f(\alpha_k) = D_k f(x_0)$$

we have

$$\text{for any } k \in \{1, ..., m\}, \qquad D_{x_k} f(\alpha_k) = D_k f(x_0) + E_k(\lambda) \quad \text{with} \quad \lim_{\lambda \to 0} E_k(\lambda) = 0. \tag{14.29}$$

Inserting (14.29) in (14.28), we get

$$f\left(b_k + \lambda y_k e_n^k\right) - f(b_k) = \lambda y_k D_k f(x_0) + \lambda y_k E_k(\lambda) \tag{14.30}$$

Inserting (14.30) in (20.3), we get

$$f(x_0 + v) - f(x_0) = \lambda \sum_{k=1}^{n} y_k D_k f(x_0) + \lambda \sum_{k=1}^{n} y_k E_k(\lambda) = Df(x_0) \cdot \lambda y + \|v\| \cdot E_{x_0}(v) \tag{14.31}$$

where

$$E_{x_0}(v) := \sum_{k=1}^{n} y_k E_k(\lambda).$$

Then from (14.29), we have

$$\lim_{v\to 0} E_{x_0}(v) = \lim_{\lambda\to 0}\sum_{k=1}^{n} y_k E_k(\lambda) = 0. \tag{14.32}$$

(14.31) and (14.32) are the definition of differentiability. ■

Example 662 *Consider* $f:\mathbb{R}^2\to\mathbb{R}^2,\quad f:(x,y)\mapsto(\sin xy,\cos xy)$. *The Jacobian of* f *is*

$$\begin{bmatrix} y\cos xy & x\cos xy \\ -y\sin xy & -x\sin xy \end{bmatrix}$$

So it is clear that f *is* C^1 *on* $\mathbb{R}^2$ *because each partial derivative exists and is continuous for any* $(x,y)\in\mathbb{R}^2$ *and therefore* f *is differentiable and its derivative is its Jacobian.*

Remark 663 *The above result is the answer to Question 3 in Remark 624. To show that* $f:\mathbb{R}^m\to\mathbb{R}^n$ *is differentiable, it is enough to verify that all its partial derivatives, i.e., the entries of the Jacobian matrix, are continuous functions.*

14.4 A sufficient condition for equality of mixed partial derivatives

Remark 664 *We may have*

$$\frac{\partial^2 f}{\partial x_i \partial x_k}(x_0) \neq \frac{\partial^2 f}{\partial x_k \partial x_i}(x_0),$$

as shown below.

Consider

$$f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=\begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & if \quad (x,y)\neq 0 \\ 0 & if \quad (x,y)=0 \end{cases}$$

We want to check that

$$\frac{\partial^2 f}{\partial x\partial y}(0,0)\neq\frac{\partial^2 f}{\partial y\partial x}(0,0)$$

Indeed,

$$if\ \ (x,y)\neq 0,\quad \frac{\partial f}{\partial x}(x,y)=\left[y\left(3x^2-y^2\right)\left(x^2+y^2\right)-2x\left(x^3-xy^2\right)y\right]\left(x^2+y^2\right)^{-2}=$$

$$=y\left(x^4-y^4+4x^2y^2\right)\left(x^2+y^2\right)^{-2},$$

and

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h\cdot 0\cdot h^2}{h^2}-0}{h}=\lim_{h\to 0}0=0;$$

therefore

$$\frac{\partial\left(\frac{\partial f(x,y)}{\partial x}\right)}{\partial y}(0,0)=\lim_{h\to 0}\frac{\frac{\partial f}{\partial x}(0,h)-\frac{\partial f}{\partial x}(0,0)}{h}=\lim_{h\to 0}\frac{-h^5}{h^5}=-1.$$

Similarly, we have that

$$if\ \ (x,y)\neq 0,\quad \frac{\partial f}{\partial y}(x,y)=\left[\left(x\left(x^2-y^2\right)-2y(xy)\right)\left(x^2+y^2\right)-2y(xy)\left(x^2-y^2\right)\right]\left(x^2+y^2\right)^{-2},$$

and

$$\frac{\partial f}{\partial y}(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{0\cdot h\cdot -h^2}{h^2}-0}{h}=\lim_{h\to 0}0=0,$$

$$\frac{\partial\left(\frac{\partial f(x,y)}{\partial y}\right)}{\partial x}(0,0)=\lim_{h\to 0}\frac{\frac{\partial f}{\partial y}(h,0)-\frac{\partial f}{\partial y}(0,0)}{h}=\lim_{h\to 0}\frac{h^5}{h^5}=1\neq -1.$$

The following Proposition gives a sufficient condition to avoid the above nuisance.

Proposition 665 *Given $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in Int(S)$, if, for any $i, k \in 1, ..., n$,*
1. *$\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial x_k}$ exist on $B(x_0, \delta) \subseteq S$, and*
2. *they are differentiable in x_0, then*

$$\frac{\partial^2 f}{\partial x_i \partial x_k}(x_0) = \frac{\partial^2 f}{\partial x_k \partial x_i}(x_0) \tag{14.33}$$

Proof. *Since we are considering only two variables (x_i and x_k) with respect to which we differentiate, and we are keeping the other variables fixed, without loss of generality, we can restrict ourselves to a function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$. Finally, for computational simplicity, we consider $x_0 = (0, 0)$. Therefore we have to prove that*

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{\partial^2 f}{\partial y \partial x}(0, 0)$$

Consider h sufficiently small so that $(h, h), (h, 0), (0, h) \in B(0, \delta)$. Define

$$\Delta(h) = f(h, h) - f(h, 0) - f(0, h) + f(0, 0)$$

and

$$\phi : (-\delta, \delta) \to \mathbb{R}, \qquad \phi(x) = f(x, h) - f(x, 0)$$

Then

$$\Delta(h) = \phi(h) - \phi(0)$$

and since, by Assumption 1., $\frac{\partial f}{\partial x}$ exists on $B(0, \delta) \subseteq S$

$$\phi'(x) = \frac{\partial f}{\partial x}(x, h) - \frac{\partial f}{\partial x}(x, 0). \tag{14.34}$$

Since, by Assumption 2., ϕ is differentiable on $(-\delta, \delta)$, we can apply the "One-dimensional Mean Value Theorem" to ϕ on $[0, h]$ therefore, $\exists \hat{\theta}_1 := \theta_1 h$ with $\theta_1 \in (0, 1)$ such that

$$\Delta(h) = \phi(h) - \phi(0) = (h - 0)\phi'(\theta_1 h) = h\left[\frac{\partial f}{\partial x}(\theta_1 h, h) - \frac{\partial f}{\partial x}(\theta_1 h, 0)\right]. \tag{14.35}$$

Define

$$g : (-h, h) \to \mathbb{R}, \qquad g(y) = \frac{\partial f}{\partial x}(\theta_1 h, y), \tag{14.36}$$

so that

$$g'(y) = \frac{\partial^2 f}{\partial x \partial y}(\theta_1 h, y). \tag{14.37}$$

Then

$$h[g(h) - g(0)] \overset{(14.36)}{=} h\left[\frac{\partial f}{\partial x}(\theta_1 h, h) - \frac{\partial f}{\partial x}(\theta_1 h, 0)\right] \overset{(14.35)}{=} \Delta(h) \tag{14.38}$$

Now we apply the "One dimensional Mean Value Theorem" to $g(y)$ on $[0, h]$, so that $\exists \hat{\theta}_2 = \theta_2 h$ with $\theta_2 \in (0, 1)$ such that

$$h \cdot g'(\theta_2 h) = g(h) - g(0) \tag{14.39}$$

and

$$\Delta(h) \overset{(14.38),(14.39)}{=} h^2 g\prime(\theta_2 h) \overset{(14.37)}{=} h^2 \frac{\partial^2 f}{\partial x \partial y}(\theta_1 h, \theta_2 h). \tag{14.40}$$

We might have started by expressing $\Delta(h)$ as follows

$$\Delta(h) = \psi(h) - \psi(0)$$

with

$$\psi(y) = f(h, y) - f(0, y)$$

Now we can follow the same procedure above to show a needed result similar to the one in (14.40). Define

$$\Delta(h) = f(h,h) - f(h,0) - f(0,h) + f(0,0)$$

and

$$\psi : (-\delta,\delta) \to \mathbb{R}, \qquad \psi(y) = f(h,y) - f(0,y)$$

Then

$$\Delta(h) = \psi(h) - \psi(0)$$

and since, by assumption 1., $\frac{\partial f}{\partial y}$ *exists on* $B(0,\delta) \subseteq S$

$$\psi'(y) = \frac{\partial f}{\partial y}(h,y) - \frac{\partial f}{\partial y}(0,y). \tag{14.41}$$

Since, by assumption 2., ψ *is differentiable on* $(-\delta,\delta)$*, we can apply the "One-dimensional Mean Value Theorem" to* ψ *on* $[0,h]$ *so that* $\exists\hat{\theta}_3 := \theta_3 h$ *with* $\theta_3 \in (0,1)$ *such that*

$$\Delta(h) = \psi(h) - \psi(0) = (h-0)\,\psi'(\theta_3 h) = h\left[\frac{\partial f}{\partial y}(h,\theta_3 h) - \frac{\partial f}{\partial y}(0,\theta_3 h)\right]. \tag{14.42}$$

Define

$$g : (-h,h) \to \mathbb{R}, \qquad g(x) = \frac{\partial f}{\partial y}(x,\theta_3 h), \tag{14.43}$$

so that

$$g'(x) = \frac{\partial^2 f}{\partial y \partial x}(x,\theta_3 h). \tag{14.44}$$

Then

$$h\,[g(h) - g(0)] \overset{14.43}{=} h\left[\frac{\partial f}{\partial y}(h,\theta_3 h) - \frac{\partial f}{\partial y}(0,\theta_3 h)\right] \overset{14.42}{=} \Delta(h) \tag{14.45}$$

Now we apply the "One dimensional Mean Value Theorem" to $g(x)$ *on* $[0,h]$*, so that* $\exists\hat{\theta}_4 = \theta_4 h$ *with* $\theta_4 \in (0,1)$ *such that*

$$h \cdot g'(\theta_4 h) = g(h) - g(0) \tag{14.46}$$

and

$$\Delta(h) \overset{(14.45),(14.46)}{=} h^2 g'(\theta_4 h) \overset{(14.44)}{=} h^2 \frac{\partial^2 f}{\partial y \partial x}(\theta_4 h, \theta_3 h) \tag{14.47}$$

From (14.40) and (14.47) we get

$$\frac{\partial^2 f}{\partial x \partial y}(\theta_1 h, \theta_2 h) = \frac{\partial^2 f}{\partial y \partial x}(\theta_4 h, \theta_3 h)$$

Taking the limits for $h \to 0$*, we have* $\theta_i h \to 0$ *for* $i = 1,2,3,4$ *and then by the continuity of* $\frac{\partial^2 f}{\partial x \partial y}$ *and* $\frac{\partial^2 f}{\partial y \partial x}$*, i.e., Assumption 2., we get* $\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0)$*, as desired* ■

14.5 Taylor's theorem for real valued functions

To get Taylor's theorem for a function $f : \mathbb{R}^m \to \mathbb{R}$, we introduce some notation in line with the definition of directional derivative, i.e., $f'(x,u) = \sum_{i=1}^n D_{x_i} f(x) \cdot u_i$.

Definition 666 *Assume* S *is an open subset of* $\mathbb{R}^m$*, the function* $f : S \to \mathbb{R}$ *admits partial derivatives at least up to order* m*,* $x \in S$*,* $u \in \mathbb{R}^m$*. Then*

$$f''(x;u) := \sum_{i=1}^n \sum_{j=1}^n D_{i,j} f(x) \cdot u_i \cdot u_j,$$

$$f'''(x;u) := \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n D_{i,j,k} f(x) \cdot u_i \cdot u_j \cdot u_k$$

and similar definition applies to $f^{(m)}(x,u)$*.*

Proposition 667 *(Taylor's formula) Assume S is an open subset of $\mathbb{R}^m$, the function $f : S \to \mathbb{R}$ admits partial derivatives at least up to order m, $x \in S$, $u \in \mathbb{R}^m$. Assume also that all its partial derivative of order $< m$ are differentiable. If y and x are such that $L(y, x) \subseteq S$, then there exists $z \in L(y, x)$ such that*

$$f(y) = f(x) + \sum_{k=1}^{m-1} \frac{1}{k!} f^{(k)}(x, y - x) + \frac{1}{m!} f^{(m)}(z, y - z).$$

Proof. . Since S is open and $L(x, y) \subseteq S$, $\exists \delta > 0$ such that $\forall t \in (-\delta, 1 + \delta)$ we have $x + t(y - x) \in S$. Define $g : (-\delta, 1 + \delta) \to \mathbb{R}$

$$g(t) = f(x + t(y - x)).$$

From standard "Calculus 1" Taylor's theorem, we have that $\exists \theta \in (0, 1)$ such that

$$f(y) - f(x) = g(1) - g(0) = \sum_{k=1}^{m-1} \frac{1}{k!} g^{(k)}(0) + \frac{1}{m!} g^{(m)}(\theta).$$

Moreover,

$$g'(t) = Df(x + t(y - x)) \cdot (y - x) = \sum_{i=1}^{n} D_{x_i} f(x + t(y - x)) \cdot (y_i - x_i) = f'(x + t(y - x), y - x),$$

$$g''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_{x_i, x_j} f(x + t(y - x)) \cdot (y_i - x_i) \cdot (y_j - x_j) = f''(x + t(y - x), y - x)$$

and similarly

$$g^{(m)}(t) = f'^{(m)}(x + t(y - x), y - x)$$

Then the desired result follow substituting 0 in the place of t where needed and choosing $z = x + \theta(y - x)$.
■

Chapter 15

Implicit function theorem

15.1 Some intuition

Below, we present an *informal discussion* of the Implicit Function Theorem. Assume that

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad (x,t) \mapsto f(x,t)$$

is at least C^1. The basic goal is to study the nonlinear equation

$$f(x,t) = 0,$$

where x can be interpreted as an endogenous variable and t as a parameter (or an exogenous variable). Assume that

$$\exists \left(x^0, t^0\right) \in \mathbb{R}^2 \text{ such that } f\left(x^0, t^0\right) = 0$$

and for some $\varepsilon > 0$

$$\exists \text{ a } C^1 \text{ function } g : \left(t^0 - \varepsilon, t^0 + \varepsilon\right) \to \mathbb{R}, t \mapsto g(t)$$

such that

$$f(g(t), t) = 0 \tag{15.1}$$

We can then say that g describes *the solution* to the equation

$$f(x,t) = 0,$$

in the unknown variable x and parameter t, in an open neighborhood of t^0. Therefore, using the Chain Rule - and in fact, Remark 642 - applied to both sides of (15.1), we get

$$\frac{\partial f(x,t)}{\partial x}_{|x=g(t)} \cdot \frac{dg(t)}{dt} + \frac{\partial f(x,t)}{\partial t}_{|x=g(t)} = 0$$

and

$$\text{assuming that } \frac{\partial f(x,t)}{\partial x}_{|x=g(t)} \neq 0$$

we have

$$\frac{dg(t)}{dt} = -\frac{\frac{\partial f(x,t)}{\partial t}_{|x=g(t)}}{\frac{\partial f(x,t)}{\partial x}_{|x=g(t)}} \tag{15.2}$$

The above expression is the derivative of the function implicitly defined by (15.1) close by to the value t^0. In other words, it is the slope of the level curve $f(x,t) = 0$ at the point $(t, g(t))$.

For example, taken

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x,t) \mapsto x^2 + t^2 - 1$$

$f(x,t) = 0$ describes the circle with center in the origin and radius equal to 1. Putting t on the horizontal axis and x on the vertical axis, we have the following picture.

Clearly
$$f((0,1))=0$$
As long as $t\in(-1,1)$, $g(t)=\sqrt{1-t^2}$ is such that
$$f(g(t),t)=0 \tag{15.3}$$
Observe that
$$\frac{d\left(\sqrt{1-t^2}\right)}{dt}=-\frac{t}{\sqrt{1-t^2}}$$
and
$$-\frac{\frac{\partial f(x,t)}{\partial t}_{|x=g(t)}}{\frac{\partial f(x,t)}{\partial x}_{|x=g(t)}}=-\frac{2t}{2x}_{|x=g(t)}=-\frac{t}{\sqrt{1-t^2}}$$
For example for $t=\frac{1}{\sqrt{2}}$, $g'(t)=-\frac{\frac{1}{\sqrt{2}}}{\sqrt{1-\frac{1}{2}}}=-1$.

Let's try to present a more detailed geometrical interpretation[1]. Consider the set $\left[\left\{(x,t)\in\mathbb{R}^2: f(x,t)=0\right\}\right]$ presented in the following picture.

`Insert picture a., page 80 in Sydsaeter (1981).`

In this case, does equation
$$f(x,t)=0 \tag{15.4}$$
define x as a function of t? Certainly, the curve presented in the picture is not the graph of a function with x as dependent variable and t as an independent variable for all values of t in $\mathbb{R}$. In fact,

1. if $t\in(-\infty,t_1]$, there is only one value of x which satisfies equation (15.4);
2. if $t\in(t_1,t_2)$, there are two values of x for which $f(x,t)=0$;
3. if $t\in(t_2,+\infty)$, there are no values satisfying the equation.

If we consider t belonging to an interval contained in (t_1,t_2), we have to to restrict the admissible range of variation of x in order to conclude that equation (15.4) defines x as a function of t in that interval. For example, we see that if the rectangle R is as indicated in the picture , the given equation defines x as a function of t, for well chosen domain and codomain - naturally associated with R. The graph of that function is indicated in the figure below.

`Insert picture b., page 80 in Sydsaeter (1981).`

The size of R is limited by the fact that we need to define a *function* and therefore one and only one value has to be associated with t. Similar rectangles and associated solutions to the equation can be constructed for all other points on the curve, *except one:* (t_2,x_2). Irrespectively of how small we choose the rectangle around that point, there will be values of t close to t_2, say t', such that there are two values of x, say x' and x'', with the property that both (t',x') and (t',x'') satisfy the equation and lie inside the rectangle. Therefore, equation (15.4) does not define x as a function of t in an open neighborhood of the point (t_2,x_2). In fact, there the slope of the tangent to the curve is infinite. If you try to use expression (15.2) to compute the slope of the curve defined by $x^2+t^2=1$ in the point $(1,0)$, you get an expression with *zero* in the denominator.

On the basis of the above discussion, we see that it is crucial to require the condition
$$\frac{\partial f(x,t)}{\partial x}_{|x=g(t)}\neq 0$$
to insure the possibility of locally writing x as a solution (to (15.4)) function of t.

We can informally, summarize what we said as follow.

If f is C^1, $f(x_0,t_0)=0$ and $\frac{\partial f(x,t)}{\partial x}_{|(x,t)=(x^0,t^0)}\neq 0$, then $f(x,t)=0$ define x as a C^1 function g of t in an open neighborhood of t^0, and $g'(t)=-\frac{\frac{\partial f(x,t)}{\partial t}}{\frac{\partial f(x,t)}{\partial x}}_{|x=g(t)}$.

Next sections provide a formal statement and proof of the Implicit Function Theorem. Some work is needed.

[1] This discussion is taken from Sydsaeter (1981), page 80-81.

15.2 Functions with full rank square Jacobian

This section gives some properties of function with nonzero Jacobian determinant at certain points. These results will be used later in the proof of the implicit function theorem.

Proposition 668 *Taken $a \in \mathbb{R}^n$, $r \in \mathbb{R}_{++}$, assume that*
1. $f := (f_i)_{i=1}^n : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\text{Cl }(B(a,r))$;
2. $\forall x \in B(a,r)$, $[Df(x)]_{n\times n}$ exists and $\det Df(x) \neq 0$;
3. $\forall x \in \mathcal{F}(B(a,r))$, $f(x) \neq f(a)$.
Then, $\exists \delta \in \mathbb{R}_{++}$ such that

$$f(B(a,r)) \supseteq B(f(a), \delta),$$

i.e., $f(a) \in Int f(B(a,r))$.

Proof. Define $B := B(a,r)$ and

$$g : \mathcal{F}(B) \to \mathbb{R}, \qquad x \mapsto \|f(x) - f(a)\|.$$

From Assumption 3, $\forall x \in \mathcal{F}(B)$, $g(x) > 0$. Moreover, since g is continuos and $\mathcal{F}(B)$ is compact, then g attains a global minimum value $m > 0$ on $\mathcal{F}(B)$. Taken $\delta = \frac{m}{2}$; we want to show that $T := B(f(a), \delta) \subseteq f(B)$, i.e., $\forall y \in T$, $y \in f(B)$. Define

$$h : \text{Cl }(B) \to \mathbb{R}, \qquad x \mapsto \|f(x) - y\|.$$

Since h is continuous and Cl (B) is compact, h attains a global minimum in a point $c \in$ Cl (B).

Claim 669 *$c \in Int(B)$.*

Proof of the Claim Since $Cl(S) = Int(S) \cup \mathcal{F}(S)$ it suffices to show that $c \notin \mathcal{F}$. Observe that, since $y \in T = B\left(f(a), \delta = \frac{m}{2}\right)$, then

$$h(a) = \|f(a) - y\| < \frac{m}{2} \tag{15.5}$$

Therefore, since c is a global minimum point for h, it must be the case that $h(c) \leq h(a) < \frac{m}{2}$. Now take $x \in \mathcal{F}(B)$; then

$$h(x) = \|f(x) - y\| = \|f(x) - f(a) - (y - f(a))\| \overset{(1)}{\geq}$$

$$\geq \|f(x) - f(a)\| - \|y - f(a)\| \overset{(2)}{\geq} g(x) - \frac{m}{2} \overset{(3)}{\geq} \frac{m}{2} \overset{(15.5)}{>} h(a),$$

where (1) follows from Remark 54, (2) from (15.5) and (3) from the fact that g has minimum value equal to m. Therefore, $[\forall x \in \mathcal{F}(B), \ \ h(x) > h(a)]$ and h does not attain its minimum on $\mathcal{F}(B)$, therefore $c \notin \mathcal{F}(B)$.

End of Proof the Claim

Then h and h^2 get their minimum at $c \in B$ [2]. Since

$$H(x) := h^2(x) = \|f(x) - y\|^2 = \sum_{i=1}^n (f_i(x) - y_i)^2$$

from Proposition 655, $DH(c) = 0$, i.e.,

$$\forall k \in \{1, ..., n\}, \quad 2\sum_{i=1}^n D_{x_k} f_i(c) \cdot (f_i(c) - y_i) = 0$$

i.e.,

$$[Df(c)] \cdot (f(c) - y) = 0.$$

Then, from Assumption 2 and from the Claim above, $Df(c)$ has full rank and therefore

$$f(c) = y,$$

i.e., since $c \in B$, $y \in f(B)$, ad desired. ■

The following pictures show a case where the hypothesis and the conclusion of the above Proposition are satisfied and some cases in which some hypothesis are not satisfied and the conclusion of the theorem is not true. The numbers under each picture indicate the hypothesis which are not satisfied.

[2] For any $x \in Cl(B)$, $h(x) \geq 0$, $c \in B$ and $h(x) \geq h(c) \geq 0$. Therefore, $h^2(x) \geq h^2(c)$.

Definition 670 *Given a set* $X \subseteq \mathbb{R}^n$ *and a set* $Y \subseteq \mathbb{R}^m$*, a function* $f : X \to Y$ *is an open function if* $\forall$ *open subset* U *of* X*,* $f(U)$ *is open in* Y*.*

Now we are going to present some conditions for openness of a function.

Proposition 671 *(1st sufficient condition for openness of a function)*
Let an open set $A \subseteq \mathbb{R}^n$ *and a function* $f : A \to \mathbb{R}^n$ *be given. If*
1. f *is continuous,*
2. f *is one-to-one,*
3. $\forall x \in A$*,* $Df(x)$ *exists and* $\det Df(x) \neq 0$*,*
then $f(A)$ *is open.*

Proof. Taken $b \in f(A)$, there exists $a \in A$ such that $f(a) = b$. Since A is open, there exists $r \in \mathbb{R}_{++}$ such that $B := B(a, r) \subseteq A$. Moreover, since f is one-to-one and since $a \notin \mathcal{F}(B)$, $\forall x \in \mathcal{F}(B)$, $f(x) \neq f(a)$.Then[3], for sufficiently small r, $\mathrm{Cl}(B(a, r)) \subseteq A$, and the assumptions of Proposition 668 are satisfied and there exists $\delta \in \mathbb{R}_{++}$ such that

$$f(A) \supseteq f(\mathrm{Cl}(B(a, r))) \supseteq B(f(a), \delta),$$

as desired. ■

Definition 672 *Given* $f : S \to T$*, and* $A \subseteq S$*, the function* $f_{||A}$ *is defined as follows*

$$f_{||A} : A \to f(A), \qquad f_{||A}(x) = f(x).$$

[4]

Proposition 673 *Let an open set* $A \subseteq \mathbb{R}^n$ *and a function* $f : A \to \mathbb{R}^n$ *be given. If*
1. f *is* C^1*,*
2. $\exists a \in A$ *such that* $\det Df(a) \neq 0$*,*
then $\exists r \in \mathbb{R}_{++}$ *such that* f *is one-to-one on* $B(a, r)$*, and, therefore,* $f_{||B(a,r)}$ *is invertible.*

Proof. Consider $(\mathbb{R}^n)^n$ with generic element $z := \left(z^i\right)_{i=1}^n$, where $\forall i \in \{1, ..., n\}$, $z^i \in \mathbb{R}^n$, and define

$$h : \mathbb{R}^{n^2} \to \mathbb{R}, \qquad \left(z^i\right)_{i=1}^n \mapsto \det \begin{bmatrix} Df_1(z^1) \\ ... \\ Df_i(z^i) \\ ... \\ Df_n(z^n) \end{bmatrix}.$$

Observe that h is continuous because f is C^1 and the determinant function is continuous in its entries. Moreover, from Assumption 2,

$$h(a, ..., a, ..., a) = \det Df(a) \neq 0.$$

Therefore, $\exists r \in \mathbb{R}_{++}$ such that

$$\forall \left(z^i\right)_{i=1}^n \in B((a, ..., a, ..., a), nr), \quad h\left(\left(z^i\right)_{i=1}^n\right) \neq 0.$$

Defined

$$\hat{B} := \left\{ \left(z^i\right)_{i=1}^n \in \mathbb{R}^{n^2} : \forall i \in \{1, ..., n\}, z^i = z^1 \right\},$$

observe that $B^* := B((a, ..., a, ..., a), nr) \cap \hat{B} \neq \varnothing$.
Defined also

$$proj : (\mathbb{R}^n)^n \to \mathbb{R}^n, proj : \left(z^i\right)_{i=1}^n \mapsto z^1,$$

[3] Simply observe that $\forall r \in \mathbb{R}_{++}$, $Cl(B)\left(x, \frac{r}{2}\right) \subseteq B(x, r)$.

[4] We distinguish between $f_{||A}$ defined above and $f_{|A}$ defined as follows:

$$f_{|A} : A \to T, \; f_{|A}(x) = f(x).$$

Then $F_{||A}$ is by definition onto, while $f_{|A}$ in not necessary so.

observe that $proj\,(B^*) = B\,(a,r) \subseteq \mathbb{R}^n$ and $\forall i \in \{1,..,n\}, \forall z_i \in B\,(a,r)$, $\left(z^1,...,z^i...,z^n\right) \in B^*$ [5] and therefore $h\left(z^1,...,z^i...,z^n\right) \neq 0$, or, summarizing,

$$\exists r \in \mathbb{R}_{++} \text{ such that } \forall i \in \{1,..,n\}, \forall z_i \in B\,(a,r), \quad h\left(z^1,...,z^i...,z^n\right) \neq 0$$

We now want to show that f is one-to-one on $B\,(a,r)$. Suppose otherwise, i.e., given $x,y \in B\,(a,r)$, $f\,(x) = f\,(y)$, but $x \neq y$. We can now apply the Mean Value Theorem (see Remark 644) to f^i for any $i \in \{1,...,n\}$ on the segment $L\,(x,y) \subseteq B\,(a,r)$. Therefore,

$$\forall i \in \{1,..,n\}, \exists z^i \in L\,(x,y) \text{ such that } 0 = f_i\,(x) - f_i\,(y) = Df\left(z^i\right)(y-x)$$

i.e.,

$$\begin{bmatrix} Df_1\left(z^1\right) \\ ... \\ Df_i\left(z^i\right) \\ ... \\ Df_n\left(z^n\right) \end{bmatrix} (y-x) = 0$$

Observe that $z^i \in B\,(a,r)\ \forall i$, and therefore $\left(z^i\right)_{i=1}^n \in B\,((a,...,a,...,a)\,, nr)$from (15.8) in the previous footnote and therefore

$$\det \begin{bmatrix} Df_1\left(z^1\right) \\ ... \\ Df_i\left(z^i\right) \\ ... \\ Df_n\left(z^n\right) \end{bmatrix} = h\left(\left(z^i\right)_{i=1}^n\right) \neq 0$$

and therefore $y = x$, a contradiction. ■

Corollary 674 *Let an open subset $A \subseteq \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^n$ be given. If:*
1. f is C^1,
2. there exists $a \in A$ such that $\det Df\,(a) \neq 0$,
then $\exists r^* \in \mathbb{R}_{++}$ such that $f_{|B(a,r^*)}$ is an open function.

Proof. *From continuity of the determinant with respect to the entries of the matrix, $\exists\ r_1 > 0$ such that $\forall x \in B\,(a,r_1)$, $\det Df\,(x) \neq 0$. From Proposition 673, $\exists\ r_2 > 0$ such that f is one-to-one on $B\,(a,r_2)$. Taken $r^* = \min\{r_1,r_2\}$, all the assumption of Proposition 671 are satisfied with respect to $f_{B(a,r^*)}$ and therefore it is open.* ■

Corollary 675 *Under the Assumption of Corollary 674, for any open neighborhood N_a of a, there exists an open neighborhood $N_{f(a)}$ of $f\,(a)$ such that $f\,(N_a) \supseteq N_{f(a)}$.*

Proof. *Since N_a is open, then there exists r_1 such that*

$$B\,(a,r_1) \subseteq N_a. \tag{15.9}$$

From Corollary 674, there exists $r_2 > 0$ such that $f_{|B(a,r_2)}$ is an open function.

---

[5]

$$\|\left(z^i\right)_{i=1}^n - (a,...,a)\,\| = \|\left(z^i - a\right)_{i=1}^n\| \overset{(1)}{\leq} \sum_{i=1}^n \|z^i - a\| \overset{\left(z^1 \in B(a,r)\right)}{<} nr,$$

where (1) follows from what said below.

Given $\left(z^1,...,z^n\right) \in \mathbb{R}^n$,

$$\|\left(z^i\right)_{i=1}^n\|^2 := \sum_{i=1}^n \sum_{j=1}^n \left(z_j^i\right)^2 = \sum_{i=1}^n \|z^i\|^2. \tag{15.6}$$

Moreover,

$$\left(\sum_{i=1}^n \|z^i\|\right)^2 = \sum_{i=1}^n \|z^i\|^2 + 2\sum_{i=1}^n \sum_{j=1}^n \|z^i\| \cdot \|z^j\| \geq \sum_{i=1}^n \|z^i\|^2, \tag{15.7}$$

and then

$$\sum_{i=1}^n \|\left(z^i\right)_{i=1}^n\| \overset{(15.6)}{=} \sqrt{\sum_{i=1}^n \|z^i\|^2} \overset{(15.7)}{\leq} \sum_{i=1}^n \|z^i\|, \tag{15.8}$$

as desidered.

Defined $r = \min\{r_1, r_2\}$, *then* $f(B(a,r))$ *is open and, obviously,* $f(a) \in f(B(a,r))$. *Then*

$$\exists \varepsilon > 0 \quad \text{such that} \quad N_{f(a)} := B(f(a), \varepsilon) \subseteq f(B(a,r)) \overset{r<r_1}{\subseteq} f(B(a,r_1)) \overset{(15.9)}{\subseteq} f(N_a), \tag{15.10}$$

as desired ■

Remark 676 *The result contained in Proposition 72 is not a global result, i.e., it is false that if* f *is* C^1 *and its Jacobian has full rank everywhere in the domain, then* f *is one-to-one. Just take the function* tan.

The next result gives a global property (in terms of openness of the function).

Proposition 677 *(2nd sufficient condition for openness of a function) Let an open set* $A \subseteq \mathbb{R}^n$ *and a function* $f : A \to \mathbb{R}^n$ *be given. If*

1. f *is* C^1,

2. $\forall x \in A$, $\det Df(x) \neq 0$,

then f *is an open function.*

Proof. Take an open set $S \subseteq A$. From Proposition 673,$\forall x \in S$ there exists $r_x \in \mathbb{R}_{++}$ such that f is one-to-one on $B(x, r_x)$. Moreover, for any $x \in S$, $\exists r'_x > 0$ such that $B(x, r'_x) \subseteq S$. Take $r^*_x := \min\{r_x, r'_x\}$. Then, from Proposition 671, $f(B(x, r^*_x))$ is open in $\mathbb{R}^n$. Since S is open, we can then write $S = \cup_{x\in S} B(x, r^*_x)$ and

$$f(S) = f(\cup_{x\in S} B(x, r^*_x)) = \cup_{x\in S} f(B(x, r^*_x))$$

where the second equality follows from Proposition 515 506.2..f), and then $f(S)$ is an open set. ■

15.3 The inverse function theorem

Proposition 673 shows that a C^1 function with full rank square Jacobian in a point a has a local inverse in an open neighborhood of a. The inverse function theorem give local differentiability properties of that local inverse function.

Lemma 678 *Let* X *and* Y *be subsets of Euclidean spaces. If* g *is the inverse function of* $f : X \to Y$ *and* $A \subseteq X$, *then* $g_{||f(A)}$ *is the inverse of* $f_{||A}$, *and*

if g *is the inverse function of* $f : X \to Y$ *and* $B \subseteq Y$, *then* $g_{||B}$ *is the inverse of* $f_{||g(B)}$.

Proof. `Proof below to be reviewed carefully.`

By assumption

$$f : X \to Y, \quad x \mapsto f(x)$$

and, from Definition 672

$$f_{||A} : A \to f(A), \qquad x \mapsto f_{||A}(x) = f(x).$$

Since f is invertible by assumption, then $f_{||A}$ is invertible: f is one-to-one and a fortiori $f_{||A}$ is one-to-one; furthermore $f_{||A}$ is also onto by definition. So the inverse function of $f_{||A}$ is

$$f^{-1}_{||A} : f(A) \to A \quad y = f(x) \mapsto f^{-1}(y) = f^{-1}(f(x)) = x. \tag{15.11}$$

By assumption g is the inverse function of f and therefore it is defined as

$$g : Y \to X, \quad y = f(x) \mapsto g(y) = g(f(x)) = x$$

and

$$g_{||f(A)} : f(A) \to g(f(A)), \quad y = f(x) \mapsto g_{||f(A)}(y) = g(y) = g(f(x)) = x. \tag{15.12}$$

Moreover,

$$g(f(A)) = A, \tag{15.13}$$

and therefore (15.11), (15.12) and (15.13) prove the first statement.

Now let's move on the second statement, whose proof is almost identical to the above one.

By assumption

$$f : X \to Y, \quad x \mapsto f(x)$$

and, from Definition 672

$$f_{||g(B)} : g(B) \to f(g(B)), \qquad x \mapsto f_{||g(B)}(x) = f(x).$$

Since f is invertible by assumption, then $f_{||g(B)}$ is invertible: f is one-to-one and a fortiori $f_{||g(B)}$ is one-to-one; furthermore $f_{||g(B)}$ is also onto by definition. So the inverse function of $f_{||g(B)}$ is

$$f^{-1}_{||g(B)} : f(g(B)) \to g(B) \qquad y = f(x) \mapsto f^{-1}(y) = f^{-1}(f(x)) = x. \tag{15.14}$$

By assumption g is the inverse function of f and therefore it is defined as

$$g : Y \to X, \quad y = f(x) \mapsto g(y) = g(f(x)) = x$$

and

$$g_{||B} : B \to g(B), \quad y = f(x) \mapsto g_{||B}(y) = g(y) = g(f(x)) = x \tag{15.15}$$

Then,

$$f(g(B)) = B \tag{15.16}$$

and therefore (15.14), (15.15) and (15.16) prove the second statement. ■

Proposition 679 *Let $f : X \to Y$ be a function from a metric space (X, d) to another metric space (Y, d'). Assume that f is one-to-one and onto. If X is compact and f is continuous, then the inverse function f^{-1} is continuous.*

Proof. It is sufficient to show that for any closed set S in X, $\left(f^{-1}\right)^{-1}(S) = f(S)$ is closed in Y. Take S closed set in X. Then, from Proposition 409, S is compact. But, being f continuous, from Proposition 521, $f(S)$ is compact and therefore it is closed from Proposition 415. ■

Proposition 680 *(Inverse function Theorem) Let an open set $S \subseteq \mathbb{R}^n$ and a function $f : S \to \mathbb{R}^n$ be given. If*

1. f is C^1, and
2. $\exists a \in S$, such that $\det Df(a) \neq 0$,
then there exist two open sets $X \subseteq S$ and $Y \subseteq f(S)$ and a unique function g such that
1. $a \in X$ and $f(a) \in Y$,
2. $Y = f(X)$,
3. f is one-to-one on X,
4. $g : Y \to X$ is the inverse of $f_{||X}$ (and $X = g(Y)$)
5. g is C^1.

Proof. Since f is C^1, $\exists r_1 \in \mathbb{R}_{++}$ such that $\forall x \in B(a, r_1)$, $\det Df(x) \neq 0$. Then, from Proposition 673, f is one-to-one on $B(a, r_1)$. Then take $r_2 \in (0, r_1)$, and define $B := B(a, r_2)$. Observe that Cl (B) $\subseteq B(a, r_1)$ .Using the fact that f is one-to-one on $B(a, r_1)$ and therefore on $B(a, r_2)$, we get that Assumption 3 in Proposition 668 is satisfied - while the other two are trivially satisfied. Then, $\exists \delta \in \mathbb{R}_{++}$ such that

$$f(B) \supseteq B(f(a), \delta) := Y.$$

Define also

$$X := f^{-1}(Y) \cap B, \tag{15.17}$$

an open set because Y and B are open sets and f is continuous. Since f is one-to-one and continuous on the compact set Cl (B) , from Proposition 78, there exists a unique continuous inverse function $\widehat{g}$: $f(\text{Cl }(B)) \to \text{Cl }(B)$ of $f_{|\text{Cl }(B)}$. From definition of Y,

$$Y \subseteq f(B) \subseteq f(\text{Cl }(B)). \tag{15.18}$$

From definition of X,

$$f(X) = Y \cap f(B) = Y. \tag{15.19}$$

Then, from Lemma 678,

$$g := \hat{g}_{||Y} \quad is\ the\ inverse\ function\ of\ f_{||X}, \tag{15.20}$$

Then, $X \overset{(15.20)}{=} g(f(X)) \overset{(15.19)}{=} g(Y)$.

The above results show conclusions 2-4 of the Proposition. About conclusion 1, observe that $a \in f^{-1}(B(f(a,\delta))) \cap B(a, r_2) = X$ and $f(a) \in f(X) = Y$.

We are then left with proving condition 5.

Following what already said in the proof of Proposition 673, we can define

$$h : \mathbb{R}^{n^2} \to \mathbb{R}, \qquad : \left(z^i\right)_{i=1}^n \mapsto \det \begin{bmatrix} Df_1(z^1) \\ ... \\ Df_i\left(z^i\right) \\ ... \\ Df_n(z^n) \end{bmatrix}.$$

and get that, from Assumption 2,

$$h(a, ..., a, ..., a) = \det Df(a) \neq 0,$$

and- see the proof of Proposition 673 for details-

$$\exists r \in \mathbb{R}_{++} \text{ such that } \forall i \in \{1, .., n\}, \forall z_i \in B(a, r), \quad h\left(z^1, ..., z^i ..., z^n\right) \neq 0, \tag{15.21}$$

and trivially also

$$\exists r \in \mathbb{R}_{++} \text{ such that } \forall z \in B(a, r), \quad h(z, ..., z..., z) = \det Df(z) \neq 0. \tag{15.22}$$

Assuming, without loss of generality that $r_1 < r$, we have that

$$\text{Cl }(B) := \text{Cl }((B)(a, r_2)) \subseteq B(a, r_1) \subseteq B(a, r). \tag{15.23}$$

Then,$\forall z^1, ..., z^n \in \text{Cl }(B)$, $h\left(z^1, ..., z^i ..., z^n\right) \neq 0$. Writing $g = \left(g^i\right)_{i=1}^n$, we want to prove that $\forall i \in \{1, .., n\}$, g^i is C^1. We go through the following two steps: 1. $\forall y \in Y$, $\forall i, k \in \{1, .., n\}$, $D_{y_k} g^i(y)$ exists, and 2. it is continuous.

Step 1.

We want to show that the following limit exists and it is finite

$$\lim_{h \to 0} \frac{g^i\left(y + he_n^k\right) - g^i(y)}{h}.$$

Define

$$\begin{aligned} x &= (x_i)_{i=1}^n = g(y) \in X \subseteq \text{Cl }(B) \\ x' &= (x'_i)_{i=1}^n = g\left(y + he_n^k\right) \in X \subseteq \text{Cl }(B) \end{aligned} \tag{15.24}$$

Then

$$f(x') - f(x) = \left(y + he_n^k\right) - y = he_n^k. \tag{15.25}$$

We can now apply the Mean Value Theorem to f^i for $i \in \{1, .., n\}$: $\exists z^i \in L(x, x') \subseteq \text{Cl }(B)$, where the inclusion follows from the fact that $x, x' \in \text{Cl }(B)$ a convex set, such that

$$\forall i \in \{1, .., n\}, \ \frac{f^i(x') - f^i(x)}{h} = \frac{Df^i\left(z^i\right)(x' - x)}{h} \tag{15.26}$$

and therefore, from (15.25) and (15.26)

$$\begin{bmatrix} Df_1(z^1) \\ ... \\ Df_i\left(z^i\right) \\ ... \\ Df_n(z^n) \end{bmatrix} \frac{1}{h}(x' - x) = e_n^k. \tag{15.27}$$

Define

$$A=\begin{bmatrix} Df_1\left(z^1\right) \\ \dots \\ Df_i\left(z^i\right) \\ \dots \\ Df_n\left(z^n\right) \end{bmatrix}_{m\times m}$$

Then, from (15.21), (15.27) admits a unique solution,

$$\frac{1}{h}\left(x'-x\right)\overset{(15.24)}{=}\frac{g\left(y+he_n^k\right)-g\left(y\right)}{h}=\frac{\varphi\left(z^1,...,z^n\right)}{h\left(z^1,...,z^n\right)}$$

where , from the Cramer rules, φ takes values which are determinants of a matrix involving entries of A (and therefore they are continuous). We are left with showing that

$$\lim_{h\to 0}\frac{\varphi\left(z^1,...,z^n\right)}{h\left(z^1,...,z^n\right)}$$

exists and it is finite, i.e., the limit of the numerator exists and its finite and the limit of the denominator exists is finite and nonzero.

Then, if $h\to 0$, $y+he_n^k\to y$, and, being g continuous, $x'\to x$ and, since $z^i\in L\left(x,x'\right)$, $z^i\to x$ for any i. Then, $h\left(z^1,...,z^n\right)\to h\left(x,...,x\right)\neq 0$, because, from 15.24, $x\in \text{Cl }(B)$ and from (15.22) and (15.23). Moreover, $\varphi\left(z^1,...,z^n\right)\to\varphi\left(x,...,x\right)$.

Step 2.

Since

$$\lim_{h\to 0}\frac{g^i\left(y+he_n^k\right)-g^i\left(y\right)}{h}=\frac{\varphi\left(x,...,x\right)}{h\left(x,...,x\right)}$$

and φ and h are continuous functions, the desired result follows. ■

15.4 The implicit function theorem

Theorem 681 *Given open sets $S\subseteq\mathbb{R}^n$ and $T\subseteq\mathbb{R}^k$ and a function*

$$f:S\times T\to\mathbb{R}^n,\quad :(x,t)\mapsto f\left(x,t\right),$$

assume that

1. *f is C^1, and*
   there exists $(x_0,t_0)\in S\times T$ such that
2. *$f(x_0,t_0)=0$,*
3. *$D_xf\left(x_0,t_0\right)_{n\times n}$ is invertible.*

Then there exists $N(t_0)\subseteq T$ open neighborhood of t_0 and a unique function

$$g:N(t_0)\to\mathbb{R}^n$$

such that

1. *g is C^1,*
2. *$g\left(t_0\right)=x_0$*
3. *$\{(x,t)\in\mathbb{R}^n\times N(t_0):f(x,t)=0\}=\{(x,t)\in\mathbb{R}^n\times N(t_0):x=g(t)\}:=graph\ g$.*

Proof. `Proof below to be reviewed carefully.`

The main idea to prove the theorem is as follows:

1. Apply the Inverse Function Theorem to $F\left(x,t\right)=\left(f\left(x,t\right),t\right)$ around (x_0,t_0);

2. Define $G = F^{-1}$ with $F(A) = B$ and $G(B) = A$;

3. Define $N(t_0) = \{t \in T : (0,t) \in B\}$, $g(t) = v(0,t)$.

We want to apply the Inverse Function Theorem to the following function

$$F : S \times T \to \mathbb{R}^{n+k} \qquad F : (x,t) \mapsto (f(x,t), t).$$

Observe that

$$\left[\begin{array}{c} D_{(x,t)}F(x,t) \end{array}\right]_{(n+k)\times(n+k)} = \left[\begin{array}{c} D_{(x,t)}f(x,t) \\ D_{(x,t)}t \end{array}\right] = \left[\begin{array}{cc} D_x f(x,t) & D_t f(x,t) \\ 0 & I \end{array}\right]$$

Therefore $\det D_{(x,t)}F(x,t) = \det D_x f(x,t)$ and in particular,

$$\det D_{(x,t)}F(x_0,t_0) = det D_x f(x_0,t_0) \neq 0. \tag{15.28}$$

Moreover,

$$F(x_0,t_0) = (f(x_0,t_0), t_0) \overset{(1)}{=} (0,t_0)$$

where

(1) follows from Assumption 2.

Because f is C^1 by Assumption 1 and the function $id : \mathbb{R}^k \to \mathbb{R}^k$, $t \mapsto t$ is C^1 as a function, because it is linear, F is C^1 on $S \times T$. Therefore we can apply the Inverse Function Theorem to F around (x_0,t_0) where $\det D_{(x,t)}F(x_0,t_0) \neq 0$ from 15.28, i.e., there exist two open sets $A \subseteq S \times T$ and $B \subseteq F(S \times T)$ and a unique function G such that

1. $(x_0,t_0) \in A$ and $F(x_0,t_0) = (0,t_0) \in B$,

2. $B = F(A)$,

3. F is one-to one on A,

4. $G : B \to A$ is the inverse function of $F_{||A}$ and $A = G(B)$,

5. G is C^1 on B.

Being $G : B \to A$ and $A, B \subseteq \mathbb{R}^{n+k}$, we can define $v : B \to S \subseteq \mathbb{R}^n$ and $w : B \to T \subseteq \mathbb{R}^k$ such that $G = (v, w)$. Therefore,

$$(x,t) = G(F(x,t)) = (v(F(x,t)), w(F(x,t)))$$

i.e.,

$$v(F(x,t)) = x \tag{15.29}$$

and

$$w(F(x,t)) = t \tag{15.30}$$

Being F restricted to A onto and one-to-one, for any $(x,t) \in B$, $\exists!\,(x',t') \in A$ such that $[F(x',t') = (x,t)]$, then,by definition of F, we have that $t' = t$, i.e., shortly,

$$\forall (x,t) \in B \quad \exists!\,(x',t) \in A \quad such \ \ that \quad (x,t) = F(x',t). \tag{15.31}$$

Therefore applying v to the both side of the equality in (15.31), we get

$$\forall (x,t) \in B, \quad v(x,t) = v(F(x',t)) \overset{(15.29)}{=} x'. \tag{15.32}$$

Now since from 1. above $(0,t_0) \in B$, $v(0,t_0)$ is well defined. Moreover,

$$v(0,t_0) \overset{1.above}{=} v(F(x_0,t_0)) \overset{(15.31)}{=} x_0. \tag{15.33}$$

We also get that

$$F(v(x,t), t) \overset{(15.32)}{=} F(x',t) \overset{(15.31)}{=} (x,t)$$

Summarizing,

$$\forall (x,t) \in B, \quad F(v(x,t), t) = (x,t). \tag{15.34}$$

Now we can define $N(t_0)$, and g of the conclusion of the theorem as follows:

$$N(t_0) = \{t \in T : (0,t) \in B\},$$

which is open in $\mathbb{R}^k$ because B is open in $\mathbb{R}^{n+k}$; see Lemma 682 below.

$$g : N(t_0) \to \mathbb{R}^n, \qquad g : t \mapsto v(0,t)$$

is C^1 because v are components of G which is C^1 on $Y \supseteq \{0\} \times N(t_0)$. Moreover

$$g(t_0) = v(0,t_0) \overset{(15.33)}{=} x_0,$$

moreover,

$$\forall (x,t) \in B \quad (f(v(x,t),t)) = F(v(x,t),t) \overset{(15.34)}{=} (x,t),$$

i.e.,

$$f(v(x,t),t) = x \tag{15.35}$$

But then since $N(t_0) := \{t \in T : (0,t) \in B\}$ we have that

$$\forall t \in N(t_0), f(v(0,t),t) = 0$$

and finally by definition of g, $f(g(t),t) = 0$. The only thing left to prove is that g is unique. Take any other h such that $f(h(t),t) = f(g(t),t)$ $\forall t \in N(t_0)$. But being f one-to-one (because F is one-to-one), this implies $h(t) = g(t)$. ■

Lemma 682 *Defined*

$$\pi_1 : X \times Y \to X, \quad (x,y) \mapsto x$$

$$\pi_2 : X \times Y \to Y, \quad (x,y) \mapsto y.$$

1.[6] *Both π_1 and π_2 are continuous and open function;*

2. For any open set $O \subset X \times Y$, and for any $\bar{x} \in X$, $O_{\bar{x}} := \{y \in Y : (\bar{x},y) \in O\} = \pi_2\left(O \cap \pi_1^{-1}(\{\bar{x}\})\right)$;

3. $O_{\bar{x}}$ is open.

Proof. 1. We prove the result only for π_1, the proof for π_2 being almost identical.

a. π_1 is continuous.

We have to prove that

$$\forall (x_0,y_0) \in X \times Y, \quad \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \textit{such that} \; \|(x,y) - (x_0,y_0)\| < \delta \Rightarrow \|x - x_0\| < \varepsilon$$

Indeed take $\delta = \varepsilon$. Then

$$\|x - x_0\| \le \|(x,y) - (x_0,y_0)\| < \delta = \varepsilon.$$

b. π_1 is open.

We have to prove that S open in $X \times Y \Rightarrow \pi_x(s)$ open in X. Take $y \in \pi_1(S)$, we have to find $\varepsilon > 0$ such that $B(y,\varepsilon) \subseteq \pi_1(S)$, i.e., we want to find $\varepsilon > 0$ such that

$$\|y' - y\| \Rightarrow y' \in \pi_1(S), \; \textit{i.e.,} \; (x,y') \in S. \tag{15.36}$$

Since $y \in \pi_1(S)$, then, by definition of π_1, $(x,y) \in S$ and since S is open then

$$\exists \delta > 0 \; \textit{such that} \; B((x,y),\delta) \subseteq S, \tag{15.37}$$

Take

$$\varepsilon = \delta > 0. \tag{15.38}$$

[6] Point 1. is not used to show the implicit function theorem.

Then

$$\|(x, y') - (x, y)\| = \|y' - y\| \overset{(??)}{<} \varepsilon.$$

Then from (15.38) and (15.37), $(x, y') \in S$ as desired.

2. $y \in \pi_2\left(O \cap \pi_1^{-1}(\{x\})\right) \Leftrightarrow \exists (x', y') \in O \cap \pi_1^{-1}(x)$ such that $\pi_2(x', y') = y \Leftrightarrow \exists (x', y') \in O$ and $(x', y') \in \pi_1^{-1}(x) := \{(x'', y'') \in X \times Y : x'' = x\}$ such that $\pi_2(x', y') = y \Leftrightarrow \exists (x', y') \in O$ and such that $x' = x\ y' = y \in Y \Leftrightarrow y \in Y,\ \ (x, y) \in O \Leftrightarrow y \in O_x$.

3. We want to show that if $\hat{y} \in O_{\bar{x}}$, then

$$\exists \delta > 0 \textit{ such that } B(\hat{y}, \delta) \subseteq O_{\bar{x}}. \tag{15.39}$$

Since $\hat{y} \in O_{\bar{x}}$, then by definition of $O_{\bar{x}}$, $(\bar{x}, \hat{y}) \in O$, and since O is open, there exists $r > 0$ such that

$$B((\bar{x}, \hat{y}), r) \subseteq O. \tag{15.40}$$

Now take $\delta = r$; to show (15.39) we are going to show that

$$\|y - \hat{y}\| < r \Rightarrow y \in O_{\bar{x}}.$$

Indeed,

$$\|y - \hat{y}\| < r \Rightarrow \|(\bar{x}, y) - (\bar{x} - \hat{y}) < r \Rightarrow$$

$$\Rightarrow (\bar{x}, y) \in B((\bar{x}, \hat{y}), r) \cap \pi_1^{-1}(\bar{x}) \subseteq O \cap \pi_1^{-1}(\bar{x}) \Rightarrow$$

$$\Rightarrow y = \pi_2(\bar{x}, y) \in \pi_2\left(O \cap \pi_1^{-1}(\bar{x})\right) = O_{\bar{x}}$$

■

Remark 683 *Conclusion 2. in the statement of Theorem 681 can be rewritten as*

$$\forall t \in N(t_0), \quad f(g(t), t) = 0 \tag{15.41}$$

Computing the Jacobian of both sides of (15.41)*, using Remark 642, we get*

$$\forall t \in N(t_0), \quad 0 = [D_x f(g(t), t)]_{n\times n} \cdot [Dg(t)]_{n\times k} + [D_t f(g(t), t)]_{n\times k} \tag{15.42}$$

and using Assumption 3 of the Implicit Function Theorem, we get

$$\forall t \in N(t_0), \quad [Dg(t)]_{n\times k} = -[D_x f(g(t), t)]_{n\times n}^{-1} \cdot [D_t f(g(t), t)]_{n\times k}$$

Observe that (15.42) *can be rewritten as the following k systems of equations: $\forall i \in \{1, ..., k\}$,*

$$[D_x f(g(t), t)]_{n\times n} \cdot [D_{t_i} g(t)]_{n\times 1} = - [D_{t_i} f(g(t), t)]_{n\times 1}$$

Example 684 [7] *Discuss the application of the Implicit Function Theorem to $f : \mathbb{R}^5 \to \mathbb{R}^2$*

$$f(x_1, x_2, t_1, t_2, t_3) \mapsto \begin{pmatrix} 2e^{x_1} + x_2 t_1 - 4t_2 + 3 \\ x_2 \cos x_1 - 6x_1 + 2t_1 - t_3 \end{pmatrix}$$

at $\left(x^0, t^0\right) = (0, 1, 3, 2, 7)$.
Let's check that each assumption of the Theorem is verified.

1. *$f(x_0, t_0) = 0$. Obvious.*

[7] The example is taken from Rudin (1976), pages 227-228.

2. *f is* C^1.

We have to compute the Jacobian of the function and check that each entry is a continuous function.

	x_1	x_2	t_1	t_2	t_3
$2e^{x_1}+x_2t_1-4t_2+3$	$2e^{x_1}$	t_1	x_2	-4	0
$x_2\cos x_1-6x_1+2t_1-t_3$	$-x_2\sin x_1-6$	$\cos x_1$	2	0	-1

3. $[D_x f(x_0,t_0)]_{n\times n}$ *is invertible.*

$$[D_x f(x_0,t_0)] = \begin{bmatrix} 2e^{x_1} & t_1 \\ -x_2\sin x_1 - 6 & \cos x_1 \end{bmatrix}_{|(0,1,3,2,7)} = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}$$

whose determinant is 20.

Therefore, we can apply the Implicit Function Theorem and compute the Jacobian of $g : N(t_0) \subseteq \mathbb{R}^2 \to N(x_0) \subseteq \mathbb{R}^3$*:*

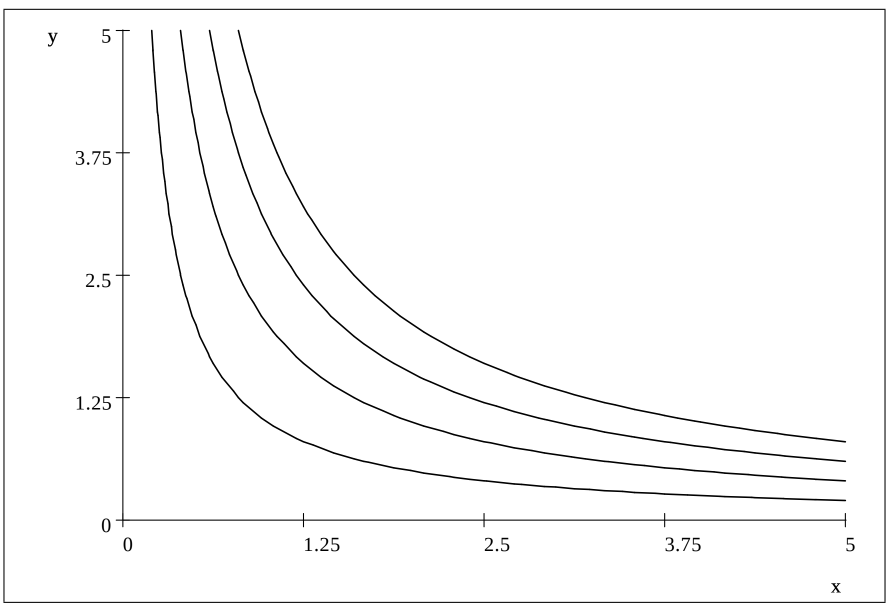
$$Dg(t) = -\begin{bmatrix} 2e^{x_1} & t_1 \\ -x_2\sin x_1 - 6 & \cos x_1 \end{bmatrix}^{-1} \begin{bmatrix} x_2 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} =$$

$$= \frac{1}{6t_1 + 2(\cos x_1)e^{x_1} + t_1x_2\sin x_1} \begin{bmatrix} 2t_1 - x_2\cos x_1 & 4\cos x_1 & -t_1 \\ -6x_2 - 4e^{x_1} - x_2^2\sin x_1 & 4x_2\sin x_1 + 24 & 2e^{x_1} \end{bmatrix}$$

Exercise 685 *Given the utility function* $u : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$, $(x,y) \mapsto u(x,y)$ *satisfying the following properties*

i. u *is* C^2, *ii.* $\forall (x,y) \in \mathbb{R}^2_{++}$, $Du(x,y) >> 0$,*iii.* $\forall (x,y) \in \mathbb{R}^2_{++}$, $D_{xx}u(x,y) < 0, D_{yy}u(x,y) < 0, D_{xy}u(x,y) > 0$,

compute the Marginal Rate of Substitution in (x_0,y_0) *and say if the graph of each indifference curve is concave.*



15.5 Some geometrical remarks on the gradient

In what follows we make some geometrical, not rigorous remarks on the meaning of the gradient, using the implicit function theorem. Consider an open subset X of $\mathbb{R}^2$, a C^1 function

$$f : X \to \mathbb{R}, \qquad : (x,y) \mapsto f(x,y)$$

where $a \in \mathbb{R}$. Assume that set

$$L(a) := \{(x,y) \in X : f(x,y) = a\}$$

is such that $\forall (x,y) \in X$, $\frac{\partial f(x,y)}{\partial y} \neq 0$ and $\frac{\partial f(x,y)}{\partial x} \neq 0$, then

1. $L(a)$ is the graph of a C^1 function from a subset of $\mathbb{R}$ to $\mathbb{R}$;

2. $(x^*, y^*) \in L(a) \Rightarrow$ the line going through the origin and the point $Df(x^*, y^*)$ is orthogonal to the line going through the origin and parallel to the tangent line to $L(a)$ at (x^*, y^*) ;or the line tangent to the curve $L(a)$ in (x^*, y^*) is orthogonal to the line to which the gradient belongs to.

3. $(x^*, y^*) \in L(a) \Rightarrow$ the directional derivative of f at (x^*, y^*) in the the direction u such that $\|u\| = 1$ is the largest one if $u = \frac{Df(x^*,y^*)}{\|Df(x^*,y^*)\|}$.

1. It follows from the Implicit Function Theorem.
2. The slope of the line going through the origin and the vector $Df(x^*, y^*)$ is

$$\frac{\frac{\partial f(x^*,y^*)}{\partial y}}{\frac{\partial f(x^*,y^*)}{\partial x}} \tag{15.43}$$

Again from the Implicit Function Theorem, the slope of the tangent line to $L(a)$ in (x^*, y^*) is

$$-\frac{\frac{\partial f(x^*,y^*)}{\partial x}}{\frac{\partial f(x^*,y^*)}{\partial y}} \tag{15.44}$$

The product between the expressions in (15.43) and (15.44) is equal to -1.
3. the directional derivative of f at (x^*, y^*) in the the direction u is

$$f'((x^*, y^*); u) = Df(x^*, y^*) \cdot u = \|Df(x^*, y^*)\| \cdot \|u\| \cdot \cos\theta$$

where θ is an angle in between the two vectors. Then the above quantity is the greatest possible iff $\cos\theta = 1$, i.e., u is colinear with $Df(x^*, y^*)$, i.e., $u = \frac{Df(x^*,y^*)}{\|Df(x^*,y^*)\|}$.

15.6 Extremum problems with equality constraints.

Given the open set $X \subseteq \mathbb{R}^n$, consider the C^1 functions

$$f : X \to \mathbb{R}, \qquad f : x \mapsto f(x),$$

$$g : X \to \mathbb{R}^m, \qquad g : x \mapsto g(x) := (g_j(x))_{j=1}^m$$

with $\underline{m \le n}$. Consider also the following "maximization problem":

$$(P) \qquad \max_{x \in X} \quad f(x) \qquad \text{subject to} \qquad g(x) = 0 \tag{15.45}$$

The set

$$C := \{x \in X : g(x) = 0\}$$

is called the constraint set associated with problem (15.45).

Definition 686 *The solution set to problem* (15.45) *is the set*

$$\{x^* \in C : \forall x \in C,\ f(x^*) \ge f(x)\},$$

and it is denoted by arg max (15.45).

The function

$$\mathcal{L} : X \times \mathbb{R}^m \to \mathbb{R}, \quad \mathcal{L} : (x, \lambda) \mapsto f(x) + \lambda^T g(x)$$

is called Lagrange function associated with problem (15.45).

Theorem 687 *Given the open set $X \subseteq \mathbb{R}^n$ and the C^1 functions*

$$f : X \to \mathbb{R}, \qquad f : x \mapsto f(x), \quad g : X \to \mathbb{R}^m, \qquad g : x \mapsto g(x) := (g_j(x))_{j=1}^m,$$

assume that

1. *f and g are C^1functions,*
2. *x_0 is a solution to problem (15.45),*[8] *and*
3. $\operatorname{rank}[Dg(x_0)]_{m\times n} = m$.

Then, there exists $\lambda_0 \in \mathbb{R}^m$, such that, $D\mathcal{L}(x_0, \lambda_0) = 0$, i.e.,

$$\begin{cases} Df(x_0) + \lambda_0 Dg(x_0) = 0 \\ g(x_0) = 0 \end{cases} \tag{15.46}$$

Proof. Define $x' := (x_i)_{i=1}^m \in \mathbb{R}^m$ and $t = (x_{m+k})_{k=1}^{n-m} \in \mathbb{R}^{n-m}$ and therefore $x = (x', t)$. From Assumption 3, without loss of generality,

$$\det[D_{x'} g(x_0)]_{m\times m} \neq 0. \tag{15.47}$$

We want to show that there exists $\lambda_0 \in \mathbb{R}^m$ which is a solution to the system

$$[Df(x_0)]_{1\times n} + \lambda_{1\times m}[Dg(x_0)]_{m\times n} = 0. \tag{15.48}$$

We can rewrite (15.48) as follows

$$\left[\; D_{x'} f(x_0)_{1\times m} \quad | \quad D_t f(x_0)_{1\times(n-m)} \;\right] + \lambda_{1\times m}\left[\; D_{x'} g(x_0)_{m\times m} \quad | \quad D_t g(x_0)_{m\times(n-m)} \;\right] = 0$$

or

$$\begin{cases} \left[\; D_{x'} f(x_0)_{1\times m} \;\right] + \lambda_{1\times m}\left[\; D_{x'} g(x_0)_{m\times m} \;\right] = 0 & (1) \\ \\ \left[\; D_t f(x_0)_{1\times(n-m)} \;\right] + \lambda_{1\times m}\left[\; D_t g(x_0)_{m\times(n-m)} \;\right] = 0 & (2) \end{cases} \tag{15.49}$$

From (15.47), there exists a unique solution λ_0 to subsystem (1) in (15.49). If $n = m$, we are done. Assume now that $n > m$. We have now to verify that λ_0 is a solution to subsystem (2) in (15.49), as well. To get the desired result, we are going to use the Implicit Function Theorem. Summarizing, observe that

$$1.\; g \text{ is } C^1, \qquad 2.\; g(x_0', t_0) = 0, \qquad 3.\quad \det[D_{x'} g(x_0', t_0)]_{m\times m} \neq 0,$$

i.e., all the assumption of the Implicit Function Theorem are verified. Then we can conclude that there exist $N(x_0) \subseteq \mathbb{R}^m$ open neighborhood of x_0' and a unique function $\varphi : N(t_0) \to \mathbb{R}^{n-m}$such that

$$1.\; \varphi \text{ is } C^1, \qquad 2.\; \varphi(t_0) = x_0', \qquad 3.\quad \forall t \in N(t_0), \quad g(\varphi(t), t) = 0. \tag{15.50}$$

Define now

$$F : N(t_0) \subseteq \mathbb{R}^{n-m} \to \mathbb{R}, \qquad : t \mapsto f(\varphi(t), t),$$

and

$$G : N(t_0) \subseteq \mathbb{R}^{n-m} \to \mathbb{R}^m, \qquad : t \mapsto g(\varphi(t), t).$$

Then, from (15.50) and from Remark 683, we have that $\forall t \in N(t_0)$,

$$0 = [DG(t)]_{m\times(n-m)} = [D_{x'} g(\varphi(t), t)]_{m\times m} \cdot [D\varphi(t)]_{m\times(n-m)} + [D_t g(\varphi(t), t)]_{m\times(n-m)}. \tag{15.51}$$

Since[9], from (15.50), $\forall t \in N(t_0), \quad g(\varphi(t), t) = 0$ and since

$$x_0 := (x_0', t_0) \overset{(15.50)}{=} (\varphi(t_0), t_0) \tag{15.52}$$

is a solution to problem (15.45), we have that $f(x_0) = F(t_0) \geq F(t)$, i.e., briefly,

$$\forall t \in N(t_0), \qquad F(t_0) \geq F(t).$$

[8] The result does apply to the case in which x_0 is a local maximum for Problem (15.45). Obviously the result apply to the case of (local) minima, as well.

[9] The only place where the proof has to be slightly changed to get the result for local maxima is here.

Then, from Proposition 655, $DF(t_0) = 0$. Then, from the definition of F and the Chain Rule, we have

$$[D_{x'} f(\varphi(t_0), t_0)]_{1\times m} \cdot [D\varphi(t_0)]_{m\times(n-m)} + [D_t f(\varphi(t_0), t_0)]_{1\times(n-m)} = 0. \tag{15.53}$$

Premultiplying (15.51) by λ, we get

$$\lambda_{1\times m} \cdot [D_{x'} g(\varphi(t), t)]_{m\times m} \cdot [D\varphi(t)]_{m\times(n-m)} + \lambda_{1\times m} \cdot [D_t g(\varphi(t), t)]_{m\times(n-m)} = 0. \tag{15.54}$$

Adding up (15.53) and (15.54) ,computed at $t = t_0$, get

$$([D_{x'} f(\varphi(t_0), t_0)] + \lambda \cdot [D_{x'} g(\varphi(t_0), t_0)]) \cdot [D\varphi(t_0)] + [D_t f(\varphi(t_0), t_0)] + +\lambda \cdot [D_t g(\varphi(t_0), t_0)] = 0,$$

and from (15.52),

$$([D_{x'} f(x_0)] + \lambda \cdot [D_{x'} g(x_0)]) \cdot [D\varphi(t_0)] + [D_t f(x_0)] + \lambda \cdot [D_t g(x_0)] = 0. \tag{15.55}$$

Then, from the definition of λ_0 as the unique solution to (1) in (15.49) ,we have that $[D_{x'} f(x_0)] + \lambda_0 \cdot [D_{x'} g(x_0)] = 0$, and then from (15.55) computed at $\lambda = \lambda_0$, we have

$$[D_t f(x_0)] + \lambda_0 \cdot [D_t g(x_0)] = 0,$$

i.e., (2) in (15.49), the desired result. ■

15.7 Exercises on part III

Problem sets: all of the problems on part III.

See Tito Pietra's file (available on line): Exercises 1 → 14 (excluding exercises 3, 5, 15).

Part IV

Nonlinear programming

Chapter 16

Convex sets

16.1 Definition

Definition 688 *A set $C \subseteq \mathbb{R}^n$ is convex if $\forall x_1,\ x_2 \in C$ and $\forall \lambda \in [0,1]$, $(1-\lambda)\, x_1 + \lambda x_2 \in C$.*

Definition 689 *A set $C \subseteq \mathbb{R}^n$ is strictly convex if $\forall x_1,\ x_2 \in C$ such that $x_1 \neq x_2$, and $\forall \lambda \in (0,1)$, $(1-\lambda)\, x_1 + \lambda x_2 \in \operatorname{Int} C$.*

Remark 690 *If C is strictly convex, then C is convex, but not vice-versa.*

Proposition 691 *The intersection of an arbitrary family of convex sets is convex.*

Proof. We want to show that given a family $\{C_i\}_{i \in I}$ of convex sets, if $x, y \in C := \cap_{i \in I} C_i$ then $(1-\lambda)\, x + \lambda y \in C$. $x, y \in C$ implies that $x, y \in C_i$, $\forall i \in I$. Since C_i is convex, $\forall i \in I$, $\forall \lambda \in [0,1]$, $(1-\lambda)\, x + \lambda y \in C_i$, and $\forall \lambda \in [0,1]$ $(1-\lambda)\, x + \lambda y \in C$. ■

Exercise 692 $\forall n \in \mathbb{N}, \forall i \in \{1, ..., n\}$, *$I_i$ is an interval in $\mathbb{R}$, then*

$$\times_{i=1}^{n} I_i$$

is a convex set.

16.2 Separation of convex sets

Definition 693 *Let H be a set in $\mathbb{R}^n$. Then,*

$$\langle \ H \text{ is a hyperplane } \rangle \Leftrightarrow$$

$$\Leftrightarrow \langle \ \exists c_0 \in \mathbb{R}, (c_i)_{i=1}^{n} \in \mathbb{R}^n \setminus \{0\} \text{ such that } H = \left\{ (x_i)_{i=1}^{n} \in \mathbb{R}^n : c_0 + c_1 x_1 + ... + c_n x_n = 0 \right\} \rangle .$$

Definition 694 *Let A and B be subsets of $\mathbb{R}^n$, and let $H := \{x \in \mathbb{R}^n : c_0 + c \cdot x = 0\}$ be a hyperplane in $\mathbb{R}^n$.*[1] *Then, H is said to*

1. *separate A and B if*

$$A \subseteq H_- := \{x \in \mathbb{R}^n : c_0 + c \cdot x \leq 0\} \quad \text{and} \quad B \subseteq H_+ := \{x \in \mathbb{R}^n : c_0 + c \cdot x \geq 0\},$$

i.e.,

$$\forall a \in A, \quad \forall b \in B, \qquad c_0 + c \cdot a \leq 0 \leq c_0 + c \cdot b,$$

i.e.,

$$\forall a \in A, \quad \forall b \in B, \qquad c \cdot a \leq -c_0 \leq c \cdot b;$$

2. *separate A and B properly if it separates them and $A \cup B \nsubseteq H$;*

[1] When we introduce a hyperplane by equation $c_0 + cx = 0$, consistently with the definition of hyperplane, we assume $c \neq 0$.

3. *separate A and B strictly if*

$$A \subseteq \{x \in \mathbb{R}^n : c_0 + c \cdot x < 0\} \quad \text{and} \quad B \subseteq \{x \in \mathbb{R}^n : c_0 + c \cdot x > 0\}.$$

Example 695 *The convex sets $\{(x, y) \in \mathbb{R}^2 : x \leq 0\}$ and $\{(x, y) \in \mathbb{R}^2 : x > 0, y > \frac{1}{x}\}$ in $\mathbb{R}^2$ cannot be strictly separated, but they are properly separated by the y-axis.*

Clearly, it is not always possible to separate two convex sets by a hyperplane. For instance, there is no line in $\mathbb{R}^2$ separating the set $\{0\}$ and the closed unit disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

Remark 696 *Let A, B be two sets in $\mathbb{R}^n$ such that at least one of them is nonempty. If they can be strictly separated, then they can also be properly separated.*

Remark 697 *Let H be a hyperplane in $\mathbb{R}^n$, and let A and B be two subsets of H. Then, H separates A and B, but does not separate them properly.*

Proposition 698 *Let a hyperplane $H := \{x \in \mathbb{R}^n : c_0 + c \cdot x = 0\}$ be given. Then,2.*

$$\langle\ H \text{ separates properly } A \text{ and } B\ \rangle \Leftrightarrow \langle\ H \text{ separates properly } \mathrm{Cl}(A) \text{ and } \mathrm{Cl}(B)\ \rangle.$$

Proof. [$\Leftarrow$]
Obvious.
[$\Rightarrow$]
We first present two proofs of the fact H separates $\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$, and then we show that the separation is proper.
1st proof.

$$\langle\ A \subseteq H_-\ \rangle \Rightarrow \langle\ \mathrm{Cl}(A) \subseteq \mathrm{Cl}(H_-) = H_- \rangle,$$

where we used the fact that H_- is closed.
Similarly, $B \subseteq H_+ \Rightarrow \mathrm{Cl}(H_+) . \subseteq H_+$
2nd proof.
Take $(a^*, b^*) \in \mathrm{Cl}(A) \times \mathrm{Cl}(B)$. Then there exists sequences $(a_n)_{n \in \mathbb{N}} \in A^\infty$ and $(b_n)_{n \in \mathbb{N}} \in B^\infty$ such that $a_n \to a^*$ and $b_n \to b^*$. By assumption,

$$\forall n \in \mathbb{N}, \quad c_0 + c \cdot a_n \leq 0 \leq c_0 + c \cdot b_n.$$

Taking limits for $n \to +\infty$, we get

$$c_0 + c \cdot a^* \leq 0 \leq c_0 + c \cdot b^*,$$

as desired.
We now show that the separation is proper:,

$$A \cup B \nsubseteq H \Rightarrow A \cup B \cap H^C \neq \varnothing \overset{S \subseteq \mathrm{Cl}(S)}{\Rightarrow} \mathrm{Cl}(A) \cup \mathrm{Cl}(B) \cap H^C \neq \varnothing.$$

■

The following three Propositions are presented without proofs. Detailed, self-contained proofs of those results are contained, for example, in Villanacci, A., (in progress), Basic Convex Analysis, mimeo, Università degli Studi di Firenze.

Proposition 699 *Let A be a closed nonempty convex set in $\mathbb{R}^n$ such that $0 \notin A$. Then, there exists a hyperplane in $\mathbb{R}^n$ that strictly separates A and $\{0\}$.*

Corollary 700 *Let B be a closed nonempty convex set in $\mathbb{R}^n$ such that $w \notin B$. Then, there exists a hyperplane in $\mathbb{R}^n$ that strictly separates B and $\{w\}$.*

Proof. Exercise. ■

Proposition 701 *Let A be a nonempty convex set in $\mathbb{R}^n$ such that $0 \notin A$. Then, there exists a hyperplane H in $\mathbb{R}^n$ such that $A \subseteq H_-$ and $\{0\} \subseteq H_+$.*

Proposition 702 *Let A be a nonempty convex set in $\mathbb{R}^n$ such that $0 \notin Int(A)$. Then, there exists a hyperplane H in $\mathbb{R}^n$ such that $A \subseteq H_-$ and $\{0\} \subseteq H_+$.*

Proposition 703 *Let A and B be nonempty convex sets in $\mathbb{R}^n$. If one of the following conditions holds, then there exists a hyperplane H such that $A \subseteq H_-$ and $B \subseteq H_+$:*

1. $0 \notin A - B$;
2. $0 \notin Int(A - B)$;
3. $IntB \neq \varnothing$ *and* $0 \notin A - Int(B)$.

Proof. 1.

Since A and B are convex, we show that $A - B$ is convex. Let $x, y \in A - B$ where $x = a_1 - b_1$, $y = a_2 - b_2$, and $a_1, a_2 \in A, b_1, b_2 \in B$. Let $\lambda \in [0, 1]$. Then,

$$(1-\lambda)x + \lambda y = (1-\lambda)(a_1 - b_1) + \lambda(a_2 - b_2) = [(1-\lambda)a_1 + \lambda a_2] - [(1-\lambda)b_1 + \lambda b_2] \in A - B$$

by convexity of A and B.

Hence, from our assumption and Lemma **??**, there exists a hyperplane $H = \{x \in \mathbb{R}^n : c \cdot x = c_0\}$ that separates $\{0\}$ and $A - B$. Without loss of generality, $A - B \subseteq H_+$ and $c_0 = 0$. Then, $\forall a \in A, b \in B$,

$$c \cdot (a - b) \geq 0 \Leftrightarrow c \cdot a \geq c \cdot b.$$

Hence, there must exists $H' = \{x' \in \mathbb{R}^n : c \cdot x' = c_1\}$ such that

$$c \cdot a \geq c_1 \geq c \cdot b,$$

i.e., such that $B \subseteq H'_-$ and $A \subseteq H'_+$.

2.

From our assumption and 702, there exists a hyperplane that separates $\{0\}$ and $A - B$, whence, following the Proof of point 1 above, we are done.

3.

From our assumption and point 1 above, there exists a hyperplane H that separates A and $Int(B)$. From Remark 698, H separates $\mathrm{Cl}(A)$ and $\mathrm{Cl}(\mathrm{Int}(B)) = \mathrm{Cl}(B)$, where last equality follows from the Assumption that $Int(B) \neq \varnothing$ and Proposition 691.7. Then since the closure of a set contains the set, the desired result follows. ■

Proposition 704 *Let A and B be subsets of $\mathbb{R}^n$. Then*

$$0 \notin A - B \Leftrightarrow A \cap B = \varnothing.$$

Proof.

$$\begin{array}{ll} 0 \notin A - B & \Leftrightarrow \\ \neg(0 \in A - B) & \Leftrightarrow \\ \neg(\exists a \in A, \exists b \in B \text{ such that } a - b = 0) & \Leftrightarrow \\ \forall a \in A, \forall b \in B, \quad a \neq b & \Leftrightarrow \\ A \cap B = \varnothing. & \end{array}$$

■

Proposition 705 *Let A and B be nonempty convex sets in $\mathbb{R}^n$. If one of the following conditions holds true, then there exists a hyperplane H such that $A \subseteq H_-$ and $B \subseteq H_+$.*

1. $A \cap B = \varnothing$;

2. $IntB \neq \varnothing$ *and* $A \cap Int(B) = \varnothing$.

16.3 Farkas' Lemma

Proposition 706 *If*[2]
1. $v_1, ..., v_m, w \in \mathbb{R}^n$*, and*
2. $v_1 x \le 0, ..., v_m x \le 0 \Rightarrow wx \le 0$*,*
then

$$\exists \lambda := (\lambda_i)_{i=1}^m \in \mathbb{R}_+^m \quad \text{such that} \quad w = \sum_{i=1}^m \lambda_i v_i.$$

Proof. Define

$$C = \left\{ y \in \mathbb{R}^n : \sum_{i=1}^m \gamma_i v_i \text{ with } (\gamma_i)_{i=1}^m \in \mathbb{R}_+^m \right\}.$$

Observe that since $0 \in \mathbb{R}_+^m$, then

$$0 \in C, \tag{16.1}$$

and

$$\forall \lambda \in \mathbb{R}_+ \text{ and } \forall y \in C, \text{ we have } \lambda y \in C. \tag{16.2}$$

We want to show that $w \in C$.
Claim 1. C is a convex, nonempty and closed set.
Proof of Claim 1.
The fact that C is a convex and nonempty is obvious. Let's check closedness. Take $(x_k)_{k \in \mathbb{N}} \in C^\infty$ such that $x_k \to x$. We want to show that $x \in C$. Indeed, $x_k \in C$ means that there exists $(\gamma_{ik})_{i=1}^m$ such that

$$x_k = \sum_{i=1}^m \gamma_{ik} v_i \to x. \tag{16.3}$$

Then

$$\forall \varepsilon > 0 \ \ \exists N_\varepsilon \in \mathbb{N} \ \text{ such that } \forall k > N_\varepsilon, \quad \left\| \sum_{i=1}^m \gamma_{ik} v_i - x \right\| < \varepsilon.$$

Now, in general, we have that $\|x\| = \|x - y + y\| \le \|x - y\| + \|y\|$, i.e.,

$$\|x\| - \|y\| \le \|x - y\|.$$

Therefore,

$$\left\| \sum_{i=1}^m \gamma_{ik} v_i \right\| - \|x\| \le \left\| \sum_{i=1}^m \gamma_{ik} v_i - x \right\| < \varepsilon, \tag{16.4}$$

and

$$0 \le \left\| \sum_{i=1}^m \gamma_{ik} v_i \right\| < \|x\| + \varepsilon. \tag{16.5}$$

Then, as verified below, $((\gamma_{ik})_{i=1}^m)_{k \in \mathbb{N}}$ is a bounded sequence. Suppose otherwise; then, since by assumption, for any $i \in \{1, ..., m\}$ and any $k \in \mathbb{N}$, $\gamma_{ik} \ge 0$, we have that there exists $i^* \in \{1, ..., m\}$ such that $\gamma_{ik^*} \to +\infty$. Then,

$$\lim_{k \to +\infty} \left\| \sum_{i=1}^m \gamma_{ik} v_i \right\| = \lim_{k \to +\infty} \sum_{i=1}^m \gamma_{ik} \|v_i\| = +\infty,$$

violating (16.5).
Then, up to a subsequence,

$$\gamma_k := (\gamma_{ik})_{i=1}^m \to \widetilde{\gamma} \in \mathbb{R}_+^m. \tag{16.6}$$

Then,

$$\sum_{i=1}^m \gamma_{ik} v^i \to \sum_{i=1}^m \widetilde{\gamma}_i v^i \overset{(16.3)}{=} x,$$

[2]In this Section, I follow very closely Section 8.1.2 in Montrucchio (1998).

and, then, from (16.6), $x \in C$, as desired.
End of the Proof of Claim 1.
Suppose now our Claim is false, i.e., $w \notin C$. We can then apply Corollary 699 to conclude that there exist $a \in \mathbb{R}^n \setminus \{0\}$ and $\gamma \in \mathbb{R}$ such that

$$\forall x \in C, \quad aw > \gamma > ax. \tag{16.7}$$

From (16.1), $0 \in C$ and then from (16.7), we have

$$\forall x \in C, \quad aw > \gamma > 0. \tag{16.8}$$

Claim 2.

$$\forall x \in C, \ ax \leq 0.$$

Proof of Claim 2.
Suppose otherwise, i.e., $\exists x^* \in C$ such that

$$ax^* > 0.$$

From (16.2), and the fact that $x^* \in C$, we have that for any $\lambda \geq 0$, $\lambda x^* \in C$. Then, from (16.7), we get

$$\forall \lambda \geq 0, \ \gamma > a\lambda x^* = \overset{(\geq 0)}{\lambda} \overset{(>0)}{(ax^*)} > 0,$$

which is clearly false: if $\lambda = 2\frac{\gamma}{ax^*} > 0$, we get $\gamma > 2\frac{\gamma}{ax^*} ax^*$, i.e., $1 > 2$.
End of the Proof of Claim 2.
Summarizing, we have shown that

$$\exists a \in \mathbb{R}^n \setminus \{0\} \text{ such that } \forall x \in C, \ ax \leq 0 \text{ and } aw > 0.$$

Since $v_1, v_2, ..., v_m \in C$, we have that

$$av_1 \leq 0, ..., av_m \leq 0 \text{ and } aw > 0,$$

contradicting the assumption. ■

Proposition 707 *(A version of Farkas Lemma) Given $A \in \mathbb{M}(m, n)$, $b \in \mathbb{R}^n$,*

either	*1.*	*$\exists x \in \mathbb{R}^n$ such that $Ax \leq 0$ and $bx > 0$,*
or	*2.*	*$\exists y \in \mathbb{R}^m$ such that $yA = b$ and $y \geq 0$,*
but not both.		

Proof. We want to show that either
1.

$$\begin{cases} Ax \leq 0 \\ bx > 0 \end{cases} \tag{16.9}$$

has a solution, or
2.

$$\begin{cases} yA = b \\ y \geq 0 \end{cases} \tag{16.10}$$

has a solution, but not both.
Claim. It suffices to show
a. $(\neg 1) \Rightarrow 2$,
b. $2 \Rightarrow (\neg 1)$.
Proof of the Claim.
Indeed, a. $\Leftrightarrow (\neg 2) \Rightarrow 1$ and b. $\Leftrightarrow 1 \Rightarrow (\neg 2)$. Therefore, showing a. and b. implies showing

$$(\neg 1) \Leftrightarrow 2, \qquad \text{and} \qquad (\neg 2) \Leftrightarrow 1. \tag{16.11}$$

Observe also that we have that $1 \vee 2$. Suppose otherwise, i.e., $(\neg 1) \wedge (\neg 2)$. But $(\neg 1)$ and (16.11) imply 2, a contradiction.

End of the proof of the Claim.

We are now left with showing a. and b.

a. Suppose that (16.9) has a no solution. Then $Ax \leq 0$ implies that $bx \leq 0$. Then from Proposition 706, identifying v_i with $R^i(A)$, the $i-th$ row of A, and w with b, we have that

$$\exists \lambda := (\lambda_i)_{i=1}^m \in \mathbb{R}_+^m \qquad \text{such that} \qquad b = \lambda A,$$

i.e., λ is a solution to (16.10).

b. By assumption, there exists $y \in \mathbb{R}_+^m$ such that $yA = b$. Then, taken $x \in \mathbb{R}^n$, we also have $yAx = bx$. Now if $bx > 0$, since $y \geq 0$, we have that $\exists i \in \{1, ..., m\}$ such that $R^i(A) \cdot x > 0$, and therefore (16.9) has no solution. ■

Chapter 17

Concave functions

Consider[1] a set $X \subseteq \mathbb{R}^n$, a set $\Pi \subseteq \mathbb{R}^k$ and the functions $f : X \times \Pi \to \mathbb{R}$, $g : X \times \Pi \to \mathbb{R}^m$, $h : X \times \Pi \to \mathbb{R}^l$. The goal of this Chapter is to study the problem:

for given f, g, h and for given $\pi \in \Pi$,

$$\max_{x \in X} f(x, \pi) \quad s.t. \quad g(x, \pi) \geq 0 \quad and \quad h(x, \pi) = 0,$$

under suitable assumptions. The role of concavity (and differentiability) of the functions f,g and h is crucial.

In what follows, unless needed, we omit the depends on π.

17.1 Different Kinds of Concave Functions

Maintained Assumptions in this Chapter. Unless otherwise stated,

$$X \text{ is an open and convex subset of } \mathbb{R}^n.$$

f is a function such that

$$f : X \to \mathbb{R}, \qquad : x \mapsto f(x).$$

For each type of concavity we study, we present

1. the definition in the case in which f is C^0 (i.e., continuous),

2. an attempt of a "partial characterization" of that definition in the case in which f is C^1 and C^2; by partial characterization, we mean a statement which is either sufficient or necessary for the concept presented in the case of continuous f;

3. the relationship between the different partial characterizations;

4. the relationship between the type of concavity and critical points and local or global extrema of f.

Finally, we study the relationship between different kinds of concavities.

The following pictures are taken from David Cass's Microeconomics Course I followed at the University of Pennsylvania (in 1985) and summarize points 1., 2. and 3. above.

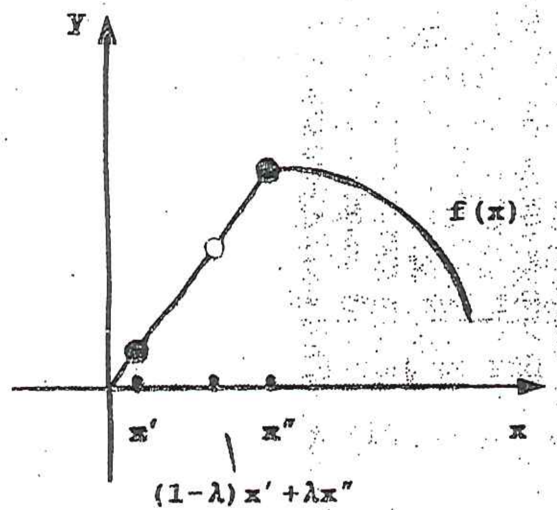
[1] This part is based on Cass (1991).

Concave and Quasi-Concave Functions

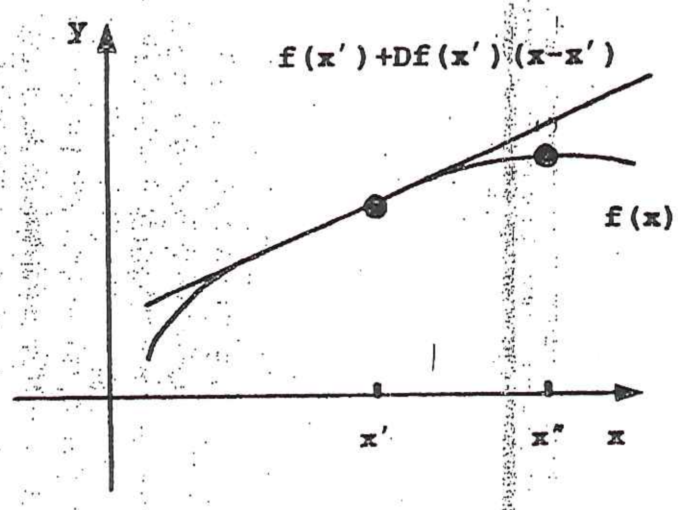
Assume $X \subset \mathbb{R}^n$ is convex and open, $f: X \to \mathbb{R}$.

- f is **concave** if

f <u>continuous</u>		f <u>differentiable</u>		f <u>twice differentiable</u>
$\{(y,x) \in \mathbb{R}^{n+1}: x \in X \;\&\; y \leq f(x)\}$ is convex				
$\Updownarrow$				
$x', x'' \in X \;\&\; 0 \leq \lambda \leq 1 \Rightarrow$ $f((1-\lambda)x'+\lambda x'') \geq (1-\lambda)f(x')+\lambda f(x'')$	$\Leftrightarrow$	$x', x'' \in X \Rightarrow$ $f(x'') \leq f(x')+Df(x')(x''-x')$	$\Leftrightarrow$	$x \in X \;\&\; \Delta x \in \mathbb{R}^n \Rightarrow$ $\Delta x^T D^2 f(x) \Delta x \leq 0$



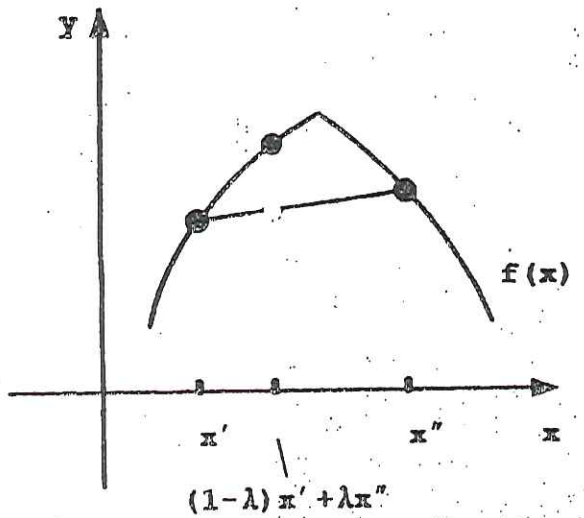
both "flats" and "kinks" are possible



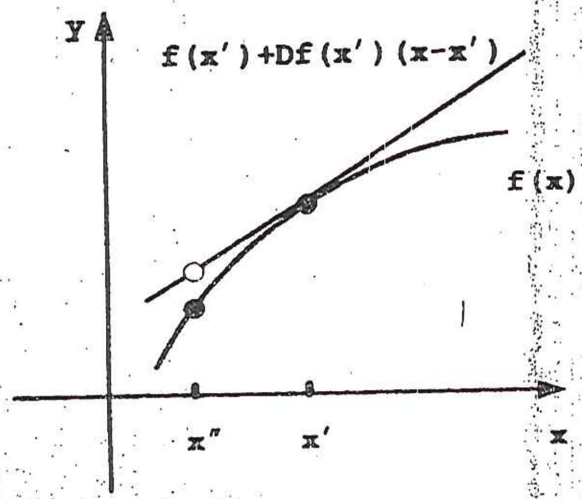
"flats" but not "kinks" are possible

- f is **strictly concave** if it is concave and

f <u>continuous</u>		f <u>differentiable</u>		f <u>twice differentiable</u>
$x', x'' \in X,\ x' \neq x''\ \&\ 0 < \lambda < 1 \Rightarrow$ $f((1-\lambda)x' + \lambda x'') > (1-\lambda)f(x') + \lambda f(x'')$	$\Leftrightarrow$	$x', x'' \in X\ \&\ x' \neq x'' \Rightarrow$ $f(x'') < f(x') + Df(x')(x''-x')$	$\Leftarrow$	$x \in X,\ \Delta x \in \mathbb{R}^n\ \&\ \Delta x \neq 0 \Rightarrow$ $\Delta x^T D^2 f(x) \Delta x < 0$



"kinks" but not "flats" are possible



neither "flats" nor "kinks" are possible

Note: When X is open (as well convex), (i) if f is concave, then it is continuous, while (ii) it is meaningful to assume, for example, that f is differentiable (i.e., that f has a first-order differential at each point in X).

f is **quasi-concave** if

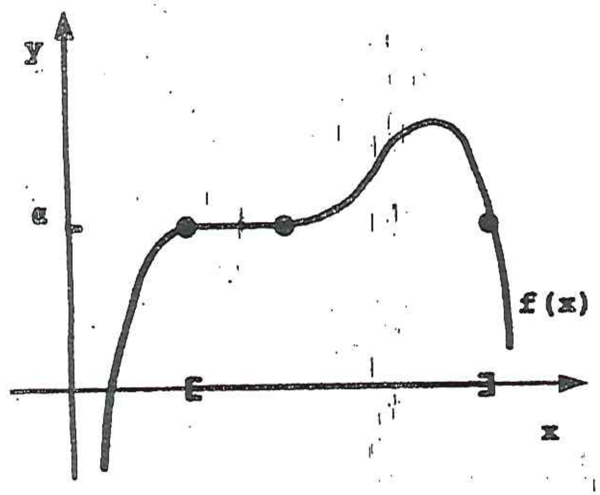
f <u>continuous</u>

$\{x \in X : f(x) \geq \alpha\}$ is convex for every $\alpha \in \mathbb{R}$

$\Updownarrow$

$x', x'' \in X$ & $0 \leq \lambda \leq 1 \Rightarrow$

$f((1-\lambda)x' + \lambda x'') \geq \min\{f(x'), f(x'')\}$

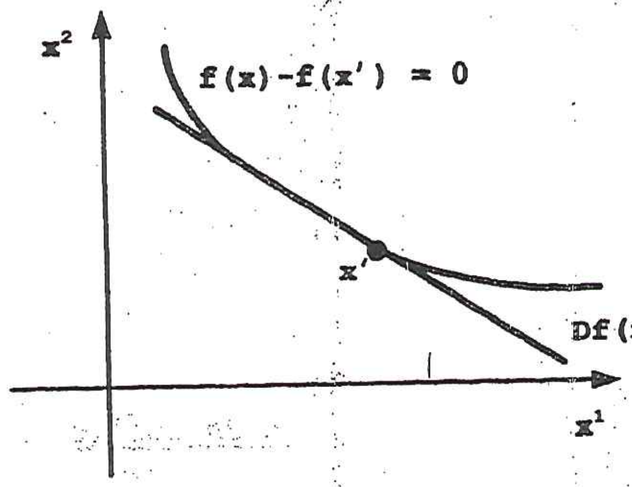


$\Leftrightarrow$

f <u>differentiable</u>

$x', x'' \in X$ & $f(x'') - f(x') \geq 0 \Rightarrow$

$Df(x')(x''-x) \geq 0$



$\Rightarrow$

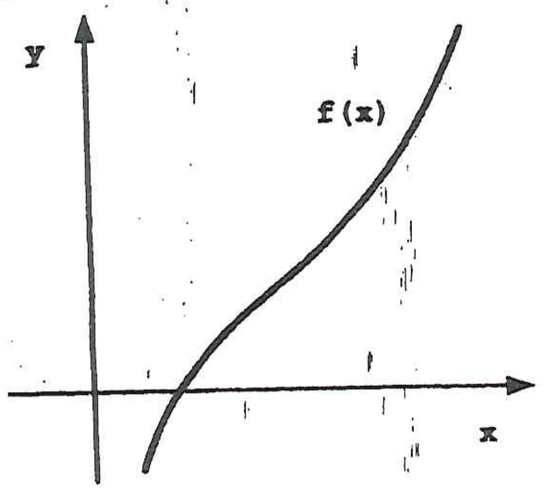
f <u>twice differentiable</u>

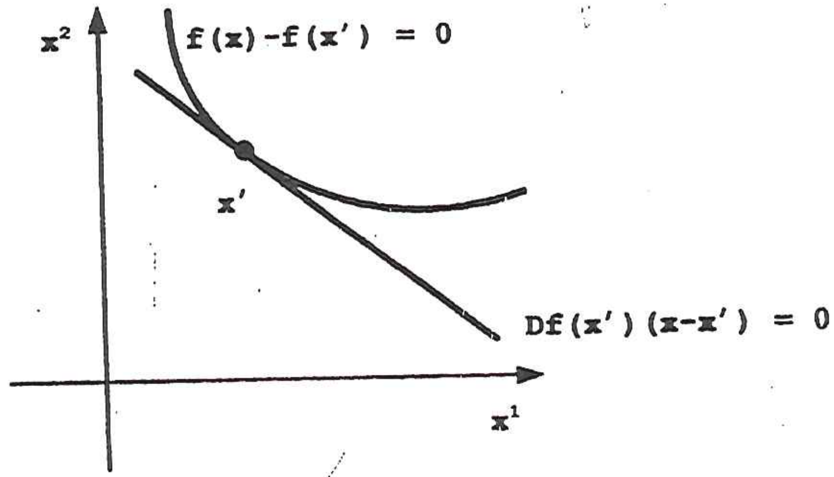
$x \in X$, $\Delta x \in \mathbb{R}^n$, & $Df(x)\Delta x = 0 \Rightarrow$

$\Delta x^T D^2 f(x) \Delta x \leq 0$

f is (at worst) "single-peaked", and may have "flats" f is **strictly quasi-concave** if it is quasi-concave and

f <u>continuous</u>		f <u>differentiable</u>		f <u>twice differentiable</u>
$x', x'' \in X,\ x' \neq x''\ \&\ 0 < \lambda < 1 \Rightarrow$		$x', x'' \in X,\ x' \neq x''\ \&$ $f(x'') - f(x') \geq 0 \Rightarrow$		$x \in X,\ \Delta x \in \mathbb{R}^n,\ \Delta x \neq 0\ \&$ $Df(x)\Delta x = 0 \Rightarrow$
$f((1-\lambda)x' + \lambda x'') > \min\{f(x'), f(x'')\}$	$\Leftarrow$	$Df(x')(x''-x') > 0$	$\Leftarrow$	$\Delta x^T D^2 f(x) \Delta x < 0$





f has no "flats"

Note: There are alternative possible definitions of strict quasi-concavity when f is differentiable. This is the definition most frequently adopted in economic applications.

17.1.1 Concave Functions.

Definition 708 *Consider a C^0 function f. f is concave iff $\forall\, x', x'' \in X$, $\forall\, \lambda \in [0,1]$,*

$$f((1-\lambda)x' + \lambda x'') \geq (1-\lambda) f(x') + \lambda f(x'').$$

Proposition 709 *Consider a C^0 function f.*
f is concave
$\Leftrightarrow$
$M = \{(x,y) \in X \times \mathbb{R} : \; y \leq f(x)\}$ is convex.

Proof.
[$\Rightarrow$]
Take $(x', y'), (x'', y'') \in M$. We want to show that

$$\forall\, \lambda \in [0,1], \; ((1-\lambda)\, x' + \lambda x'', (1-\lambda)\, y' + \lambda y'') \in M.$$

But, from the definition of M, we get that

$$(1-\lambda) y' + \lambda y'' \leq (1-\lambda) f(x') + \lambda f(x'') \leq f((1-\lambda)x' + \lambda x'').$$

[$\Leftarrow$]
From the definition of M, $\forall\, x', x'' \in X$, $(x', f(x')) \in M$ and $(x'', f(x'')) \in M$.
Since M is convex,

$$((1-\lambda)\, x' + \lambda x'', \; (1-\lambda)\, f(x') + \lambda f(x'')) \in M$$

and from the definition of M,

$$(1-\lambda)\, f(x') + \lambda f(x'') \leq f(\lambda x' + (1-\lambda)\, x'')$$

as desired.
■

Proposition 710 *(Some properties of concave functions).*
1. If $f, g : X \to \mathbb{R}$ are concave functions and $a, b \in \mathbb{R}_+$, then the function $af + bg : X \to \mathbb{R}$, $af + bg : x \mapsto af(x) + bg(x)$ is a concave function.
2. If $f : X \to \mathbb{R}$ is a concave function and $F : A \to \mathbb{R}$, with $A \supseteq Im\, f$, is nondecreasing and concave, then $F \circ f$ is a concave function.

Proof.
1. This result follows by a direct application of the definition.
2. Let $x', x'' \in X$ and $\lambda \in [0,1]$. Then

$$(F \circ f)((1-\lambda)\, x' + \lambda x'') \overset{(1)}{\geq} F((1-\lambda)\, f(x') + \lambda f(x'')) \overset{(2)}{\geq} (1-\lambda) \cdot (F \circ f)(x') + \lambda \cdot (F \circ f)(x''),$$

where (1) comes from the fact that f is concave and F is non decreasing, and
(2) comes from the fact that F is concave.
■

Remark 711 *(from Sydsæter (1981)). With the notation of part 2 of the above Proposition, the assumption that F is concave cannot be dropped, as the following example shows. Take $f, F : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $f(x) = \sqrt{x}$ and $F(y) = y^3$. Then f is concave and F is strictly increasing, but $F \circ f(x) = x^{\frac{3}{2}}$ and its second derivative is $\frac{3}{4} x^{-\frac{1}{2}} > 0$. Then, from Calculus I, we know that $F \circ f$ is strictly convex and therefore it is not concave.*
Of course, the monotonicity assumption cannot be dispensed either. Consider $f(x) = -x^2$ and $F(y) = -y$. Then, $(F \circ f)(x) = x^2$, which is not concave.

Proposition 712 *Consider a differentiable function f.*
f is concave
$\Leftrightarrow$
$\forall x', x'' \in X, \;\; f(x'') - f(x') \leq Df(x')(x'' - x')$.

Proof.
[⇒]
From the definition of concavity, we have that for $\lambda \in (0,1)$,

$$\begin{aligned} &(1-\lambda) f(x') + \lambda f(x'') \le f(x' + \lambda(x'' - x')) && \Rightarrow \\ &\lambda (f(x'') - f(x')) \le f(x' + \lambda(x'' - x')) - f(x') && \Rightarrow \\ &f(x'') - f(x') \le \tfrac{f\left(x' + \lambda\left(x'' - x'\right)\right) - f(x')}{\lambda}. \end{aligned}$$

Taking limits of both sides of the lasts inequality for $\lambda \to 0$, we get the desired result.
[⇐]
Consider $x', x'' \in X$ and $\lambda \in (0,1)$. For $\lambda \in \{0,1\}$, the desired result is clearly true. Since X is convex, $x^{\lambda} := (1-\lambda) x' + \lambda x'' \in X$. By assumption,

$$f(x'') - f(x^{\lambda}) \le Df(x^{\lambda})(x'' - x^{\lambda}) \text{ and}$$

$$f(x') - f(x^{\lambda}) \le Df(x^{\lambda})(x' - x^{\lambda})$$

Multiplying the first expression by λ, the second one by $(1-\lambda)$ and summing up, we get

$$\lambda(f(x'') - f(x^{\lambda})) + (1-\lambda)(f(x') - f(x^{\lambda})) \le Df(x^{\lambda})(\lambda(x'' - x^{\lambda}) + (1-\lambda)(x' - x^{\lambda}))$$

Since

$$\lambda(x'' - x^{\lambda}) + (1-\lambda)(x' - x^{\lambda}) = x^{\lambda} - x^{\lambda} = 0,$$

we get

$$\lambda f(x'') + (1-\lambda) f(x') \le f(x^{\lambda}),$$

i.e., the desired result.
■

Definition 713 *Given a symmetric matrix $A_{n \times n}$, A is negative semidefinite if $\forall x \in \mathbb{R}^n$, $xAx \le 0$. A is negative definite if $\forall x \in \mathbb{R}^n \setminus \{0\}$, $xAx < 0$.*

Proposition 714 *Consider a C^2 function f.*
f is concave
⇔
$\forall x \in X$, $D^2 f(x)$ is negative semidefinite.

Proof.
[⇒]
We want to show that $\forall u \in \mathbb{R}^n$, $\forall x_0 \in X$, it is the case that $u^T D^2 f(x_0) u \le 0$. Since X is open, $\forall\, x_0 \in X$ $\exists\, a \in \mathbb{R}_{++}$ such that $|h| < a \Rightarrow (x_0 + hu) \in X$. Taken $I := (-a, a) \subseteq \mathbb{R}$, define

$$g : I \to \mathbb{R}, \quad g : h \mapsto f(x_0 + hu) - f(x_0) - Df(x_0) hu.$$

Observe that

$$g'(h) = D_x f(x_0 + hu) \cdot u + Df(x_0) \cdot u$$

and

$$g''(h) = u \cdot D^2 f(x_0 + hu) \cdot u$$

Since f is a concave function, from Proposition 712, we have that $\forall\, h \in I$, $g(h) \le 0$. Since $g(0) = 0$, $h = 0$ is a maximum point. Then, $g'(0) = 0$ and

$$g''(0) \le 0 \quad (1).$$

Moreover, $\forall h \in I$, $g'(h) = Df(x_0 + hu)u - Df(x_0)u$ and $g''(h) = u^T D^2 f(x_0 + hu)u$. Then,

$$g''(0) = u \cdot D^2 f(x_0) \cdot u \quad (2).$$

(1) and (2) give the desired result.
[⇐]
Consider $x, x^0 \in X$. From Taylor's Theorem (see Proposition 667), we get

$$f(x) = f(x^0) + Df(x^0)(x - x^0) + \frac{1}{2}\left(x - x^0\right)^T D^2 f(\overline{x})(x - x^0)$$

where $\overline{x} = x^0 + \theta(x - x^0)$, for some $\theta \in (0,1)$. Since, by assumption, $\left(x - x^0\right)^T D^2 f(\overline{x})(x - x^0) \leq 0$, we have that

$$f(x) - f(x^0) \leq Df(x^0)(x - x^0),$$

the desired result.
■

Some Properties.

Proposition 715 *Consider a concave function f. If x_0 is a local maximum point, then it is a global maximum point.*

Proof.
By definition of local maximum point, we know that $\exists \delta > 0$ such that $\forall x \in B(x_0, \delta),\ f(x_0) \geq f(x)$. Take $y \in X$; we want to show that $f(x^0) \geq f(y)$.
Since X is convex,

$$\forall \lambda \in [0,1],\ (1-\lambda) x^0 + \lambda y \in X.$$

Take $\lambda^0 > 0$ and sufficiently small to have $\left(1 - \lambda^0\right) x^0 + \lambda^0 y \in B(x^0, \delta)$. To find such λ^0, just solve the inequality $\left\|\left(1 - \lambda^0\right) x^0 + \lambda^0 y - x^0\right\| = \left\|\lambda^0 \left(y - x^0\right)\right\| = \left|\lambda^0\right| \left\|\left(y - x^0\right)\right\| < \delta$, where, without loss of generality, $y \neq x^0$.
Then,

$$f\left(x^0\right) \geq f\left(\left(1 - \lambda^0\right) x^0 + \lambda^0 y\right) \overset{f \text{ concave}}{\geq} \left(1 - \lambda^0\right) f(x^0) + \lambda^0 f(y),$$

or $\lambda^0 f(x^0) \geq \lambda^0 f(y)$. Dividing both sides of the inequality by $\lambda^0 > 0$, we get $f(x^0) \geq f(y)$.
■

Proposition 716 *Consider a differentiable and concave function f. If $Df(x^0) = 0$, then x^0 is a global maximum point.*

Proof.
From Proposition 712, if $Df(x^0) = 0$, we get that $\forall\, x \in X,\ f\left(x^0\right) \geq f(x)$, the desired result.
■

17.1.2 Strictly Concave Functions.

Definition 717 *Consider a C^0 function f. f is strictly concave iff $\forall\, x', x'' \in X$ such that $x' \neq x''$, $\forall\, \lambda \in (0,1)$,*

$$f((1-\lambda)x' + \lambda x'') > (1-\lambda) f(x') + \lambda f(x'').$$

Proposition 718 *Consider a C^1 function f.*
f is strictly concave
$\Leftrightarrow \forall x', x'' \in X$ such that $x' \neq x''$,

$$f(x'') - f(x') < Df(x')(x'' - x').$$

Proof.
[⇒]
Since strict concavity implies concavity, it is the case that

$$\forall x', x'' \in X,\ f(x'') - f(x') \leq Df(x')(x'' - x'). \tag{17.1}$$

By contradiction, suppose f is not strictly concave. Then, from 17.1, we have that

$$\exists\, x', x'' \in X,\ x' \neq x'' \text{ such that } f(x'') = f(x') + Df(x')(x'' - x'). \tag{17.2}$$

From the definition of strict concavity and 17.2, for $\lambda \in (0,1)$,

$$f((1-\lambda)x' + \lambda x'') > (1-\lambda) f(x') + \lambda f(x') + \lambda Df(x')(x'' - x')$$

or

$$f((1-\lambda)x' + \lambda x'') > f(x') + \lambda Df(x')(x'' - x'). \tag{17.3}$$

Applying 17.1 to the points $x(\lambda) := (1-\lambda)x' + \lambda x''$ and x', we get that for $\lambda \in (0,1)$,

$$f((1-\lambda)x' + \lambda x'') \leq f(x') + Df(x')((1-\lambda)x' + \lambda x'' - x')$$

or

$$f((1-\lambda)x' + \lambda x'') \leq f(x') + \lambda Df(x')(x'' - x'). \tag{17.4}$$

And 17.4 contradicts 17.3.
[⇐] The proof is very similar to that one in Proposition 709.
■

Proposition 719 *Consider a C^2 function f. If*

$$\forall x \in X, \quad D^2 f(x) \text{ is negative definite,}$$

then f is strictly concave.

Proof.
The proof is similar to that of Proposition 714.
■

Remark 720 *In the above Proposition, the opposite implication does not hold. The standard counterexample is $f : \mathbb{R} \to \mathbb{R}, f : x \mapsto -x^4$.*

Some Properties.

Proposition 721 *Consider a strictly concave, C^0 function f. If x^0 is a local maximum point, then it is a strict global maximum point, i.e., the unique global maximum point.*

Proof.
First, we show that a. it is a global maximum point, and then b. the desired result.
a. It follows from the fact that strict concavity is stronger than concavity and from Proposition 715.
b. Suppose otherwise, i.e., $\exists x', x^0 \in X$ such that $x' \neq x^0$ and both of them are global maximum points. Then, $\forall \lambda \in (0,1)$, $(1-\lambda)x' + \lambda x^0 \in X$, since X is convex, and

$$f\left((1-\lambda)x' + \lambda x^0\right) > (1-\lambda) f(x') + \lambda f\left(x^0\right) = f(x') = f\left(x^0\right),$$

a contradiction.
■

Proposition 722 *Consider a strictly concave, differentiable function f. If $Df\left(x^0\right) = 0$, then x^0 is a strict global maximum point.*

Proof.
Take an arbitrary $x \in X$ such that $x \neq x^0$. Then from Proposition 718, we have that $f(x) < f(x^0) + Df(x^0)(x - x^0) = f\left(x^0\right)$, the desired result.
■

17.1.3 Quasi-Concave Functions.

Definitions.

Definition 723 *Consider a C^0 function f. f is quasi-concave iff $\forall x', x'' \in X$, $\forall \lambda \in [0,1]$,*

$$f((1-\lambda)x' + \lambda x'') \geq \min \{f(x'), f(x'')\}.$$

Proposition 724 *If $f : X \to \mathbb{R}$ is a quasi-concave function and $F : \mathbb{R} \to \mathbb{R}$ is non decreasing, then $F \circ f$ is a quasi-concave function.*

Proof.
Without loss of generality, assume

$$f(x'') \geq f(x') \quad (1).$$

Then, since f is quasi-concave, we have

$$f((1-\lambda)x' + \lambda x'') \geq f(x') \quad (2).$$

Then,

$$F(f((1-\lambda)x' + \lambda x'')) \overset{(a)}{\geq} F(f(x')) \overset{(b)}{=} \min \{F(f(x')), F(f(x''))\},$$

where (a) comes from (2) and the fact that F is nondecreasing, and
(b) comes from (1) and the fact that F is nondecreasing.
■

Proposition 725 *Consider a C^0 function f. f is quasi-concave $\Leftrightarrow$ $\forall \alpha \in \mathbb{R}$, $B(\alpha) := \{x \in X : f(x) \geq \alpha\}$ is convex.*

Proof.
[$\Rightarrow$] [Strategy: write what you want to show].
We want to show that $\forall \alpha \in \mathbb{R}$ and $\forall \lambda \in [0,1]$, we have that

$$\langle x', x'' \in B(a) \rangle \Rightarrow \langle (1-\lambda)x' + \lambda x'' \in B(\alpha) \rangle,$$

i.e.,

$$\langle f(x') \geq \alpha \text{ and } f(x'') \geq \alpha \rangle \Rightarrow \langle f((1-\lambda)x' + \lambda x'') \geq \alpha \rangle.$$

But by Assumption,

$$f((1-\lambda)x' + \lambda x'') \geq \min \{f(x'), f(x'')\} \overset{\text{def } x', x''}{\geq} \alpha.$$

[$\Leftarrow$]
Consider arbitrary $x', x'' \in X$. Define $\alpha := \min \{f(x'), f(x'')\}$. Then $x', x'' \in B(\alpha)$. By assumption, $\forall \lambda \in [0,1]$, $(1-\lambda)x' + \lambda x'' \in B(\alpha)$, i.e.,

$$f((1-\lambda)x' + \lambda x'') \geq \alpha := \min \{f(x'), f(x'')\}.$$

■

Proposition 726 *Consider a differentiable function f. f is quasi-concave $\Leftrightarrow$ $\forall x', x'' \in X$,*

$$f(x'') - f(x') \geq 0 \quad \Rightarrow \quad Df(x')(x'' - x') \geq 0.$$

Proof.
[$\Rightarrow$] [Strategy: Use the definition of directional derivative.]
Take x', x'' such that $f(x'') \geq f(x')$. By assumption,

$$f((1-\lambda)x' + \lambda x'') \geq \min \{f(x'), f(x'')\} = f(x')$$

and

$$f((1-\lambda)x'+\lambda x'')-f(x')\geq 0.$$

Dividing both sides of the above inequality by $\lambda>0$, and taking limits for $\lambda\to 0^+$, we get

$$\lim_{\lambda\to 0^+}\frac{f(x'+\lambda(x''-x'))-f(x')}{\lambda}=Df(x')(x''-x')\geq 0.$$

[$\Leftarrow$]
Without loss of generality, take

$$f(x')=\min\{f(x'),f(x'')\}\quad (1).$$

Define

$$\varphi:[0,1]\to\mathbb{R},\ \varphi:\lambda\mapsto f((1-\lambda)x'+\lambda x'').$$

We want to show that

$$\forall\lambda\in[0,1],\ \varphi(\lambda)\geq\varphi(0).$$

Suppose otherwise, i.e., $\exists\ \lambda^*\in[0,1]$ such that $\varphi(\lambda^*)<\varphi(0)$. Observe that in fact it cannot be $\lambda^*\in\{0,1\}$: if $\lambda^*=0$, we would have $\varphi(0)<\varphi(0)$, and if $\lambda^*=1$, we would have $\varphi(1)<\varphi(0)$, i.e., $f(x'')<f(x')$, contradicting (1). Then, we have that

$$\exists\lambda^*\in(0,1)\ \text{such that}\ \varphi(\lambda^*)<\varphi(0)\quad (2).$$

Observe that from (1), we also have that

$$\varphi(1)\geq\varphi(0)\quad (3).$$

Therefore, see Lemma 727, $\exists\ \lambda^{**}>\lambda^*$ such that

$$\varphi'(\lambda^{**})>0\quad (4),\ \text{and}$$
$$\varphi(\lambda^{**})<\varphi(0)\quad (5).$$

From (4), and using the definition of φ', and the Chain Rule,[2] we get

$$0<\varphi'(\lambda^{**})=[Df((1-\lambda^{**})x'+\lambda^{**}x'')](x''-x')\quad (6).$$

Define $x^{**}:=(1-\lambda^{**})x'+\lambda^{**}x''$. From (5), and the assumption, we get that

$$f(x^{**})<f(x').$$

Therefore, by assumption,

$$0\leq Df(x^{**})(x'-x^{**})=Df(x^{**})(-\lambda^{**})(x''-x'),$$

i.e.,

$$[Df(x^{**})]^T(x''-x')\leq 0\quad (7).$$

But (7) contradicts (6).
■

Lemma 727 *Consider a function $g:[a,b]\to\mathbb{R}$ with the following properties:*
1. g is differentiable on (a,b);
2. there exists $c\in(a,b)$ such that $g(b)\geq g(a)>g(c)$.
Then, $\exists\ t\in(c,b)$ such that $g'(t)>0$ and $g(t)<g(a)$.

[2] Defined $v:[0,1]\to X\subseteq\mathbb{R}^n,\lambda\mapsto(1-\lambda)x'+\lambda x''$, we have that $\varphi=f\circ v$. Therefore, $\varphi'(\lambda^*)=Df(v(\lambda^*))\cdot Dv(\lambda^*)$.

Proof.

Without loss of generality and to simplify notation, assume $g(a)=0$. Define $A := \{x \in [c,b] : g(x) = 0\}$.

Observe that $A = [c,b] \cap g^{-1}(0)$ is closed; and it is non empty, because g is continuous and by assumption $g(c) < 0$ and $g(b) \geq 0$.

Therefore, A is compact, and we can define $\xi := \min A$.

Claim. $x \in [c,\xi) \Rightarrow g(x) < 0$.

Suppose not, i.e., $\exists y \in (c,\xi)$ such that $g(y) \geq 0$. If $g(y) = 0$, ξ could not be $\min A$. If $g(y) > 0$, since $g(c) < 0$ and g is continuous, there exists $x' \in (c,y) \subseteq (c,\xi)$, again contradicting the definition of ξ. End of the proof of the Claim.

Finally, applying Lagrange Theorem to g on $[c,\xi]$, we have that $\exists t \in (c,\xi)$ such that $g'(t) = \frac{g(\xi)-g(c)}{\xi - c}$. Since $g(\xi) = 0$ and $g(c) < 0$, we have that $g'(t) < 0$. From the above Claim, the desired result then follows.
■

Proposition 728 *Consider a C^2 function f. If f is quasi-concave then*

$$\forall x \in X, \forall \Delta \in \mathbb{R}^n \text{ such that } Df(x) \cdot \Delta = 0, \quad \Delta^T D^2 f(x) \Delta \leq 0.$$

Proof.

`for another proof- see Laura Carosi' s file`

Suppose otherwise, i.e., $\exists\, x \in X$, and $\exists \Delta \in \mathbb{R}^\kappa$ such that $Df(x) \cdot \Delta = 0$ and $\Delta^T D^2 f(x^0) \Delta > 0$.

Since the function $h : X \to R$, $h : x \mapsto \Delta^T D^2 f(x) \Delta$ is continuous and X is open, $\forall\, \lambda \in [0,1]$, $\exists \varepsilon > 0$ such that if $\| x - x^0 \| < \varepsilon$, then

$$\Delta \cdot D^2 f\left(\lambda x + (1-\lambda) x^0\right) \cdot \Delta > 0 \qquad (1).$$

Define $\overline{x} := x^0 + \mu \frac{\Delta}{\|\Delta\|}$, with $0 < \mu < \varepsilon$. Then,

$$\|\overline{x} - x^0\| = \|\mu \frac{\Delta}{\|\Delta\|}\| = \mu < \varepsilon$$

and $\overline{x}$ satisfies (1). Observe that

$$\Delta = \frac{\| \Delta \|}{\mu}(\overline{x} - x^0).$$

Then, we can rewrite (1) as

$$\left(\overline{x} - x^0\right)^T D^2 f\left(\lambda \overline{x} + (1-\lambda) x^0\right)\left(\overline{x} - x^0\right) > 0$$

From Taylor Theorem, $\exists\, \lambda \in (0,1)$ such that

$$f(\overline{x}) = f(x^0) + (\overline{x} - x^0)^T Df\left(x^0\right) + \frac{1}{2}(\overline{x} - x^0)^T D^2 f(\lambda \overline{x} + (1-\lambda) x^0)(\overline{x} - x^0).$$

Since $Df(x^0)\Delta = 0$ and from (1), we have

$$f(\overline{x}) > f(x^0) \qquad (2).$$

Letting $\widetilde{x} = x^0 + \mu(-\Delta / \| \Delta \|)$, using the same procedure as above, we can conclude that

$$f(\widetilde{x}) > f(x^0) \ (3).$$

But, since $x^0 = \frac{1}{2}(\overline{x} + \widetilde{x})$, (2) and (3) contradict the Definition of quasi-concavity.
■

Remark 729 *In the above Proposition, the opposite implication does not hold. Consider $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto x^4$.From Proposition 725, that function is clearly not quasi-concave. Take $\alpha > 0$. Then $B(\alpha) = \left\{x \in \mathbb{R} : x^4 \geq \alpha\right\} = (-\infty, -\sqrt{\alpha}) \cup (\sqrt{\alpha}, +\infty)$ which is not convex.*

On the other hand observe the following. $f'(x) = 4x^3$ and $4x^3\Delta = 0$ if either $x = 0$ or $\Delta = 0$. In both cases $\Delta 12 x^2 \Delta = 0$. (This is example is taken from Avriel M. and others (1988), page 91).

Some Properties.

Remark 730 *Consider a quasi concave function f. It is NOT the case that*

if x_0 is a local maximum point , then it is a global maximum point. To see that, consider the following function.

$$f : \mathbb{R} \to \mathbb{R}, \quad f : x \mapsto \begin{cases} -x^2 + 1 & if \quad x < 1 \\ 0 & if \quad x \geq 1 \end{cases}$$

Proposition 731 *Consider a C^0 quasi-concave function f. If x_0 is a strict local maximum point, then it is a strict global maximum point.*

Proof.

By assumption, $\exists \delta > 0$ such that if $x \in B(x_0, \delta) \cap X$ and $x_0 \neq x$, then $f(x_0) > f(x)$.

Suppose the conclusion of the Proposition is false; then $\exists x' \in X$ such that $f(x') \geq f(x_0)$.

Since f is quasi-concave,

$$\forall \lambda \in [0,1], \quad f((1-\lambda) x_0 + \lambda x') \geq f(x_0). \quad (1)$$

For sufficiently small λ, $(1-\lambda) x_0 + \lambda x' \in B(x_0, \delta)$ and (1) above holds, contradicting the fact that x_0 is the *strict* local maximum point.

■

Proposition 732 *Consider $f : (a,b) \to \mathbb{R}$. f monotone $\Rightarrow$ f quasi-concave.*

Proof.

Without loss of generality, take $x'' \geq x'$.

Case 1. f is increasing. Then $f(x'') \geq f(x')$. If $\lambda \in [0,1]$, then $(1-\lambda) x' + \lambda x'' = x' + \lambda (x'' - x') \geq x'$ and therefore $f((1-\lambda) x' + \lambda x'') \geq f(x')$.

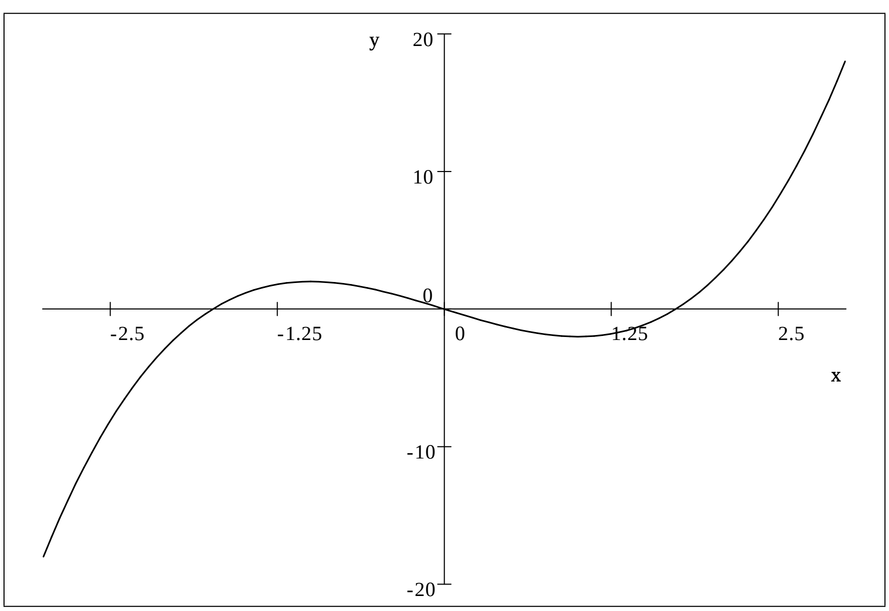
Case 2. f is decreasing. Then $f(x'') \leq f(x')$. If $\lambda \in [0,1]$, then $(1-\lambda) x' + \lambda x'' = (1-\lambda) x' - (1-\lambda) x'' + x'' = x'' - (1-\lambda)(x'' - x') \leq x''$and therefore $f((1-\lambda) x' + \lambda x'') \geq f(x'')$.

■

Remark 733 *The following statement is false: If f_1 and f_2 are quasi-concave and $a, b \in \mathbb{R}_+$, then $af_1 + bf_2$ is quasi-concave.*

It is enough to consider $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^3 + x$, and $f_2(x) = -4x$. Since $f_1' > 0$, then f_1 and, of course, f_2 are monotone and then, from Proposition 732, they are quasi-concave. On the other hand, $g(x) = f_1(x) + f_2(x) = x^3 - x$ has a strict local maximum in $x = -1$which is not a strict global maximum, and therefore, from Proposition 731, g is not quasi-concave.

$x^3 - 3x$



Remark 734 *Consider a differentiable quasi-concave function f. It is NOT the case that if $Df(x^0) = 0$, then x^0 is a global maximum point.*
Just consider $f : \mathbb{R} \to \mathbb{R}, f : x \mapsto x^3$ and $x_0 = 0$, and use Proposition 732.

17.1.4 Strictly Quasi-concave Functions.

Definitions.

Definition 735 *Consider a C^0 function f. f is strictly quasi-concave iff $\forall\ x', x'' \in X$, such that $x' \neq x''$, and $\forall\ \lambda \in (0, 1)$, we have that*

$$f((1-\lambda)x' + \lambda x'') > \min\{f(x'), f(x'')\}.$$

Proposition 736 *Consider a C^0 function f. f is strictly quasi-concave $\Rightarrow \forall \alpha \in \mathbb{R},\ \ B(\alpha) := \{x \in X :\ f(x) \geq \alpha\}$ is strictly convex.*

Proof.
Taken an arbitrary α and $x', x'' \in B(\alpha)$, with $x' \neq x''$, we want to show that $\forall \lambda \in (0, 1)$, we have that

$$x^\lambda := (1-\lambda)x' + \lambda x'' \in \text{Int}\, B(\alpha)$$

Since f is strictly quasi-concave,

$$f\left(x^\lambda\right) > \min\{f(x'), f(x'')\} \geq \alpha$$

Since f is C^0, there exists $\delta > 0$ such that $\forall x \in B\left(x^\lambda, \delta\right)$

$$f(x) > \alpha$$

i.e., $B\left(x^\lambda, \delta\right) \subseteq B(\alpha)$, as desired. Of course, we are using the fact that $\{x \in X :\ f(x) > \alpha\} \subseteq B(\alpha)$.
■

Remark 737 *Observe that in Proposition 736, the opposite implication does not hold true: just consider $f : \mathbb{R} \to \mathbb{R},\ f : x \mapsto 1$.*
Observe that $\forall \alpha \leq 1,\ B(\alpha) = \mathbb{R}$, and $\forall \alpha > 1,\ B(\alpha) = \varnothing$. On the other hand, f is not strictly quasi-concave.

Definition 738 *Consider a differentiable function f. f is differentiable-strictly-quasi-concave iff $\forall\ x', x'' \in X$, such that $x' \neq x''$, we have that*

$$f(x'') - f(x') \geq 0 \ \ \Rightarrow \ \ Df(x')(x'' - x') > 0.$$

Proposition 739 *Consider a differentiable function f.*
If f is differentiable-strictly-quasi-concave, then f is strictly quasi-concave.

Proof.
The proof is analogous to the case of quasi concave functions.
■

Remark 740 *Given a differentiable function, it is not the case that strict-quasi-concavity implies differentiable-strict-quasi-concavity.*
$f : \mathbb{R} \to \mathbb{R},\ f : x \mapsto x^3$ a. is differentiable and strictly quasi concave and b. it is not differentiable-strictly-quasi-concave.
a. f is strictly increasing and therefore strictly quasi concave - see Fact below.
b. Take $x' = 0$ and $x'' = 1$. Then $f(1) = 1 > 0 = f(0)$. But $Df(x')(x'' - x') = 0 \cdot 1 = 0 \not> 0$.

Remark 741 *If we restrict the class of differentiable functions to whose with non-zero gradients everywhere in the domain, then differentiable-strict-quasi-concavity and strict-quasi-concavity are equivalent (see Balasko (1988), Math. 7.2.).*

Fact. Consider $f : (a, b) \to \mathbb{R}$. f strictly monotone $\Rightarrow f$ strictly quasi concave.

Proof.
By assumption, $x' \neq x''$, say $x' < x''$ implies that $f(x') < f(x'')$ (or $f(x') > f(x'')$). If $\lambda \in (0,1)$, then $(1-\lambda)x' + \lambda x'' > x'$ and therefore $f((1-\lambda)x' + \lambda x'') > \min\{f(x'), f(x'')\}$.
■

Proposition 742 *Consider a C^2 function f. If*

$$\forall x \in X, \forall \Delta \in \mathbb{R}^n \setminus \{0\}, \text{ we have that } \left\langle Df(x)\Delta = 0 \Rightarrow \Delta^T D^2 f(x) \Delta < 0 \right\rangle,$$

then f is differentiable-strictly-quasi-concave.

Proof.

Suppose otherwise, i.e., there exist $x', x'' \in X$ such that

$$x' \neq x'',\ f(x'') \geq f(x') \quad \text{and} \quad Df(x')(x'' - x') \leq 0.$$

Since X is an open set, $\exists a \in \mathbb{R}_{++}$ such the following function is well defined:

$$g : [-a, 1] \to \mathbb{R}, g : h \mapsto f((1-h)x' + hx'').$$

Since g is continuous, there exists $h_m \in [0,1]$ which is a global minimum. We now proceed as follows. Step 1. $h_m \notin \{0,1\}$. Step 2. h_m is a strict local maximum point, a contradiction.
Preliminary observe that

$$g'(h) = Df(x' + h(x'' - x')) \cdot (x'' - x')$$

and

$$g''(h) = (x'' - x')^T \cdot D^2 f(x' + h(x'' - x')) \cdot (x'' - x').$$

Step 1. If $Df(x')(x'' - x') = 0$, then, by assumption,

$$(x'' - x')^T \cdot D^2 f(x' + h(x'' - x')) \cdot (x'' - x') < 0.$$

Therefore, *zero* is a strict local maximum (see, for example, Theorem 13.10, page 378, in Apostol (1974)). Therefore, there exists $h^* \in \mathbb{R}$ such that $g(h^*) = f(x' + h^*(x'' - x')) < f(x') = g(0)$.
If

$$g'(0) = Df(x')(x'' - x') < 0,$$

then there exists $h^{**} \in \mathbb{R}$ such that

$$g(h^{**}) = f(x' + h^*(x'' - x')) < f(x') = g(0).$$

Moreover, $g(1) = f(x'') \geq f(x')$. In conclusion, neither *zero* nor *one* can be global minimum points for g on $[0,1]$.
Step 2. Since the global minimum point $h_m \in (0,1)$, we have that

$$0 = g'(h_m) = Df(x' + h_m(x'' - x'))(x'' - x').$$

Then, by assumption,

$$g''(0) = (x'' - x')^T \cdot D^2 f(x' + h_m(x'' - x')) \cdot (x'' - x') < 0,$$

but then h_m is a strict local maximum point, a contradiction.
■

Remark 743 *Differentiable-strict-quasi-concavity does not imply the condition presented in Proposition 742. $f : \mathbb{R} \to \mathbb{R},\ f : x \mapsto -x^4$ is differentiable-strictly-quasi-concave (in next section we will show that strict-concavity implies differentiable-strict-quasi-concavity). On the other hand, take $x^* = 0$. Then $Df(x^*) = 0$. Therefore, for any $\Delta \in \mathbb{R}^n \setminus \{0\}$, we have $Df(x^*)\Delta = 0$, but $\Delta^T D^2 f(x^*)\Delta = 0 \nless 0$.*

Some Properties.

Proposition 744 *Consider a differentiable-strictly-quasi-concave function f.*
x^* *is a strict global maximum point* $\Leftrightarrow Df(x^*) = 0$.

Proof.
[$\Rightarrow$] Obvious.
[$\Leftarrow$] From the contropositive of the definition of differentiable-strictly-quasi-concave function, we have:
$\forall\ x^*, x'' \in X$, such that $x^* \neq x''$, it is the case that $Df(x^*)(x'' - x^*) \leq 0 \Rightarrow f(x'') - f(x^*) < 0$ or $f(x^*) > f(x'')$. Since $Df(x^*) = 0$, then the desired result follows.
■

Remark 745 *Obviously, we also have that if f is differentaible-strictly-quasi-concave, it is the case that:*
x^* *local maximum point* $\Rightarrow$ x^* *is a strict maximum point.*

Remark 746 *The above implication is true also for continuous strictly quasi concave functions. (Suppose otherwise, i.e., $\exists\ x' \in X$ such that $f(x') \geq f(x^*)$. Since f is strictly quasi-concave, $\forall \lambda \in (0,1)$, $f((1-\lambda)x^* + \lambda x') > f(x^*)$, which for sufficiently small λ contradicts the fact that x^* is a local maximum point.*

Is there a definition of $?-$concavity weaker than concavity and such that:
If f is a $?-$concave function, then
x^* is a global maximum point iff $Df(x^*) = 0$.
The answer is given in the next section.

17.1.5 Pseudo-concave Functions.

Definition 747 *Consider a differentiable function f. f is pseudo-concave iff*

$$\forall x', x'' \in X, \quad f(x'') > f(x') \Rightarrow Df(x')(x'' - x') > 0,$$

or

$$\forall x', x'' \in X, \quad Df(x')(x'' - x') \leq 0 \Rightarrow f(x'') \leq f(x').$$

Proposition 748 *If f is a pseudo-concave function, then*

$$x^* \text{ is a global maximum point} \Leftrightarrow Df(x^*) = 0.$$

Proof.
[$\Rightarrow$] Obvious.
[$\Leftarrow$] $Df(x^*) = 0 \Rightarrow \forall x \in X, \quad Df(x^*)(x - x^*) \leq 0 \Rightarrow f(x) \leq f(x^*)$.
■

Remark 749 *Observe that the following "definition of pseudo-concavity" will not be useful:*

$$\forall x', x'' \in X, \quad Df(x')(x'' - x') \geq 0 \Rightarrow f(x'') \leq f(x') \tag{17.5}$$

For such a definition the above Proposition would still apply, but it is not weaker than concavity. Simply consider the function $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto -x^2$. That function is concave, but it does not satisfy condition (17.5). Take $x' = -2$ and $x'' = -1$. Then, $f'(x')(x'' - x') = 4(-1 - (-2)) = 4 > 0$, but $f(x'') = -1 > f(x') = -4$.

We summarize some of the results of this subsection in the following tables.

Class of function	Fundamental properties		
	C $\Rightarrow$ G max	L max $\Rightarrow$ G max	Uniqueness of G. max
Strictly concave	Yes	Yes	Yes
Concave	Yes	Yes	No
Diff.ble-str.-q.-conc.	Yes	Yes	Yes
Pseudoconcave	Yes	Yes	No
Quasiconcave	No	No	No

where C stands for property of being a critical point, and L and G stand for local and global, respectively. Observe that the first, the second and the last row of the second column apply to the case of C^0 and not necessarily differentiable functions.

17.2 Relationships among Different Kinds of Concavity

The relationships among different definitions of concavity in the case of *differentiable functions* are summarized in the following table.

$$\begin{array}{lllllll} & & & & \text{strict concavity} & & \\ & & & & \Downarrow & \searrow & \\ \text{linearity} & \Rightarrow & \text{affinity} & \Rightarrow & \text{concavity} & & \\ & & & & \Downarrow & & \\ & & & & \text{pseudo-concavity} & \Leftarrow & \text{differentiable-strict-quasi-concavity} \\ & & & & \Downarrow & & \\ & & & & \text{quasi-concavity} & & \end{array}$$

All the implications which are not implied by those explicitly written do not hold true.

In what follows, we prove the truth of each implication described in the table and we explain why the other implications do no hold.

Recall that

1. $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function iff $\forall x', x'' \in \mathbb{R}^n,\ \forall a, b \in \mathbb{R}\ f(ax' + bx'') = af(x') + bf(x'')$;

2. $g : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function iff there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ and $c \in \mathbb{R}^m$ such that $\forall x \in \mathbb{R}^n,\ \ g(x) = f(x) + c$.

$\boxed{SC \Rightarrow C}$

Obvious ("$a > b \Rightarrow a \geq b$").

$\boxed{C \Rightarrow PC}$

From the assumption and from Proposition 712, we have that $f(x'') - f(x') \leq Df(x')(x'' - x')$. Then $f(x'') - f(x') > 0 \Rightarrow Df(x')(x'' - x') > 0$.

$\boxed{PC \Rightarrow QC}$

Suppose otherwise, i.e., $\exists\ x', x'' \in X$ and $\exists \lambda^* [0,1]$ such that

$$f((1 - \lambda^*) x' + \lambda^* x'') < \min\{f(x'), f(x'')\}.$$

Define $x(\lambda) := (1 - \lambda) x' + \lambda x''$. Consider the segment $L(x', x'')$ joining x' to x''. Take $\overline{\lambda} \in \arg\min_\lambda f(x(\lambda))\ \ s.t.\ \lambda \in [0,1]$. $\overline{\lambda}$ is well defined from the Extreme Value Theorem. Observe that $\overline{\lambda} \neq 0, 1$, because $f(x(\lambda^*)) < \min\{f(x(0)) = f(x'), f(x(1)) = f(x'')\}$.

Therefore, $\forall \lambda \in [0,1]$ and $\forall \mu \in (0,1)$,

$f\left(x\left(\overline{\lambda}\right)\right) \leq f\left((1 - \mu) x\left(\overline{\lambda}\right) + \mu x(\lambda)\right)$.

$$\begin{array}{ccccc} & & (1-\mu)\,x\left(\overline{\lambda}\right) + \mu x(\lambda) & & \\ & & \downarrow & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \uparrow & \uparrow & & \uparrow & \uparrow \\ x' & x(\lambda) & & x\left(\overline{\lambda}\right) & x'' \end{array}$$

Then,

$$\forall \lambda \in [0,1], \quad 0 \leq \lim_{\mu \to 0^+} \frac{f\left((1 - \mu) x\left(\overline{\lambda}\right) + \mu x(\lambda)\right) - f\left(x\left(\overline{\lambda}\right)\right)}{\mu} = Df\left(x\left(\overline{\lambda}\right)\right)\left(x(\lambda) - x\left(\overline{\lambda}\right)\right).$$

Taking $\lambda = 0, 1$ in the above expression, we get:

$$Df\left(x\left(\overline{\lambda}\right)\right)\left(x' - x\left(\overline{\lambda}\right)\right) \geq 0 \quad (1)$$

and

$$Df\left(x\left(\overline{\lambda}\right)\right)\left(x'' - x\left(\overline{\lambda}\right)\right) \geq 0 \quad (2).$$

Since

$$x' - x\left(\overline{\lambda}\right) = x' - \left(1 - \overline{\lambda}\right)x' - \overline{\lambda}x'' = -\overline{\lambda}\left(x'' - x'\right) \quad (3),$$

and

$$x'' - x\left(\overline{\lambda}\right) = x'' - \left(1 - \overline{\lambda}\right)x' - \overline{\lambda}x'' = \left(1 - \overline{\lambda}\right)\left(x'' - x'\right) \quad (4),$$

substituting (3) in (1), and (4) in (2), we get

$$\overset{(-)}{-\overline{\lambda}} \cdot \left[Df\left(x\left(\overline{\lambda}\right)\right) \cdot \left(x'' - x'\right)\right] \geq 0,$$

and

$$\overset{(+)}{\left(1 - \overline{\lambda}\right)} \cdot \left[Df\left(x\left(\overline{\lambda}\right)\right) \cdot \left(x'' - x'\right)\right] \geq 0.$$

Therefore,

$$0 = Df\left(x\left(\overline{\lambda}\right)\right) \cdot \left(x'' - x'\right) = Df\left(x\left(\overline{\lambda}\right)\right) \cdot \left(1 - \overline{\lambda}\right) \cdot \left(x'' - x'\right) \overset{(4)}{=}$$

$$= Df\left(x\left(\overline{\lambda}\right)\right) \cdot \left(x'' - x\left(\overline{\lambda}\right)\right).$$

Then, by pseudo-concavity,

$$f\left(x''\right) \leq f\left(x\left(\overline{\lambda}\right)\right) \quad (5).$$

By assumption,

$$f\left(x\left(\lambda^*\right)\right) < f\left(x''\right) \quad (6).$$

(5) and (6) contradict the definition of $\overline{\lambda}$.

$\boxed{DSQC \Rightarrow PC}$

Obvious.

$\boxed{SC \Rightarrow DSQC}$

Obvious.

$\boxed{L \Rightarrow C}$

Obvious.

$\boxed{C \nRightarrow SC}$

$f : \mathbb{R} \to \mathbb{R},\ f : x \mapsto x.$

$\boxed{QC \nRightarrow PC}$

$$f : \mathbb{R} \to \mathbb{R},\ f : x \mapsto \begin{cases} 0 & if \quad x \leq 0 \\ e^{-\frac{1}{x^2}} & if \quad x > 0 \end{cases}$$

f is clearly nondecreasing and therefore, from Lemma 732, quasi-concave.
f is not pseudo-concave: $0 < f(-1) > f(1) = 0$, but
$f'(1)(-1-1) = 0 \cdot (-2) = 0$.

$\boxed{PC \nRightarrow C}$, $\boxed{DSQC \nRightarrow C}$ and $\boxed{DSQC \nRightarrow SC}$
Take $f : (1, +\infty) \to \mathbb{R},\ f : x \mapsto x^3$.

Take $x' < x''$. Then $f(x'') > f(x')$. Moreover, $\overset{(>0)}{Df}(x')\overset{(>0)}{(x'' - x')} > 0$. Therefore, f is $DSQC$ and therefore PC. Since $f''(x) > 0$, f is strictly convex and therefore it is not concave and, a fortiori, it is not strictly concave.

$\boxed{PC \nRightarrow DSQC}$, $\boxed{C \nRightarrow DSQC}$
Consider $f : \mathbb{R} \to \mathbb{R},\ f : x \mapsto 1$. f is clearly concave and PC, as well ($\forall x', x'' \in \mathbb{R}, Df(x')(x'' - x') \geq 0$). Moreover, any point in $\mathbb{R}$ is a critical point, but it is not the unique global maximum point. Therefore, from Proposition 744, f is not differentiable - strictly - quasi - concave.

$\boxed{QC \nRightarrow DSQC}$
If so, we would have $QC \Rightarrow DSQC \Rightarrow PC$, contradicting the fact that $QC \nRightarrow PC$.

$\boxed{C \nRightarrow L}$ and $\boxed{SC \nRightarrow L}$
$f : \mathbb{R} \to \mathbb{R},\ f : x \mapsto -x^2$.

17.2.1 Hessians and Concavity.

In this subsection, we study the relation between submatrices of a matrix involving the Hessian matrix of a C^2 function and the concavity of that function.

Definition 750 *Consider a matrix $A_{n \times n}$. Let $1 \leq k \leq n$.*

A $k-th$ order principal submatrix (minor) of A is the (determinant of the) square submatrix of A obtained deleting $(n-k)$ rows and $(n-k)$ columns in the same position. Denote these matrices by $\widetilde{D}_k^i$.

The $k-th$ order leading principal submatrix (minor) of A is the (determinant of the) square submatrix of A obtained deleting the last $(n-k)$ rows and the last $(n-k)$ columns. Denote these matrices by D_k.

Example 751 *Consider*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then

$$\widetilde{D}_1^1 = a_{11},\ \widetilde{D}_1^2 = a_{22},\ \widetilde{D}_1^3 = a_{33}, \quad D_1 = \widetilde{D}_1^1 = a_{11};$$

$$\widetilde{D}_2^1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ \widetilde{D}_2^2 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix},\ \widetilde{D}_2^3 = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix},$$

$$D_2 = \widetilde{D}_2^1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix};$$

$$D_3 = \widetilde{D}_3^1 = A.$$

Definition 752 *Consider a C^2 function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$. The bordered Hessian of f is the following matrix*

$$B_f(x) = \begin{bmatrix} 0 & Df(x) \\ [Df(x)]^T & D^2 f(x) \end{bmatrix}_{(n+1)\times(n+1)}.$$

Theorem 753 *(Simon, (1985), Theorem 1.9.c, page 79 and Sydsaeter (1981), Theorem 5.17, page 259). Consider a C^2 function $f : X \to \mathbb{R}$.*

1. If $\forall x \in X$, $\forall k \in \{1, ..., n\}$,
$sign\left(k - \text{leading principal minor of } D^2 f(x)\right) = sign\ (-1)^k$,
then f is strictly concave.
2. $\forall x \in X$, $\forall k \in \{1, ..., n\}$,
$sign\left(\text{non zero } k - \text{ principal minor of } D^2 f(x)\right) = sign\ (-1)^k$,
iff f is concave.
3. If $n \geq 2$ and $\forall x \in X$, $\forall k \in \{3, ..., n+1\}$,
$sign\left(k - \text{leading principal minor of } Bf(x)\right) = sign\ (-1)^{k-1}$,
then f is pseudo concave and, therefore, quasi-concave.
4. If f is quasi-concave, then $\forall x \in X$, $\forall k \in \{2, ..., n+1\}$,
$sign\left(\text{non zero } k - \text{leading principal minors of } Bf(x)\right) = sign\ (-1)^{k-1}$

Remark 754 *It can be proved that Conditions in part 1 and 2 of the above Theorem are sufficient for $D^2 f(x)$ being negative definite and equivalent to $D^2 f(x)$ being negative semidefinite, respectively.*

Remark 755 *(From Sydsaetter (1981), page 239) It is tempting to conjecture that a function f is concave iff*

$$\forall x \in X, \forall k \in \{1, ..., n\},\ sign\left(\text{non zero } k - \textbf{leading} \text{ principal minor of } D^2 f(x)\right) = sign\ (-1)^k, \quad (17.6)$$

That conjecture is false. Consider

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f : (x_1, x_2, x_3) \mapsto -x_2^2 + x_3^2.$$

Then $Df(x) = (0, -2x_2, 2x_3)$ and

$$D^2 f(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

All the leading principal minors of the above matrix are zero, and therefore Condition 17.6 is satisfied, but f is not a concave function. Take $x' = (0,0,0)$ and $x''(0,0,1)$. Then

$$\forall \lambda \in (0,1),\ \ f((1-\lambda)x' + \lambda x'') = \lambda^2 < (1-\lambda) f(x') + \lambda f(x'') = \lambda$$

Example 756 *Consider $f : \mathbb{R}^2_{++} \to \mathbb{R}, f : (x,y) \mapsto x^\alpha y^\beta$, with $\alpha, \beta \in \mathbb{R}_{++}$. Observe that $\forall (x,y) \in \mathbb{R}^2_{++}$, $f(x,y) > 0$. Verify that*

1. *if $\alpha + \beta < 1$, then f is strictly concave;*

2. *$\forall \alpha, \beta \in \mathbb{R}_{++}$, f is quasi-concave;*

3. *$\alpha + \beta \leq 1$ if and only if f is concave.*

1.
$D_x f(x,y) = \alpha x^{\alpha-1} y^\beta = \frac{\alpha}{x} f(x,y)$;
$D_y f(x,y) = \beta x^\alpha y^{\beta-1} = \frac{\beta}{y} f(x,y)$;
$D^2_{x,x} f(x,y) = \alpha(\alpha-1) x^{\alpha-2} y^\beta = \frac{\alpha(\alpha-1)}{x^2} f(x,y)$;
$D^2_{y,y} f(x,y) = \beta(\beta-1) x^\alpha y^{\beta-2} = \frac{\beta(\beta-1)}{y^2} f(x,y)$;
$D^2_{x,.y} f(x,y) = \alpha\beta x^{\alpha-1} y^{\beta-1} = \frac{\alpha}{x}\frac{\beta}{y} f(x,y)$.

$$D^2 f(x,y) = f(x,y) \begin{bmatrix} \frac{\alpha(\alpha-1)}{x^2} & \frac{\alpha}{x}\frac{\beta}{y} \\ \frac{\alpha}{x}\frac{\beta}{y} & \frac{\beta(\beta-1)}{y^2} \end{bmatrix}.$$

a. $\frac{\alpha(\alpha-1)}{x^2} < 0 \Leftrightarrow \alpha \in (0,1)$.
b. $\frac{\alpha(\alpha-1)\beta(\beta-1) - \alpha^2\beta^2}{x^2 y^2} = \frac{1}{x^2y^2}\left(\alpha\beta(\alpha\beta - \alpha - \beta + 1) - \alpha^2\beta^2\right) =$
$= \frac{1}{x^2y^2}\alpha\beta(1 - \alpha - \beta) > 0 \overset{\alpha,\beta>0}{\Leftrightarrow} \alpha + \beta < 1$.
In conclusion, if $\alpha, \beta \in (0,1)$ and $\alpha + \beta < 1$, then f is strictly concave.

2.

Observe that

$$f(x,y) = g(h(x,y))$$

where

$$h : \mathbb{R}^2_{++} \to \mathbb{R}, \quad (x,y) \mapsto \alpha \ln x + \beta \ln y$$

$$g : \mathbb{R} \to \mathbb{R}, \quad z \mapsto e^z$$

Since h is strictly concave (why?) and therefore quasi-concave and g is strictly increasing, the desired result follows from Proposition 724.

3.

Obvious from above results.

Chapter 18

Maximization Problems

Let the following objects be given:

1. an open convex set $X \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$;
2. $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^m$, $h : X \to \mathbb{R}^k$, $m, k \in \mathbb{N}$, with f, g, h at least differentiable.

The goal of this Chapter is to study the problem.

$$\begin{array}{lll} \max_{x \in X} & f(x) & \\ s.t. & g(x) \geq 0 & (1) \\ & h(x) = 0 & (2) \end{array} \tag{18.1}$$

f is called **objective function**; x **choice variable** vector; (1) and (2) in (18.1) **constraints**; g and h **constraint functions**;

$$C := \{x \in X : g(x) \geq 0 \quad \text{and} \quad h(x) = 0\}$$

is the constraint set.

To solve the problem (18.1) means to describe the following set

$$\{x^* \in C : \forall x \in C, \ f(x^*) \geq f(x)\}$$

which is called solution set to problem (18.1) and it is also denoted by $\arg\max$ (18.1). We will proceed as follows.

1. We will analyze in detail the problem with inequality constraints, i.e.,

$$\begin{array}{lll} \max_{x \in X} & f(x) & \\ s.t. & g(x) \geq 0 & (1) \end{array}$$

2. We will analyze in detail the problem with equality constraints, i.e.,

$$\begin{array}{lll} \max_{x \in X} & f(x) & \\ s.t. & h(x) = 0 & (2) \end{array}$$

3. We will describe how to solve the problem with both equality and inequality constraints, i.e.,

$$\begin{array}{lll} \max_{x \in X} & f(x) & \\ s.t. & g(x) \geq 0 & (1) \\ & h(x) = 0 & (2) \end{array}$$

18.1 The case of inequality constraints: Kuhn-Tucker theorems

Consider the open and convex set $X \subseteq \mathbb{R}^n$ and the differentiable functions $f : X \to \mathbb{R}$, $g := \left(g^j\right)_{j=1}^m : X \to \mathbb{R}^m$. The problem we want to study is

$$\max_{x \in X} f(x) \qquad s.t. \qquad g(x) \geq 0. \tag{18.2}$$

Definition 757 *The Kuhn-Tucker system (or conditions) associated with problem 18.2 is*

$$\begin{cases} Df(x) + \lambda Dg(x) & = & 0 & (1) \\ \lambda & \geq & 0 & (2) \\ g(x) & \geq & 0 & (3) \\ \lambda g(x) & = & 0 & (4) \end{cases} \tag{18.3}$$

Equations (1) *are called first order conditions; equations* (2), (3) *and* (4) *are called complementary slackness conditions.*

Remark 758 $(x, \lambda) \in X \times \mathbb{R}^m$ *is a solution to Kuhn-Tucker system iff it is a solution to any of the following systems:*

1.
$$\begin{cases} \frac{\partial f(x)}{\partial x_i} + \sum_{j=1}^m \lambda_j \frac{\partial g_j(x)}{\partial x_i} & = & 0 & for\ i = 1, ..., n & (1) \\ \lambda_j & \geq & 0 & for\ j = 1, ..., m & (2) \\ g_j(x) & \geq & 0 & for\ j = 1, ..., m & (3) \\ \lambda_j g_j(x) & = & 0 & for\ j = 1, ..., m & (4) \end{cases}$$

2.
$$\begin{cases} Df(x) + \lambda Dg(x) & = & 0 & & (1) \\ \min\{\lambda_j, g_j(x)\} & = & 0 & for\ j = 1, ..., m & (2) \end{cases}$$

Moreover, $(x, \lambda) \in X \times \mathbb{R}^m$ is a solution to Kuhn-Tucker system iff it is a solution to

$$Df(x) + \lambda Dg(x) \quad = \quad 0 \qquad (1)$$

and for each $j = 1, ..., m$, to **one** of the following conditions

$$\begin{array}{lllll} \text{either} & (& \lambda_j > 0 & \text{and} & g_j(x) = 0 &) \\ \text{or} & (& \lambda_j = 0 & & g_j(x) > 0 &) \\ \text{or} & (& \lambda_j = 0 & & g_j(x) = 0 &) \end{array}$$

Definition 759 *Given* $x^* \in C$, *we say that* j *is a binding constraint at* x^* *if* $g_j(x^*) = 0$. *Let*

$$J^*(x^*) := \{j \in \{1, ..., m\} : g_j(x^*) = 0\},$$

$$g^* := (g_j)_{j \in J^*(x^*)}, \quad \widehat{g} := (g_j)_{j \notin J^*(x^*)}$$

and

$$m^* = \#J^*(x^*).$$

Definition 760 $x^* \in \mathbb{R}^n$ *satisfies the constraint qualifications associated with problem 18.2 if it is a solution to*

$$\max_{x \in \mathbb{R}^n} Df(x^*)\, x \qquad s.t. \qquad Dg^*(x^*)(x - x^*) \quad \geq \quad 0 \tag{18.4}$$

The above problem is obtained from 18.2

1. replacing g with g^*;
2. linearizing f and g^* around x^*, i.e., substituting f and g^* with $f(x^*) + Df(x^*)(x - x^*)$ and $g(x^*) + Dg(x^*)(x - x^*)$, respectively;
3. dropping redundant terms, i.e., the term $f(x^*)$ in the objective function, and the term $g^*(x^*) = 0$ in the constraint.

Theorem 761 *Suppose* x^* *is a solution to problem 18.2 and to problem 18.4, then there exists* $\lambda^* \in \mathbb{R}^m$ *such that* (x^*, λ^*) *satisfies Kuhn-Tucker conditions.*

The proof of the above theorem requires the following lemma.

Lemma 762 *(Farkas) Given a matrix* $A_{m\times n}$ *and a vector* $a \in \mathbb{R}^n$*,*
either 1. there exists $\lambda \in \mathbb{R}^m_+$ *such that* $a = \lambda A$*,*
or 2. there exists $y \in \mathbb{R}^n$ *such that* $Ay \geq 0$ *and* $ay < 0$*,*
but not both.

Proof. It follows immediately from Proposition 707. ■
Proof. of Theorem 761
(main steps: 1. use the fact x^* is a solution to problem 18.4; 2. apply Farkas Lemma; 3. choose $\lambda^* = (\lambda \text{ from Farkas }, 0)$).

Since x^* is a solution to problem 18.4, for any $x \in \mathbb{R}^n$ such that $Dg^*(x^*)(x - x^*) \geq 0$ it is the case that $Df(x^*)x^* \geq Df(x^*)x$ or

$$Dg^*(x^*)(x - x^*) \geq 0 \Rightarrow [-Df(x^*)](x - x^*) \geq 0. \tag{18.5}$$

Applying Farkas Lemma identifying

$$a \quad \text{with} \quad -Df(x^*)$$

and

$$A \quad \text{with} \quad Dg^*(x^*)$$

we have that either
1. there exists $\lambda \in \mathbb{R}^m_+$ such that

$$-Df(x^*) = \lambda Dg^*(x^*) \tag{18.6}$$

or 2. there exists $y \in \mathbb{R}^n$ such that

$$Dg^*(x^*)y \geq 0 \quad \text{and} \quad -Df(x^*)y < 0 \tag{18.7}$$

but not both 1 and 2.

Choose $x = y + x^*$ and therefore you have $y = x - x^*$. Then, 18.7 contradicts 18.5. Therefore, 1. above holds.

Now, choose $\lambda^* := (\lambda, 0) \in \mathbb{R}^{m^*} \times \mathbb{R}^{m-m^*}$, we have that

$$Df(x^*) + \lambda^* Dg(x^*) = Df(x^*) + (\lambda, 0)\begin{pmatrix} Dg^*(x^*) \\ D\widehat{g}(x^*) \end{pmatrix} = Df(x^*) + \lambda Dg(x^*) = 0$$

where the last equality follows from 18.6;
$\lambda^* \geq 0$ by Farkas Lemma;
$g(x^*) \geq 0$ from the assumption that x^* solves problem 18.2;
$\lambda^* g(x^*) = (\lambda, 0)\begin{pmatrix} g^*(x^*) \\ \widehat{g}(x^*) \end{pmatrix} = \lambda g^*(x) = 0$, where the last equality follows from the definition of g^*. ■

Theorem 763 *If* x^* *is a solution to problem* (18.2) *and*
either for $j = 1, .., m$*,* g_j *is pseudo-concave and* $\exists x^{++} \in X$ *such that* $g(x^{++}) \gg 0$*,*
or rank $Dg^*(x^*) = m^* := \#J^*(x^*)$*,*
then x^* *solves problem* (18.4)*.*

Proof. We prove the conclusion of the theorem under the first set of conditions.

Main steps: 1. suppose otherwise: $\exists\, \widetilde{x}$ ... ; 2. use the two assumptions; 3. move from x^* in the direction $x^\theta := (1-\theta)\widetilde{x} + \theta x^{++}$.

Suppose that the conclusion of the theorem is false. Then there exists $\widetilde{x} \in \mathbb{R}^n$ such that

$$Dg^*(x^*)(\widetilde{x} - x^*) \geq 0 \quad \text{and} \quad Df(x^*)(\widetilde{x} - x^*) > 0 \tag{18.8}$$

Moreover, from the definition of g^* and x^{++}, we have that

$$g^*\left(x^{++}\right) >> 0 = g^*(x^*)$$

Since for $j = 1, .., m$, g_j is pseudo-concave we have that

$$Dg^*(x^*)\left(x^{++} - x^*\right) \gg 0 \tag{18.9}$$

Define
$$x^{\theta} := (1-\theta)\,\widetilde{x} + \theta x^{++}$$
with $\theta \in (0,1)$. Observe that
$$x^{\theta} - x^{*} = (1-\theta)\,\widetilde{x} + \theta x^{++} - (1-\theta)\,x^{*} - \theta x^{*} = (1-\theta)\,(\widetilde{x} - x^{*}) + \theta\left(x^{++} - x^{*}\right)$$
Therefore,
$$Dg^{*}(x^{*})\left(x^{\theta} - x^{*}\right) = (1-\theta)\,Dg^{*}(x^{*})\,(\widetilde{x} - x^{*}) + \theta Dg^{*}(x^{*})\left(x^{++} - x^{*}\right) \gg 0 \tag{18.10}$$
where the last equality come from 18.8 and 18.9.
Moreover,
$$Df(x^{*})\left(x^{\theta} - x^{*}\right) = (1-\theta)\,Df(x^{*})\,(\widetilde{x} - x^{*}) + \theta Df(x^{*})\left(x^{++} - x^{*}\right) \gg 0 \tag{18.11}$$
where the last equality come from 18.8 and a choice of θ sufficiently small.[1]
Observe that from Remark 626, 18.10 and 18.11 we have that
$$(g^{*})'\left(x^{*}, x^{\theta}\right) \gg 0$$
and
$$f'\left(x^{*}, x^{\theta}\right) > 0$$
Therefore, using the fact that X is open, and that $\widehat{g}(x^{*}) \gg 0$, there exists γ such that
$$\begin{array}{l} x^{*} + \gamma\left(x^{\theta} - x^{*}\right) \in X \\ g^{*}\left(x^{*} + \gamma\left(x^{\theta} - x^{*}\right)\right) \gg g^{*}(x^{*}) = 0 \\ f^{*}\left(x^{*} + \gamma\left(x^{\theta} - x^{*}\right)\right) > f(x^{*}) \\ \widehat{g}\left(x^{*} + \gamma\left(x^{\theta} - x^{*}\right)\right) \gg 0 \end{array} \tag{18.12}$$
But then 18.12 contradicts the fact that x^{*} solves problem (18.2). ■
From Theorems 761 and 763, we then get the following corollary.

Theorem 764 *Suppose x^{*} is a solution to problem 18.2, and one of the following constraint qualifications hold:*
a. for $j = 1, ..., m$, g_j is pseudo-concave and there exists $x^{++} \in X$ such that $g\left(x^{++}\right) \gg 0$
b. rank $Dg^{*}(x^{*}) = \#J^{*}$,
Then there exists $\lambda^{*} \in \mathbb{R}^{m}$ such that (x^{*}, λ^{*}) solves the system 18.3.

Theorem 765 *If f is pseudo-concave, and for $j = 1, ..., m$, g_j is quasi-concave, and (x^{*}, λ^{*}) solves the system 18.3, then x^{*} solves problem 18.2.*

Proof. Main steps: 1. suppose otherwise and use the fact that f is pseudo-concave; 2. for $j \in J^{*}(x^{*})$, use the quasi-concavity of g_j; 3. for $j \in J^{*}(x^{*})$, use (second part of) kuhn-Tucker conditions; 4. Observe that 2. and 3. above contradict the first part of Kuhn-Tucker conditions.)
Suppose otherwise, i.e., there exists $\widehat{x} \in X$ such that
$$g(\widehat{x}) \geq 0 \quad \text{and} \quad f(\widehat{x}) > f(x^{*}) \tag{18.13}$$
From 18.13 and the fact that f pseudo-concave, we get
$$Df(x^{*})\,(\widehat{x} - x^{*}) > 0 \tag{18.14}$$

[1] Assume that $\theta \in (0,1)$, $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$. We want to show that there exist $\theta^{*} \in (0,1)$ such that
$$(1-\theta)\,\alpha + \theta\beta > 0$$
i.e.,
$$\alpha > \theta\,(\alpha - \beta)$$
If $(\alpha - \beta) = 0$, the claim is true.
If $(\alpha - \beta) > 0$, any $\theta < \frac{\alpha}{\alpha-\beta}$ will work (observe that $\frac{\alpha}{\alpha-\beta} > 0$).
If $(\alpha - \beta) < 0$, the claim is clearly true because $0 < \alpha$ and $\theta\,(\alpha - \beta) < 0$.

From 18.13, the fact that $g^*(x^*) = 0$ and that g_j is quasi-concave, we get that

$$\text{for } \ j \in J^*(x^*), \quad Dg^j(x^*)(\widehat{x} - x^*) \geq 0$$

and since $\lambda^* \geq 0$,

$$\text{for } \ j \in J^*(x^*), \quad \lambda_j^* Dg^j(x^*)(\widehat{x} - x^*) \geq 0 \tag{18.15}$$

For $j \in \widehat{J}(x^*)$, from Kuhn-Tucker conditions, we have that $g_j(x^*) > 0$ and $\lambda_j^* = 0$, and therefore

$$\text{for } \ j \in \widehat{J}(x^*), \quad \lambda_j^* Dg^j(x^*)(\widehat{x} - x^*) = 0 \tag{18.16}$$

But then from 18.14, 18.15 and 18.16, we have

$$Df(x^*)(\widehat{x} - x^*) + \lambda^* Dg(x^*)(\widehat{x} - x^*) > 0$$

contradicting Kuhn-Tucker conditions. ■

We can summarize the above results as follows. Call (M) the problem

$$\max_{x \in X} f(x) \qquad s.t. \qquad g(x) \ \geq \ 0 \tag{18.17}$$

and define

$$M := \arg\max(M) \tag{18.18}$$

$$S := \{x \in X : \exists \lambda \in \mathbb{R}^m \text{ such that } (x, \lambda) \text{ is a solution to Kuhn-Tucker system } (18.3)\} \tag{18.19}$$

1. Assume that one of the following conditions hold:
   (a) for $j = 1, ..., m$, g_j is pseudo-concave and there exists $x^{++} \in X$ such that $g(x^{++}) \gg 0$
   (b) rank $Dg^*(x^*) = \#J^*$.
   Then
   $$x^* \in M \Rightarrow x^* \in S$$

2. Assume that both the following conditions hold:
   (a) f is pseudo-concave, and
   (b) for $j = 1, ..., m$, g_j is quasi-concave.
   Then
   $$x^* \in S \Rightarrow x^* \in M.$$

18.1.1 On uniqueness of the solution

The following proposition is a useful tool to show uniqueness.

Proposition 766 *The solution to problem*

$$\max_{x \in X} f(x) \quad s.t. \ g(x) \geq 0 \qquad (P)$$

either does not exist or it is unique if one of the following conditions holds

1. *f is strictly quasi-concave, and*
   for $j \in \{1, ..., m\}$, g_j is quasi-concave;

2. *f is quasi-concave and locally non-satiated (i.e., $\forall x \in X, \forall \varepsilon > 0$, there exists $x' \in B(x, \varepsilon)$ such that $f(x') > f(x)$), and*
   for $j \in \{1, ..., m\}$, g_j is strictly quasi-concave.

Proof. 1.

Since $g_j$is quasi concave $V^j := \{x \in X : g_j(x) \geq 0\}$ is convex. Since the intersection of convex sets is convex $V = \cap_{j=1}^m V^j$ is convex.

Suppose that both x' and x'' are solutions to problem (P) and $x' \neq x''$. Then for any $\lambda \in (0,1)$,

$$(1-\lambda)x' + \lambda x'' \in V \tag{18.20}$$

because V is convex, and

$$f((1-\lambda)x' + \lambda x'') > \min\{f(x'), f(x'')\} = f(x') = f(x'') \tag{18.21}$$

because f is strictly-quasi-concave.

But (18.20) and (18.21) contradict the fact that x' and x'' are solutions to problem (P).

2.

Observe that V is strictly convex because each V^j is strictly convex. Suppose that both x' and x'' are solutions to problem (P) and $x' \neq x''$. Then for any $\lambda \in (0,1)$,

$$x(\lambda) := (1-\lambda)x' + \lambda x'' \in \text{Int } V$$

i.e., $\exists \varepsilon > 0$ such that $B(x(\lambda), \varepsilon) \subseteq V$. Since f is locally non-satiated, there exists $x' \in B(x(\lambda), \varepsilon) \subseteq V$ such that

$$f(\widehat{x}) > f(x(\lambda)) \tag{18.22}$$

Since f is quasi-concave,

$$f(x(\lambda)) \geq f(x') = f(x'') \tag{18.23}$$

(18.22) and (18.23) contradict the fact that x' and x'' are solutions to problem (P). ■

Remark 767 *1. If f is strictly increasing (i.e., $\forall x', x'' \in X$ such that $x' > x''$, we have that $f(x') > f(x'')$) or strictly decreasing, then f is locally non-satiated.*

2. If f is affine and not constant, then f is quasi-concave and Locally NonSatiated.

Proof of 2.

$f : \mathbb{R}^n \to \mathbb{R}$ affine and not constant means that there exists $a \in \mathbb{R}$ and $b \in \mathbb{R}\backslash\{0\}$ such that $f : x \mapsto a + b^T x$. Take an arbitrary $\overline{x}$ and $\varepsilon > 0$. For $i \in \{1, ..., n\}$, define $\alpha_i := \frac{\varepsilon}{k} \cdot (sign\ b_i)$ and $\widetilde{x} := \overline{x} + (\alpha_i)_{i=1}^n$, with $k \neq 0$ and which will be computed below. Then

$f(\widetilde{x}) = a + b\overline{x} + \sum_{i=1}^n \frac{\varepsilon}{k}|b_i| > f(\overline{x})$;

$\|\widetilde{x} - \overline{x}\| = \left\|\frac{\varepsilon}{k} \cdot ((sign\ b_i) \cdot b_i)_{i=1}^n\right\| = \frac{\varepsilon}{k} \cdot \left\|\sqrt{\sum_{i=1}^n (b_i)^2}\right\| = \frac{\varepsilon}{k} \cdot \|b\| < \varepsilon$ *if* $k > \frac{1}{\|b\|}$.

Remark 768 *In part 2 of the statement of the Proposition f **has** to be both quasi-concave and Locally NonSatiated.*

a. Example of f quasi-concave (and g_j strictly-quasi-concave) with more than one solution:

$$\max_{x \in \mathbb{R}} 1 \quad s.t. \quad x + 1 \geq 0 \quad 1 - x \geq 0$$

The set of solution is $[-1, +1]$

a. Example of f Locally NonSatiated (and g_j strictly-quasi-concave) with more than one solution:

$$\max_{x \in \mathbb{R}} x^2 \quad s.t. \quad x + 1 \geq 0 \quad 1 - x \geq 0$$

The set of solutions is $\{-1, +1\}$.

18.2 The Case of Equality Constraints: Lagrange Theorem.

Consider the C^1 functions

$$f : X \to \mathbb{R}, \qquad f : x \mapsto f(x),$$

$$g : X \to \mathbb{R}^m, \qquad g : x \mapsto g(x) := (g_j(x))_{j=1}^m$$

with $\underline{m \leq n}$. Consider also the following “”maximization problem:

$$(P) \qquad \max_{x \in X} \quad f(x) \qquad s.t. \qquad g(x) = 0 \tag{18.24}$$

$$\mathcal{L} : X \times \mathbb{R}^m \to \mathbb{R}, \quad \mathcal{L} : (x, \lambda) \mapsto f(x) + \lambda g(x)$$

is called Lagrange function associated with problem (15.45).
We recall below the statement of Theorem 687.

Theorem 769 *(Necessary Conditions)*
Assume that $\operatorname{rank}[Dg(x^*)] = m$.
Under the above condition, we have that
x^* *is a local maximum for* (P)
$\Rightarrow$
there exists $\lambda^* \in \mathbb{R}^m$, *such that*

$$\begin{cases} Df(x^*) + \lambda^* Dg(x^*) = 0 \\ g(x^*) = 0 \end{cases} \tag{18.25}$$

Remark 770 *The full rank condition in the above Theorem cannot be dispensed. The following example shows a case in which* x^* *is a solution to maximization problem (18.24),* $Dg(x^*)$ *does not have full rank and there exists no* λ^* *satisfying Condition 18.25. Consider*

$$\max_{(x,y) \in \mathbb{R}^2} x \quad s.t. \quad \begin{array}{l} x^3 - y = 0 \\ x^3 + y = 0 \end{array}$$

The constraint set is $\{(0,0)\}$ and therefore the solution is just $(x^*, y^*) = (0,0)$. The Jacobian matrix of the constraint function is

$$\left[\begin{array}{cc} 3x^2 & -1 \\ 3x^2 & 1 \end{array}\right]_{|(x^*,y^*)} = \left[\begin{array}{cc} 0 & -1 \\ 0 & 1 \end{array}\right]$$

which does have full rank.

$$(0,0) = Df(x^*, y^*) + (\lambda_1, \lambda_2) Dg(x^*, y^*) =$$
$$= (1,0) + (\lambda_1, \lambda_2) \left[\begin{array}{cc} 0 & -1 \\ 0 & 1 \end{array}\right] = (1, -\lambda_1 + \lambda_2),$$

from which it follows that there exists no λ^* solving the above system.

Theorem 771 *(Sufficient Conditions)*
Assume that
1. f *is pseudo-concave,*
2. for $j = 1, ..., m$, g_j *is quasi concave.*
Under the above conditions, we have what follows.
[there exist $(x^*, \lambda^*) \in X \times \mathbb{R}^m$ *such that*
3. $\lambda^* \geq 0$,
4. $Df(x^*) + \lambda^* Dg(x^*) = 0$, *and*
5. $g(x^*) = 0$ *]*
$\Rightarrow$
x^* *solves* (P).

Proof.
Suppose otherwise, i.e., there exists $\widehat{x} \in X$ such that

$$for\ j = 1, ..., m, \quad g_j(\widehat{x}) = g_j(x^*) = 0 \quad (1), \text{ and}$$

$$f(\widehat{x}) > f(x^*) \qquad (2).$$

Quasi-concavity of g_j and (1) imply that

$$Dg^j(x^*)(\widehat{x}-x^*) \geq 0 \quad (3).$$

Pseudo concavity of f and (2) imply that

$$Df(x^*)(\widehat{x}-x^*) > 0 \quad (4).$$

But then

$$0 \overset{Assumption}{=} [Df(x^*) + \lambda Dg(x^*)](\widehat{x}-x^*) = \overset{(>0)}{Df(x^*)(\widehat{x}-x^*)} + \overset{(\geq 0)}{\lambda} \overset{(\geq 0)}{Dg(x^*)(\widehat{x}-x^*)} > 0,$$

a contradiction.
■

18.3 The Case of Both Equality and Inequality Constraints.

Consider
the open and convex set $X \subseteq \mathbb{R}^n$ and the differentiable functions $f: X \to \mathbb{R}$, $g: X \to \mathbb{R}^m$, $h: X \to \mathbb{R}^l$.
Consider the problem

$$\max_{x\in X} f(x) \qquad s.t. \qquad \begin{array}{rcl} g(x) & \geq & 0 \\ h(x) & = & 0. \end{array} \tag{18.26}$$

Observe that

$$h(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow \text{ for } k = 1, ..., l, \;\; h^k(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow \text{ for } k = 1, ..., l, \;\; h^{k1}(x) := h^k(x) \geq 0 \text{ and } h^{k2}(x) := -h^k(x) \geq 0.$$

Defined $h^{\cdot 1}(x) := \left(h^{k1}(x)\right)_{k=1}^l$ and $h^{\cdot 2}(x) := \left(h^{k2}(x)\right)_{k=1}^l$, problem 18.26 with associated multipliers can be rewritten as

$$\max_{x\in X} f(x) \qquad s.t. \qquad \begin{array}{rcl l} g(x) & \geq & 0 & \lambda \\ h^{\cdot 1}(x) & \geq & 0 & \mu_1 \\ h^{\cdot 2}(x) & \geq & 0 & \mu_2 \end{array} \tag{18.27}$$

The Lagrangian function of the above problem is

$$\mathcal{L}(x; \lambda, \mu_1, \mu_2) = f(x) + \lambda^T g(x) + (\mu_1 - \mu_2)^T h(x) =$$

$$= f(x) + \sum_{j=1}^m \lambda_j g_j(x) + \sum_{k=1}^l \left(\mu_1^k - \mu_2^k\right)^T h(x),$$

and the Kuhn-Tucker Conditions are:

$$\begin{array}{ll} Df(x) + \lambda^T Dg(x) + (\mu_1 - \mu_2)^T Dh(x) = 0 & \\ g_j(x) \geq 0, \; \lambda_j \geq 0, \; \lambda_j g_j(x) = 0, & \text{for } j = 1, ..., m, \\ h^k(x) = 0, \; \left(\mu_1^k - \mu_2^k\right) := \mu \gtreqless 0, & \text{for } \; k = 1, ..., l. \end{array} \tag{18.28}$$

Theorem 772 *Assume that f, g and h are C^2 functions and that*
either rank $\begin{bmatrix} Dg^*(x^*) \\ Dh(x^*) \end{bmatrix} = m^* + l$,
or for $j = 1, ..., m$, $-g_j$ is pseudoconcave, and $\forall k$, h_k and $-h_k$ are pseudoconcave
Under the above conditions,
if x^* solves 18.26, then $\exists (x^*, \lambda^*, \mu^*) \in X \times \mathbb{R}^m \times \mathbb{R}^l$ which satisfies the associated Kuhn-Tucker conditions.

Proof. The above conditions are called "Weak reverse convex constraint qualification" (Mangasarian (1969)) or "Reverse constraint qualification" (Bazaraa and Shetty (1976)). The needed result is presented and proved in

Mangasarian[2],- see 4, page 172 and Theorem 6, page 173, and Bazaraa and Shetty (1976) - see 7 page 148, and theorems 6.2.3, page 148 and Theorem 6.2.4, page 150.

See also El-Hodiri (1991), Theorem 1, page 48 and Simon (1985), Theorem 4.4. (iii), page 104. ■

■

Remark 773 *For other conditions, see Theorem 5.8, page 124, in Jahn (1996).*

Theorem 774 *Assume that*

f is pseudo-concave, and

for $j = 1, ..., m$, g_j is quasi-concave, and for $k = 1, ..., l$, h^k is quasi-concave and $-h^k$ is quasi-concave.

Under the above conditions,

if $(x^*, \lambda^*, \mu^*) \in X \times \mathbb{R}^m \times \mathbb{R}^l$ satisfies the Kuhn-Tucker conditions associated with 18.26, then x^* solves 18.26.

Proof.

This follows from Theorems proved in the case of inequality constraints.

■

Similarly, to what we have done in previous sections, we can summarize what said above as follows.

Call (M_2) the problem

$$\max_{x\in X} f(x) \qquad s.t. \qquad \begin{array}{rcl} g(x) & \geq & 0 \\ h(x) & = & 0. \end{array} \tag{18.29}$$

and define

$$M_2 := \arg\max (M_2)$$

$$S_2 := \{x \in X : \exists \lambda \in \mathbb{R}^m \text{ such that } (x, \lambda) \text{ is a solution to Kuhn-Tucker system } (18.28)\}$$

1. Assume that one of the following conditions hold:

   (a) $rank \begin{bmatrix} Dg^*(x^*) \\ Dh(x^*) \end{bmatrix} = m^* + l$,or

   (b) for $j = 1, ..., m$, g_j is linear, and $h(x)$ is affine.
   Then
   $$M_2 \subseteq S_2$$

2. Assume that both the following conditions hold:

   (a) f is pseudo-concave, and

   (b) for $j = 1, ..., m$, g_j is quasi-concave, and for $k = 1, ..., l$, h^k is quasi-concave and $-h^k$ is quasi-concave.
   Then
   $$M_2 \supseteq S_2$$

18.4 Main Steps to Solve a (Nice) Maximization Problem

We have studied the problem

$$\max_{x\in X} f(x) \qquad s.t. \qquad g(x) \geq 0 \qquad (M)$$

which we call a maximization problem in the "canonical form", i.e., a maximization problem with constraints in the form of "$\geq$", and we have defined

$$M := \arg\max (M)$$

[2] What Mangasarian calls a linear function is what we call an affine function.

$$C := \{x \in X : g(x) \geq 0\}$$

$$S := \{x \in X : \exists \lambda \in \mathbb{R}^m \text{ such that } (x, \lambda) \text{ satisfies Kuhn-Tucker Conditions } (18.3)\}$$

Recall that X is an open, convex subset of $\mathbb{R}^n$, $f : X \to \mathbb{R}$, $\forall j \in \{1, ..., m\}$, $g_j : X \to \mathbb{R}$ and $g := (g_j)_{j=1}^m : X \to \mathbb{R}^m$.

In many cases, we have to study the following problem

$$\max_x f(x) \quad s.t. \quad g(x) \geq 0, \tag{M'}$$

in which the set X is not specified.

We list the main steps to try to solve (M').

1. Canonical form.

Write the problem in the (in fact, our definition of) canonical form. Sometimes the problem contains a parameter $\pi \in \Pi$ an open subset of $\mathbb{R}^k$. Then we should write: for given $\pi \in \Pi$

$$\max_{x \in X} f(x, \pi) \quad s.t. \quad g(x, \pi) \geq 0.$$

2. The set X and the functions f and g.

a. Define the functions $\widetilde{f}$, $\widetilde{g}$ naturally arising from the problem with domain equal to their definition set, where the definition set of a function φ is the largest set $\mathcal{D}_\varphi$ which can be the domain of that function.

b. Determine X. A possible choice for X is the intersection of the "definition set" of each function, , i.e.,

$$X = \mathcal{D}_f \cap \mathcal{D}_{g_1} \cap ... \cap \mathcal{D}_{g_m}$$

c. Check if X is open and convex.

d. To apply the analysis described in the previous sections, show, if possible, that f and g are of class C^2 or at least C^1.

3. Existence.

Try to apply the Extreme Value Theorem. If f is at least C^1, then f is continuous and therefore we have to check if the constraint set C is non-empty and compact. Recall that a set S in $\mathbb{R}^n$ is compact if and only if S is $\mathbb{R}^n$closed and $\mathbb{R}^n$ bounded.

Boundedness has to be shown "brute force", i.e., using the specific form of the maximization problem.

If $X = \mathbb{R}^n$, then

$$C := \{x \in X : g(x) \geq 0\}$$

is $\mathbb{R}^n$ closed, because of the following well-known argument: $C = \cap_{j=1}^m g_j^{-1}([0, +\infty))$; since g_j is C^2 (or at least C^1) and therefore continuous, and $[0, +\infty)$ closed, $g_j^{-1}([0, +\infty))$ is closed in $X = \mathbb{R}^n$; then C is closed because intersection of closed sets.

A problem may arise if X is an open proper subset of $\mathbb{R}^n$. In that case the above argument shows that C is a X closed set and therefore it is not necessarily $\mathbb{R}^n$ closed. A possible way out is the following one. Consider the set

$$\widetilde{C} := \{x \in \mathbb{R}^n : \widetilde{g}(x) \geq 0\}$$

which is $\mathbb{R}^n$ closed - see argument above. Then, we are left with showing that

$$C = \widetilde{C}$$

If $\widetilde{C}$ is compact, C is compact as well.[3]

4. Number of solutions.

See subsection 18.1.1. In fact, summarizing what said there, we know that the solution to (M), if any, is unique if

1. f is strictly-quasi-concave, and for $j \in \{1, ..., m\}$, g_j is quasi-concave; **or**

2. for $j \in \{1, ..., m\}$, g_j is strictly-quasi-concave and

either a. f is quasi-concave and locally non-satiated,

or b. f is affine and non-costant,

[3] See Example below for an application of the above presented strategy.

or c. f is quasi-concave and strictly monotone,
or d. f is quasi-concave and $\forall x \in X$, $Df(x) >> 0$,
or e. f is quasi-concave and $\forall x \in X$, $Df(x) << 0$.

5. Necessity of K-T conditions.

Check if the conditions which insure that $M \subseteq S$ hold, i.e.,
either a. for $j = 1, ..., m$, g_j is pseudo-concave and there exists $x^{++} \in X$ such that $g(x^{++}) \gg 0$,
or b. rank $Dg^*(x^*) = \# J^*$.

If those conditions holds, each property we show it holds for elements of S does hold *a fortiori* for elements of M.

6. Sufficiency of K-T conditions.

Check if the conditions which insure that $M \supseteq S$ hold, i.e., that
f is pseudo-concave and for $j = 1, ..., m$, g_j is quasi-concave.

If those conditions holds, each property we show it does *not* hold for elements of S does not hold *a fortiori* for elements of M.

7. K-T conditions.

Write the Lagrangian function and then the Kuhn-Tucker conditions.

8. Solve the K-T conditions.

Try to solve the system of Kuhn-Tucker conditions in the unknown variables (x, λ). To do that;
either, analyze all cases,
or, try to get a "good conjecture" and check if the conjecture is correct.

Example 775 *Discuss the problem*

$$\max_{(x_1,x_2)} \tfrac{1}{2}\log(1+x_1) + \tfrac{1}{3}\log(1+x_2) \qquad s.t. \qquad \begin{array}{rcl} x_1 & \geq & 0 \\ x_2 & \geq & 0 \\ x_1 + x_2 & \leq & w \end{array}$$

with $w > 0$.

1. Canonical form.

For given $w \in \mathbb{R}_{++}$,

$$\max_{(x_1,x_2)} \tfrac{1}{2}\log(1+x_1) + \tfrac{1}{3}\log(1+x_2) \qquad s.t. \qquad \begin{array}{rcl} x_1 & \geq & 0 \\ x_2 & \geq & 0 \\ w - x_1 - x_2 & \geq & 0 \end{array} \tag{18.30}$$

2. The set X and the functions f and g.

a.

$$\begin{array}{ll} \widetilde{f}: (-1,+\infty)^2 \to \mathbb{R}, & (x_1,x_2) \mapsto \tfrac{1}{2}\log(1+x_1) + \tfrac{1}{3}\log(1+x_2) \\ \widetilde{g}_1: \mathbb{R}^2 \to \mathbb{R} & (x_1,x_2) \mapsto x_1 \\ \widetilde{g}_2: \mathbb{R}^2 \to \mathbb{R} & (x_1,x_2) \mapsto x_2 \\ \widetilde{g}_3: \mathbb{R}^2 \to \mathbb{R} & (x_1,x_2) \mapsto w - x_1 - x_2 \end{array}$$

b.

$$X = (-1,+\infty)^2$$

and therefore f and g are just $\widetilde{f}$ and $\widetilde{g}$ restricted to X.

c. X is open and convex because Cartesian product of open intervals which are open, convex sets.

d. Let's try to compute the Hessian matrices of f, g_1, g_2, g_3. Gradients are

$$\begin{array}{rcl} Df(x_1,x_2), & = & \left(\frac{1}{2(x_1+1)}, \frac{1}{3(x_2+1)}\right) \\ D\widetilde{g}_1(x_1,x_2) & = & (1,0) \\ D\widetilde{g}_2(x_1,x_2) & = & (0,1) \\ D\widetilde{g}_3(x_1,x_2) & = & (-1,-1) \end{array}$$

Hessian matrices are

$$\begin{array}{rcl} D^2 f(x_1,x_2), & = & \begin{bmatrix} -\frac{1}{2(x_1+1)^2} & 0 \\ 0 & -\frac{1}{3(x_2+1)^2} \end{bmatrix} \\ D^2\widetilde{g}_1(x_1,x_2) & = & 0 \\ D^2\widetilde{g}_2(x_1,x_2) & = & 0 \\ D^2\widetilde{g}_3(x_1,x_2) & = & 0 \end{array}$$

In fact, g_1, g_2, g_3 are affine functions. In conclusion, f and g_1, g_2, g_3 are C^2. In fact, g_1 and g_2 are linear and g_3 is affine.

3. Existence.

C is clearly bounded: $\forall x \in C$,

$$(0,0) \leq (x_1, x_2) \leq (w, w)$$

In fact, the first two constraint simply say that $(x_1, x_2) \geq (0,0)$. Moreover, from the third constraint $x_1 \leq w - x_2 \leq w$, simply because $x_2 \geq 0$; similar argument can be used to show that $x_2 \leq w$.

To show closedness, use the strategy proposed above.

$$\widetilde{C} := \{x \in \mathbb{R}^n : g(x) \geq 0\}$$

is obviously closed. Since $\widetilde{C} \subseteq \mathbb{R}^2_+$, because of the first two constraints, $\widetilde{C} \subseteq X := (-1, +\infty)^2$ and therefore $C = \widetilde{C} \cap X = \widetilde{C}$ is closed.

We can then conclude that C is compact and therefore $\arg\max(18.30) \neq \varnothing$.

4. Number of solutions.

From the analysis of the Hessian and using Theorem 753, parts 1 ad 2, we have that f is strictly concave:

$$-\frac{1}{2(x_1+1)^2} < 0$$

$$\det \begin{bmatrix} -\frac{1}{2(x_1+1)^2} & 0 \\ 0 & -\frac{1}{3(x_2+1)^2} \end{bmatrix} = \frac{1}{2(x_1+1)^2} \cdot \frac{1}{3(x_2+1)^2} > 0$$

Moreover g_1, g_2, g_3 are affine and therefore concave. From Proposition 766, part 1, the solution is unique.

5. Necessity of K-T conditions.

Since each g_j is affine and therefore pseudo-concave, we are left with showing that there exists $x^{++} \in X$ such that $g(x^{++}) >> 0$. Just take $\left(x_1^{++}, x_2^{++}\right) = \frac{w}{4}(1,1)$:

$$\begin{array}{lcl} \frac{w}{4} & > & 0 \\ \frac{w}{4} & > & 0 \\ w - \frac{w}{4} - \frac{w}{4} = \frac{w}{2} & > & 0 \end{array}$$

Therefore

$$M \subseteq S$$

6. Sufficiency of K-T conditions.

f is strictly concave and therefore pseudo-concave, and each g_j is linear and therefore quasi-concave. Therefore

$$M \supseteq S$$

7. K-T conditions.

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \mu; w) = \frac{1}{2}\log(1+x_1) + \frac{1}{3}\log(1+x_2) + \lambda_1 x_1 + \lambda_2 x_2 + \mu(w - x_1 - x_2)$$

$$\begin{cases} \frac{1}{2(x_1+1)} + \lambda_1 - \mu & = & 0 \\ \frac{1}{3(x_2+1)} + \lambda_2 - \mu & = & 0 \\ \min\{x_1, \lambda_1\} & = & 0 \\ \min\{x_2, \lambda_2\} & = & 0 \\ \min\{w - x_1 - x_2, \mu\} & = & 0 \end{cases}$$

8. Solve the K-T conditions.

Conjecture: $x_1 > 0$ and therefore $\lambda_1 = 0$; $x_2 > 0$ and therefore $\lambda_2 = 0$; $w - x_1 - x_2 = 0$. The Kuhn-Tucker system becomes:

$$\begin{cases} \frac{1}{2(x_1+1)} - \mu & = & 0 \\ \frac{1}{3(x_2+1)} - \mu & = & 0 \\ w - x_1 - x_2 & = & 0 \\ \mu & \geq & 0 \\ x_1 > 0, x_2 > 0 & & \\ \lambda_1 = 0, \lambda_2 = 0 & & \end{cases}$$

Then,

$$\begin{cases} \frac{1}{2(x_1+1)} & = & \mu \\ \frac{1}{3(x_2+1)} & = & \mu \\ w - x_1 - x_2 & = & 0 \\ \mu & > & 0 \\ x_1 > 0, x_2 > 0 \\ \lambda_1 = 0, \lambda_2 = 0 \end{cases}$$

$$\begin{cases} x_1 & = & \frac{1}{2\mu} - 1 \\ x_2 & = & \frac{1}{3\mu} - 1 \\ w - \left(\frac{1}{2\mu} - 1\right) - \left(\frac{1}{3\mu} - 1\right) & = & 0 \\ \mu & > & 0 \\ x_1 > 0, x_2 > 0 \\ \lambda_1 = 0, \lambda_2 = 0 \end{cases}$$

$0 = w - \left(\frac{1}{2\mu} - 1\right) - \left(\frac{1}{3\mu} - 1\right) = w - \frac{5}{6\mu} + 2$; *and* $\mu = \frac{5}{6(w+2)} > 0$. *Then* $x_1 = \frac{1}{2\mu} - 1 = \frac{6(w+2)}{2 \cdot 5} - 1 = \frac{3w+6-5}{5} = \frac{3w+1}{5}$ *and* $x_2 = \frac{1}{3\mu} - 1 = \frac{6(w+2)}{3 \cdot 5} - 1 = \frac{2w+4-5}{5} = \frac{2w-1}{5}$.

Summarizing

$$\begin{cases} x_1 & = & \frac{3w+1}{5} > 0 \\ x_2 & = & \frac{2w-1}{5} > 0 \\ \mu & = & \frac{5}{6(w+2)} > 0 \\ \lambda_1 = 0, \lambda_2 = 0 \end{cases}$$

Observe that while $x_1 > 0$ *for any value of* w, $x_2 > 0$*iff* $w > \frac{1}{2}$. *Therefore, for* $w \in \left(0, \frac{1}{2}\right]$, *the above one is not a solution, and we have to come up with another conjecture;*

$x_1 = w$ *and therefore* $\lambda_1 = 0$; $x_2 = 0$ *and* $\lambda_2 \geq 0$; $w - x_1 - x_2 = 0$ *and* $\mu \geq 0$.*The Kuhn-Tucker conditions become*

$$\begin{cases} \frac{1}{2(w+1)} - \mu & = & 0 \\ \frac{1}{3} + \lambda_2 - \mu & = & 0 \\ \lambda_1 & = & 0 \\ x_2 & = & 0 \\ \lambda_2 & \geq & 0 \\ x_1 & = & w \\ \mu & \geq & 0 \end{cases}$$

and

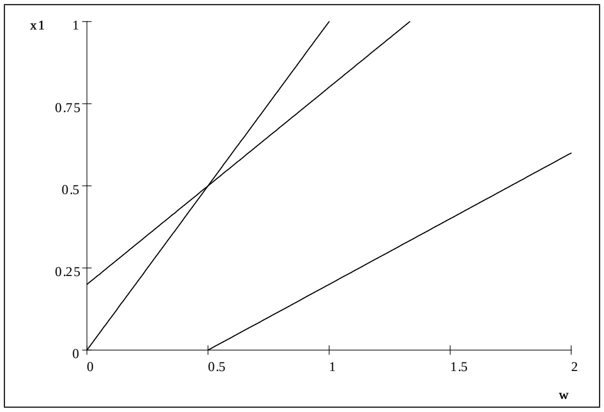
$$\begin{cases} \mu & = & \frac{1}{2(w+1)} > 0 \\ \lambda_2 & = & \frac{1}{2(w+1)} - \frac{1}{3} = \frac{3-2w-2}{6(w+1)} = \frac{1-2w}{6(w+1)} \\ \lambda_1 & = & 0 \\ x_2 & = & 0 \\ \lambda_2 & \geq & 0 \\ x_1 & = & w \end{cases}$$

$\lambda_2 = \frac{1-2w}{6(w+1)} = 0$ *if* $w = \frac{1}{2}$,*and* .$\lambda_2 = \frac{1-2w}{6(w+1)} > 0$ *if* $w \in \left(0, \frac{1}{2}\right)$

Summarizing, the unique solution x^* *to the maximization problem is*

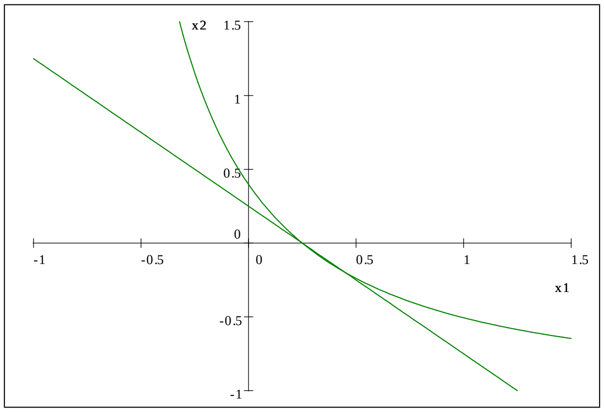
if	$w \in \left(0, \frac{1}{2}\right)$,	*then*	$x_1^* = w$,	$\lambda_1^* = 0$	*and*	$x_2^* = 0$,	$\lambda_2^* > 0$
if	$w = \frac{1}{2}$,	*then*	$x_1^* = w$,	$\lambda_1^* = 0$		$x_2^* = 0$,	$\lambda_2^* = 0$
if	$w \in \left(\frac{1}{2}, +\infty\right)$,	*then*	$x_1^* = \frac{3w+1}{5} > 0$	$\lambda_1^* = 0$	*and*	$x_2^* = \frac{2w-1}{5} > 0$	$\lambda_2^* = 0$

The graph of x_1^* *as a function of* w *is presented below* `(please, complete the picture)`

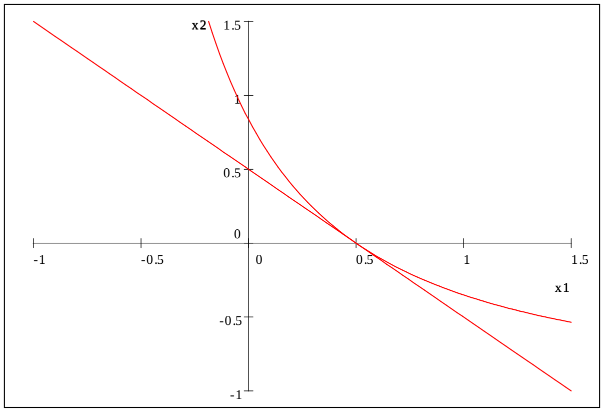


The graph below shows constraint sets for different "important" values of w and some significant level curve of the objective function.

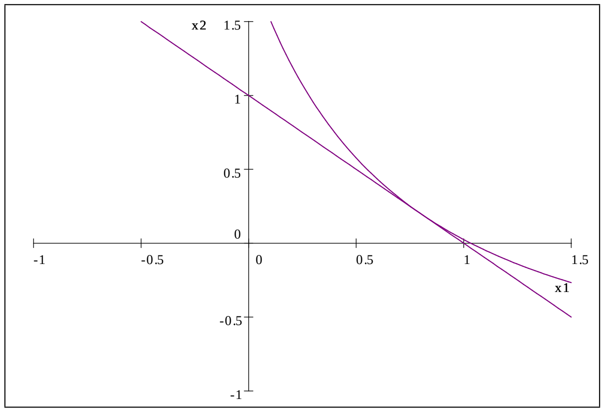
$w \in \left(0, \frac{1}{2}\right)$:



$w = \frac{1}{2}$:



$w > \frac{1}{2}$:



Observe that in the example, we get that if $\lambda_2^* = 0$, the associated constraint $x_2 \geq 0$ is not significant. See Subsection 18.6.2, for a discussion of that statement.

Of course, several problems may arise in applying the above procedure. Below, we describe some commonly encountered problems and some possible (partial) solutions.

18.4.1 Some problems and some solutions

1. The set X.

X is not open.

Rewrite the problem in terms of an open set X' and some added constraints. A standard example is the following one.

$$\max_{x \in \mathbb{R}^n_+} f(x) \quad s.t. \quad g(x) \geq 0$$

which can be rewritten as

$$\max_{x \in \mathbb{R}^n} f(x) \quad s.t. \quad \begin{array}{l} g(x) \geq 0 \\ x \geq 0 \end{array}$$

2. Existence.

a. The constraint set is not compact. Consider again the problem.

$$\max_{x \in \mathbb{R}^n_+} f(x) \quad s.t. \quad g(x) \geq 0. \qquad (P)$$

If the constraint set is not compact, it is sometimes possible to find another maximization problem such that

i. its constraint set is compact and nonempty, and

ii. whose solution set is contained in the solution set of the problem we are analyzing.

A way to try to achieve both i. and ii. above is to "restrict the constraint set (to make it compact) without eliminating the solution of the original problem". Sometimes, a problem with the above properties is the following one.

$$\max_{x \in X} f(x) \qquad s.t. \qquad \begin{array}{rcl} g(x) & \geq & 0 \\ f(x) - f(\widehat{x}) & \geq & 0 \end{array} \qquad (P1)$$

where $\widehat{x}$ is an element of X such that $g(\widehat{x}) \geq 0$.

Define

$$M := \arg\max (P) \quad M^1 := \arg\max (P1)$$

and V and V^1 the constraint sets of Problems (P) and $(P1)$, respectively. Observe that

$$V^1 \subseteq V \tag{18.31}$$

If V^1 is compact, then $M^1 \neq \varnothing$ and the only thing left to show is that $M^1 \subseteq M$, which is always insured as proved below.

Proposition 776 *1.* $M^1 \subseteq M$ *and 2.* $M \subseteq M^1$*.*

Proof. 1.

If $M^1 = \varnothing$, we are done. Suppose that $M^1 \neq \varnothing$, and that the conclusion of the Proposition is false, i.e., there exists $x^1 \in M^1$ such that

a. $x^1 \in M^1$, and b. $x^1 \notin M$, or

a. $\forall x \in X$ such that $g(x) \geq 0$ and $f(x) \geq f(\widehat{x})$, we have $f\left(x^1\right) \geq f(x)$;

and

b. either i. $x^1 \notin V$,

or ii. $\exists \widetilde{x} \in X$ such that

$$g(\widetilde{x}) \geq 0 \tag{18.32}$$

and

$$f(\widetilde{x}) > f\left(x^1\right) \tag{18.33}$$

Let's show that i. and ii. cannot hold.

i.

It cannot hold simply because $V^1 \subseteq V$, from 18.31.

ii.

Since $x^1 \in V^1$,

$$f\left(x^1\right) \geq f(\widehat{x}) \tag{18.34}$$

From (18.33) and (18.34), it follows that

$$f(\widetilde{x}) > f(\widehat{x}) \tag{18.35}$$

But (18.32), (18.35) and (18.33) contradict the definition of x^1, i.e., a. above.

2.

If $M = \varnothing$, then we are done. Suppose now that $M \neq \varnothing$ and take $x^* \in M$. We want to show that a. $x^* \in V^1$ and b. for any $x \in V^1$, we have $f(x^*) \geq f(x)$.

a. Since $x^* \in M \subseteq V$, if our claim is false, then we have $x \in V \setminus V^1$, i.e., $f(x^*) < f(\widehat{x})$, with $\widehat{x} \in V$, contradicting the fact that $x^* \in M$.

b. If $x \in V^1$, then $x \in V$ - simply because $V^1 \subseteq V$. Therefore $f(x^*) > f(x)$ by the assumption that $x^* \in M$. ■

b. Existence without the Extreme Value Theorem If you are not able to show existence, but
i. sufficient conditions to apply Kuhn-Tucker conditions hold, and
ii. you are able to find a solution to the Kuhn-Tucker conditions,
then a solution exists.

18.5 The Implicit Function Theorem and Comparative Statics Analysis

The Implicit Function Theorem can be used to study how solutions ($x \in X \subseteq \mathbb{R}^n$) to maximizations problems and, if needed, associated Lagrange or Kuhn-Tucker multipliers ($\lambda \in \mathbb{R}^m$) change when parameters ($\pi \in \Pi \subseteq \mathbb{R}^k$) change. That analysis can be done if the solutions to the maximization problem (and the multipliers) are solution to a system of equation of the form

$$F_1(x, \pi) = 0$$

with (# choice variables) = (# dimension of the codomain of F_1), or

$$F_2(\xi, \pi) = 0$$

where $\xi := (x, \lambda)$, and (# choice variables *and* multipliers) = (# dimension of the codomain of F_2),
To apply the Implicit Function Theorem, it must be the case that the following conditions do hold.

1. (# choice variables x) = (# dimension of the codomain of F_1), or
   (# choice variables *and* multipliers) = (# dimension of the codomain of F_2).

2. F_i has to be at least C^1.That condition is insured if the above systems are obtained from maximization problems characterized by functions f, g which are at least C^2: usually the above systems contain some form of first order conditions, which are written using first derivatives of f and g.

3. $F_1(x^*, \pi_0) = 0$ or $F_2(\xi^*, \pi_0) = 0$. The existence of a solution to the system is usually the result of the strategy to describe how to solve a maximization form - see above Section 18.4.

4. $\det [D_x F_1(x^*, \pi_0)]_{n\times n} \neq 0$ or $\det [D_\xi F_2(\xi^*, \pi_0)]_{(n+m)\times(n+m)} \neq 0$. That condition has to be verified directly on the problem.

If the above conditions are verified, the Implicit Function Theorem allow to conclude what follows (in reference to F_2).

There exist an open neighborhood $N(\xi^*) \subseteq X$ of ξ^*, an open neighborhood $N(\pi_0) \subseteq \Pi$ of π_0 and a unique C^1 function $g : N(\pi_0) \subseteq \Pi \subseteq \mathbb{R}^p \to N(\xi^*) \subseteq X \subseteq \mathbb{R}^n$ such that $\forall \pi \in N(\pi_0)$, $F(g(\pi), \pi) = 0$ and

$$Dg(\pi) \quad = - \left[D_\xi F(\xi, \pi)_{|\xi = g(\pi)} \right]^{-1} \cdot \left[D_\pi F(\xi, \pi)_{|\xi = g(\pi)} \right]$$

Therefore, using the above expression, we may be able to say if the increase in any value of any parameter implies an increase in the value of any choice variable (or multiplier).

Three significant cases of application of the above procedure are presented below. We are going to consider C^2 functions defined on open subsets of Euclidean spaces.

18.5.1 Maximization problem without constraint

Assume that the problem to study is

$$\max_{x \in X} f(x, \pi)$$

and that

1. f is concave;
2. There exists a solution x^* to the above problem associated with π_0.

Then, from Proposition 716, we know that x^* is a solution to

$$Df(x, \pi_0) = 0$$

Therefore, we can try to apply the Implicit Function Theorem to

$$F_1(x, \pi) = Df(x, \pi_0)$$

Example 777 *Consider the maximization problem of a firm, described as follows. Let the following objects be given.*

Price $p \in \mathbb{R}_{++}$ of output; quantity $x \in \mathbb{R}$ of output; index $t \in \mathbb{R}$ of technological change; production function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$, such that $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$, $f' > 0$ and $f'' < 0$; cost function $c : \mathbb{R}^2 \to \mathbb{R}$, $(x, t) \mapsto c(x, t)$ such that $c \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$, $D_x c > 0$, $D_{xx} c > 0 > 0$, $D_t c < 0$.

Then, the maximization problem we want to analyze is

$$\max_{x \in \mathbb{R}} \ \pi(x, t) := pf(x) - c(x, t).$$

We omit here the analysis of the existence problems. Observe that

$$D_x \pi = pf'(x) - D_x(x, t);$$

$$D_{xx} \pi = pf''(x) - D_{xx}(x, t) < 0.$$

Therefore, the objective function is strictly concave and solutions to the maximization problem are solution to the equation

$$F(x; p, t) := pf'(x) - D_x(x, t) = 0.$$

To study the effect of a change in t on the solution value of x, we can apply the Implicit Function Theorem to the above equation, to get that

$$D_t(p, t) = -\frac{D_t F(x; p, t)}{D_x F(x; p, t)} = -\frac{-D_{xt}(x, t)}{pf''(x) - D_{xx}(x, t)}.$$

Since $pf''(x) - D_{xx}(x,t) < 0$, then the sign of $D_t(p,t)$ is equal to the sign of $-D_{xt}(x,t)$, where $D_{xt}(x,t)$ is the derivative of the marginal cost with respect to t.

Another example of application of the strategy illustrated above is presented in Section 19.3.

18.5.2 Maximization problem with equality constraints

Consider a maximization problem

$$\max_{x \in X} \quad f(x,\pi) \qquad s.t. \qquad g(x,\pi) = 0,$$

Assume that necessary and sufficient conditions to apply Lagrange Theorem hold and that there exists a vector (x^*, λ^*) which is a solution (not necessarily unique) associated with the parameter π_0. Therefore, we can try to apply the Implicit Function Theorem to

$$F_2(\xi,\pi) = \begin{pmatrix} Df(x^*,\pi_0) + \lambda^* Dg(x^*,\pi_0) \\ g(x^*,\pi_0). \end{pmatrix} \tag{18.36}$$

18.5.3 Maximization problem with Inequality Constraints

Consider the following maximization problems with inequality constraints. For given $\pi \in \Pi$,

$$\max_{x \in X} f(x,\pi) \quad s.t. \quad g(x,\pi) \geq 0 \tag{18.37}$$

Moreover, assume that the set of solutions of that problem is nonempty and characterized by the set of solutions of the associated Kuhn-Tucker system, i.e., using the notation of Subsection 18.1,

$$M = S \neq \varnothing.$$

We have seen that we can write Kuhn-Tucker conditions in one of the two following ways, beside some other ones,

$$\begin{cases} Df(x,\pi) + \lambda Dg(x,\pi) & = & 0 & (1) \\ \lambda & \geq & 0 & (2) \\ g(x,\pi) & \geq & 0 & (3) \\ \lambda g(x,\pi) & = & 0 & (4) \end{cases} \tag{18.38}$$

or

$$\begin{cases} Df(x,\pi) + \lambda Dg(x,\pi) & = & 0 & & (1) \\ \min\{\lambda_j, g_j(x,\pi)\} & = & 0 & \text{for } j \in \{1,...,m\} & (2) \end{cases} \tag{18.39}$$

The Implicit Function Theorem *cannot* be applied to either system (18.38) or system (18.39): system (18.38) contains *inequalities*; system (18.39) involves functions which are *not differentiable.* We present below conditions under which the Implicit Function Theorem can be anyway applied to allow to make comparative statics analysis. Take a solution (x^*, λ^*, π_0) to the above system(s). Assume that

$$\text{for each } j, \text{ either } \lambda_j^* > 0 \text{ or } g_j(x^*,\pi_0) > 0.$$

In other words, there is no j such that $\lambda_j = g_j(x^*,\pi_0) = 0$. Consider a partition $J^*, \widehat{J}$ of $\{1,..,m\}$, and the resulting Kuhn-Tucker conditions.

$$\begin{cases} Df(x^*,\pi_0) + \lambda^* Dg(x^*,\pi_0) \; = \; 0 & \\ \lambda_j^* > 0 & \text{for } j \in J^* \\ g_j(x^*,\pi_0) = 0 & \text{for } j \in J^* \\ \lambda_j^* = 0 & \text{for } j \in \widehat{J} \\ g_j(x^*,\pi_0) > 0 & \text{for } j \in \widehat{J} \end{cases} \tag{18.40}$$

Define

$$\begin{aligned} & g^*(x^*,\pi_0) := (g_j(x^*,\pi_0))_{j \in J^*} \\ & \widehat{g}(x^*,\pi_0) := (g_j(x^*,\pi_0))_{j \in \widehat{J}} \\ & \lambda^{**} := \left(\lambda_j^*\right)_{j \in J^*} \\ & \widehat{\lambda}^* := \left(\lambda_j^*\right)_{j \in \widehat{J}} \end{aligned}$$

Write the system of equations obtained from system (18.40) eliminating strict inequality constraints and substituting in the zero variables:

$$\begin{cases} Df\left(x^*, \pi_0\right) + \lambda^{**} Dg^*\left(x^*, \pi_0\right) & = & 0 \\ g^*\left(x^*, \pi_0\right) & = & 0 \end{cases} \tag{18.41}$$

Observe that the number of equations is equal to the number of "remaining" unknowns and they are

$$n + \# J^*$$

i.e., Condition 1 presented at the beginning of the present Section 18.5 is satisfied. Assume that the needed rank condition does hold and we therefore can apply the Implicit Function Theorem to

$$F_2\left(\xi, \pi\right) = \begin{pmatrix} Df\left(x^*, \pi_0\right) + \lambda^{**} Dg^*\left(x^*, \pi_0\right) \\ g^*\left(x^*, \pi_0\right) \end{pmatrix} = 0$$

Then. we can conclude that there exists a unique C^1 function φ defined in an open neighborhood N_1 of π_0 such that

$$\forall \pi \in N_1, \qquad \varphi\left(\pi\right) := \left(x^*\left(\pi\right), \lambda^{**}\left(\pi\right)\right)$$

is a solution to system (18.41) at π.

Therefore, by definition of φ,

$$\begin{cases} Df\left(x^*\left(\pi\right), \pi\right) + \lambda^{**}\left(\pi\right)^T Dg^*\left(x^*\left(\pi\right), \pi\right) & = & 0 \\ g^*\left(x^*\left(\pi\right), \pi\right) & = & 0 \end{cases} \tag{18.42}$$

Since φ is continuous and $\lambda^{**}\left(\pi_0\right) > 0$ and $\widehat{g}\left(x^*\left(\pi_0\right), \pi_0\right) > 0$, there exist an open neighborhood $N_2 \subseteq N_1$ of π_0 such that $\forall \pi \in N_2$

$$\begin{cases} \lambda^{**}\left(\pi\right) & > & 0 \\ \widehat{g}\left(x\left(\pi\right), \pi\right) & > & 0 \end{cases} \tag{18.43}$$

Take also $\forall \pi \in N_2$

$$\left\{ \widehat{\lambda}^*\left(\pi\right) \quad = \quad 0 \right. \tag{18.44}$$

Then, systems (18.42), (18.43) and (18.44) say that $\forall \pi \in N_2$, $\left(x\left(\pi\right), \lambda^*\left(\pi\right), \widehat{\lambda}\left(\pi\right)\right)$ satisfy Kuhn-Tucker conditions for problem (18.37) and therefore, since $C = M$, they are solutions to the maximization problem.

The above conclusion *does not hold true* if Kuhn-Tucker conditions are of the following form

$$\begin{cases} Df\left(x, \pi\right) + \lambda^T Dg\left(x, \pi\right) \quad = \quad 0 & \\ \lambda_j = 0, \quad g_j\left(x, \pi\right) = 0 & \text{for } j \in J' \\ \lambda_j > 0, \quad g_j\left(x, \pi\right) = 0 & \text{for } j \in J'' \\ \lambda_j = 0, \quad g_j\left(x, \pi\right) > 0 & \text{for } j \in \widehat{J} \end{cases} \tag{18.45}$$

where $J' \neq \varnothing$, J'' and $\widehat{J}$ is a partition of J.

In that case, applying the same procedure described above, i.e., eliminating strict inequality constraints and substituting in the zero variables, leads to the following systems in the unknowns $x \in \mathbb{R}^n$ and $\left(\lambda_j\right)_{j \in J''} \in \mathbb{R}^{\# J''}$:

$$\begin{cases} Df\left(x, \pi\right) + \left(\lambda_j\right)_{j \in J''} D\left(g_j\right)_{j \in J''}\left(x, \pi\right) \quad = \quad 0 & \\ g_j\left(x, \pi\right) = 0 & \text{for } j \in J' \\ g_j\left(x, \pi\right) = 0 & \text{for } j \in J'' \end{cases}$$

and therefore the number of equation is $n + \# J'' + \# J' > n + \# J''$,simply because we are considering the case $J' \neq \varnothing$. Therefore the crucial condition

$$(\#\text{ choice variables } \textit{and} \text{ multipliers}) = (\#\text{ dimension of the codomain of } F_2)$$

is violated.

Even if the Implicit Function Theorem could be applied to the equations contained in (18.45), in an open neighborhood of π_0 we could have

$$\lambda_j\left(\pi\right) < 0 \text{ and/or } \quad g_j\left(x\left(\pi\right), \pi\right) < 0 \quad \text{for } j \in J'$$

Then $\varphi\left(\pi\right)$ would be solutions to a set of equations and inequalities which are not Kuhn-Tucker conditions of the maximization problem under analysis, and therefore $x\left(\pi\right)$ would not be a solution to the that maximization problem.

An example of application of the strategy illustrated above is presented in Section 19.1.

18.6 The Envelope Theorem and the meaning of multipliers

18.6.1 The Envelope Theorem

Consider the problem (M) : for given $\pi \in \Pi$,

$$\max_{x \in X} f(x, \pi) \qquad s.t. \quad g(x, \pi) = 0$$

Assume that for every π, the above problem admits a unique solution characterized by Lagrange conditions and that the Implicit function theorem can be applied. Then, there exists an open set $\mathcal{O} \subseteq \Pi$ such that

$$\begin{array}{ll} x : \mathcal{O} \to X, \ x : \pi \mapsto \arg\max(P) \ , & \\ v : \mathcal{O} \to \mathbb{R}, \ v : \pi \mapsto \max(P) & \text{and} \\ \lambda : \mathcal{O} \to \mathbb{R}^m, \ \pi \mapsto \ \text{unique Lagrange multiplier vector} & \end{array}$$

are differentiable functions.

Theorem 778 *For any $\pi^* \in \mathcal{O}$ and for any pair of associated $(x^*, \lambda^*) := (x(\pi^*), \lambda(\pi^*))$, we have*

$$D_\pi v(\pi^*) = D_\pi \mathcal{L}(x^*, \lambda^*, \pi^*)$$

i.e.,

$$D_\pi v(\pi^*) = D_\pi f(x^*, \pi^*) + \lambda^* D_\pi g(x^*, \pi^*)$$

Remark 779 *Observe that the above analysis applies also to the case of inequality constraints, as long as the set of binding constraints does not change.*

Proof. of Theorem 778 By definition of $v(.)$ and $x(.)$, we have that

$$\forall \pi \in \mathcal{O}, \quad v(\pi) = f(x(\pi), \pi). \quad (1)$$

Consider an arbitrary value π^* and the unique associate solution $x^* = x(\pi^*)$ of problem (P). Differentiating both sides of (1) with respect to π and computing at π^*, we get

$$[D_\pi v(\pi^*)]_{1\times k} = \left[D_x f(x, \pi)_{|(x^*, \pi^*)}\right]_{1\times n} \cdot \left[D_\pi x(\pi)_{|\pi=\pi^*}\right]_{n\times k} + \left[D_\pi f(x, \pi)_{|(x^*, \pi^*)}\right]_{1\times k} \quad (2)$$

From Lagrange conditions

$$D_x f(x, \pi)_{|(x^*, \pi^*)} = -\lambda^* D_x g(x, \pi)_{|(x^*, \pi^*)} \quad (3),$$

where λ^* is the unique value of the Lagrange multiplier. Moreover

$$\forall \pi \in \mathcal{O}, \quad g(x(\pi), \pi) = 0. \quad (4)$$

Differentiating both sides of (4) with respect to π and computing at π^*, we get

$$\left[D_x g(x, \pi)_{|(x^*, \pi^*)}\right]_{m\times n} \cdot \left[[D_\pi x(\pi)]_{|\pi=\pi^*}\right]_{n\times k} + \left[D_\pi g(x, \pi)_{|(x^*, \pi^*)}\right]_{m\times k} = 0 \quad (5).$$

Finally,

$$[D_\pi v(\pi^*)]_{1\times k} \overset{(2),(3)}{=} -\lambda^* D_x g(x, \pi)_{|(x^*, \pi^*)} D_\pi x(\pi)_{|\pi=\pi^*} + D_\pi f(x, \pi)_{|(x^*, \pi^*)} \overset{(5)}{=}$$

$$= D_\pi f(x, \pi)_{|(x^*, \pi^*)} + \lambda^* D_\pi g(x, \pi)_{|(x^*, \pi^*)}$$

■

18.6.2 On the meaning of the multipliers

The main goal of this subsection is to try to formalize the following statements.

1. The fact that $\lambda_j = 0$ indicates that the associated constraint $g_j(x) \geq 0$ is not significant - see Proposition 780 below.
2. The fact that $\lambda_j > 0$ indicates that a way to increase the value of the objective function is to violate the associated constraint $g_j(x) \geq 0$ - see Proposition 781 below.

For simplicity, consider the case $m = 1$. Let (CP) be the problem

$$\max_{x \in X} f(x) \quad s.t. \quad g(x) \geq 0$$

and (UP) the problem

$$\max_{x \in X} f(x)$$

with f strictly quasi-concave, g is quasi-concave and solutions to both problem exist . Define $x^* := \arg\max (CP)$ with associated multiplier λ^*, and $x^{**} := \arg\max (UP)$.

Proposition 780 *If $\lambda^* = 0$, then $x^* = \arg\max (UP) \Leftrightarrow x^* = \arg\max (CP)$.*

Proof. By the assumptions of this section, the solution to (CP) exists, is unique, it is equal to x^* and there exists λ^* such that

$$\begin{aligned} Df(x^*) + \lambda^* Dg(x^*) &= 0 \\ \min\{g(x^*), \lambda^*\} &= 0. \end{aligned}$$

Moreover, the solution to (UP) exists, is unique and it is the solution to

$$Df(x) = 0.$$

Since $\lambda^* = 0$, the desired result follows. ■

Take $\varepsilon > 0$ and $k \in (-\varepsilon, +\infty)$. Let (CPk) be the problem

$$\max_{x \in X} f(x) \quad s.t. \quad g(x) \geq k$$

Let

$$\widehat{x} : (-\varepsilon, +\infty) \to X, \quad k \mapsto \arg\max (CPk)$$

$$\widehat{v} : (-\varepsilon, +\infty) \to \mathbb{R}, \quad k \mapsto \max (CPk) := f(\widehat{x}(k))$$

Let $\widehat{\lambda}(k)$ be such that $\left(\widehat{x}(k), \widehat{\lambda}(k)\right)$ is the solution to the associated Kuhn-Tucker conditions.

Observe that

$$x^* = \widehat{x}(0), \quad \lambda^* = \widehat{\lambda}(0) \tag{18.46}$$

Proposition 781 *If $\lambda^* > 0$, then $\widehat{v}'(0) < 0$.*

Proof. From the envelope theorem,

$$\forall k \in (-\varepsilon, +\infty), \widehat{v}'(k) = \frac{\partial (f(x) + \lambda (g(x) - k))}{\partial k}_{|\widehat{x}(k), \widehat{\lambda}(k)} = -\widehat{\lambda}(k)$$

and from (18.46)

$$\widehat{v}'(0) = -\widehat{\lambda}(0) = -\lambda^* < 0.$$

■

Remark 782 *Consider the following problem. For given $a \in \mathbb{R}$,*

$$\max_{x \in X} f(x) \quad s.t. \quad g(x) - a \geq 0 \tag{18.47}$$

Assume that the above problem is "well-behaved" and that $x(a) = \arg\max$ (18.47), $v(a) = f(x(a))$ and $(x(a), \lambda(a))$ is the solution of the associated Kuhn-Tucker conditions. Then, applying the Envelope Theorem we have

$$v'(a) = \lambda(a)$$

Chapter 19

Applications to Economics

19.1 A Walrasian Consumer Problems

In this section, we present two versions of the Walrasian consumer's problem.

19.1.1 Zero consumption is allowed.

We consider again the problem presented in Section 10.3, whose basic characteristics we repeat below. Let the following objects be given: price vector $p \in \mathbb{R}^n_{++}$, consumption vector $x \in \mathbb{R}^n$, consumer's wealth $w \in \mathbb{R}_{++}$, continuous utility function $u : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto u(x)$. The consumer solves the following problem. For given, $p \in \mathbb{R}^n_{++}$,$w \in \mathbb{R}_{++}$, solve

$$
\begin{array}{lllllll}
(P0) & \max_{x\in\mathbb{R}^n} & u(x) & s.t. & w - px & \geq 0 & \mu \\
 & & & & x_1 & \geq 0 & \lambda_1 \\
 & & & & ... & & \\
 & & & & x_n & \geq 0. & \lambda_n
\end{array}
$$

where we wrote multipliers next to each constraint. To be able to easily apply Kuhn-Tucker theorems, we also assume that

1. u is C^2;
2. for any $x \in \mathbb{R}^n$, $Du(x) >> 0$, (and therefore u is strictly increasing), and
3. or any $x \in \mathbb{R}^n$, $D^2u(x)$ is negative definite, (and therefore u is strictly concave).

It is not difficult to show that all the steps in the "Recipe to solve a nice maximization problem" (see Section 18.4) do go through. Let's write and discuss the system of Kuhn-Tucker conditions, which is presented below.

$$
\begin{cases}
D_{x^1}u(x) - \mu p^1 + \lambda_1 & = 0 \\
... & \\
D_{x^n}u(x) - \mu p^n + \lambda_n & = 0 \\
\min\{\mu, w - px\} & = 0 \\
\min\{\lambda_1, x_1\} & = 0 \\
... & \\
\min\{\lambda_n, x_n\} & = 0
\end{cases}
\tag{19.1}
$$

Let (x^*, μ^*, λ^*) denote the solution to the above system associated to a given (p, w).

Claim1. $\mu^* > 0$ and therefore $w - px^* = 0$.

Proof of the Claim.

Assume otherwise; then, from system (19.1), we get $D_{x^1}u(x) + \lambda_1 = 0$; but, $\lambda_1 \geq 0$ and, by assumption $Du(x) >> 0$,

we also have $D_{x^1}u(x) + \lambda_1 > 0$, a contradiction.

Claim1. Assume $n = 2$. Then,

(a) if

$$\frac{D_{x_1}u\left(\frac{w}{p_1},0\right)}{D_{x_2}u\left(\frac{w}{p_1},0\right)}<\frac{p_1}{p_2}, \tag{19.2}$$

then $x_2^*>0$ and $\lambda_2^*=0$;
(b) if

$$\frac{p_1}{p_2}<\frac{D_{x_1}u\left(0,\frac{w}{p_2}\right)}{D_{x_2}u\left(0,\frac{w}{p_2}\right)},$$

then $x_1^*>0$ and $\lambda_1^*=0$.
Proof of the Claim.
(a)
Assume otherwise, i.e., $x_2^*=0$ and therefore $\lambda_2^*\geq 0$. From Claim 1, we have that $x_1^*=\frac{w}{p_1}$ and therefore $\lambda_1^*=0$. Then, Kuhn-Tucker system becomes, omitting stars,

$$\begin{cases} D_{x^1}u\left(\frac{w}{p_1},0\right)-\mu p^1 & = & 0\\ D_{x^2}u\left(\frac{w}{p_1},0\right)-\mu p^2+\lambda_2 & = & 0\\ \lambda_2 & \geq & 0.\end{cases}$$

Then,

$$0=D_{x^2}u\left(\frac{w}{p_1},0\right)-\mu p^2+\lambda_2=D_{x^2}u\left(\frac{w}{p_1},0\right)-\frac{D_{x^1}u\left(\frac{w}{p_1},0\right)}{p^1}p^2+\lambda_2,$$

and multiplying both sides by $\frac{p^1}{p^2\cdot D_{x^2}u\left(\frac{w}{p_1},0\right)}$, we get

$$0=\frac{p_1}{p_2}-\frac{D_{x^1}u\left(\frac{w}{p_1},0\right)}{D_{x^2}u\left(\frac{w}{p_1},0\right)}+\lambda_2\frac{p^1}{p^2\cdot D_{x^2}u\left(\frac{w}{p_1},0\right)},$$

and therefore

$$\frac{p_1}{p_2}-\frac{D_{x^1}u\left(\frac{w}{p_1},0\right)}{D_{x^2}u\left(\frac{w}{p_1},0\right)}=-\lambda_2\frac{p^1}{p^2\cdot D_{x^2}u\left(\frac{w}{p_1},0\right)}\leq 0,$$

contradicting assumption (19.2).
(b))
The proof is similar to the above one.

19.1.2 Zero consumption is not allowed.

In this case, we assume the domain of the utility function is $\mathbb{R}^n_{++}$. We also make the following assumptions:
1. u is a $\mathcal{C}^2$ function;
2. u is differentiably strictly increasing, i.e., $\forall x\in\mathbb{R}^n_{++},\ Du(x)\gg 0$;
3. u is differentiably strictly quasi-concave, i.e., $\forall x\in\mathbb{R}^n_{++},\ \Delta x\neq 0$ and $Du(x)\Delta x=0\Rightarrow\Delta x^T D^2u(x)\Delta x<0$; (this assumption is more general of the corresponding assumption used in the previous section);
4. for any $\underline{u}\in\mathbb{R}$, $\left\{x\in\mathbb{R}^n_{++}:u(x)\geq\underline{u}\right\}$ is closed in $\mathbb{R}^n$.

Remark 783 *Assumption 4 means that "indifference curves do not touch the axes".*

The maximization problem for household h is

$$(P1)\qquad \max_{x\in\mathbb{R}^n_{++}}\quad u(x)\qquad s.t.\qquad px-w\leq 0.$$

The budget set of the above problem is clearly not compact. But below, we show that the solution set to $(P1)$ is the same as the solution set to $(P2)$ and $(P3)$ below: indeed, the constraint set of $(P3)$ is compact.

$$(P2) \qquad \max_{x\in\mathbb{R}^n_{++}} \quad u(x) \quad s.t. \quad px - w = 0;$$

$$(P3) \qquad \max_{x\in\mathbb{R}^n_{++}} \quad u(x) \quad s.t. \quad \begin{array}{c} px - w \le 0; \\ u(x) \ge u(e^*), \end{array}$$

where $e^* \in \left\{x \in \mathbb{R}^n_{++} : px \le w\right\}$.

Denote by C_1, C_2 and C_3 the constraint sets of Problem $(P1), (P2)$ and $(P3)$, respectively. Define $M_i = \arg\max(Pi)$ for $i \in \{1,2,3\}$.

Proposition 784 *1.* $M_3 \neq \varnothing$*;*
2. $M_1 = M_2 = M_3$*.*

Proof. 1.
From the Extreme Value Theorem, it suffices to show that C_3 is compact. Indeed,

$$C_{31} := \left\{x \in \mathbb{R}^n_{++} : px - w \le 0\right\}$$

is closed in $\mathbb{R}^n$, because affine functions on $\mathbb{R}^n$ are continuous. Moreover,

$$C_{32} := \left\{x \in \mathbb{R}^n_{++} : u(x) \ge u(e^*)\right\}$$

is closed in $\mathbb{R}^n$, by assumption. Then C is closed in $\mathbb{R}^n$ because intersection of two closed sets in $\mathbb{R}^n$.
C_3 is bounded below by $0 \in \mathbb{R}^n$ and above by $\left(\frac{w}{p_i}\right)_{i=1}^n \in \mathbb{R}^n$.
2.
a. $M_1 = M_3$.
It follows directly from the analysis presented in Subsection 2 in Section 18.4.1.
b. $M_1 = M_2$.
We want to show that

$$x^* \in M_1 \quad \overset{def.}{\Leftrightarrow} \quad x^* \in C_1 \text{ and for any } x \in C_1, \text{ we have } u(x^*) \ge u(x)$$

$$\Rightarrow$$

$$x^* \in M_2 \quad \overset{def.}{\Leftrightarrow} \quad x^* \in C2 \text{ and for any } x \in C_2, \text{ we have } u(x^*) \ge u(x)$$

[$\Rightarrow$]
If $x^* \in M_1$, then $x^* \in C_2$: suppose not; then $x^* \in C_1 \setminus C_2$, i.e., $px^* < w$. But then, by strict monotonicity of u, $x^* \notin M_1$.
Moreover, since $C_2 \subseteq C_1$, then $u(x^*) \ge u(x)$ for any $x \in C_2$, as desired.
[$\Leftarrow$]
If $x^* \in M_2$, then $x^* \in C_2 \subseteq C_1$. Now, suppose that there exists $x \in C_1 \setminus C_2$ such that

$$u(x) > u(x^*). \tag{19.3}$$

Since $px < w$, then $p\left(x + (w - px)\cdot e^1_n\right) = w$, where $e^1_n = (1, 0, ..., 0)$ is the first element in the canonical basis of $\mathbb{R}^n$. Therefore, $\widetilde{x} := x + (w - px)\cdot e^1_n \in C_2$ and, from strict monotonicity of u and from (19.3), we have

$$u(\widetilde{x}) > (x) > u(x^*),$$

contradicting the fact that $x^* \in M_2$. ■

Theorem 785 *Under the above Assumptions,*

$$\xi_h : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \to\to \mathbb{R}^C_{++}, \ (p, w) \mapsto\mapsto \arg\max(P)$$

is indeed a $\mathcal{C}^1$ *function.*

Proof.

Observe that, from it can be easily shown that, ξ is a function.

We want to show that $(P2)$ satisfies necessary and sufficient conditions to Lagrange Theorem, and then apply the Implicit Function Theorem to the First Order Conditions of that problem.
The necessary condition is satisfied because $D_x[px-w]=p\neq 0$;
Define also
$\mu:\mathbb{R}^n_{++}\times\mathbb{R}^{n-1}_{++}\to\mathbb{R}^n_{++}$,
$\mu:(p,w)\mapsto$ Lagrange multiplier for $(P2)$.
The sufficient conditions are satisfied because: from Assumptions (smooth 4), u is differentiably strictly quasi-concave; the constraint is linear; the Lagrange multiplier μ is strictly positive -see below.

The Lagrangian function for problem $(P2)$ and the associated First Order Conditions are described below.

$$\mathcal{L}(x,\mu,p,w)=u(x)+\mu\cdot(-px+w)$$

$$\begin{array}{llll}(FOC) & (1) & Du(x)-\mu p & = & 0\\ & (2) & -px+w & = & 0\end{array}$$

Define

$$F:\mathbb{R}^n_{++}\times\mathbb{R}_{++}\times\mathbb{R}^n_{++}\times\mathbb{R}_{++}\to\mathbb{R}^n\times\mathbb{R},$$

$$F:(x,\mu,p,w)\mapsto\begin{pmatrix}Du(x)-\mu p\\ -px+w\end{pmatrix}.$$

As an application of the Implicit Function Theorem, it is enough to show that $D_{(x,\mu)}F(x,\mu,p,w)$ has full row rank $(n+1)$.
Suppose $D_{(x,\mu)}F$ does not have full rank; then there would exist
$\Delta x\in\mathbb{R}^n$ and $\Delta\mu\in\mathbb{R}$ such that $\Delta:=(\Delta x,\Delta\mu)\neq 0$ and $D_{(x,\mu)}F\cdot(\Delta x,\Delta\mu)=0$, or

$$\begin{bmatrix}D^2u(x) & -p^T\\ -p & 0\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta\mu\end{bmatrix}=0,$$

or

$$\begin{array}{lll}(a) & D^2u(x)\,\Delta x-p^T\Delta\mu & =0\\ (b) & -p\Delta x & =0\end{array}$$

.

The idea of the proof is to contradict Assumption u3.

Claim 1. $\Delta x\neq 0$.
By assumption it must be $\Delta\neq 0$ and therefore, if $\Delta x=0$, $\Delta\mu\neq 0$. Since $p\in\mathbb{R}^n_{++}$, $p^T\Delta\mu\neq 0$. Moreover, if $\Delta x=0$, from (a), we would have $p^T\Delta\mu=0$, a contradiction. .

Claim 2. $Du\cdot\Delta x=0$.
From $(FOC1)$, we have $Du\cdot\Delta x-\mu_h p\cdot\Delta x=0$; using (b) the desired result follows .

Claim 3. $\Delta x^T D^2u\cdot\Delta x=0$.
Premultiplying (a) by Δx^T, we get $\Delta x^T D^2u(x)\,\Delta x-\Delta x^T p^T\Delta\mu=0$. Using (b), the result follows.

Claims 1, 2 and 3 contradict Assumption u3.

■

The above result gives also a way of computing $D_{(p,w)}x(p,w)$, as an application of the Implicit Function Theorem .

Since

	x	μ	p	w
$Du(x)-\mu p$	D^2u	$-p^T$	$-\mu I_n$	0
$-px+w$	$-p$	0	$-x$	1

$$\left[D_{(p,w)}(x,\mu)(p,w)\right]_{(n+1)\times(n+1)}=\begin{bmatrix}D_px & D_wx\\ D_p\mu & D_p\mu\end{bmatrix}=$$

$$=-\begin{bmatrix}D^2u & -p^T\\ -p & 0\end{bmatrix}^{-1}_{(n+1)\times(n+1)}\begin{bmatrix}-\mu I_n & 0\\ -x & 1\end{bmatrix}_{(n+1)\times(n+1)}$$

To compute the inverse of the above matrix, we can use the following fact about the inverse of partitioned matrix (see for example, Goldberger, (1963), page 26)

Let A be an $n \times n$ nonsingular matrix partitioned as

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

where $E_{n_1 \times n_1}$, $F_{n_1 \times n_2}$, $G_{n_2 \times n_1}$, $H_{n_2 \times n_2}$ and $n_1 + n_2 = n$. Suppose that E and $D := H - GE^{-1}F$ are non singular. Then

$$A^{-1} = \begin{bmatrix} E^{-1}\left(I + FD^{-1}GE^{-1}\right) & -E^{-1}FD^{-1} \\ -D^{-1}GE^{-1} & D^{-1} \end{bmatrix}.$$

If we assume that D^2u is negative definite and therefore invertible, we have

$$\begin{bmatrix} D^2u & -p^T \\ -p & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \left(D^2\right)^{-1}\left(I + \delta^{-1}p^Tp\left(D^2\right)^{-1}\right) & \delta^{-1}\left(D^2\right)^{-1}p^T \\ \delta^{-1}p\left(D^2\right)^{-1} & \delta^{-1} \end{bmatrix}$$

where $\delta = -p\left(D^2\right)^{-1}p^T \in \mathbb{R}_{++}$.
And

$$[D_p x(p,w)]_{n \times n} = -\begin{bmatrix} \left(D^2\right)^{-1}\left(I + \delta^{-1}p^Tp\left(D^2\right)^{-1}\right) & \delta^{-1}\left(D^2\right)^{-1}p^T \end{bmatrix} \begin{bmatrix} \mu I_n \\ x \end{bmatrix} =$$
$$= -\mu\left(D^2\right)^{-1}\left(I + \delta^{-1}p^Tp\left(D^2\right)^{-1}\right) - \delta^{-1}\left(D^2\right)^{-1}p^Tx = -\delta^{-1}\left(D^2\right)^{-1}\left[\mu\left(\delta I + p^Tp\left(D^2\right)^{-1}\right) + p^Tx\right]$$

$$[D_w x(p,w)]_{n \times 1} = -\begin{bmatrix} \left(D^2\right)^{-1}\left(I + \delta^{-1}p^Tp\left(D^2\right)^{-1}\right) & \delta^{-1}\left(D^2\right)^{-1}p^T \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \delta^{-1}\left(D^2\right)^{-1}p^T$$

$$[D_p \mu(p,w)]_{1 \times n} = -\begin{bmatrix} \delta^{-1}p\left(D^2\right)^{-1} & \delta^{-1} \end{bmatrix} \begin{bmatrix} \mu I_n \\ x \end{bmatrix} = -\delta^{-1}\left(\mu p\left(D^2\right)^{-1} + x\right).$$

$$[D_w \mu(p,w)]_{1 \times 1} = -\begin{bmatrix} \delta^{-1}p\left(D^2\right)^{-1} & \delta^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \delta^{-1}.$$

.

.

As a simple application of the Envelope Theorem, we also have that, defined the indirect utility function as

$$v : \mathbb{R}^{n+1}_{++} \to \mathbb{R}, \; v : (p,w) \mapsto u(x(p,w)),$$

we have that

$$D_{(p,w)}v(p,w) = \lambda \begin{bmatrix} -x^T & 1 \end{bmatrix}.$$

19.2 Production

Definition 786 *A production vector (or input-output or netput vector) is a vector* $y := (y^c)_{c=1}^C \in \mathbb{R}^C$ *which describes the net outputs of* C *commodities from a production process. Positive numbers denote outputs, negative numbers denote inputs, zero numbers denote commodities neither used nor produced.*

Observe that, given the above definition, py is the profit of the firm.

Definition 787 *The set of all feasible production vectors is called the production set* $Y \subseteq \mathbb{R}^C$*. If* $y \in Y$*, then* y *can be obtained as a result of the production process; if* $y \notin Y$*,that is not the case.*

Definition 788 *The Profit Maximization Problem (PMP) is*

$$\max_y py \quad s.t. \quad y \in Y.$$

It is convenient to describe the production set Y using a function $F : \mathbb{R}^C \to \mathbb{R}$ called *the transformation function*. That is done as follows:

$Y = \{y \in \mathbb{R}^C : F(y) \geq 0\}$.

We list below a smooth version of the assumptions made on Y, using the transformation function.

Some assumption on $F(.)$.

(1) $\exists y \in \mathbb{R}^C$ such that $F(y) \geq 0$.

(2) F is C^2.

(3) (No Free Lunch) If $y \geq 0$, then $F(y) < 0$.

(4) (Possibility of Inaction) $F(0) = 0$.

(5) (F is differentiably strictly decreasing) $\forall y \in \mathbb{R}^C$, $DF(y) \ll 0$

(6) (Irreversibility) If $y \neq 0$ and $F(y) \geq 0$, then $F(-y) < 0$.

(7) (F is differentiably strictly concave) $\forall \Delta \in \mathbb{R}^C \setminus \{0\}$, $\Delta^T D^2 F(y) \Delta < 0$.

Definition 789 *Consider a function $F(.)$ satisfying the above properties and a strictly positive real number N. The Smooth Profit Maximization Problem (SPMP) is*

$$\max_y py \quad s.t. \quad F(y) \geq 0 \text{ and } \|y\| \leq N. \tag{19.4}$$

Remark 790 *For any solution to the above problem it must be the case that $F(y) = 0$. Suppose there exists a solution y' to (SPMP) such that $F(y') > 0$. Since F is continuous, in fact C^2, there exists $\varepsilon > 0$ such that $z \in B(y', \varepsilon) \Rightarrow F(z) > 0$. Take $z' = y' + \frac{\varepsilon \cdot \mathbf{1}}{C}$. Then, $d(y', z') := \left(\sum_{c=1}^C \left(\frac{\varepsilon}{C}\right)^2\right)^{\frac{1}{2}} = \left(C\left(\frac{\varepsilon}{C}\right)^2\right)^{\frac{1}{2}} = \left(\frac{\varepsilon^2}{C}\right)^{\frac{1}{2}} = \frac{\varepsilon}{\sqrt{C}} < \varepsilon$. Therefore $z' \in B(y', \varepsilon)$ and*

$$F(z') > 0 \quad (1).$$

But,

$$pz' = py' + p\frac{\varepsilon \cdot \mathbf{1}}{C} > py' \quad (2).$$

(1) and (2) contradict the fact that y' solves (SPMP).

Proposition 791 *If a solution with $\|y\| < N$ to $(SPMP)$ exists . Then $y : \mathbb{R}^C_{++} \to \mathbb{R}^C$, $p \mapsto \arg\max(19.4)$ is a well defined C^1function.*

Proof.

Let's first show that $y(p)$ is single valued.

Suppose there exist $y, y' \in y(p)$ with $y \neq y'$. Consider $y^\lambda := (1-\lambda)y + \lambda y'$. Since $F(.)$ is strictly concave, it follows that $F(y^\lambda) > (1-\lambda)F(y) + \lambda F(y') \geq 0$, where the last inequality comes from the fact that $y, y' \in y(p)$. But then $F(y^\lambda) > 0$. Then following the same argument as in Remark 790, there exists $\varepsilon > 0$ such that $z' = y^\lambda + \frac{\varepsilon \cdot \mathbf{1}}{C}$ and $F(z') > 0$. But $pz' > py^\lambda = (1-\lambda)py + \lambda py' = py$, contradicting the fact that $y \in y(p)$.

Let's now show that y is C^1

From Remark 790 and from the assumption that $\|y\| < N$, (SPMP) can be rewritten as $\max_y py \; s.t. \; F(y) = 0$. We can then try to apply Lagrange Theorem.

Necessary conditions: $DF(y) \ll 0$;

sufficient conditions: py is linear and therefore pseudo-concave; $F(.)$ is differentiably strictly concave and therefore quasi-concave; the Lagrange multiplier λ is strictly positive -see below.

Therefore, the solutions to $(SPMP)$ are characterized by the following First Order Conditions, i.e., the derivative of the Lagrangian function with respect to y and λ equated to zero:

$$\mathcal{L}(y,p) = py + \lambda F(y). \qquad \begin{matrix} y \\ p + \lambda DF(y) = 0 \end{matrix} \qquad \begin{matrix} p \\ F(y) = 0 \end{matrix}$$

.

Observe that $\lambda = -\frac{p1}{D_{y^1} F(y)} > 0$.

As usual to show differentiability of the choice function we take derivatives of the First Order Conditions.

$$\begin{matrix} & y & \lambda \\ p + \lambda DF(y) = 0 & D^2 F(y) & [DF(y)]^T \\ F(y) = 0 & DF(y) & 0 \end{matrix}$$

We want to show that the above matrix has full rank. By contradiction, assume that there exists $\Delta := (\Delta y, \Delta\lambda) \in \mathbb{R}^C \times \mathbb{R}$, $\Delta \neq 0$ such that

$$\begin{bmatrix} D^2F(y) & [DF(y)]^T \\ DF(y) & 0 \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta\lambda \end{bmatrix} = 0,$$

i.e.,

$$D^2F(y) \cdot \Delta y + [DF(y)]^T \cdot \Delta\lambda = 0 \quad (a),$$

$$DF(y) \cdot \Delta y = 0 \qquad (b).$$

Premultiplying (a) by Δy^T, we get $\Delta y^T \cdot D^2F(y) \cdot \Delta y + \Delta y^T \cdot [DF(y)]^T \cdot \Delta\lambda = 0$. From (b), it follows that $\Delta y^T \cdot D^2F(y) \cdot \Delta y = 0$, contradicting the differentiably strict concavity of $F(.)$.

(3)

From the Envelope Theorem, we know that if $(\overline{y}, \overline{\lambda})$ is the unique pair of solution-multiplier associated with $\overline{p}$, we have that

$D_p\pi(p)_{|\overline{p}} = D_p(py)_{|(\overline{p},\overline{y})} + \overline{\lambda} D_pF(y)_{|(\overline{p},\overline{y})}$.

Since $D_p(py)_{|(\overline{p},\overline{y})} = \overline{y}$, $D_pF(y) = 0$ and, by definition of $\overline{y}$, $\overline{y} = y(\overline{p})$, we get $D_p\pi(p)_{|\overline{p}} = y(\overline{p})$, as desired.

(4)

From (3), we have that $D_py(p) = D_p^2\pi(p)$. Since $\pi(.)$ is convex -see Proposition **??** (2)- the result follows.

(5)

From Proposition **??** (4) and the fact that $y(p)$ is single valued, we know that $\forall \alpha \in \mathbb{R}_{++}$, $y(p) - y(\alpha p) = 0$. Taking derivatives with respect to α, we have $D_py(p)_{|(\alpha p)} \cdot p = 0$. For $\alpha = 1$, the desired result follows.

■

19.3 The demand for insurance

Consider an individual whose wealth is

$$\begin{array}{lll} W - d & \text{with probability} & \pi\text{, and} \\ W & \text{with probability} & 1 - \pi, \end{array}$$

where $W > 0$ and $d > 0$.

Let the function

$$u : A \to \mathbb{R},\ u : c \to u(c)$$

be the individual's Bernoulli function.

Assumption 1. $\forall c \in \mathbb{R}$, $u'(c) > 0$ and $u''(c) < 0$.

Assumption 2. u is bounded above.

An insurance company offers a contract with following features: the potentially insured individual pays a premium p in each state and receives d if the accident occurs. The (potentially insured) individual can buy a quantity $a \in \mathbb{R}$ of the contract. In the case, she pays a premium $(a \cdot p)$ in each state and receives a reimbursement $(a \cdot d)$ if the accident occurs. Therefore, if the individual buys a quantity a of the contract, she get a wealth described as follows

$$\begin{array}{lll} W_1 := W - d - ap + ad & \text{with probability} & \pi\text{, and} \\ W_2 := W - ap & \text{with probability} & 1 - \pi. \end{array} \tag{19.5}$$

Remark 792 *It is reasonable to assume that* $p \in (0, d)$.

Define

$$U:\mathbb{R}\to\mathbb{R},\ \ U:a\mapsto \pi u\left(W-d-ap+ad\right)+\left(1-\pi\right)u\left(W-ap\right).$$

Then the individual solves the following problem. For given, $W\in\mathbb{R}_{++},d\in\mathbb{R}_{++},p\in(0,d)\,,\ \pi\in(0,1)$

$$\max_{a\in\mathbb{R}}\quad U\left(a\right)\quad (M) \tag{19.6}$$

To show existence of a solution, we introduce the problem presented below. For given $W\in\mathbb{R}_{++},d\in\mathbb{R}_{++},p\in(0,d)\,,\ \pi\in(0,1)$

$$\max_{a\in\mathbb{R}} U\left(a\right)\quad s.t.\quad U\left(a\right)\geq U\left(0\right)\quad (M')$$

Defined$A^*:=\arg\max\left(M\right)$ and $A':=\arg\max\left(M'\right)$, the existence of solution to (M), follows from the Proposition below.

Proposition 793 *1. $A'\subset A^*$. 2. $A'\neq\varnothing$.*

Proof.
Exercise
■

To show that the solution is unique, observe that

$$U'\left(a\right)=\pi u'\left(W-d+a\left(d-p\right)\right)\left(d-p\right)+\left(1-\pi\right)u'\left(W-ap\right)\left(-p\right) \tag{19.7}$$

and therefore

$$U''\left(a\right)=\overset{(+)}{\pi}u''(W-d+\overset{(-)}{a}\left(d-p\right))\overset{(+)}{(d-p)^2}+\overset{(+)}{(1-\pi)}\overset{(-)}{u''\left(W-ap\right)}\overset{(+)}{p^2}<0.$$

Summarizing, the unique solution of problem (M) is the unique solution of the equation:

$$U'\left(a\right)=0.$$

Definition 794 $a^*:\mathbb{R}_{++}\times(0,1)\times(0,d)\times\mathbb{R}_{++}\to\mathbb{R}$,
$a^*:(d,\pi,p,W)\mapsto\arg\max\left(M\right)$.
$U^*:\Theta\to\mathbb{R}$,
$U^*:\theta\mapsto\pi u\left(W-d+a^*\left(\theta\right)\left(d-p\right)\right)+\left(1-\pi\right)u\left(W-a^*\left(\theta\right)p\right)$

Proposition 795 *The signs of the derivatives of a^* and U^* with respect to θ are presented in the following table*[1]*:*

		d	π	p	W
a^*		>0 *if* $a^*\in[0,1]$	>0	$\gtreqless 0$	≤ 0 *if* $a^*\leq 1$
U^*		≤ 0 *if* $a^*\in[0,1]$	≤ 0 *if* $a^*\in[0,1]$	≤ 0 *if* $a^*\geq 0$	>0

Proof. Exercise. ■

19.4 Exercises on part IV

See Tito Pietra's file (available on line): Exercises 15.1 → 15.6.

[1] Conditions on $a^*(\theta)$ contained in the table can be expressed in terms of exogenous variables.

Part V

Problem Sets

Chapter 20

Exercises

20.1 Linear Algebra

1.

Show that the set of pair of real numbers is **not** a vector space with respect to the following operations:
(i). $(a,b)+(c,d)=(a+c,b+d)$ and $k(a,b)=(ka,b)$;
(ii) $(a,b)+(c,d)=(a+c,b)$ and $k(a,b)=(ka,kb)$.

2.
Show that W is *not* a vector subspace of $\mathbb{R}^3$ on $\mathbb{R}$ if
(i) $W=\left\{(x,y,z)\in\mathbb{R}^3 : z\geq 0\right\}$;
(ii) $W=\left\{x\in\mathbb{R}^3 : \|x\|\leq 1\right\}$,
(iii) $W=\mathbb{Q}^3$.

3.
Let V be the vector space of all functions $f:\mathbb{R}\to\mathbb{R}$. Show the W is a vector subspace of Vif
(i) $W=\{f\in V : f(1)=0\}$;
(ii) $W=\{f\in V : f(1)=f(2)\}$.

4.

Show that
(i).

$$V=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1+x_2+x_3=0\right\}$$

is a vector subspace of $\mathbb{R}^3$;
(ii).

$$S=\{(1,-1,0),(0,1,-1)\}$$

is a basis for V.

5.
Show the following fact.
Proposition. Let a matrix $A\in\mathbb{M}(n,n)$, with $n\in\mathbb{N}$ be given. The set

$$\mathcal{C}_A := \{B\in\mathbb{M}(n,n) : BA=AB\}$$

is a vector subspace of $\mathbb{M}(n,n)$ (with respect to the field $\mathbb{R}$).

6.
Let U and V be vector subspaces of a vector space W. Show that

$$U+V := \{w\in W : \exists u\in U \text{ and } v\in V \text{ such that } w=u+v\}$$

is a vector subspace of W.

7.
Show that the following set of vectors is linearly independent:

$$\{(1,1,1,),(0,1,1),(0,0,1)\}.$$

8. Using the definition, find the change-of-basis matrix from

$$S=\{u_1=(1,2),\ u_2=(3,5)\}$$

to

$$E=\{e_1=(1,0),\ e_2=(0,1)\}$$

and from E to S. Check the conclusion of Proposition **??**, i.e., that one matrix is the inverse of the other one.

9. Say if the determinant of the following matrix is different from zero.

$$C=\begin{bmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{bmatrix}$$

10. Say for which values of $k\in\mathbb{R}$ the following matrix has rank a. 4, b. 3:

$$A:=\begin{bmatrix} k+1 & 1 & -k & 2 \\ -k & 1 & 2-k & k \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

11.
Show that

$$V=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1-x_2=0\right\}$$

is a vector subspace of $\mathbb{R}^3$ and find a basis for V.

12.
Given

$$l:\mathbb{R}^4\to\mathbb{R}^4,\quad l(x_1,x_2,x_3,x_4)=\begin{pmatrix} x_1, & x_1+x_2, & x_1+x_2+x_3, & x_1+x_2+x_3+x_4 \end{pmatrix}$$

show it is linear, compute the associated matrix with respect to canonical bases, and compute $\ker l$ and $\mathrm{Im} l$.

13.
Complete the text below.
Proposition. Assume that $l\in L(V,U)$ and $\ker l=\{0\}$. Then,

$$\forall u\in \mathrm{Im}\, l,\quad \text{there exists a unique } v\in V \text{ such that } l(v)=u.$$

Proof.
Since ...., by definition, there exists $v\in V$ such that

$$l(v)=u. \tag{20.1}$$

Take $v'\in V$ such that $l(v')=u$. We want to show that

$$\text{........................} \tag{20.2}$$

Observe that

$$l(v) - l(v') \overset{(a)}{=} \text{.........................} \tag{20.3}$$

where (a) follows from ....
Moreover,

$$l(v) - l(v') \overset{(b)}{=} \text{.........................}, \tag{20.4}$$

where (b) follows from ....
Therefore,

$$l(v - v') = 0,$$

and, by definition of $\ker l$,

$$\text{.........................} \tag{20.5}$$

Since, ..., from (20.5), it follows that

$$v - v' = 0.$$

14.
Let the following sets be given:

$$V = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - x_2 + x_3 - x_4 = 0\right\}$$

and

$$W = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\right\}$$

If possible, find a basis of $V \cap W$.

15.
Say if the following statement is true or false.
Let V and U be vector spaces on $\mathbb{R}$, W a vector subspace of U and $l \in \mathcal{L}(V, U)$. Then $l^{-1}(W)$ is a vector subspace of V.

16.
Let the following full rank matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

be given. Say for which values of $k \in \mathbb{R}$, the following linear system has solutions.

$$\begin{bmatrix} 1 & a_{11} & a_{12} & 0 & 0 & 0 \\ 2 & a_{21} & a_{22} & 0 & 0 & 0 \\ 3 & 5 & 6 & b_{11} & b_{12} & 0 \\ 4 & 7 & 8 & b_{21} & b_{22} & 0 \\ 1 & a_{11} & a_{12} & 0 & 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} k \\ 1 \\ 2 \\ 3 \\ k \end{bmatrix}$$

17.
Consider the following Proposition contained in Section 8.1 in the class Notes:
Proposition .$\forall v \in V$,

$$[l]_{\mathcal{V}}^{\mathcal{U}} \cdot [v]_{\mathcal{V}} = [l(v)]_{\mathcal{U}} \tag{20.6}$$

Verify the above equality in the case in which
a.

$$l : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

b. the basis $\mathcal{V}$ of the domain of l is

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

c. the basis $\mathcal{U}$ of the codomain of l is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\},$$

d.

$$v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

18.

Complete the following proof.
Proposition. Let
$n, m \in \mathbb{N}$ such that $m > n$, and
a vector subspace L of $\mathbb{R}^m$ such that $\dim L = n$
be given. Then, there exists $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that $\operatorname{Im} l = L$.
Proof. Let $\{v^i\}_{i=1}^n$ be a basis of $L \subseteq \mathbb{R}^m$. Take $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\forall i \in \{1, ..., n\}, \qquad l_2(e_n^i) = v^i,$$

where e_n^i is the i–th element in the canonical basis in $\mathbb{R}^n$. Such function does exists and, in fact, it is unique as a consequence of a Proposition in the Class Notes that we copy below:
....
Then, from the Dimension theorem

$$\dim \operatorname{Im} l = \text{...}$$

Moreover,

$$L = \text{..........} \{v^i\}_{i=1}^n \subseteq \text{..............}$$

Summarizing,

$$L \subseteq \operatorname{Im}\ l\ ,\ \dim L = n \text{ and } \dim \operatorname{Im} l \leq n,$$

and therefore

$$\dim \operatorname{Im} l = n.$$

Finally, from Proposition ....in the class Notes since $L \subseteq \operatorname{Im}\ l$, $\dim L = n$ and $\dim \operatorname{Im} l = n$, we have that $\operatorname{Im} l = L$, as desired.
Proposition .... in the class Notes says what follows:
....

19.

Say for which value of the parameter $a \in \mathbb{R}$ the following system has one, infinite or no solutions

$$\begin{cases} ax_1 + x_2 = 1 \\ x_1 + x_2 = a \\ 2x_1 + x_2 = 3a \\ 3x_1 + 2x_2 = a \end{cases}$$

20.

Say for which values of k,the system below admits one, none or infinite solutions.

$$A(k) \cdot x = b(k)$$

where $k \in \mathbb{R}$, and

$$A(k) \equiv \begin{bmatrix} 1 & 0 \\ 1-k & 2-k \\ 1 & k \\ 1 & k-1 \end{bmatrix}, \qquad b(k) \equiv \begin{bmatrix} k-1 \\ k \\ 1 \\ 0 \end{bmatrix}.$$

21.

Let $\mathcal{V} = \left\{v^1, v^2, ..., v^n\right\}$ be a set of vectors in $\mathbb{R}^n$ such that for any $i, j \in \{1, ..., n\}$,

$$v^i \cdot v^j = \begin{cases} 0 & if \quad i \neq j \\ \\ 1 & if \quad i = j. \end{cases} \tag{20.7}$$

Show that $\mathcal{V}$ is a basis of $\mathbb{R}^n$.

22.

Given a vector space V on a field F, sets $A, B \subseteq V$ and $k \in F$, we define

$$A + B := \{v \in V : \text{ there exist } a \in A \text{ and } b \in B \text{ such that } v = a + b\},$$

$$kA := \{v \in V : \text{ there exist } a \in A \text{ such that } v = ka\}.$$

Given a vector space V on a field F, a linear function $T \in \mathcal{L}(V, V)$ and W vector subspace of V, W is said to be T-invariant if

$$T(W) \subseteq W.$$

Let W be both S-invariant and T-invariant and let $k \in F$. Show that
a. W is $S + T$-invariant;
b. W is $S \circ T$-invariant;
c. W is kT-invariant.

23.

Show that the set of all 2×2 symmetric real matrices is a vector subspace of $\mathbb{M}(2, 2)$ and compute its dimension.

24.

Let V be a vector space on a field F and W a vector subspace of V. Show that
a. $W + W = W$, and
b. for any $\alpha \in F \setminus \{0\}$, $\alpha W = W$.

25.

Let $\mathcal{P}_n(\mathbb{R})$ be the set polynomials of degree smaller or equal than $n \in \mathbb{N}_+$ on the set of real numbers , i.e.,

$$\mathcal{P}_n(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \text{ such that } \exists a_0, a_1, ..., a_n \in \mathbb{R} \text{ such that for any } t \in \mathbb{R}, f(t) = \sum_{i=0}^{n} a_i t^i \right\}.$$

Show that $\mathcal{P}_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^{n+1}$.

20.2 Some topology in metric spaces

20.2.1 Basic topology in metric spaces

1.

Do Exercise 341: Let d be a metric on a non-empty set X. Show that

$$d' : X \times X \to \mathbb{R}, d(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

is a metric on X.

2.

Let X be the set of continuous real valued functions with domain $[0,1] \subseteq \mathbb{R}$ and

$$d(f,g) = \int_0^1 f(x) - g(x)\, dx,$$

where the integral is the Riemann Integral (that one you learned in Calculus 1). Show that (X,d) is not a metric space.

3.

Do Exercise 358 for $n = 2$: $\forall n \in \mathbb{N}, \forall i \in \{1,...,n\}, \forall a_i, b_i \in \mathbb{R}$ with $a_i < b_i$,

$$\times_{i=1}^{n} (a_i, b_i)$$

is $(\mathbb{R}^n, d_2)$ open.

4.

Show the second equality in Remark 366:

$$\cap_{n=1}^{+\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

5.

Say if the following set is $(\mathbb{R}, d_2)$ open or closed:

$$S := \left\{x \in \mathbb{R} : \exists\ n \in \mathbb{N} \text{ such that } x = (-1)^n\ \frac{1}{n}\right\}$$

6.

Say if the following set is $(\mathbb{R}, d_2)$ open or closed:

$$A := \cup_{n=1}^{+\infty} \left(\frac{1}{n}, 10 - \frac{1}{n}\right).$$

7.

Do Exercise 376: show that $\mathcal{F}(S) = \mathcal{F}\left(S^C\right)$.

8.

Do Exercise 377: show that $\mathcal{F}(S)$ is a closed set.

9.

Let the metric space $(\mathbb{R}, d_2)$ be given. Find Int S, Cl $(S), \mathcal{F}(S), D(S), Is(S)$ and say if S is open or closed for $S = \mathbb{Q}$, $S = (0,1)$ and $S = \left\{x \in \mathbb{R} : \exists n \in \mathbb{N}_+ \text{ such that } x = \frac{1}{n}\right\}$.

10.

Show that the following statements are **false**:

a. Cl (Int S) $= S$,
b. Int Cl $(S) = S$.
11.
Given $S \subseteq \mathbb{R}$, say if the following statements are true or false.
a. S is an open bounded interval $\Rightarrow S$ is an open set;
b. S is an open set $\Rightarrow S$ is an open bounded interval;
c. $x \in \mathcal{F}(S) \Rightarrow x \in D(S)$;
d. $x \in D(S) \Rightarrow x \in \mathcal{F}(S)$.

12.
Using the definition of convergent sequences, show that the following sequences do converge:
a. $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ such that $\forall n \in \mathbb{N}$, $x_n = 1$;
b. $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\infty$ such that $\forall n \in \mathbb{N}$, $x_n = \frac{1}{n}$.

13.
Using Proposition 403, show that $[0,1]$ is $(\mathbb{R}, d_2)$ closed.

14.
Show the following result: A subset of a discrete space, i.e., a metric space with the discrete metric, is compact if and only if it is finite.

15.
Say if the following statement is true: An open set is not compact.

16.
Using the definition of compactness, show the following statement: Any open ball in $(\mathbb{R}^2, d_2)$ is not compact.

17.
Show that $f(A \cup B) = f(A) \cup f(B)$.

18.
Show that $f(A \cap B) \neq f(A) \cap (B)$.

19.
Using the characterization of continuous functions in terms of open sets, show that for any metric space (X, d) the constant function is continuous.

20.
a. Say if the following sets are $(\mathbb{R}^n, d_2)$ compact:
i.

$$\mathbb{R}^n_+,$$

ii.

$$\forall x \in \mathbb{R}^n \text{ and} \forall r \in \mathbb{R}_{++},, \; Cl \; B(x,r).$$

b. Say if the following set is $(\mathbb{R}, d_2)$ compact:

$$\left\{ x \in \mathbb{R} : \exists n \in \mathbb{N} \text{ such that } x = \frac{1}{n} \right\}.$$

21.
Given the continuous functions

$$g : \mathbb{R}^n \to \mathbb{R}^m$$

show that the following set is closed

$$\{x \in \mathbb{R}^n : g(x) \geq 0\}$$

22.

Assume that $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous. Say if

$$X = \{x \in \mathbb{R}^m : f(x) = 0\}$$

is (a) closed, (b) is compact.

23.

Using the characterization of continuous functions in terms of open sets, show that the following function is not continuous

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$$

24.

Using the Extreme Value Theorem, say if the following maximization problems have solutions (with $\| \ \|$ being the Euclidean norm).

$$\max_{x \in \mathbb{R}^n} \sum_{i=1}^n x_i \quad s.t. \quad \|x\| \leq 1$$

$$\max_{x \in \mathbb{R}^n} \sum_{i=1}^n x_i \quad s.t. \quad \|x\| < 1$$

$$\max_{x \in \mathbb{R}^n} \sum_{i=1}^n x_i \quad s.t. \quad \|x\| \geq 1$$

25.

Let $(E, \|\|_E)$, $(F, \|\|_F)$ be normed vector spaces. A function $f : (E, \|\|_E)$, $(F, \|\|_F)$ is bounded if

$$\exists M \in \mathbb{R}_{++} \text{ such that } \forall x \in E \quad \|f(x)\|_F \leq M.$$

Show that given a linear function $l : E \to F$,

$$l \text{ is bounded} \Leftrightarrow l = 0.$$

26.

$f : (X, d) \to \mathbb{R}$ is upper semicontinuous at $x_0 \in X$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d(x - x_0) < \delta \Rightarrow f(x) < f(x_0) + \varepsilon.$$

f is upper semicontinuous if is upper semicontinuous at any $x_0 \in X$.

Show that the following statements are equivalent:

a. f is upper semicontinuous;

b. for any $\alpha \in \mathbb{R}$, $\{x \in X : f(x) < \alpha\}$ is (X, d) open;

c. for any $\alpha \in \mathbb{R}$, $\{x \in X : f(x) \geq \alpha\}$ is (X, d) closed.

27.

Let A be a subset of $(\mathbb{R}^n, d)$ where d is the Euclidean distance. Show that if A is $(\mathbb{R}^n, d)$ open, then for any $x \in X$, $\{x\} + A$ is $(\mathbb{R}^n, d)$ open.

Hint: use the fact that for any $x, y \in \mathbb{R}^n$, $d(x, y) = \|x - y\|$ and therefore, for any $a \in \mathbb{R}^n$, $d(a + x, a + y) = \|a + x - a - y\| = \|x - y\| = d(x, y)$.

28.

Let (X, d) be a metric space. Show that if K_1 and K_2 are compact subsets of X, then $K_1 + K_2$ is compact.

29.

Given two metric spaces (E, d_1) and (F, d_2), a function $f : E \to F$ is an isometry with respect to d_1 and d_2 if $\forall x_1, x_2 \in E$,

$$d_2(f(x_1), f(x_2)) = d_1(x_1, x_2).$$

Show that if $f : E \to F$ is an isometry then

a. f is one-to-one;

b. $\hat{f} : E \to f(E)$ is invertible;

c. f is continuous.

20.2.2 Correspondences

To solve the following exercises on correspondences, we need some preliminary definitions.[1]

A set $C \subseteq \mathbb{R}^n$ is convex if $\forall x_1,\, x_2 \in C$ and $\forall \lambda \in [0,1]$, $(1-\lambda)\, x_1 + \lambda x_2 \in C$.

A set $C \subseteq \mathbb{R}^n$ is strictly convex if $\forall x_1,\, x_2 \in C$ and $\forall \lambda \in (0,1)$, $(1-\lambda)\, x_1 + \lambda x_2 \in \operatorname{Int} C$.

Consider an open and convex set $X \subseteq \mathbb{R}^n$ and a continuous function $f : X \to \mathbb{R}$, f is quasi-concave iff $\forall$ $x', x'' \in X$, $\forall\, \lambda \in [0,1]$,

$$f((1-\lambda)x' + \lambda x'') \geq \min \{f(x'), f(x'')\}.$$

f is strictly quasi-concave

Definition 796 *iff $\forall\ x', x'' \in X$, such that $x' \neq x''$, and $\forall\ \lambda \in (0,1)$, we have that*

$$f((1-\lambda)x' + \lambda x'') > \min \{f(x'), f(x'')\}.$$

We define the budget correspondence as

Definition 797

$$\beta : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \to\to \mathbb{R}^C,\ \beta(p,w) = \left\{x \in \mathbb{R}^C_+ : px \leq w\right\}.$$

The Utility Maximization Problem (UMP) is

Definition 798

$$\max\nolimits_{x \in \mathbb{R}^C_+} \quad u(x) \qquad s.t. \quad px \leq w,\ \ or\ x \in \beta(p,w)$$

$\xi : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \to\to \mathbb{R}^C,\ \xi(p,w) = \arg\max(UMP)$ *is the demand correspondence.*

The Profit Maximization Problem (PMP) is

$$\max_y py \quad s.t. \quad y \in Y.$$

Definition 799 *The supply correspondence is*

$$y : \mathbb{R}^C_{++} \to\to \mathbb{R}^C,\ y(p) = \arg\max(PMP).$$

We can now solve some exercises. (`the numbering has to be changed`)

1.

Show that ξ is non-empty valued.

2.

Show that for every $(p,w) \in \mathbb{R}^C_{++} \times \mathbb{R}_{++}$,

(a) if u is quasiconcave, ξ is convex valued;

(b) if u is strictly quasiconcave, ξ is single valued, i.e., it is a function.

3.

Show that β is closed.

4.

If a solution to (PMP) exists, show the following properties hold.

(a) If Y is convex, $y(.)$ is convex valued;

(b) If Y is strictly convex (i.e., $\forall \lambda \in (0,1)$, $y^\lambda := (1-\lambda)\, y' + \lambda y'' \in \operatorname{Int} Y$), $y(.)$ is single valued.

5

[1] The definition of quasi-concavity and strict quasi-concavity will be studied in detail in Chapter .

Consider $\phi_1, \phi_2 : [0,2] \rightarrow\rightarrow \mathbb{R}$,

$$\phi_1(x) = \begin{cases} \left[-1 + 0.25 \cdot x, x^2 - 1\right] & if \quad x \in [0,1) \\ [-1,1] & if \quad x = 1 \\ \left[-1 + 0.25 \cdot x, x^2 - 1\right] & if \quad x \in (1,2] \end{cases},$$

and

$$\phi_2(x) = \begin{cases} \left[-1 + 0.25 \cdot x, x^2 - 1\right] & if \quad x \in [0,1) \\ [-0.75, -0.25] & if \quad x = 1 \\ \left[-1 + 0.25 \cdot x, x^2 - 1\right] & if \quad x \in (1,2] \end{cases}.$$

Say if ϕ_1 and ϕ_2 are LHC, UHC, closed, convex valued, compact valued.

6.

Consider $\phi : \mathbb{R}_+ \rightarrow\rightarrow \mathbb{R}$,

$$\phi(x) = \begin{cases} \left\{\sin \frac{1}{x}\right\} & if \quad x > 0 \\ [-1,1] & if \quad x = 0 \end{cases},$$

Say if ϕ is LHC, UHC, closed.

7.

Consider $\phi : [0,1] \rightarrow\rightarrow [-1,1]$

$$\phi(x) = \begin{cases} [0,1] & if \quad x \in \mathbb{Q} \cap [0,1] \\ [-1,0] & if \quad x \in [0,1] \setminus \mathbb{Q} \end{cases}.$$

Say if ϕ is LHC, UHC, closed.

8.

Consider $\phi_1, \phi_2 : [0,3] \rightarrow\rightarrow \mathbb{R}$,

$$\phi_1(x) = \left[x^2 - 2, x^2\right],$$

and

$$\phi_2(x) = \left[x^2 - 3, x^2 - 1\right],$$

$$\phi_3(x) := (\phi_1 \cap \phi_2)(x) := \phi_1(x) \cap \phi_2(x).$$

Say if ϕ_1, ϕ_2 and ϕ_3 are LHC, UHC, closed.

20.3 Differential Calculus in Euclidean Spaces

1 .

Using the definition, compute the partial derivative of the following function in an arbitrary point (x_0, y_0) :

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = 2x^2 - xy + y^2.$$

2 .

If possible, compute partial derivatives of the following functions.

a. $f(x,y) = x \cdot \arctan \frac{y}{x}$;

b. $f(x,y) = x^y$;

c. $f(x,y) = (\sin(x+y))^{\sqrt{x+y}}$ in $(0,3)$

3,

Given the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & if \quad x^2 + y^2 \neq 0 \\ \\ 0 & \text{otherwise} \end{cases},$$

show that it admits both partial derivatives in $(0,0)$ and it is not continuous in $(0,0)$.

4 .
Using the definition, compute the directional derivative $f'((1,1);(\alpha_1,\alpha_2))$ with $\alpha_1,\alpha_2 \neq 0$ for $f:\mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \frac{x+y}{x^2+y^2+1}.$$

5 .
Let the following function be given.

$$f:\mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \begin{cases} \frac{x^2y^2}{x^3+y^3} & \text{if} \quad x \neq 0 \\ y & \text{if} \quad x = 0. \end{cases}$$

Using the definition of a directional derivative, compute, if possible $f'(0;u)$ for every $u \in \mathbb{R}^2$.

6 .
Using the definition, show that the following functions are differentiable.
a. $l \in L(\mathbb{R}^n,\mathbb{R}^m)$;
b. the projection function $f:\mathbb{R}^n \to \mathbb{R}$, $f:\left(x^i\right)_{i=1}^n \mapsto x^1$.
7.
Show the following result which was used in the proof of Proposition 620. A linear function $l:\mathbb{R}^n \to \mathbb{R}^m$ is continuous.

8 .
Compute the Jacobian matrix of $f:\mathbb{R}^2 \to \mathbb{R}^3$,

$$f(x,y) = (\sin x \cos y, \qquad \sin x \sin y, \qquad \cos x \cos y)$$

9 .
Given differentiable functions $g,h:\mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}\backslash\{0\}$, compute the Jacobian matrix of $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = \left(g(x)\cdot h(z), \quad \frac{g(h(x))}{y}, \quad e^{x\cdot g(h(x))}\right)$

10 .
Compute total derivative and directional derivative at x_0 in the direction u.
a.

$$f:\mathbb{R}^3_{++} \to \mathbb{R}, \qquad f(x_1,x_2,x_3) = \frac{1}{3}\log x_1 + \frac{1}{6}\log x_2 + \frac{1}{2}\log x_3$$

$x_0 = (1,1,2)$, $u = \frac{1}{\sqrt{3}}(1,1,1)$;
b.

$$f:\mathbb{R}^3 \to \mathbb{R}, \qquad f(x_1,x_2,x_3) = x_1^2 + 2x_2^2 - x_3^2 - 2x_1x_2 - 6x_2x_3$$

$x_0 = (1,0,-1)$, $u = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$;
c.

$$f:\mathbb{R}^2 \to \mathbb{R}, \qquad f(x_1,x_2) = x_1 \cdot e^{x_1x_2}$$

$x_0 = (0,0)$, $u = (2,3)$.

11 .
Given

$$f(x,y,z) = \left(x^2+y^2+z^2\right)^{\frac{-1}{2}},$$

show that if $(x,y,z) \neq 0$, then

$$\frac{\partial^2 f(x,y,z)}{\partial x^2} + \frac{\partial^2 f(x,y,z)}{\partial y^2} + \frac{\partial^2 f(x,y,z)}{\partial z^2} = 0$$

12 .
Given the C^2 functions $g, h : \mathbb{R} \to \mathbb{R}_{++}$, compute the Jacobian matrix of

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \qquad f(x,y,z) = \left(\tfrac{g(x)}{h(z)}, \qquad g(h(x)) + xy, \qquad \ln(g(x) + h(x)) \right)$$

13 .
Given the functions

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(x,y) = \begin{pmatrix} e^x + y \\ e^y + x \end{pmatrix}$$

$$g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto g(x)$$

$$h : \mathbb{R} \to \mathbb{R}^2, \qquad h(x) = f(x, g(x))$$

Assume that g is C^2. a. compute the differential of h in 0; b. check the conclusion of the Chain Rule.

14 .
Let the following differentiable functions be given.

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad (x_1,x_2,x_3) \mapsto f(x_1,x_2,x_3)$$

$$g : \mathbb{R}^3 \to \mathbb{R} \qquad (x_1,x_2,x_3) \mapsto g(x_1,x_2,x_3)$$

$$a : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x_1,x_2,x_3) \mapsto \begin{pmatrix} f(x_1,x_2,x_3) \\ g(x_1,x_2,x_3) \\ x_1 \end{pmatrix}$$

$$b : \mathbb{R}^3 \to \mathbb{R}^2, \qquad (y_1,y_2,y_3) \mapsto \begin{pmatrix} g(y_1,y_2,y_3) \\ f(y_1,y_2,y_3) \end{pmatrix}$$

Compute the directional derivative of the function $b \circ a$ in the point $(0,0,0)$ in the direction $(1,1,1)$.

15 .
Using the theorems of Chapter 16, show that the function in (**??**) is differentiable.

16 .
Given

$$f : \mathbb{R}^3 \to \mathbb{R}^1, \qquad (x,y,z) \mapsto z + x + y^3 + 2x^2y^2 + 3xyz + z^3 - 9,$$

say if you can apply the Implicit Function Theorem to the function in $(x_0,y_0,z_0) = (1,1,1)$ and, if possible, compute $\frac{\partial x}{\partial z}$ and $\frac{\partial y}{\partial z}$ in $(1,1,1)$.

17 .
Using the notation of the statement of the Implicit Function Theorem presented in the Class Notes, say if that Theorem can be applied to the cases described below; if it can be applied, compute the Jacobian of g. (Assume that a solution to the system $f(x,t) = 0$ does exist).

a. $f : \mathbb{R}^4 \to \mathbb{R}^2$,

$$f(x_1,x_2,t_1,t_2) \mapsto \begin{pmatrix} x_1^2 - x_2^2 + 2t_1 + 3t_2 \\ x_1x_2 + t_1 - t_2 \end{pmatrix}$$

b. $f : \mathbb{R}^4 \to \mathbb{R}^2$,

$$f(x_1,x_2,t_1,t_2) \mapsto \begin{pmatrix} 2x_1x_2 + t_1 + t_2^2 \\ x_1^2 + x_2^2 + t_1^2 - 2t_1t_2 + t_2^2 \end{pmatrix}$$

c. $f : \mathbb{R}^4 \to \mathbb{R}^2$,

$$f(x_1, x_2, t_1, t_2) \mapsto \begin{pmatrix} t_1^2 - t_2^2 + 2x_1 + 3x_2 \\ t_1 t_2 + x_1 - x_2 \end{pmatrix}$$

18.
Say under which conditions, if $z^3 - xz - y = 0$, then

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{3z^2 + x}{(3z^2 - x)^3}$$

19. Do Exercise 685: Let the utility function $u : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$, $(x, y) \mapsto u(x, y)$ be given. Assume that it satisfies the following properties i. u is C^2, ii. $\forall (x, y) \in \mathbb{R}^2_{++}$, $Du(x, y) >> 0$,iii. $\forall (x, y) \in \mathbb{R}^2_{++}$, $D_{xx}u(x, y) < 0, D_{yy}u(x, y) < 0, D_{xy}u(x, y) > 0$. Compute the Marginal Rate of Substitution in (x_0, y_0) and say if the graph of each indifference curve is concave.

20.
Let the function $f, g : \mathbb{R}^n \to \mathbb{R}$ be given and assume that

for every $x_0 \in \mathbb{R}^n$ and every $u \in \mathbb{R}^n$, the directional derivatives $f'(x_0; u)$ and $g'(x_0; u)$ do exist.

Define $h : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x) \cdot g(x)$. If possible, compute $h'(x_0; u)$ for every $x_0 \in \mathbb{R}^n$ and every $u \in \mathbb{R}^n$.

21.
A function $f : \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto f(x)$ is homogenous of degree $n \in \mathbb{N}_+$ if

$$\text{for every } x = (x_1, x_2) \in \mathbb{R}^2 \text{ and every } a \in \mathbb{R}_+,\ f(ax_1, ax_2) = a^n f(x_1, x_2).$$

Show that if f is homogenous of degree n and $f \in C^1(\mathbb{R}^2, \mathbb{R})$, then

for every $x = (x_1, x_2) \in \mathbb{R}^2$, $\quad x_1 \cdot D_{x_1} f(x_1, x_2) + x_2 \cdot D_{x_2} f(x_1, x_2) = nf(x_1, x_2).$

20.4 Nonlinear Programming

1. [2] Determine, if possible, the nonnegative parameter values for which the following functions $f : X \to \mathbb{R}$, $f : (x_i)_{i=1}^n := x \mapsto f(x)$ are concave, pseudo-concave, quasi-concave, strictly concave.

(a) $X = \mathbb{R}_{++}$, $f(x) = \alpha x^\beta$;

(b) $X = \mathbb{R}^n_{++}$, $n \geq 2$, $f(x) = \sum_{i=1}^n \alpha_i (x_i)^{\beta_i}$ (for pseudo-concavity and quasi-concavity consider only the case $n = 2$).

(c) $X = \mathbb{R}$, $f(x) = \min\{\alpha, \beta x - \gamma\}$.

2.
a. Discuss the following problem. For given $\pi \in (0, 1)$, $a \in (0, +\infty)$,

$$\max_{(x_1, x_2)} \pi \cdot u(x) + (1 - \pi) u(y) \quad \text{s.t.} \quad \begin{array}{l} y \leq a - \frac{1}{2}x \\ y \leq 2a - 2x \\ x \geq 0 \\ y \geq 0 \end{array}$$

where $u : \mathbb{R} \to \mathbb{R}$ is a C^2 function such that $\forall z \in \mathbb{R}$, $u'(z) > 0$ and $u''(z) < 0$.

b. Say if there exist values of (π, a) such that $(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{2}{3}a, \frac{2}{3}a, \lambda_1, 0, 0, 0\right)$, with $\lambda_1 > 0$, is a solution to Kuhn-Tucker conditions, where for $j \in \{1, 2, 3, 4\}$, λ_j the multiplier associated with constraint j.

c. "Assuming" that the first, third and fourth constraint hold with a strict inequality, and the multiplier associated with the second constraint is strictly positive, describe in detail how to compute the effect of a change of a or π on a solution of the problem.

3.

[2] Exercise 1 is taken from David Cass' problem sets for his Microeconomics course at the University of Pennsylvania.

a. Discuss the following problem. For given $\pi \in (0,1), w_1, w_2 \in \mathbb{R}_{++}$,

$$\max_{(x,y,m)\in\mathbb{R}^2_{++}\times\mathbb{R}} \quad \pi \log x + (1-\pi)\log y \qquad s.t \quad \begin{array}{l} w_1 - m - x \geq 0 \\ w_2 + m - y \geq 0 \end{array}$$

b. Compute the effect of a change of w_1 on the component x^* of the solution.
c. Compute the effect of a change of π on the objective function computed at the solution of the problem.

4.
a. Discuss the following problem.

$$\min_{(x,y)\in\mathbb{R}^2} \quad x^2 + y^2 - 4x - 6y \qquad s.t. \quad \begin{array}{l} x + y \leq 6 \\ y \leq 2 \\ x \geq 0 \\ y \geq 0 \end{array}$$

Let (x^*, y^*) be a solution to the the problem.
b. Can it be $x^* = 0$?
c. Can it be $(x^*, y^*) = (2,2)$?.

5.
Characterize the solutions to the following problems.
(a) (consumption-investment)

$$\max_{(c_1,c_2,k)\in\mathbb{R}^3} \quad \begin{array}{l} u(c_1) + \delta u(c_2) \\ s.t. \\ c_1 + k \leq e \\ c_2 \leq f(k) \\ c_1, c_2, k \geq 0, \end{array}$$

where $u : \mathbb{R} \to \mathbb{R},\ u' > 0, u'' < 0;\ f : \mathbb{R}_+ \to \mathbb{R},\ f' > 0, f'' < 0$ and such that $f(0) = 0;\ \delta \in (0,1), e \in \mathbb{R}_{++}$. After having written Kuhn Tucker conditions, consider just the case in which $c_1, c_2, k > 0$.
(b) (labor-leisure)

$$\max_{(x,l)\in\mathbb{R}^2} \quad \begin{array}{l} u(x,l) \\ s.t. \\ px + wl \leq w\bar{l} \\ l \leq \bar{l} \\ x, l \geq 0, \end{array}$$

where $u : \mathbb{R}^2$ is C^2, $\forall (x,l)\ Du(x,l) \gg 0, u$ is differentiably strictly quasi-concave, i.e.,$\forall (x,l)$, if $\Delta \neq 0$ and $Du(x,l)\cdot\Delta = 0$, then $\Delta^T D^2 u\ \Delta < 0;\ p > 0,\ w > 0$ and $\bar{l} > 0$.
Describe solutions for which $x > 0$ and $0 < l < \bar{l}$,

6.
(a) Consider the model described in Exercise 6. (a). What would be the effect on consumption (c_1, c_2) of an increase in initial endowment e?

What would be the effect on (the value of the objective function computed at the solution of the problem) of an increase in initial endowment e?

Assume that $f(k) = ak^\alpha$, with $a \in \mathbb{R}_{++}$ and $\alpha \in (0,1)$. What would be the effect on consumption (c_1, c_2) of an increase in a?

(b) Consider the model described in Exercise 6. (b).What would be the effect on leisure l of an increase in the wage rate w? in the price level p?

What would be the effect on (the value of the objective function computed at the solution of the problem) of an increase in the wage rate w? in the price level p?.

7.
Show that if $f : \mathbb{R}^2 \to \mathbb{R},\ x \mapsto f(x)$ is homogenous of degree 1, then

$$f \text{ is concave} \Leftrightarrow \text{ for any } x, y \in \mathbb{R}^2,\ f(x+y) \geq f(x) + f(y).$$

Chapter 21

Solutions

21.1 Linear Algebra

1.

We want to prove that the following sets are **not** vector spaces. Thus, it is enough to find a counter-example violating one of the conditions defining vector spaces.

(i) The definition violates the so-called M2 distributive assumption since for $k = 1$ and $a = b = 1$,

$$(1+1)\cdot(1,1) = 2\cdot(1,1) = (2,1) : \text{while} : 1\cdot(1,1) + 1\cdot(1,1) = (2,2)$$

(ii) The definition violates the so-called A4 commutative property since for $a = b = c = 1$ and $d = 0$,

$$(1,1) + (1,0) = (2,1) \neq (1,0) + (1,1) = (2,0)$$

2.

(i) Take any $w := (x, y, z) \in W$ with $z > 0$, and $\alpha \in \mathbb{R}$ with $\alpha < 0$; then $\alpha w = (\alpha x, \alpha y, \alpha z)$ with $\alpha z < 0$ and therefore $w \notin W$.

(ii) Take any nonzero $w \in W$ and define $\alpha = 2/||w||$. Observe that $||\alpha w|| = 2 > 1$ and therefore $\alpha w \notin W$.

(iii) Multiplication of any nonzero element of $\mathbb{Q}^3$ by $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ will give an element of $\mathbb{R}^3 \setminus \mathbb{Q}^3$ instead of $\mathbb{Q}^3$.

3.

We use Proposition 139. Therefore, we have to check that

a. $0 \in W$; b. $\forall u, v \in W, \forall \alpha, \beta \in F,\ \alpha u + \beta v \in W$.

Define simply by 0 the function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R},\ f(x) = 0$.

(i) a. Since $0(1) = 0$, $0 \in W$.

b.

$$\alpha u(1) + \beta v(1) = \alpha \cdot 0 + \beta \cdot 0 = 0,$$

where the first equality follows from the assumption that $u, v \in W$. Then, indeed, $\alpha u + \beta v \in W$.

(ii) a. Since $0(1) = 0 = 0(2)$, we have that $0 \in W$.

b.

$$\alpha u(1) + \beta v(1) = \alpha u(2) + \beta v(2),$$

where the equality follows from the assumption that $u, v \in W$ and therefore $u(1) = u(2)$ and $v(1) = v(2)$.

4.

Again we use Proposition 139 and we have to check that

a. $0 \in W$; b. $\forall u, v \in W, \forall \alpha, \beta \in F,\ \alpha u + \beta v \in W$.

a. $(0,0,0) \in W$ simply because $0 + 0 + 0 = 0$.

b. Given $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in V$, i.e., such that

$$u_1 + u_2 + u_3 = 0 \quad \text{and} \quad v_1 + v_2 + v_3 = 0, \tag{21.1}$$

we have

$$\alpha u+\beta v=(\alpha u_1+\beta v_1,\alpha u_2+\beta v_2,\alpha u_3+\beta v_3).$$

Then,

$$\alpha u_1+\beta v_1+\alpha u_2+\beta v_2+\alpha u_3+\beta v_3=\alpha(u_1+u_2+u_3)+\beta(v_1+v_2+v_3)\overset{(21.1)}{=}0.$$

(ii) We have to check that a. S is linearly independent and b. *span* $S=V$.
a.

$$\alpha(1,-1,0)+\beta(0,1,-1)=\begin{pmatrix}\alpha & -\alpha+\beta & -\beta\end{pmatrix}=0$$

implies that $\alpha=\beta=0$.
b. Taken $(x_1,x_2,x_3)\in V$, we want to find $\alpha,\beta\in\mathbb{R}$ such that $(x_1,x_2,x_3)=\alpha(1,-1,0)+\beta(0,1,-1)=\begin{pmatrix}\alpha & -\alpha+\beta & -\beta\end{pmatrix}$, i.e., we want to find $\alpha,\beta\in\mathbb{R}$ such that

$$\begin{cases}x_1 &= \alpha\\ x_2 &= -\alpha+\beta\\ x_3 &= -\beta\\ x_1+x_2+x_3 &= 0\end{cases}$$

$$\begin{cases}-x_2-x_3 &= \alpha\\ x_2 &= -\alpha+\beta\\ x_3 &= -\beta\end{cases}$$

Then, $\alpha=-x_2-x_3$, $\beta=-x_3$ is the (unique) solution to the above system.

5.
1. $0\in\mathbb{M}(n,n):A0=0A=0$.
2. $\forall\alpha,\beta\in\mathbb{R}$ and $\forall B,B'\in\mathcal{C}_A$,

$$(\alpha B+\beta B')A=\alpha BA+\beta B'A=\alpha AB+\beta AB'=A\alpha B+A\beta B'=A(\alpha B+\beta B').$$

6.
i. $0\in U+V$, because $0\in U$ and $0\in V$.
ii. Take $\alpha,\beta\in F$ and $w^1,w^2\in U+V$. Then there exists $u^1,u^2\in U$ and $v^1,v^2\in V$ such that $w^1=u^1+v^1$ and $w^2=u^2+v^2$. Therefore,

$$\alpha w^1+\beta w^2=\alpha\left(u^1+v^1\right)+\beta\left(u^2+v^2\right)=\left(\alpha u^1+\beta u^2\right)+\left(\alpha v^1+\beta v^2\right)\in U+V,$$

because U and V are vector spaces and therefore $\alpha u^1+\beta v^1\in U$ and $\alpha u^2+\beta v^2\in V$.
7.
We want to show that if $\sum_{i=1}^3\beta_i v_i=0$, then $\beta_i=0$ for all i. Note that $\sum_{i=1}^3\beta_i v_i=(\beta_1,\beta_1+\beta_2,\beta_1+\beta_2+\beta_3)=0$, which implies the desired result.
8.
We want to apply Definition **??**: Consider a vector space V of dimension n and two bases $\mathcal{V}=\left\{v^i\right\}_{i=1}^n$ and $\mathcal{U}=\left\{u^k\right\}_{k=1}^n$ of V. Then,

$$P=\begin{bmatrix}\left[u^1\right]_{\mathcal{V}} & \dots & \left[u^k\right]_{\mathcal{V}} & \dots & \left[u^n\right]_{\mathcal{V}}\end{bmatrix}\in\mathbb{M}(n,n),$$

is called the change-of-basis matrix from the basis $\mathcal{V}$ to the basis $\mathcal{U}$. Then in our case, the change-of-basis matrix from S to E is

$$P=\begin{bmatrix}\left[e^1\right]_S & \left[e^2\right]_S\end{bmatrix}\in\mathbb{M}(2,2).$$

Moreover,using also Proposition **??**, the change-of-basis matrix from S to E is

$$Q=\begin{bmatrix}\left[v^1\right]_E & \left[v^2\right]_E\end{bmatrix}=P^{-1}.$$

Computation of $\left[e^1\right]_S$. We want to find α and β such that $e_1=\alpha u_1+\beta u_2$, i.e., $(1,0)=\alpha(1,2)+\beta(3,5)=\begin{pmatrix}\alpha+3\beta & 2\alpha+5\beta\end{pmatrix}$, i.e.,

$$\begin{cases}\alpha+3\beta &= 1\\ \\ 2\alpha+5\beta &= 0\end{cases}$$

whose solution is $\alpha = -5, \beta = 2$.

Computation of $\left[e^2\right]_S$. We want to find α and β such that $e_2 = \alpha u_1 + \beta u_2$, i.e., $(0,1) = \alpha(1,2) + \beta(3,5) = \begin{pmatrix} \alpha + 3\beta & 2\alpha + 5\beta \end{pmatrix}$, i.e.,

$$\begin{cases} \alpha + 3\beta & = & 0 \\ \\ 2\alpha + 5\beta & = & 1 \end{cases}$$

$$\begin{cases} \alpha + 3\beta = 0 \\ 2\alpha + 5\beta = 1 \end{cases}$$

whose solution is $\alpha = 3, \beta = -1$. Therefore,

$$P = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}.$$

Since E is the canonical basis, we have

$$S = \{u_1 = (1,2),\ u_2 = (3,5)\}$$

$$Q = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}.$$

Finally

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

as desired.

9.

Easiest way: use row and column operations to change C to a triangular matrix.

$$\det C = \det \begin{bmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{bmatrix} = \left[R^1 \leftrightarrow R^3\right] = -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 2 & 1 & 1 & -2 & 1 \\ 6 & 2 & 1 & 0 & 5 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 2 & 1 & 1 & -2 & 1 \\ 6 & 2 & 1 & 0 & 5 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -2R^1 + R^2 \to R^2 \\ -6R^1 + R^3 \to R^3 \\ -3R^1 + R^4 \to R^4 \\ R^1 + R^5 \to R^5 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & -4 & -11 & 12 & -13 \\ 0 & -3 & -4 & 9 & -10 \\ 0 & 0 & -1 & 2 & 5 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & -4 & -11 & 12 & -13 \\ 0 & -3 & -4 & 9 & -10 \\ 0 & 0 & -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 4R^2 + R^3 \to R^3 \\ 3R^2 + R^4 \to R^4 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 5 & 3 & 5 \\ 0 & 0 & -1 & 2 & 5 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 5 & 3 & 5 \\ 0 & 0 & -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} -5R^3 + R^4 \to R^4 \\ R^3 + R^5 \to R^5 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & -17 & -30 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & -17 & -30 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix} = \left[\frac{6}{17}R^4 + R^5 \to R^5\right] = -\det \begin{bmatrix} 1 & 1 & 2 & -2 & 3 \\ 0 & -1 & -3 & 2 & -5 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & -17 & -30 \\ 0 & 0 & 0 & 0 & \frac{24}{17} \end{bmatrix}$$

$$= -\det\begin{bmatrix}1 & 1 & 2 & -2 & 3\\ 0 & -1 & -3 & 2 & -5\\ 0 & 0 & 1 & 4 & 7\\ 0 & 0 & 0 & -17 & -30\\ 0 & 0 & 0 & 0 & \frac{24}{17}\end{bmatrix} = -(1)(-1)(1)(-17)(\frac{24}{17}) \Longrightarrow \det C = -24$$

10.

Observe that it cannot be 4 as $rank(A) \leq \min\{\#rows, \#collums\}$. It's easy to check that $rank(A) = 3$ by using elementary operations on rows and columns of A :

$-2R^3 + R^1 \to R^1$, $C^4 + C^1 \to C^1$, $C^4 + C^3 \to C^3$, $C^1 + C^3 \to C^3$, $-C^2 + C^3 \to C^3$,

to get

$$\begin{bmatrix}k-1 & 1 & 0 & 0\\ 0 & 0 & 1 & k\\ 0 & 0 & 0 & -1\end{bmatrix}$$

which has the last three columns independent for any k.

11.

Defined

$$l : \mathbb{R}^3 \to \mathbb{R}, \quad (x_1, x_2, x_3) \mapsto x_1 - x_2,$$

it is easy to check that l is linear and $V = \ker l$,a vector space. Moreover,

$$[l] = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}.$$

Therefore, $\dim\ker l = 3 - \text{rank } [l] = 3 - 1 = 2$. $u_1 = (1,1,0)$ and $u_2 = (0,0,1)$ are independent elements of V. Therefore, from Remark 189, u_1, u_2 are a basis for V.

12.

Linearity is easy. By definition, l is linear if $\forall u, v \in \mathbb{R}^4$ and $\forall \alpha, \beta \in \mathbb{R}, l(\alpha u + \beta v) = \alpha l(u) + \beta l(v)$. Then,

$$\alpha l(u) + \beta l(v) =$$

$$=\alpha(u_1, u_1+u_2, u_1+u_2+u_3, u_1+u_2+u_3+u_4)++\beta(v_1, v_1+v_2, v_1+v_2+v_3, v_1+v_2+v_3+v_4) =$$

$$=(\alpha u_1+\beta v_1, \alpha u_1+\alpha u_2+\beta v_1+\beta v_2, \alpha u_1+\alpha u_2+\alpha u_3+\beta v_1+\beta v_2+\beta v_3, \alpha u_1+\alpha u_2+\alpha u_3+\alpha u_4+\beta v_1+\beta v_2+\beta v_3+\beta v_4) =$$

$$= l(\alpha u + \beta v)$$

and then l is linear.

$$[l] = \begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1\end{bmatrix}.$$

Therefore, $\dim\ker l = 4 - \text{rank } [l] = 4 - 4 = 0$. Moreover, $\dim \operatorname{Im} l = 4$ and the column vectors of $[l]$ are a basis of $\operatorname{Im} l$.

13.

Proposition. Assume that $l \in L(V, U)$ and $\ker l = \{0\}$. Then,

$$\forall u \in Iml, \quad \text{there exists a unique } v \in V \text{ such that } l(v) = u.$$

Proof.

Since $u \in \mathtt{Im}l$, by definition, there exists $v \in V$ such that

$$l(v) = u. \tag{21.2}$$

Take $v' \in V$ such that $l(v') = u$. We want to show that

$$v = v'. \tag{21.3}$$

Observe that

$$l(v) - l(v') \overset{(a)}{=} u - u = 0, \tag{21.4}$$

where (a) follows from (21.2) and (21.3).
Moreover,

$$l(v) - l(v') \overset{(b)}{=} l(v - v'), \tag{21.5}$$

where (b) follows from the assumption that $l \in \mathcal{L}(V, U)$.
Therefore,

$$l(v - v') = 0,$$

and, by definition of $\ker l$,

$$v - v' \in \ker l. \tag{21.6}$$

Since, by assumption, $\ker l = \{0\}$, from (21.6), it follows that

$$v - v' = 0.$$

14.
Both V and W are ker of linear function; therefore V, W and $V \cap W$ are vector subspaces of $\mathbb{R}^4$. Moreover

$$V \cap W = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \begin{cases} x_1 - x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \end{cases} \right\}$$

$$\operatorname{rank} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 2$$

Therefore, $\dim \ker l = \dim V \cap W = 4 - 2 = 2$.
Let's compute a basis of $V \cap W$:

$$\begin{cases} x_1 - x_2 = -x_3 + x_4 \\ x_1 + x_2 = -x_3 - x_4 \end{cases}$$

After taking sum and subtraction we get following expression

$$\begin{cases} x_1 = -x_3 \\ x_2 = -x_4 \end{cases}$$

A basis consists two linearly independent vectors. For example,

$$\{(-1, 0, 1, 0), (0, -1, 0, 0)\}.$$

15.
By proposition 139, we have to show that
1. $0 \in l^{-1}(W)$,
2. $\forall \alpha, \beta \in \mathbb{R}$ and $v^1, v^2 \in l^{-1}(W)$ we have that $\alpha v^1 + \beta v^2 \in l^{-1}(W)$.
1.

$$l(0) \overset{l \in \mathcal{L}(V,U)}{=} 0 \overset{W \text{ vector space}}{\in} W$$

,

2.Since $v^1, v^2 \in l^{-1}(W)$,

$$l(v^1), l(v^2) \in W. \tag{21.7}$$

Then

$$l(\alpha v^1 + \beta v^2) \overset{l \in \mathcal{L}(V,U)}{=} \alpha l(v^1) + \beta l(v^2) \overset{(a)}{\in} W$$

where (a) follows from (21.7) and the fact that W is a vector space.

16.

Observe that

$$\det\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 5 & 6 & b_{11} & b_{12} & 0 \\ 7 & 8 & b_{21} & b_{22} & 0 \\ a_{11} & a_{12} & 0 & 0 & k \end{bmatrix} = \det A \cdot \det B \cdot k.$$

Then, if $k \neq 0$,then the rank of both matrix of coefficients and augmented matrix is 5 and the set of solution to the system is an affine subspace of $\mathbb{R}^6$ of dimension 1. If $k = 0$,then the system is

$$\begin{bmatrix} 1 & a_{11} & a_{12} & 0 & 0 & 0 \\ 2 & a_{21} & a_{22} & 0 & 0 & 0 \\ 3 & 5 & 6 & b_{11} & b_{12} & 0 \\ 4 & 7 & 8 & b_{21} & b_{22} & 0 \\ 1 & a_{11} & a_{12} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \end{bmatrix},$$

which is equivalent to the system

$$\begin{bmatrix} 1 & a_{11} & a_{12} & 0 & 0 & 0 \\ 2 & a_{21} & a_{22} & 0 & 0 & 0 \\ 3 & 5 & 6 & b_{11} & b_{12} & 0 \\ 4 & 7 & 8 & b_{21} & b_{22} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix},$$

whose set of solution is an affine subspace of $\mathbb{R}^6$ of dimension 2.

17.

$$[l]_{\mathcal{V}}^{\mathcal{U}} := \left[\left[l \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_{\mathcal{U}}, \left[l \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_{\mathcal{U}} \right] = \left[\left[\begin{array}{c} 1 \\ 1 \end{array} \right]_{\mathcal{U}}, \left[\begin{array}{c} 1 \\ -1 \end{array} \right]_{\mathcal{U}} \right] = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix},$$

$$[v]_{\mathcal{V}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$[l(v)]_{\mathcal{U}} = \left[\begin{pmatrix} 7 \\ -1 \end{pmatrix} \right]_{\mathcal{U}} = \begin{bmatrix} -9 \\ 8 \end{bmatrix}$$

$$[l]_{\mathcal{V}}^{\mathcal{U}} \cdot [v]_{\mathcal{V}} = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \end{bmatrix}$$

18.

Let
$n, m \in \mathbb{N}$ such that $m > n$, and
a vector subspace L of $\mathbb{R}^m$ such that $\dim L = n$
be given. Then, there exists $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that $\operatorname{Im} l = L$.
Proof. Let $\{v^i\}_{i=1}^n$ be a basis of $L \subseteq \mathbb{R}^m$. Take $l \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\forall i \in \{1, ..., n\}, l_2(e_n^i) = v^i,$$

where e_n^i is the i–th element in the canonical basis in $\mathbb{R}^n$. Such function does exists and, in fact, it is unique as a consequence of a Proposition in the Class Notes that we copy below:

Let V and U be finite dimensional vectors spaces such that $S = \{v^1, ..., v^n\}$ is a basis of V and $\{u^1, ..., u^n\}$ is a set of arbitrary vectors in U. Then there exists a unique linear function $l : V \to U$ such that $\forall i \in \{1, ..., n\}$, $l(v^i) = u^i$ - see Proposition 273, page 82.

Then, from the Dimension theorem

$$\dim \operatorname{Im} l = n - \dim \ker l \leq n.$$

Moreover, $L = \operatorname{span}\{v^i\}_{i=1}^n \subseteq \operatorname{Im} l$. Summarizing,

$$L \subseteq \mathtt{Im}\ l\ ,\ \dim L = n \text{ and } \dim \mathtt{Im} l \leq n,$$

and therefore

$$\dim \mathrm{Im} l = n.$$

Finally, from Proposition 179 in the class Notes since $L \subseteq \mathtt{Im}\ l$, $\dim L = n$ and $\dim \mathrm{Im} l = n$, we have that $\mathtt{Im} l = L$, as desired.

Proposition 179 in the class Notes says what follows: Proposition. Let W be a subspace of an $n-$dimensional vector space V. Then, 1. $\dim W \leq n$; 2. If $\dim W = n$, then $W = V$.

19.

$$[A \mid b] = \begin{bmatrix} a & 1 & 1 \\ 1 & 1 & a \\ 2 & 1 & 3a \\ 3 & 2 & a \end{bmatrix} \Longrightarrow \left[R^1 \leftrightarrow R^2\right] \Longrightarrow \begin{bmatrix} 1 & 1 & a \\ a & 1 & 1 \\ 2 & 1 & 3a \\ 3 & 2 & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & a \\ a & 1 & 1 \\ 2 & 1 & 3a \\ 3 & 2 & a \end{bmatrix} \Longrightarrow \begin{matrix} -aR^1 + R^2 \to R^2 \\ -2R^1 + R^3 \to R^3 \\ -3R^1 + R^4 \to R^4 \end{matrix} \Longrightarrow \begin{bmatrix} 1 & 1 & a \\ 0 & 1-a & 1-a^2 \\ 0 & -1 & a \\ 0 & -1 & -2a \end{bmatrix} := \left[A'(a) \mid b'(a)\right]$$

Since

$$\det \begin{bmatrix} 1 & 1 & a \\ 0 & -1 & a \\ 0 & -1 & -2a \end{bmatrix} = 3a,$$

We have that if $a \neq 0$, then rank $A \leq 2 < 3 =$ rank $\left[A'(a) \mid b'(a)\right]$, and the system has no solutions. If $a = 0$, $\left[A'(a) \mid b'(a)\right]$ becomes

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

whose rank is 3 and again the system has no solutions.

20.

$$\left[A(k) \left| b(k)\right.\right] \equiv \left[\begin{array}{ccc} 1 & 0 & k-1 \\ 1-k & 2-k & k \\ 1 & k & 1 \\ 1 & k-1 & 0 \end{array}\right]$$

$$\det \left[\begin{array}{ccc} 1 & 0 & k-1 \\ 1 & k & 1 \\ 1 & k-1 & 0 \end{array}\right] = 2 - 2k$$

If $k \neq 1$, the system has no solutions. If $k = 1$,

$$\left[A(1) \left| b(1)\right.\right] \equiv \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

Then, if $k = 1$, there exists a unique solution.

21.

The following Proposition is contained in the class Notes.

Proposition 800 *Let V be a vector space of dimension n.*

1. *If $S = \{u^1, ..., u^n\} \subseteq V$ is a linearly independent set, then it is a basis of V;*
2. *If* $\operatorname{span}(u^1, ..., u^n) = V$*, then $\{u^1, ..., u^n\}$ is a basis of V.*

From that Proposition, it suffices to show that $\mathcal{V}$ is linearly independent, i.e., given $(\alpha_i)_{i=1}^n \in \mathbb{R}^n$, if

$$\sum_{i=1}^{n} \alpha_i v^i = 0 \tag{21.8}$$

then

$$\boxed{(\alpha_i)_{i=1}^n = 0.}$$

Now, for any $j \in \{1, ..., n\}$, we have

$$0 \overset{(1)}{=} \left(\sum_{i=1}^{n} \alpha_i v^i \right) v^j \overset{(2)}{=} \sum_{i=1}^{n} \alpha_i v^i v^j \overset{(3)}{=} \alpha_j,$$

where (1) follows from (21.8),
(2) follows from properties of the scalar product;
(3) follows from (20.7).

22.

a. Let $w \in W$. By assumption, $S(w) \in W$ and $T(w) \in W$ and since W is a subspace, $S(w)+T(w) \in W$[1]. Therefore, $(S+T)(w) = S(w) + T(w) \in W$, as desired.

b. Let $w \in W$. By assumption, $T(w) \in W$. Then $(S \circ T)(w) = S(T(w)) \in W$ since W is S-invariant.

c. Let $w \in W$. By assumption, $(kT)(w) = kT(w) \in W$.

23.

The set of 2×2 symmetric real matrices is

$$\mathcal{S} = \left\{ A \in \mathbb{M}(2,2) : A = A^T \right\}.$$

We want to show that
i. $0 \in \mathcal{S}$ and
ii. for any $\alpha, \beta \in \mathbb{R}$, for any $A, B \in \mathcal{S}$, $\alpha A + \beta B \in \mathcal{S}$.
i. $0 = 0^T -$
2.$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T = \alpha A + \beta B$.
We want to show that

$$\mathcal{B} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

is a basis of $\mathcal{S}$ and therefore $\dim \mathcal{S} = 3$. $\mathcal{B}$ is clearly linearly independent. Moreover,

$$\mathcal{S} = \left\{ A \in \mathbb{M}(2,2) : \exists a,b,c \in \mathbb{R} \text{ such that } A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right\},$$

and

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

i.e., span $(\mathcal{B})) = \mathcal{S}$, as desired.

24.

a. $[\supseteq]$ Taken $w \in W$,we want to find $w^1, w^2 \in W$ such that $w = w^1 + w^2$.take $w^1 = w$ and $w^2 = 0 \in W$.
$[\subseteq]$ Take $w^1, w^2 \in W$. Then $w^1 + w^2 \in W$ by definition of vector space.

b. $[\supseteq]$ Let $w \in W$. Then $\frac{1}{\alpha} w \in W$ and $\alpha \left(\frac{1}{\alpha} w \right) w \in \alpha W$.
$[\subseteq]$ It follows from the definition of vector space.

25.

[1] see Proposition 138 in Villanacci(20 September, 2012)

The isomorphism is $\varphi : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^{n+1}$

$$f(t) = \sum_{i=0}^{n} a_i t^i \mapsto (a_i)_{i=0}^{n}.$$

Indeed , φ is linear
...
and defined $\psi : \mathbb{R}^{n+1} \to \mathcal{P}_n(\mathbb{R})$,

$$(a_i)_{i=0}^{n} \to f(t) = \sum_{i=0}^{n} a_i t^i,$$

we have that $\varphi \circ \psi = id_{|\mathcal{P}_n(\mathbb{R})}$ and $\psi \circ \varphi = id_{|\mathbb{R}^{n+1}}$
....

21.2 Some topology in metric spaces

21.2.1 Basic topology in metric spaces

1.

To prove that d' is a metric, we have to check the properties listed in Definition 327.

a. $d'(x,y) \geq 0$, $d'(x,y) = 0 \Leftrightarrow x = y$

By definition of $d'(x,y)$, it is always going to be positive as $d(x,y) \geq 0$. Furthermore, $d'(x,y) = 0 \iff d(x,y) = 0 \iff x = y$.

b. $d'(x,y) = d'(y,x)$

Applying the definition

$$d'(x,y) = d'(y,x) \iff \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)}$$

but $d(x,y) = d(y,x)$ so we have

$$\frac{d(x,y)}{1+d(x,y)} = \frac{d(x,y)}{1+d(x,y)}$$

c. $d'(x,z) \leq d'(x,y) + d'(y,z)$

Applying the definition

$$d'(x,z) \leq d'(x,y) + d'(y,z) \iff \frac{d(x,z)}{1+d(x,z)} \leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$

Multiplying both sides by $[1+d(x,z)][1+d(x,y)][1+d(y,z)]$

$$d(x,z)[1+d(x,y)][1+d(y,z)] \leq d(x,y)[1+d(x,z)][1+d(y,z)] + d(y,z)[1+d(x,z)][1+d(x,y)]$$

Simplifying we obtain

$$\boxed{d(x,z) \leq d(x,y) + d(y,z)} + [[1+d(x,z)][1+d(x,y)][1+d(y,z)] + 2[1+d(x,y)][1+d(y,z)]]$$

which concludes the proof.

2.

It is enough to show that one of the properties defining a metric does not hold.

It can be $d(f,g) = 0$ and $f \neq g$. Take

$$f(x) = 0, \forall x \in [0,1],$$

and

$$g(x) = -2x + 1$$

Then,

$$\int_0^1 (-2x+1)\, dx = 0.$$

It can be $d(f,g) < 0$.Consider the null function and the function that take value 1 for all x in $[0;1]$. Then $d(0,1) = -\int_0^1 1\, dx$. by linearity of the Riemann integral, which is equal to -1. Then, $d(0,1) < 0$.

3.

Define $S = (a_1, b_1) \times (a_2, b_2)$ and take $x^0 := (x_1^0, x_2^0) \in S$. Then, for $i \in \{1,2\}$, there exist $\varepsilon_i > 0$ such that $x_i^0 \in B(x_i^0, \varepsilon_i) \subseteq (a_i, b_i)$. Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then, for $i \in \{1,2\}$, $x_i^0 \in B(x_i^0, \varepsilon) \subseteq (a_i, b_i)$ and, defined $B = .B(x_1^0, \varepsilon) \times B(x_2^0, \varepsilon)$, we have that $x^0 \in B \subseteq S$. It then suffices to show that $B(x^0, \varepsilon) \subseteq B$. Observe that

$$x \in B\left(x^0, \varepsilon\right) \Leftrightarrow d\left(x, x^0\right) < \varepsilon,$$

$$d\left(\left(x_1^0, 0\right), (x_1, 0)\right) = \sqrt{\left(x_1^0 - x_1\right)^2} = \left|x_1^0 - x_1\right|,$$

and

$$d\left(\left(x_1^0, 0\right), (x_1, 0)\right) = \sqrt{\left(x_1^0 - x_1\right)} \leq \sqrt{\left(x_1^0 - x_1\right)^2 + \left(x_1^0 - x_1\right)^2} = d\left(x, x^0\right).$$

4.
Show the second equality in Remark 366:

$$\cap_{n=1}^{+\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

5.

$$S = \left\{-1, +\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, ...\right\}$$

The set is not open: it suffices to find $x \in S$ and such that $x \notin \text{Int } S$; take for example -1. We want to show that it false that

$$\exists \varepsilon > 0 \text{ such that } (-1-\varepsilon, -1+\varepsilon) \subseteq S.$$

In fact, $\forall \varepsilon > 0$, $-1 - \frac{\varepsilon}{2} \in (-1-\varepsilon, -1+\varepsilon)$, but $-1 - \frac{\varepsilon}{2} \notin S$. The set is not closed. It suffices to show that $\mathcal{F}(S)$ is not contained in S, in fact that $0 \notin S$ (obvious) and $0 \in \mathcal{F}(S)$. We want to show that $\forall \varepsilon > 0$, $B(0, \varepsilon) \cap S \neq \varnothing$.In fact, $(-1)^n \frac{1}{n} \in B(0, \varepsilon)$ if n is even and $(-1)^n \frac{1}{n} = \frac{1}{n} < \varepsilon$. It is then enough to take n even and $n > \frac{1}{\varepsilon}$.

6.

$$A = (0, 10)$$

The set is $(\mathbb{R}, d_2)$ open, as a union of infinite collection of open sets. The set is not closed, because A^c is not open. 10 or 0 do not belongs to $\text{Int}(A^C)$

7.
The solution immediately follow from Definition of boundary of a set: Let a metric space (X, d) and a set $S \subseteq X$ be given. x is an boundary point of S if
any open ball centered in x intersects both S and its complement in X, i.e., $\forall r \in \mathbb{R}_{++}, \quad B(x, r) \cap S \neq \varnothing \quad \wedge \quad B(x, r) \cap S^C \neq \varnothing$.
As you can see nothing changes in definition above if you replace the set with its complement.
8.

$$\begin{aligned}
x \in (\mathcal{F}(S))^C \quad & \Leftrightarrow x \notin (\mathcal{F}(S)) \\
& \Leftrightarrow \neg\left(\forall r \in \mathbb{R}_{++}, B(x,r) \cap S \neq \varnothing \ \wedge \ B(x,r) \cap S^C \neq \varnothing\right) \\
& \Leftrightarrow \exists r \in \mathbb{R}_{++} \text{ such that } B(x,r) \cap S = \varnothing \ \vee \ B(x,r) \cap S^C \neq \varnothing \\
& \Leftrightarrow \exists r \in \mathbb{R}_{++} \text{ such that } B(x,r) \subseteq S^C \ \vee \ B(x,r) \subseteq S \\
& \Leftrightarrow x \in \text{Int } S^C \ \vee x \in \text{Int } S \\
& \overset{(1)}{\Leftrightarrow} \exists r_x^* \in \mathbb{R}_{++} \text{ such that either a. } B(x, r_x^*) \subseteq \text{Int } S^C \text{ or b. } B(x, r_x^*) \subseteq \text{Int } S.
\end{aligned} \tag{21.9}$$

where (1) follows from the fact that the Interior of a set is an open set.
If case a. in (21.9) holds true, then, using Lemma 461, $B(x, r_x^*) \subseteq (\mathcal{F}(S))^C$ and similarly for case b., as desired.

9.

	Int S	$\mathrm{Cl}(S)$	$\mathcal{F}(S)$	$D(S)$	$Is(S)$	open or closed
$S=\mathbb{Q}$	$\varnothing$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\varnothing$	neither open nor closed
$S=(0,1)$	$(0,1)$	$[0,1]$	$\{0,1\}$	$[0,1]$	$\varnothing$	open
$S=\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}_+}$	$\varnothing$	$S\cup\{0\}$	$S\cup\{0\}$	$\{0\}$	S	neither open nor closed

10.
a.
Take $S=\mathbb{N}$. Then, Int $S=\varnothing$, Cl $(\varnothing)=\varnothing$, and Cl (Int S) $=\varnothing\neq\mathbb{N}=S$.
b.
Take $S=\mathbb{N}$. Then, Cl $(S)=\mathbb{N}$, Int $\mathbb{N}=\varnothing$, and Int Cl $(S)=\varnothing\neq\mathbb{N}=S$.

11.
a.
True. If S is an open bounded interval, then $\exists a,b\in\mathbb{R}$, $a<b$ such that $S=(a,b)$. Take $x\in S$ and $\delta=\min\{|x-a|,|x-b|\}$. Then $I(x,\delta)\subseteq(a,b)$.
b.
False. $(0,1)\cup(2,3)$ is an open set, but it is not an open interval.
c.
False. Take $S:=\{0,1\}$. $0\in\mathcal{F}(S)$, but $0\notin D(S)$
d. .
False. Take $S(0,1)$. $\frac{1}{2}\in D(S)\,0$, but $\frac{1}{2}\notin\mathcal{F}(S)$.
12.
Recall that: A sequence $(x_n)_{n\in\mathbb{N}}\in X^\infty$ is said to be (X,d) convergent to $x_0\in X$ (or convergent with respect to the metric space (X,d)) if $\forall\varepsilon>0,\exists n_0\in\mathbb{N}$ such that $\forall n>n_0,\quad d(x_n,x_0)<\varepsilon$.
a.
$(x_n)_{n\in N}\in\mathbb{R}^\infty$ such that $\forall n\in N,: x_n=1$
Let $\epsilon>0$ then by definition of $(x_n)_{n\in N}$, $\forall n>0$, $d(x_n,1)=0<\epsilon$. So that

$$\lim_{n\to\infty}x_n=1$$

b.
$(x_n)_{n\in N}\in\mathbb{R}^\infty$ such that $\forall n\in N,: x_n=\frac{1}{n}$
Let $\epsilon>0$. Because N is unbounded, $\exists n_0\in N$, such that $n_0>\frac{1}{\epsilon}$. Then $\forall n>n_0$, $d(x_n,0)=\frac{1}{n}<\frac{1}{n_0}<\epsilon$. Then, by definition of a limit, we proved that

$$\lim_{n\to\infty}x_n=0$$

13.
Take $(x_n)_{n\in\mathbb{N}}\in[0,1]^\infty$ such that $x_n\to x_0$; we want to show that $x_0\in[0,1]$. Suppose otherwise, i.e., $x_0\notin[0,1]$.
Case 1. $x_0<0$. By definition of convergence, chosen $\varepsilon=-\frac{x_0}{2}>0$, there exists $n_\varepsilon\in\mathbb{N}$ such that $\forall n>n_\varepsilon$, $d(x_n,x_0)<\varepsilon$, i.e., $|x_n-x_0|<\varepsilon=-\frac{x_0}{2}$, i.e., $x_0+\frac{x_0}{2}<x_n<x_0-\frac{x_0}{2}=\frac{x_0}{2}<0$. Summarizing, $\forall n>n_\varepsilon$, $x_n\notin[0,1]$, contradicting the assumption that $(x_n)_{n\in\mathbb{N}}\in[0,1]^\infty$.
Case 2. $x_0>1$. Similar to case 1.

14.
This is Example 7.15, page 150, Morris (2007):
1. In fact, we have the following result: Let (X,d) be a metric space and $A=\{x_1,...,x_n\}$ any finite subset of X.Then A is compact, as shown below.
Let O_i, $i\in I$ be any family of open sets such that $A\subseteq\cup_{i\in I}O_i$. Then for each $x_j\in A$, there exists O_{i_j} such that $x_j\in O_{i_j}$. Then $A\subseteq O_{i_1}\cup O_{i_2}\cup...\cup O_{i_n}$. Therefore A is compact.

2. Conversely, let A be compact. Then the family of singleton sets $O_x = \{x\}$, $x \in A$ is such that each O_x is open and $A \subseteq \cup_{x\in A} O_x$. Since A is compact, there exists $O_{x_1}, O_{x_2}, ..., O_{x_n}$ such that $A \subseteq O_{x_1} \cup O_{x_2} \cup ... \cup O_{x_n}$, that is, $A \subseteq \{x_1, ..., x_n\}$. Hence, A is finite.

15.

In general it is false. For example in a discrete metric space: see previous exercise.

16.

Take an open ball $B(x,r)$. Consider $\mathcal{S} = \left\{B\left(x, r\left(1-\frac{1}{n}\right)\right)\right\}_{n\in\mathbb{N}\setminus\{0,1\}}$. Observe that S is an open cover of $B(x,r)$; in fact $\cup_{n\in\mathbb{N}\setminus\{0,1\}} B\left(x, r\left(1-\frac{1}{n}\right)\right) = B(x,r)$,as shown below.

$[\subseteq]$ $x' \in \cup_{n\in\mathbb{N}\setminus\{0,1\}} B\left(x, r\left(1-\frac{1}{n}\right)\right) \Leftrightarrow \exists n_{x'}\mathbb{N}\setminus\{0,1\}$ such $x \in B\left(x, r\left(1-\frac{1}{n_{x'}}\right)\right) \subseteq B(x,r)$.

$[\supseteq]$ Take $x' \in B(x,r)$. Then, $d(x,x') < r$. Take n such that $d(x',x) < r\left(1-\frac{1}{n_{x'}}\right)$, i.e., $n > \frac{r}{r-d(x',x)}$ (and $n > 1$), then $x' \in B\left(x, r\left(1-\frac{1}{n}\right)\right)$.

Consider an arbitrary subcover of $\mathcal{S}$, i.e.,

$$\mathcal{S}' = \left\{B\left(x, r\left(1-\frac{1}{n}\right)\right)\right\}_{n\in\mathcal{N}}$$

with $\#\mathcal{N} = N \in \mathbb{N}$. Define $n^* = \min\{n \in \mathcal{N}\}$. Then $\cup_{n\in\mathcal{N}} B\left(x, r\left(1-\frac{1}{n}\right)\right) = B\left(x, r\left(1-\frac{1}{n^*}\right)\right)$, and if $d(x',x) \in \left(r\left(1-\frac{1}{n^*}\right), r\right)$, then $x' \in B(x,r)$ and $x' \notin \cup_{n\in\mathcal{N}} B\left(x, r\left(1-\frac{1}{n}\right)\right)$.

17.

1st proof.

We have to show that $f(A\cup B) \subseteq f(A) \cup f(B)$ and $f(A) \cup f(B) \subseteq f(A\cup B)$.

To prove the first inclusion, take $y \in f(A\cup B)$; then $\exists x \in A\cup B$ such that $f(x) = y$. Then either $x \in A$ or $x \in B$ that implies $f(x) = y \in A$ or $f(x) = y \in B$. In both case $y \in f(A) \cup f(B)$

We now show the opposite e inclusion. Let $y \in f(A) \cup f(B)$, then $y \in f(A)$ or $y \in f(B)$, but $y \in f(A)$ implies that $\exists x \in A$ such that $f(x) = y$. The same implication for $y \in f(B)$. As results, $y = f(x)$ in either case with $x \in A \cup B$ i.e. $y \in f(A\cup B)$.

2nd proof.

$$\begin{aligned}
&y \in f(A\cup B) \Leftrightarrow \\
&\Leftrightarrow \exists x \in A\cup B \text{ such that } f(x) = y \\
&\Leftrightarrow (\exists x \in A \text{ such that } f(x) = y) \vee (\exists x \in B \text{ such that } f(x) = y) \\
&\Leftrightarrow (y \in f(A)) \vee (y \in f(B)) \\
&\Leftrightarrow y \in f(A) \cup f(B)
\end{aligned}$$

18.:

First proof. Take $f = \sin$, $A = [-2\pi, 0]$, $B = [0, 2\pi]$.

Second proof. Consider

$$f : \{0,1\} \to \mathbb{R}, \qquad x \mapsto 1$$

Then take $A = \{0\}$ and $B = \{1\}$. Then $A \cap B = \varnothing$, so $f(A\cap B) = \varnothing$. But as $f(A) = f(B) = \{1\}$, we have that $f(A) \cap f(B) = \{1\} \neq \varnothing$.

19.

Take $c \in \mathbb{R}$ and define the following function

$$f : X \to Y, \quad f(x) = c.$$

It suffices to show that the preimage of every open subset of the domain is open in the codomain. The inverse image of any open set K is either X (if $c \in K$) or $\varnothing$ (if $c \notin K$), which are both open sets.

20.

a.
i. $\mathbb{R}^n_+$ is not bounded, then by Proposition 423 it is not compact.
ii. $Cl\ B(x,r)$ is compact.
From Proposition 423, it suffices to show that the set is closed and bounded.
$Cl\ B(x,r)$ is closed from Proposition 379.
$Cl\ B(x,r)$ is bounded because $Cl\ B(x,r) = \{y \in \mathbb{R}^n : \ d(x,y) \leq r\} \subseteq B(x, 2r)$.
Let's show in detail the equality.
i. $Cl\ B(x,r) \subseteq C$.

The function $d_x : \mathbb{R}^n \to \mathbb{R}$, $d_x(y) = d(x,y) := \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{1}{2}}$ is continuous. Therefore, $C = d_x^{-1}([0,r])$ is closed. Since $B(x,r) \subseteq C$, by definition of closure, the desired result follows.

ii. $Cl\ B(x,r) \supseteq C$.

From Corollary 464, it suffices to show that $Ad\ B(x,r) \supseteq C$. If $d(y,x) < r$, we are done. Suppose that $d(y,x) = r$.We want to show that for every $\varepsilon > 0$, we have that $B(x,r) \cap B(y,\varepsilon) \neq \varnothing$.If $\varepsilon > r$, then $x \in B(x,r) \cap B(y,\varepsilon)$. Now take, $\varepsilon \leq r$. It is enough to take a point "very close to y inside $B(x,r)$"". For example, we can verify that $z \in B(x,r) \cap B(y,\varepsilon)$, where $z = x + \left(1 - \frac{\varepsilon}{2r}\right)(y-x)\big)$ .Indeed,

$$d(x,z) = \left(1 - \frac{\varepsilon}{2r})d(y,x)\right) = (1 - \frac{\varepsilon}{2r})r = r - \frac{\varepsilon}{2} < r,$$

and

$$d(y,z) = \frac{\varepsilon}{2r} d(y,x) = \frac{\varepsilon}{2r} r = \frac{\varepsilon}{2} < \varepsilon.$$

c.

See solution to Exercise 5, where it was shown that S is not closed and therefore using Proposition 423, we can conclude S is not compact.

21.

Observe that given for any $j \in \{1,...,m\}$, the continuous functions $g_j : \mathbb{R}^n \to \mathbb{R}$ and $g = (g_j)_{j=1}^m$, we can define

$$C := \{x \in \mathbb{R}^n : g(x) \geq 0\}.$$

Then C is closed, because of the following argument:

$C = \cap_{j=1}^m g_j^{-1}([0,+\infty))$; since g_j is continuous, and $[0,+\infty)$ is closed, then $g_j^{-1}([0,+\infty))$ is closed in $\mathbb{R}^n$; then C is closed because intersection of closed sets.

22.

The set is closed, because $X = f^{-1}(\{0\})$.
The set is not compact: take f as the constant function.

23.

Let$V = B_Y(1,\varepsilon)$, be an open ball around the value 1 of the codomain, with $\varepsilon < 1$. $f^{-1}(V) = \{0\} \cup B_X(1,\varepsilon)$ is the union of an open set and a closed set, so is neither open nor closed.

24.

To apply the Extreme Value Theorem, we first have to check if the function to be maximized is continuous. Clearly, the function $\sum_{i=1}^n x_i$ is continuous as is the sum of affine functions. Therefore, to check for the existence of solutions for the problems we only have to check for the compactness of the restrictions.

The first set is closed, because it is the inverse image of the closed set $[0,1]$ via the continuous function $\|\|$. The first set is bounded as well by definition. Therefore the set is compact and the function is continuous, we can apply Extreme Value theorem. The second set is not closed, therefore it is not compact and Extreme Value theorem can not be applied. The third set is unbounded, and therefore it is not compact and the Extreme Value theorem can not be applied.

25.

[$\Leftarrow$]
Obvious.
[$\Rightarrow$]
We want to show that $l \neq 0 \Rightarrow l$ is not bounded, i.e., $\forall M \in \mathbb{R}_{++}, \exists x \in E$ such that $\|l(x)\|_F > M$.

Since $l \neq 0, \exists y \in E \backslash \{0\}$ such that $l(y) \neq 0$. Define $x = \frac{2M}{\|l(x)\|_F} y$. Then

$$\|l(y)\|_F = \left\| l\left(\frac{2M}{\|l(x)\|_F} x \right) \right\|_F = \frac{2M}{\|l(x)\|_F} \|l(x)\|_F = 2M > M,$$

as desired.

26.

$a \Rightarrow b$.

Take an arbitrary $\alpha \in \mathbb{R}$; if $\{x \in X : f(x) < \alpha\} = f^{-1}((-\infty, \alpha)) = \varnothing$, we are done. Otherwise, take $x_0 \in f^{-1}((-\infty, \alpha))$, Then, $\alpha - f(x_0) := \varepsilon > 0$ and by definition of upper semicontinuity , we have

$$\exists \delta > 0 \text{ such that } d(x - x_0) < \delta \Rightarrow f(x) < f(x_0) + \varepsilon = f(x_0) + \alpha - f(x_0) = \alpha,$$

i.e., $B(x_0, \delta) \subseteq f^{-1}((-\infty, \alpha))$, i.e., the desired result.

$b \Leftrightarrow c$.

$\{x \in X : f(x) < \alpha\} = X \setminus \{x \in X : f(x) \geq \alpha\}$.

$b \Rightarrow a$.

We want to show that

$$\forall x_0 \in X, \ \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } B(x_0, \delta) \subseteq f^{-1}((-\infty, f(x_0) + \varepsilon)).$$

But by assumption $f^{-1}((-\infty, f(x_0) + \varepsilon))$ is an open set and contains x_0 and therefore the desired result follows.

27.

Take $y \in \{x\} + A$. Then there exists $a \in A$ such that $y = x + a$ and since A is open there exists $\varepsilon > 0$ such that

$$a \in B(a, \varepsilon) \subseteq A. \tag{21.10}$$

We want to show that

i. $\{x\} + B(a, \varepsilon)$ is an open ball centered at $y = x + a$, i.e., $\{x\} + B(a, \varepsilon) = B(x + a, \varepsilon)$, and

ii. $B(x + a, \varepsilon) \subseteq \{x\} + A$.

i.

[$\subseteq$] $y \in \{x\} + B(a, \varepsilon) \Leftrightarrow \exists z \in X$ such that $d(z, a) < \varepsilon$ and $y = x + z \Rightarrow d(y, x + a) = d(x + z, x + a) \overset{Hint}{=} d(z, a) < \varepsilon \Rightarrow y \in B(x + a, \varepsilon)$.

[$\supseteq$] $y \in B(x + a, \varepsilon) \Leftrightarrow d(y, x + a) < \varepsilon$. Now since $y = x + (y - x)$ and $d(y - x, a) = \|y - x - a\| = \|y - (x + a)\| = d(y, x + a) < \varepsilon$, we get the desired conclusion.

ii.

$y \in B(x + a, \varepsilon) \Leftrightarrow \|y - (x + a)\| < \varepsilon$. Since $y = x + (y - x)$ and $\|(y - x) - a\| < \varepsilon$, i.e., $y - x \in B(a, \varepsilon) \overset{(21.10)}{\subseteq} A$, we get the desired result.

28.

By assumption, for any $i \in \{1, 2\}$ and for any $\{x_i^n\}_n \subseteq K_i$, there exists $x_i \in K_i$ such that, up to a subsequence $x_i^n \to x_i$.Take $\{y^n\} \subseteq K_1 + K_2 = K$. Then $\forall n$, $y^n = x_1^n + x_2^n$ with $x_i^n \in K_i$, $i = 1, 2$.Thus taking converging subsequences of $(x_i^n)_n$, $i \in \{1, 2\}$, we get $y^n \to x_1 + x_2 \in K$ as desired.

29.

a. We want to show that $\forall x_1, x_2 \in E$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Indeed

$$0 = d(f(x_1), f(x_2)) = d(x_1, x_2) \Rightarrow x_1 = x_2.$$

b. It follows from a.

c. We want to show that $\forall x_0 \in E, \forall \varepsilon > 0, \exists \delta > 0$ such that

$$d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \varepsilon.$$

Take $\delta = \varepsilon$. Then,

$$d_1(x, x_0) < \varepsilon \Rightarrow d_2(f(x), f(x_0)) < \varepsilon.$$

21.2.2 Correspondences

1.

Since u is a continuous function, from the Extreme Value Theorem , we are left with showing that for every (p,w) , $\beta(p,w)$ is non empty and compact,i.e., β is non empty valued and compact valued.

$x = \left(\frac{w}{Cp^c}\right)_{c=1}^{C} \in \beta(p,w)$.

$\beta(p,w)$ is closed because is the intersection of the inverse image of two closed sets via continuous functions.

$\beta(p,w)$ is bounded below by zero.

$\beta(p,w)$ is bounded above because for every c, $x^c \leq \frac{w-\sum_{c'\neq c} p^{c'}x^{c'}}{p^c} \leq \frac{w}{p^c}$, where the first inequality comes from the fact that $px \leq w$, and the second inequality from the fact that $p \in \mathbb{R}^C_{++}$ and $x \in \mathbb{R}^C_{+}$.

2.

(a)Consider $x', x'' \in \xi(p,w)$. We want to show that $\forall \lambda \in [0,1]$, $x^\lambda := (1-\lambda)x' + \lambda x'' \in \xi(p,w)$. Observe that $u(x') = u(x'') := u^*$. From the quasiconcavity of u, we have $u(x^\lambda) \geq u^*$. We are therefore left with showing that $x^\lambda \in \beta(p,w)$, i.e., β is convex valued. To see that, simply, observe that $px^\lambda = (1-\lambda)px' + \lambda px'' \leq (1-\lambda)w + \lambda w = w$.

(b) Assume otherwise. Following exactly the same argument as above we have $x', x'' \in \xi(p,w)$, and $px^\lambda \leq w$. Since u is strictly quasi concave, we also have that $u(x^\lambda) > u(x') = u(x'') := u^*$, which contradicts the fact that $x', x'' \in \xi(p,w)$.

3.

We want to show that for every (p,w) the following is true. For every sequence $\{(p_n, w_n)\}_n \subseteq \mathbb{R}^C_{++} \times \mathbb{R}_{++}$ such that

$(p_n, w_n) \to (p,w)$, $x_n \in \beta(p_n, w_n)$, $x_n \to x$,

it is the case that $x \in \beta(p,w)$.

Since $x_n \in \beta(p_n, w_n)$, we have that $p_n x_n \leq w_n$. Taking limits of both sides, we get $px \leq w$,i.e., $x \in \beta(p,w)$.

4.

(a) We want to show that $\forall y', y'' \in y(p)$, $\forall \lambda \in [0,1]$, it is the case that $y^\lambda := (1-\lambda)y' + \lambda y'' \in y(p)$, i.e., $y^\lambda \in Y$ and $\forall y \in Y$, $py^\lambda \geq py$.

$y^\lambda \in Y$ simply because Y is convex.

$py^\lambda := (1-\lambda)py' + \lambda py'' \overset{y',y'' \in y(p)}{\geq} (1-\lambda)py + \lambda py = py.$

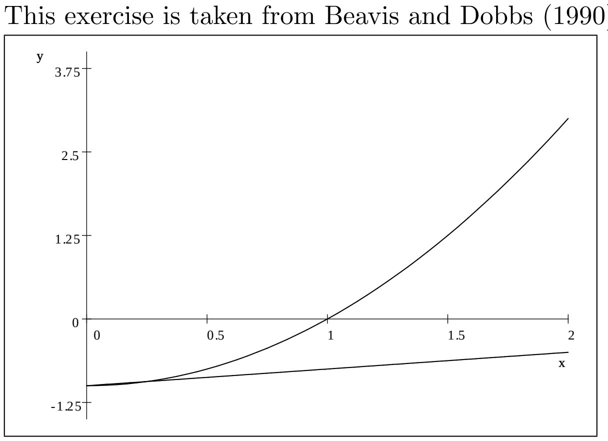
(b)Suppose not; then $\exists y', y'' \in Y$ such that $y' \neq y''$ and such that

$$\forall y \in Y,\ py' = py'' > py \quad (1)\,.$$

Since Y is strictly convex,$\forall \lambda \in (0,1)$, $y^\lambda := (1-\lambda)y' + \lambda y'' \in \text{Int}\, Y$. Then, $\exists \varepsilon > 0$ such that $B(y^\lambda, \varepsilon) \subseteq Y$. Consider $y^* := y^\lambda + \frac{\varepsilon}{2C}\mathbf{1}$, where $\mathbf{1} := (1, ..., 1) \in \mathbb{R}^C$. $d(y^*, y^\lambda) = \sqrt{\sum_{c=1}^{C}\left(\frac{\varepsilon}{2C}\right)^2} = \frac{\varepsilon}{2\sqrt{C}}$. Then, $y^* \in B(y^\lambda, \varepsilon) \subseteq Y$ and, since $p \gg 0$, we have that $py^* > py^\lambda = py' = py''$, contradicting (1) .

5.

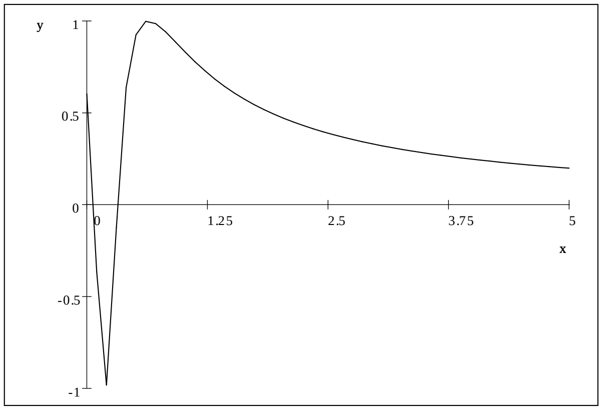
This exercise is taken from Beavis and Dobbs (1990), pages 74-78.



For every $x \in [0,2]$, both $\phi_1(x)$ and $\phi_2(x)$ are closed, bounded intervals and therefore convex and compact sets. Clearly ϕ_1 is closed and ϕ_2 is not closed.

$\phi_1$and ϕ_2 are clearly UHC and LHC for $x \neq 1$. Using the definitions, it is easy to see that for $x = 1$, ϕ_1 is UHC, and not LHC and ϕ_2 is LHC and not UHC.

6.



For every $x > 0$, ϕ is a continuous function. Therefore, for those values of x, ϕ is both UHC and LHC.

ϕ is UHC in 0. For every neighborhood of $[-1, 1]$ and for any neighborhood of $\{0\}$ in $\mathbb{R}^+$, $\phi(x) \subseteq [-1, 1]$.

ϕ is not LHC in 0. Take the open set $V = \left(\frac{1}{2}, \frac{3}{2}\right)$;we want to show that $\forall \varepsilon > 0$ $\exists z^* \in (0, \varepsilon)$ such that $\phi(z^*) \notin \left(\frac{1}{2}, \frac{3}{2}\right)$. Take $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and $z^* = \frac{1}{n\pi}$. Then $0 < z^* < \varepsilon$ and $\sin z^* = \sin n\pi = 0 \notin \left(\frac{1}{2}, \frac{3}{2}\right)$.

Since ϕ is UHC and closed valued, from Proposition 16 is closed.

7.

ϕ is not closed. Take $x_n = \frac{\sqrt{2}}{2n} \in [0, 1]$ for every $n \in \mathbb{N}$. Observe that $x_n \to 0$. For every $n \in \mathbb{N}$, $y_n = -1 \in \phi(x_n)$ and $y_n \to -1$. But $-1 \in \phi(0) = [0, 1]$.

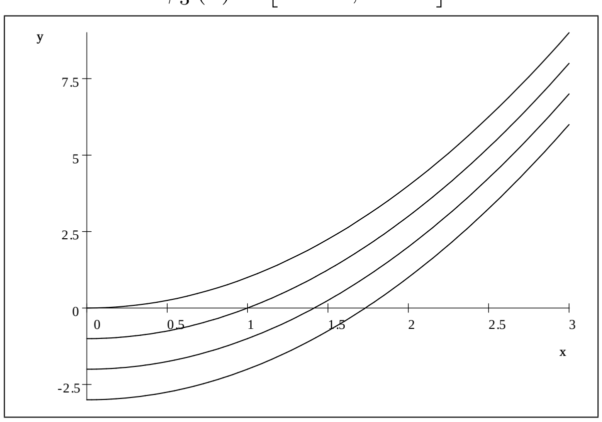
ϕ is not UHC. Take $x = 0$ and a neighborhood $V = \left(-\frac{1}{2}, \frac{3}{2}\right)$ of $\phi(0) = [0, 1]$. Then $\forall \varepsilon > 0$, $\exists x^* \in (0, \varepsilon) \setminus \mathbb{Q}$. Therefore, $\phi(x^*) = [-1, 0] \nsubseteq V$.

ϕ is not LHC. Take $x = 0$ and the open set $V = \left(\frac{1}{2}, \frac{3}{2}\right)$.Then $\phi(0) \cap \left(\frac{1}{2}, \frac{3}{2}\right) = [0, 1] \cap \left(\frac{1}{2}, \frac{3}{2}\right) = \left(\frac{1}{2}, 1\right] \neq \varnothing$. But, as above, $\forall \varepsilon > 0$, $\exists x^* \in (0, \varepsilon) \setminus \mathbb{Q}$. . Then $\phi(x^*) \cap V = [-1, 0] \cap \left(\frac{1}{2}, \frac{3}{2}\right) = \varnothing$.

8.

(This exercise is taken from Klein, E. (1973), *Mathematical Methods in Theoretical Economics, Academic Press*, New York, NY, page 119).

Observe that $\phi_3(x) = \left[x^2 - 2, x^2 - 1\right]$.



$\phi_1([0, 3]) = \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, \times \leq 3, y \geq x^2 - 2, y \leq x^2\right\}$. $\phi_1([0, 3])$ is defined in terms of weak inequalities and continuous functions and it is closed and therefore ϕ_1 is closed. Similar argument applies to ϕ_2 and ϕ_3.

Since $[-10, 10]$ is a compact set such that $\phi_1([0, 3]) \subseteq [-10, 10]$, from Proposition 17, ϕ_1 is UHC. Similar argument applies to ϕ_2 and ϕ_3.

ϕ_1 is LHC. Take an arbitrary $\overline{x} \in [0, 3]$ and a open set V with non-empty intersection with $\phi_1(\overline{x}) = \left[\overline{x}^2 - 2, \overline{x}^2\right]$. To fix ideas, take $V = \left[\varepsilon, \overline{x}^2 + \varepsilon\right]$, with $\varepsilon \in \left(0, \overline{x}^2\right)$. Then, take $U = \left(\sqrt{\varepsilon}, \sqrt{\overline{x}^2 + \varepsilon}\right)$. Then for every $x \in U$, $\left\{x^2\right\} \subseteq V \cap \phi_1(x)$.

Similar argument applies to ϕ_2 and ϕ_3.

21.3 Differential Calculus in Euclidean Spaces

1 .

The partial derivative of f with respect to the first coordinate at the point (x_0, y_0), is - if it exists and is finite -

$$\lim_{x\to x_0} \frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

$$\begin{aligned}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0} &= \frac{2x^2-xy_0+y_0^2-(2x_0^2-x_0y_0+y_0^2)}{x-x_0}\\ &= \frac{2x^2-2x_0^2-(xy_0-x_0y_0)}{x-x_0}\\ &= 2(x+x_0)-y_0\end{aligned}$$

Then

$$D_1f(x_0,y_0) = 4x_0 - y_0$$

The partial derivative of f with respect to the second coordinate at the point (x_0, y_0), is - if it exists and is finite -

$$\lim_{y\to y_0} \frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

$$\begin{aligned}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0} &= \frac{2x_0^2-x_0y+y^2-(2x_0^2-x_0y_0+y_0^2)}{y-y_0}\\ &= \frac{-x_0(y-y_0)+y^2-y_0^2}{y-y_0}\\ &= -x_0+(y+y_0)\end{aligned}$$

$$D_2f(x_0,y_0) = -x_0 + 2y_0.$$

2 .

a.

The domain of f is $\mathbb{R}\backslash\{0\}\times\mathbb{R}$. As arctan is differentiable over the whole domain, we may compute the partial derivative over the whole domain of f at the point (x, y) - we omit from now on the superscript $_0$

$$\begin{aligned}D_1f(x,y) &= \arctan\frac{y}{x} + x(-\frac{y}{x^2})\frac{1}{1+(\frac{y}{x})^2}\\ &= \arctan\frac{y}{x} - y\frac{1}{1+(\frac{y}{x})^2}\\ D_2f(x,y) &= x\frac{1}{x}\frac{1}{1+(\frac{y}{x})^2} = \frac{1}{1+(\frac{y}{x})^2}\end{aligned}$$

b.

The function is defined on $\mathbb{R}_{++}\times\mathbb{R}$. and

$$\forall (x,y)\in\mathbb{R}_{++}\times\mathbb{R},\quad f(x,y) = e^{y\ln x}$$

Thus as exp and ln are differentiable over their whole respective domain, we may compute the partial derivatives :

$$\begin{aligned}D_1f(x,y) &= \frac{y}{x}e^{y\ln x} = yx^{y-1}\\ D_2f(x,y) &= \ln(x)e^{ylnx} = \ln(x)x^y\end{aligned}$$

c.

$f(x,y) = (\sin(x+y))^{\sqrt{x+y}} = e^{\sqrt{x+y}\ln[\sin(x+y)]}$ in $(0,3)$.
We check that $\sin(0+3) > 0$ so that the point belongs to the domain of the function. Both partial derivatives in (x,y) have the same expression since f is symmetric with respect to x and y.

$$D_1 f(x,y) = D_2 f(x,y) = \left[\frac{1}{2\sqrt{x+y}}\ln[\sin(x+y)] + \sqrt{x+y}\cot(x+y)\right](\sin(x+y))^{\sqrt{x+y}},$$

and

$$D_1 f(0,3) = D_2 f(0,3) = \left[\frac{1}{2\sqrt{3}}\ln[\sin(3)] + \sqrt{3}\tan(3)\right](\sin(3))^{\sqrt{3}}$$

3.

a.

$$D_x f(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0;$$

b.

$$D_y f(0,0) = \lim_{k\to 0}\frac{f(0,k) - f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0;$$

c.

Consider the sequences $(x_n)_{n\in N}$ and $(y_n)_{n\in N}$ such that $\forall n \in N,: x_n = y_n = \frac{1}{n}$. We have that $\lim_{n\to 0}(x_n, y_n) = 0_{\mathbb{R}^2}$, but

$$f(x_n, y_n) = \frac{\frac{1}{n^2}}{\frac{1}{n^2} + \frac{1}{n^2}}.$$

Then

$$\lim_{n\to 0} f(x_n, y_n) = \frac{1}{2} \neq f(0,0) = 0$$

Thus, f is not continuous in $(0,0)$.

4 .

$$f'((1,1);(\alpha_1,\alpha_2)) = \lim_{h\to 0}\frac{f(1+h\alpha_1, 1+h\alpha_2) - f(1,1)}{h} =$$

$$\lim_{h\to 0}\frac{1}{h}\left[\frac{2 + h(\alpha_1+\alpha_2)}{(1+h\alpha_1)^2 + (1+h\alpha_2)^2 + 1} - \frac{2}{3}\right] = \cdots$$

$$= -\frac{\alpha_1+\alpha_2}{9}$$

5.

The directional derivative of f at the point $x_0 = (0,0)$ in the direction $u \in \mathbb{R}$ is given by

$$\lim_{h\to 0}\frac{f(x_0+hu) - f(x_0)}{h}.$$

Take any $u = (u_1, u_2) \in R^2$. Then

$$\lim_{h\to 0}\frac{f(x_0+hu) - f(x_0)}{h} = \lim_{h\to 0}\frac{f((hu_1, hu_2)) - f((0,0))}{h} = \lim_{h\to 0}\frac{f((hu_1, hu_2))}{h}$$

If $u_1 \neq 0$, then

$$\lim_{h\to 0}\frac{\frac{(hu_1)^2(hu_2)^2}{(hu_1)^2 + (hu_2)^2}}{h} = \lim_{h\to 0}\frac{\frac{h^4(u_1^2u_2^2)}{h^2(u_1^2+u_2^2)}}{h} = \lim_{h\to 0}\frac{hu_1^2u_2^2}{u_1^2+u_2^2} = 0$$

If $u_1 = 0$:

$$\lim_{h\to 0}\frac{f((hu_1, hu_2))}{h} = \lim_{h\to 0}\frac{hu_2}{h} = \lim_{h\to 0} u_2 = u_2.$$

6 .

a) given $x_0 \in \mathbb{R}^n$ we need to find $T_{x_0} \colon \mathbb{R}^n \to \mathbb{R}^m$ linear and E_{x_0} with $\lim_{v \to 0} E_{x_0}(v) = 0$. Take $T_{x_0} = l$ and $E_{x_0} \equiv 0$. Then, the desired result follows.

b) projection is linear, so by a) is differentiable.

7.

From the definition of continuity, we want to show that $\forall x_0 \in \mathbb{R}^n$, $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\|x - x_0\| < \delta \Rightarrow \|l(x) - l(x_0)\| < \varepsilon$. Defined $[l] = A$, we have that

$$\begin{aligned}
&\|l(x) - l(x_0)\| = \|A \cdot x - x_0\| = \\
&= \left\|R^1(A) \cdot (x - x_0), ..., R^m(A) \cdot (x - x_0)\right\| \overset{(1)}{\leq} \textstyle\sum_{i=1}^m \left|R^i(A) \cdot (x - x_0)\right| \overset{(2)}{\leq} \\
&\leq \textstyle\sum_{i=1}^m \left\|R^i(A)\right\| \cdot \|x - x_0\| \leq m \cdot \left(\max_{i \in \{1,\ldots,m\}} \left\{\left\|R^i(A)\right\|\right\}\right) \cdot \|x - x_0\|,
\end{aligned} \tag{21.11}$$

where (1) follows from Remark 56 and (2) from Proposition 53.4, i.e., Cauchy-Schwarz inequality. Take

$$\delta = \frac{\varepsilon}{m \cdot \left(\max_{i \in \{1,\ldots,m\}} \{R^i(A)\}\right)}.$$

Then we have that $\|x - x_0\| < \delta$ implies that $\|x - x_0\| \cdot m \cdot \left(\max_{i \in \{1,\ldots,m\}} \left\{R^i(A)\right\}\right) < \varepsilon$, and from (21.11), $\|l(x) - l(x_0)\| < \varepsilon$, as desired.

8 .

$$Df(x,y) = \begin{bmatrix} \cos x \cos y & -\sin x \sin y \\ \cos x \sin y & \sin x \cos y \\ -\sin x \cos y & -\cos x \sin y \end{bmatrix}.$$

9 .

$$Df(x,y,z) = \begin{bmatrix} g'(x)h(z) & 0 & g(x)h'(z) \\ g'(h(x))h'(x)/y & -g(h(x))/y^2 & 0 \\ \exp(xg(h(x))((g(h(x)) + g'(h(x))h'(x)x)) & 0 & 0 \end{bmatrix}.$$

10 .

a.

f is differentiable over its domain since $x \mapsto \log x$ is differentiable over $\mathbb{R}_{++}$. Then from Proposition 625, we know that

$$[df_x] = Df(x) = \begin{bmatrix} \frac{1}{3x_1}, & \frac{1}{6x_2}, & \frac{1}{2x_3} \end{bmatrix}$$

Then

$$[df_{x_0}] = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{4} \end{bmatrix}$$

By application of Remark 626, we have that

$$f'(x_0, u) = Df(x_0).u = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{4} \end{bmatrix} . \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{\sqrt{3}}{4}$$

b.

As a polynomial expression, f is differentiable over its domain.

$$[df_x] = Df(x) = \begin{bmatrix} 2x_1 - 2x_2, & 4x_2 - 2x_1 - 6x_3, & -2x_3 - 6x_2 \end{bmatrix}$$

Then

$$[df_{x_0}] = \begin{bmatrix} 2 & 4 & 2 \end{bmatrix}$$

and

$$f'(x_0, u) = Df(x_0).u = \begin{bmatrix} 2 & 4 & 2 \end{bmatrix} . \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0$$

c.

$$[df_x] = Df(x) = \begin{bmatrix} e^{x_1 x_2} + x_2 x_1 e^{x_1 x_2}, & x_1^2 e^{x_1 x_2} \end{bmatrix}$$

Then

$$[df_{x_0}] = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$f'(x_0, u) = Df(x_0).u = \begin{bmatrix} 1, & 0 \end{bmatrix} . \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2$$

11 .

Then since f is in $C^\infty(\mathbb{R}\backslash\{0\})$, we know that f admits partial derivative *functions* and that these functions admit themselves partial derivatives in (x, y, z). Since, the function is symmetric in its arguments, it is enough to compute explicitly $\frac{\partial^2 f(x,y,z)}{\partial x^2}$.

$$\frac{\partial f}{\partial x}(x, y, z) = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

Then

$$\frac{\partial^2 f}{\partial x^2}(x, y, z) = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} + 3x^2(x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

Then $\forall (x, y, z) \in \mathbb{R}^3\backslash\{0\}$,

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x, y, z) + \frac{\partial^2 f}{\partial y^2}(x, y, z) + \frac{\partial^2 f}{\partial z^2}(x, y, z) &= -3(x^2 + y^2 + z^2)^{-\frac{3}{2}} + (3x^2 + 3y^2 + 3z^2)(x^2 + y^2 + z^2)^{-\frac{5}{2}} \\ &= -3(x^2 + y^2 + z^2)^{-\frac{3}{2}} + 3(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ &= 0 \end{aligned}$$

12 .

$g, h \in C^2(\mathbb{R}, \mathbb{R}_{++})^2$

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(x, y, z) = \left(\tfrac{g(x)}{h(z)}, \quad g(h(x)) + xy, \quad \ln(g(x) + h(x)) \right)$$

Since $Im(g) \subseteq \mathbb{R}_{++}$, $(x, y, z) \mapsto \frac{g(x)}{h(z)}$ is differentiable as the ratio of differentiable functions. And since $Im(g+h) \subseteq \mathbb{R}_{++}$ and that ln is differentiable over $\mathbb{R}_{++}$, $x \mapsto \ln(g(x)+h(x))$ is differentiable by proposition 619. Then

$$Df(x, y, z) = \begin{bmatrix} \frac{g'(x)}{h(z)} & 0 & -h'(z)\frac{g(x)}{h(z)^2} \\ h'(x)g'(h(x)) + y & x & 0 \\ \frac{g'(x)+h'(x)}{g(x)+h(x)} & 0 & 0 \end{bmatrix}$$

13 .

a.

Since

$$h(x) = \begin{pmatrix} e^x + g(x) \\ e^{g(x)} + x \end{pmatrix},$$

then

$$[dh_x] = Dh(x) = \begin{pmatrix} e^x + g'(x) \\ g'(x) e^{g(x)} + 1 \end{pmatrix},$$

$$[dh_0] = Dh(0) = \begin{pmatrix} 1 + g'(0) \\ g'(0) e^{g(0)} + 1 \end{pmatrix},$$

b.

Let us define $l : \mathbb{R} \to \mathbb{R}^2$, $l(x) = (x, g(x))$. Then $h = f \circ l$. As l is differentiable on $\mathbb{R}$ and f is differentiable on $\mathbb{R}^2$ we may apply the “chain rule".

$$dh_0 = df_{l(0)} \circ dl_0$$

$$[dl_x] = Dl(x) = \begin{bmatrix} 1 \\ g'(x) \end{bmatrix}$$

$$[df_{(x,y)}] = Dl(x,y) = \begin{bmatrix} e^x & 1 \\ 1 & e^y \end{bmatrix}$$

Then

$$[dh_0] = \begin{bmatrix} e^0 & 1 \\ 1 & e^{g(0)} \end{bmatrix} \begin{bmatrix} 1 \\ g'(0) \end{bmatrix} = \begin{bmatrix} 1+g'(0) \\ 1+g'(0)e^{g(0)} \end{bmatrix}$$

14 .

$$D_x\left(b \circ a\right)(x) = D_y b\left(y\right)_{|y=a(x)} \cdot D_x a\left(x\right).$$

$$D_y b\left(y\right) = \left[\begin{array}{ccc} D_{y_1} g\left(y\right) & D_{y_2} g\left(y\right) & D_{y_3} g\left(y\right) \\ D_{y_1} f\left(y\right) & D_{y_2} f\left(y\right) & D_{y_3} f\left(y\right) \end{array}\right]_{|y=a(x)}$$

$$D_x a\left(x\right) = \left[\begin{array}{ccc} D_{x_1} f\left(x\right) & D_{x_2} f\left(x\right) & D_{x_3} f\left(x\right) \\ D_{x_1} g\left(x\right) & D_{x_2} g\left(x\right) & D_{x_3} g\left(x\right) \\ 1 & 0 & 0 \end{array}\right]$$

$$D_x\left(b \circ a\right)(x) =$$

$$\left[\begin{array}{ccc} D_{y_1} g(f(x),g(x),x_1) & D_{y_2} g(f(x),g(x),x_1) & D_{y_3} g(f(x),g(x),x_1) \\ D_{y_1} f(f(x),g(x),x_1) & D_{y_2} f(f(x),g(x),x_1) & D_{y_3} f(f(x),g(x),x_1) \end{array}\right] \left[\begin{array}{ccc} D_{x_1} f(x) & D_{x_2} f(x) & D_{x_3} f(x) \\ D_{x_1} g(x) & D_{x_2} g(x) & D_{x_3} g(x) \\ 1 & 0 & 0 \end{array}\right] =$$

$$= \left[\begin{array}{ccc} D_{y_1} g & D_{y_2} g & D_{y_3} g \\ D_{y_1} f & D_{y_2} f & D_{y_3} f \end{array}\right] \left[\begin{array}{ccc} D_{x_1} f & D_{x_2} f & D_{x_3} f \\ D_{x_1} g & D_{x_2} g & D_{x_3} g \\ 1 & 0 & 0 \end{array}\right] =$$

$$= \left[\begin{array}{ccc} D_{y_1} g \cdot D_{x_1} f + D_{y_2} g D_{x_1} g + D_{y_3} g & D_{y_1} g \cdot D_{x_2} f + D_{y_2} g \cdot D_{x_2} g & D_{y_1} g \cdot D_{x_3} f + D_{y_2} g \cdot D_{x_3} g \\ D_{y_1} f \cdot D_{x_1} f + D_{y_2} f \cdot D_{x_1} g + D_{y_3} f & D_{y_1} f \cdot D_{x_2} f + D_{y_2} f \cdot D_{x_2} g & D_{y_1} f \cdot D_{x_3} f + D_{y_2} f \cdot D_{x_3} g \end{array}\right].$$

15 .

By the sufficient condition of differentiability, it is enough to show that the function $f \in C^1$. Partial derivatives are $\dfrac{\partial f}{\partial x} = 2x + y$ and $\dfrac{\partial f}{\partial y} = -2y + x$ – both are indeed continuous, so f is differentiable.

16 .

i) $f \in C^1$ as $Df(x,y,z) = (1 + 4xy^2 + 3yz, 3y^2 + 4x^2y + 3z, 1 + 3xy + 3z^2)$ has continuous entries (everywhere, in particular around (x_0, y_0, z_0))).
ii) $f(x_0, y_0, z_0) = 0$ by direct calculation.
iii) $f'_z = \frac{\partial f}{\partial z}|_{(x_0,y_0,z_0)} = 7 \neq 0$, $f'_y = \frac{\partial f}{\partial y}|_{(x_0,y_0,z_0)} = 10 \neq 0$ and $f'_x = \frac{\partial f}{\partial x}|_{(x_0,y_0,z_0)} = 8 \neq 0$.
Therefore we can apply Implicit Function Theorem around $(x_0, y_0, z_0) = (1, 1, 1)$ to get

$$\frac{\partial x}{\partial z} = -\frac{f'_z}{f'_x} = -7/8,$$

$$\frac{\partial y}{\partial z} = -\frac{f'_z}{f'_y} = -7/10.$$

17 .

a)

$$Df = \begin{bmatrix} 2x_1 & -2x_2 & 2 & 3 \\ x_2 & x_1 & 1 & -1 \end{bmatrix}$$

and each entry of Df is continuous; then f is C^1. $\det D_x f(x,t) = 2x_1^2 + 2x_2^2 \neq 0$ except for $x_1 = x_2 = 0$. Finally,

$$Dg(t) = -\begin{bmatrix} 2x_1 & -2x_2 \\ x_2 & x_1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = -\frac{1}{2x_1^2 + 2x_2^2} \begin{bmatrix} 2x_1 + 2x_2 & 3x_1 - 2x_2 \\ -2x_2 + 2x_1 & -2x_1 - 3x_2 \end{bmatrix}.$$

b)

$$Df = \begin{bmatrix} 2x_2 & 2x_1 & 1 & 2t_2 \\ 2x_1 & 2x_2 & 2t_1 - 2t_2 & -2t_1 + 2t_2 \end{bmatrix}$$

continuous, $\det D_x f(x,t) = 4x_2^2 - 4x_1^2 \neq 0$ except for $|x_1| = |x_2|$. Finally

$$Dg(t) = -\begin{bmatrix} 2x_2 & 2x_1 \\ 2x_1 & 2x_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2t_2 \\ 2t_1 - 2t_2 & -2t_1 + 2t_2 \end{bmatrix} =$$

$$= -\tfrac{1}{4x_2^2 - 4x_1^2} \begin{bmatrix} -4x_1t_1 + 4x_1t_2 + 2x_2 & 4x_1t_1 - 4x_1t_2 + 4x_2t_2 \\ -2x_1 + 4x_2t_1 - 4x_2t_2 & -4x_1t_2 - 4x_2t_1 + 4x_2t_2 \end{bmatrix}.$$

c)

$$Df = \begin{bmatrix} 2 & 3 & 2t_1 & -2t_2 \\ 1 & -1 & t_2 & t_1 \end{bmatrix}$$

continuous, $\det D_x f(x,t) = -5 \neq 0$ always. Finally

$$Dg(t) = -\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2t_1 & -2t_2 \\ t_2 & t_1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2t_1 - 3t_2 & 2t_2 - 3t_1 \\ -2t_1 + 2t_2 & 2t_2 + 2t_1 \end{bmatrix}.$$

18.

As an application of the Implicit Function Theorem, we have that

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial(z^3 - xz - y)}{\partial x}}{\frac{\partial(z^3 - xz - y)}{\partial z}} = -\frac{-z}{3z^2 - x}$$

if $3z^2 - x \neq 0$. Then,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial\left(\frac{z(x,y)}{3(z(x,y))^2 - x}\right)}{\partial y} = \frac{\frac{\partial z}{\partial y}\left(3z^2 - x\right) - 6\frac{\partial z}{\partial y} \cdot z^2}{\left(3z^2 - x\right)^2}$$

Since,

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial(z^3 - xz - y)}{\partial y}}{\frac{\partial(z^3 - xz - y)}{\partial z}} = -\frac{-1}{3z^2 - x},$$

we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\frac{1}{3z^2 - x}\left(3z^2 - x\right) - 6\frac{1}{3z^2 - x} \cdot z^2}{\left(3z^2 - x\right)^2} = \frac{-3z^2 - x}{\left(3z^2 - x\right)^3}$$

19.

As an application of the Implicit Function Theorem, we have that the Marginal Rate of Substitution in (x_0, y_0) is

$$\frac{dy}{dx}_{|(x,y)=(x_0,y_0)} = -\frac{\frac{\partial(u(x,y)-k)}{\partial x}}{\frac{\partial(u(x,y)-k)}{\partial y}}_{|(x,y)=(x_0,y_0)} < 0$$

$$\frac{d^2 y}{dx^2} = -\frac{\partial\left(\frac{D_x u(x,y(x))}{D_y u(x,y(x))}\right)}{\partial x} = -\frac{\left(\overset{(-)}{D_{xx}u} + \overset{(+)}{D_{xy}u}\overset{(-)}{\frac{dy}{dx}}\right)\overset{(+)}{D_y u} - \left(\overset{(+)}{D_{xy}u} + \overset{(-)}{D_{yy}u}\overset{(-)}{\frac{dy}{dx}}\right)\overset{(+)}{D_x u}}{\underset{(+)}{\left(D_y u\right)^2}} > 0$$

and therefore the function $y(x)$ describing indifference curves is convex.

20.

Adapt the proof for the case of the derivative of the product of functions from $\mathbb{R}$ to $\mathbb{R}$.

21.

Differentiate both sides of

$$f(ax_1, ax_2) = a^n f(x_1, x_2)$$

with respect to a and then replace a with 1.

21.4 Nonlinear Programming

1.

(a)

If $\beta = 0$, then $f(x) = \alpha$. The constant function is concave and therefore pseudo-concave, quasi-concave, not strictly concave.

If $\beta > 0$, $f'(x) = \alpha\beta x^{\beta-1}$, $f''(x) = \alpha\beta(\beta-1)x^{\beta-2}$.

$f''(x) \le 0 \Leftrightarrow \alpha\beta(\beta-1) \ge 0 \overset{\alpha\ge 0, \beta\ge 0}{\Leftrightarrow} 0 \le \beta \le 1 \Leftrightarrow f$ concave $\Rightarrow f$ quasi-concave.

$f''(x) < 0 \Leftrightarrow (\alpha > 0$ and $\beta \in (0,1)) \Rightarrow f$ strictly concave.

(b)

The Hessian matrix of f is

$$D^2 f(x) = \begin{bmatrix} \alpha_1\beta_1(\beta_1-1)x^{\beta_1-2} & & 0 \\ & \ddots & \\ 0 & & \alpha_n\beta_n(\beta_n-1)x^{\beta_n-2} \end{bmatrix}$$

.

$D^2 f(x)$ is negative semidefinite $\Leftrightarrow (\forall i,\ \beta_i \in [0,1]) \Rightarrow f$ is concave.

$D^2 f(x)$ is negative definitive $\Leftrightarrow (\forall i,\ \alpha_i > 0$ and $\beta_i \in (0,1)) \Rightarrow f$ is strictly concave.

The border Hessian matrix is

$$B(f(x)) = \begin{bmatrix} 0 & \alpha_1\beta_1 x^{\beta_1-1} & - & \alpha_n\beta_n x^{\beta_n-1} \\ \alpha_1\beta_1 x^{\beta_1-1} & \alpha_1\beta_1(\beta_1-1)x^{\beta_1-2} & & 0 \\ | & & \ddots & \\ \alpha_n\beta_n x^{\beta-1} & 0 & & \alpha_n\beta_n(\beta_n-1)x^{\beta_n-2} \end{bmatrix}$$

.

The determinant of the significant leading principal minors are

$$\det\begin{bmatrix} 0 & \alpha_1\beta_1 x^{\beta_1-1} \\ \alpha_1\beta_1 x^{\beta_1-1} & \alpha_1\beta_1(\beta_1-1)x^{\beta_1-2} \end{bmatrix} = -\alpha_1^2\beta_1^2\left(x^{\beta_1-1}\right)^2 < 0$$

$$\det\begin{bmatrix} 0 & \alpha_1\beta_1 x^{\beta_1-1} & \alpha_2\beta_2 x^{\beta_2-1} \\ \alpha_1\beta_1 x^{\beta_1-1} & \alpha_1\beta_1(\beta_1-1)x^{\beta_1-2} & 0 \\ \alpha_2\beta_2 x^{\beta_1-1} & 0 & \alpha_2\beta_2(\beta_2-1)x^{\beta_2-2} \end{bmatrix} =$$

$$= -\left[\alpha_1\beta_1 x^{\beta_1-1}\alpha_1\beta_1 x^{\beta_1-1}\alpha_2\beta_2(\beta_2-1)x^{\beta_2-2}\right] - \left[\alpha_2\beta_2 x^{\beta_1-1}\alpha_2\beta_2 x^{\beta_2-1}\alpha_1\beta_1(\beta_1-1)x^{\beta_1-2}\right] =$$

$$= -\left(\alpha_1\beta_1\alpha_2\beta_2 x^{\beta_1+\beta_2-4}\right)\left[\alpha_1\beta_1 x^{\beta_2}(\beta_2-1) + \alpha_2\beta_2 x^{\beta_1}(\beta_1-1)\right] =$$

$$= -\alpha_1\beta_1\alpha_2\beta_2 x^{\beta_1+\beta_2-4}\left[\alpha_1\beta_1 x^{\beta_2}(\beta_2-1) + \alpha_2\beta_2 x^{\beta_1}(\beta_1-1)\right] > 0$$

iff for $i = 1, 2$, $\alpha_i > 0$ and $\beta_i \in (0,1)$.

(c)

If $\beta = 0$, then $f(x) = \min\{\alpha, -\gamma\} = 0$.

If $\beta > 0$, we have

The intersection of the two line has coordinates $x^* : \left(= \frac{\alpha+\gamma}{\beta}, \alpha\right)$.

f is clearly not strictly concave, because it is constant in a subset of its domain. Let's show it is concave and therefore pseudo-concave and quasi-concave.

Given $x', x'' \in X$, 3 cases are possible.

Case 1. $x', x'' \le x^*$.

Case 2. $x', x'' \ge x^*$.

Case 3. $x' \le x^*$ and $x'' \ge x^*$.

The most difficult case is case 3: we want to show that $(1-\lambda)f(x') + \lambda f(x'') \le f((1-\lambda)x' + \lambda x'')$.

Then, we have

$$(1-\lambda) f(x') + \lambda f(x'') = (1-\lambda)\min\{\alpha, \beta x' - \gamma\} + \lambda \min\{\alpha, \beta x'' - \gamma\} = (1-\lambda)(\beta x' - \gamma) + \lambda\alpha.$$

Since, by construction $\alpha \geq \beta x' - \gamma$,

$$(1-\lambda)(\beta x' - \gamma) + \lambda\alpha \leq \alpha;$$

since, by construction $\alpha \leq \beta x'' - \gamma$,

$$(1-\lambda)(\beta x' - \gamma) + \lambda\alpha \leq (1-\lambda)(\beta x' - \gamma) + \lambda((\beta x'' - \gamma) = \beta[(1-\lambda)x' + \lambda x''] - \gamma.$$

Then

$$(1-\lambda) f(x') + \lambda f(x'') \leq \min\{\alpha, \beta[(1-\lambda)x' + \lambda x''] - \gamma\} = f((1-\lambda)x' + \lambda x''),$$

as desired.

2.

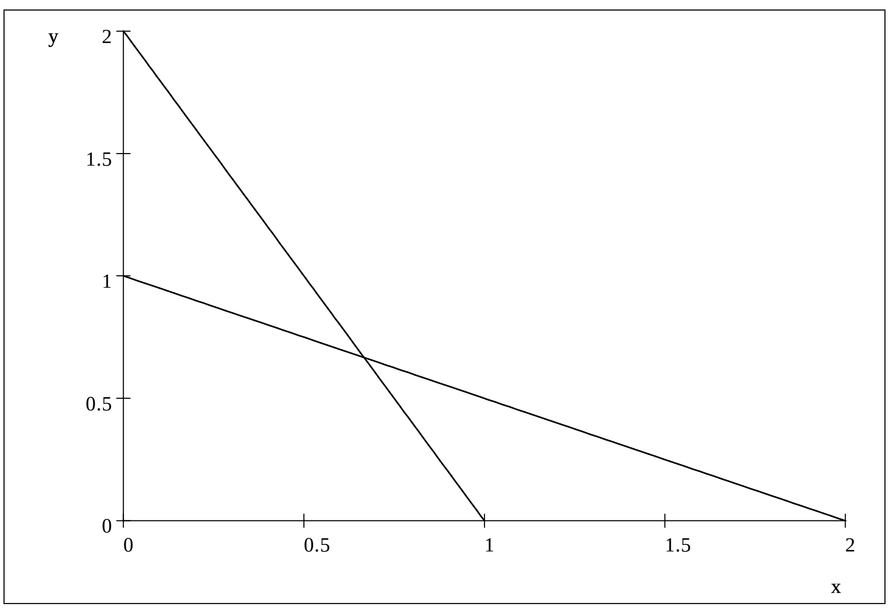
a.

i. Canonical form.

For given $\pi \in (0,1)$, $a \in (0,+\infty)$,

$$\begin{array}{llll} \max_{(x,y)\in\mathbb{R}^2} & \pi \cdot u(x) + (1-\pi)u(y) \quad \text{s.t.} & a - \frac{1}{2}x - y \geq 0 & \lambda_1 \\ & & 2a - 2x - y \geq 0 & \lambda_2 \\ & & x \geq 0 & \lambda_3 \\ & & y \geq 0 & \lambda_4 \end{array}$$

$$\left\{ \begin{array}{l} y = a - \frac{1}{2}x \\ y = 2a - 2x \end{array} \right., \text{ solution is } \left[x = \frac{2}{3}a, y = \frac{2}{3}a\right]$$



ii. The set X and the functions f and g (X open and convex; f and g at least differentiable).

a. The domain of all function is $\mathbb{R}^2$. Take $X = \mathbb{R}^2$ which is open and convex.

b. $Df(x,y) = (\pi \cdot u'(x), (1-\pi)u'(y))$.The Hessian matrix is

$$\left[\begin{array}{cc} \pi \cdot u''(x) & 0 \\ 0 & (1-\pi)u''(y) \end{array} \right]$$

Therefore, f and g are C^2 functions and f is strictly concave and the functions g^j are affine.

iii. Existence.

C is closed and bounded below by $(0,0)$ and above by (a,a) :

$y \le a - \frac{1}{2}x \le a$
$2x \le 2a - y \le 2a.$
iv. Number of solutions.
The solution is unique because f is strictly concave and the functions g^j are affine and therefore concave.
v. Necessity of K-T conditions.
The functions g^j are affine and therefore concave.
$x^{++} = \left(\frac{1}{2}a, \frac{1}{2}a\right)$

$$\begin{array}{l} a - \frac{1}{2}\frac{1}{2}a - \frac{1}{2}a = \frac{1}{4}a > 0 \\ 2a - 2\frac{1}{2}a - \frac{1}{2}a = \frac{1}{2}a > 0 \\ \frac{1}{2}a > 0 \\ \frac{1}{2}a > 0 \end{array}$$

vi. Sufficiency of K-T conditions.
The objective function is strictly concave and the functions g^j are affine
vii. K-T conditions.

$$\mathcal{L}(x, y, \lambda_1, ..., \lambda_4; \pi, a) = \pi \cdot u(x) + (1-\pi)u(y) + \lambda_1\left(a - \frac{1}{2}x - y\right) + \lambda_2(2a - 2x - y) + \lambda_3 x + \lambda_4 y.$$

$$\begin{cases} \pi \cdot u'(x) - \frac{1}{2}\lambda_1 - 2\lambda_2 + \lambda_3 & = 0 \\ (1-\pi)u'(y) - \lambda_1 - \lambda_2 + \lambda_4 & = 0 \\ \min\left\{\lambda_1, a - \frac{1}{2}x - y\right\} & = 0 \\ \min\{\lambda_2, 2a - 2x - y\} & = 0 \\ \min\{\lambda_3, x\} & = 0 \\ \min\{\lambda_4, y\} & = 0 \end{cases}$$

b. Inserting $(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{2}{3}a, \frac{2}{3}a, \lambda_1, 0, 0, 0\right)$, with $\lambda_1 > 0$, in the Kuhn-Tucker conditions we get:

$$\begin{cases} \pi \cdot u'\left(\frac{2}{3}a\right) - \frac{1}{2}\lambda_1 & = 0 \\ (1-\pi)u'\left(\frac{2}{3}a\right) - \lambda_1 & = 0 \\ a - \frac{1}{2}\frac{2}{3}a - \frac{2}{3}a & = 0 \\ \min\left\{0, 2a - 2\frac{2}{3}a - \frac{2}{3}a\right\} & = 0 \\ \min\left\{0, \frac{2}{3}a\right\} & = 0 \\ \min\left\{0, \frac{2}{3}a\right\} & = 0 \end{cases}$$

and

$$\begin{cases} \lambda_1 & = 2\pi \cdot u'\left(\frac{2}{3}a\right) > 0 \\ \lambda_1 & = (1-\pi)u'\left(\frac{2}{3}a\right) \ge 0 \\ a - \frac{1}{2}\frac{2}{3}a - \frac{2}{3}a & = 0 \\ \min\{0, 0\} & = 0 \\ \min\left\{0, \frac{2}{3}a\right\} & = 0 \\ \min\left\{0, \frac{2}{3}a\right\} & = 0 \end{cases}$$

Therefore, the proposed vector is a solution if

$$2\pi \cdot u'\left(\frac{2}{3}a\right) = (1-\pi)u'\left(\frac{2}{3}a\right) > 0,$$

i.e.,

$$2\pi = 1 - \pi \quad \text{or} \quad \pi = \frac{1}{3} \text{ and for any } a \in \mathbb{R}_{++}.$$

c. If the first, third and fourth constraint hold with a strict inequality, and the multiplier associated with the second constraint is strictly positive, Kuhn-Tucker conditions become:

$$\begin{cases} \pi \cdot u'(x) - 2\lambda_2 & = 0 \\ (1-\pi)u'(y) - \lambda_2 & = 0 \\ a - \frac{1}{2}x - y & > 0 \\ 2a - 2x - y & = 0 \\ x & > 0 \\ y & > 0 \end{cases}$$

$$\begin{cases} \pi \cdot u'(x) - 2\lambda_2 & = 0 \\ (1-\pi) u'(y) - \lambda_2 & = 0 \\ 2a - 2x - y & = 0 \end{cases}$$

	x	y	λ_2	π	a
$\pi \cdot u'(x) - 2\lambda_2$	$\pi \cdot u''(x)$	0	-2	$u'(x)$	0
$(1-\pi) u'(y) - \lambda_2$	0	$(1-\pi) u''(y)$	-1	$-u'(y)$	0
$2a - 2x - y$	-2	-1	0	0	2

$$\det \begin{bmatrix} \pi \cdot u''(x) & 0 & -2 \\ 0 & (1-\pi) u''(y) & -1 \\ -2 & -1 & 0 \end{bmatrix} =$$

$$= \pi u''(x) \det \begin{bmatrix} (1-\pi) u''(y) & -1 \\ -1 & 0 \end{bmatrix} - 2 \det \begin{bmatrix} 0 & -2 \\ (1-\pi) u''(y) & -1 \end{bmatrix} =$$

$$= -\pi u''(x) - 4(1-\pi) u''(y) > 0$$

$$D_{(\pi,a)}(x, y, \lambda_2) = - \begin{bmatrix} \pi \cdot u''(x) & 0 & -2 \\ 0 & (1-\pi) u''(y) & -1 \\ -2 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} u'(x) & 0 \\ -u'(y) & 0 \\ 0 & 2 \end{bmatrix}$$

Using maple:

$$\begin{bmatrix} \pi \cdot u''(x) & 0 & -2 \\ 0 & (1-\pi) u''(y) & -1 \\ -2 & -1 & 0 \end{bmatrix}^{-1} =$$

$$= \frac{1}{-\pi u''(x) - 4(1-\pi) u''(y)} \begin{bmatrix} -1 & 2 & 2u''(y) - 2\pi u''(y) \\ 2 & -4 & \pi u''(x) \\ 2u''(y) - 2\pi u''(y) & \pi u''(x) & \pi u''(x) u''(y) - \pi^2 u''(x) u''(y) \end{bmatrix}$$

$$D_{(\pi,a)}(x, y, \lambda_2) =$$

$$= -\frac{1}{-\pi u''(x) - 4(1-\pi) u''(y)} \begin{bmatrix} -1 & 2 & 2u''(y)(1-\pi) \\ 2 & -4 & \pi u''(x) \\ 2u''(y)(1-\pi) & \pi u''(x) & \pi u''(x) u''(y)(1-\pi) \end{bmatrix} \begin{bmatrix} u'(x) & 0 \\ -u'(y) & 0 \\ 0 & 2 \end{bmatrix} =$$

$$= \frac{1}{\pi u''(x) + 4(1-\pi) u''(y)} \begin{bmatrix} -u'(x) - 2u'(y) & 4u''(y)(1-\pi) \\ 2u'(x) - 4u'(y) & 2\pi u''(x) \\ 2u''(y)(1-\pi) \cdot u'(x) + \pi u''(x)(-u'(y)) & 2\pi u''(x) u''(y)(1-\pi) \end{bmatrix}$$

3.

i. Canonical form.

For given $\pi \in (0,1)$, $w_1, w_2 \in \mathbb{R}_{++}$,

$$\max_{(x,y,m) \in \mathbb{R}^2_{++} \times \mathbb{R}} \quad \pi \log x + (1-\pi) \log y \quad s.t. \quad \begin{array}{ll} w_1 - m - x \geq 0 & \lambda_x \\ w_2 + m - y \geq 0 & \lambda_y \end{array}$$

where λ_x and λ_y are the multipliers associated with the first and the second constraint respectively.

ii. The set X and the functions f and g (X open and convex; f and g at least differentiable)

The set $X = \mathbb{R}^2_{++} \times \mathbb{R}$ is open and convex. The constraint functions are affine and therefore $\mathcal{C}^2$. The gradient and the Hessian matrix of the objective function are computed below:

	x	y	m
$\pi \log x + (1-\pi) \log y$	$\frac{\pi}{x}$	$\frac{1-\pi}{y}$	0

	x	y	m
$\frac{\pi}{x}$	$-\frac{\pi}{x^2}$	0	0
$\frac{1-\pi}{y}$	0	$-\frac{1-\pi}{y^2}$	0
0	0	0	0

Therefore, the objective function is $\mathcal{C}^2$ and concave, but not strictly concave.

iii. Existence.

The problem has the same solution set as the following problem:

$$\begin{array}{lll} \max_{(x,y,m)\in\mathbb{R}^2_{++}\times\mathbb{R}} \quad \pi \log x + (1-\pi)\log y & s.t & \\ & w_1 - m - x \geq 0 & \lambda_x \\ & w_2 + m - y \geq 0 & \lambda_y \\ & \pi \log x + (1-\pi)\log y \geq \pi \log w_1 + (1-\pi)\log w_2 & \end{array}$$

whose constraint set is compact (details left to the reader).

iv. Number of solutions.

The objective function is concave and the constraint functions are affine; uniqueness is not insured on the basis of the sufficient conditions presented in the notes.

v. Necessity of K-T conditions.

Constraint functions are affine and therefore pseudo-concave. Choose $(x,y,m)^{++} = \left(\frac{w_1}{2}, \frac{w_2}{2}, 0\right)$.

vi. Sufficiency of Kuhn-Tucker conditions.

f is concave and therefore pseudo-concave and constraint functions are affine and therefore quasi-concave..

vii. K-T conditions.

$$\begin{array}{ll} D_x L = 0 \Rightarrow & \frac{\pi}{x} - \lambda_x = 0 \\ D_y L = 0 \Rightarrow & \frac{1-\pi}{y} - \lambda_y = 0 \\ D_m L = 0 \Rightarrow & -\lambda_x + \lambda_y = 0 \\ & \min\{w_1 - m - x, \lambda_x\} = 0 \\ & \min\{w_2 + m - y, \lambda_y\} = 0 \end{array}$$

viii. Solve the K-T conditions.

Constraints are binding: $\lambda_x = \frac{\pi}{x} > 0$ and $\lambda_y = \frac{1-\pi}{y} > 0$. Then, we get

$$\begin{array}{l} \lambda_x = \frac{\pi}{x} \text{ and } x = \frac{\pi}{\lambda_x} \\ \lambda_y = \frac{1-\pi}{y} \text{and } y = \frac{1-\pi}{\lambda_y} \\ \lambda_x = \lambda_y := \lambda \\ w_1 - m - x = 0 \\ w_2 + m - y = 0 \end{array}$$

$$\begin{array}{l} \lambda_x = \frac{\pi}{x} \text{ and } x = \frac{\pi}{\lambda_x} \\ \lambda_y = \frac{1-\pi}{y} \text{and } y = \frac{1-\pi}{\lambda_y} \\ \lambda_x = \lambda_y \\ w_1 - m - \frac{\pi}{\lambda} = 0 \\ w_2 + m - \frac{1-\pi}{\lambda} = 0 \end{array}$$

Then $w_1 - \frac{\pi}{\lambda} = -w_2 + \frac{1-\pi}{\lambda}$ and $\lambda = \frac{1}{w_1+w_2}$.Therefore

$$\begin{array}{l} \lambda_x = \frac{1}{w_1+w_2} \\ \lambda_y = \frac{1}{w_1+w_2} \\ x = \pi\left(w_1+w_2\right) \\ y = (1-\pi)\left(w_1+w_2\right) \end{array}$$

b., c.

Computations of the desired derivatives are straightforward.

4.

i. Canonical form.

$$\max_{(x,y)\in\mathbb{R}^2} \quad -x^2-y^2+4x+6y \qquad s.t. \qquad \begin{array}{ll} -x-y+6\geq 0 & \lambda_1 \\ 2-y\geq 0 & \lambda_2 \\ x\geq 0 & \mu_x \\ y\geq 0. & \mu_y \end{array}$$

ii. The set X and the functions f and g (X open and convex; f and g at least differentiable)

$X=\mathbb{R}^2$ is open and convex. The constraint functions are affine and therefore C^2. The gradient and Hessian matrix of the objective function are computed below.

	x	y
$-x^2-y^2+4x+6y$	$-2x+4$	$-2y+6$

,

	x	y
$-2x+4$	-2	0
$-2y+6$	0	-2

Therefore the objective function is C^2 and strictly concave.

iii. Existence.

The constraint set C is nonempty (0 belongs to it) and closed. It is bounded below by 0. y is bounded above by 2. x is bounded above because of the first constraint:: $x\leq 6-y \overset{y\geq 0}{\leq} 6$. Therefore C is compact.

iv. Number of solutions.

Since the objective function is strictly concave (and therefore strictly quasi-concave) and the constraint function are affine and therefore quasi-concave, the solution is unique.

v. Necessity of K-T conditions.

Constraints are affine and therefore pseudo-concave. Take $(x^{++},y^{++})=(1,1)$.

vi. Sufficiency of K-T conditions.

The objective function is strictly concave and therefore pseudo-concave. Constraints are affine and therefore quasi-concave.

vii. K-T conditions.

$\mathcal{L}\left(x,y,\lambda_1,\lambda_2,\mu_x,\mu_y\right)=-x^2-y^2+4x+6y+\lambda_1\cdot(-x-y+6)++\lambda_2\cdot(2-y)+\mu_x x+\mu_y y.$

$$\begin{array}{l} -2x+4-\lambda_1+\mu_x=0 \\ -2y+6-\lambda_1-\lambda_2+\mu_y=0 \\ \min\{-x-y+6,\lambda_1\}=0 \\ \min\{2-y,\lambda_2\}=0 \\ \min\{x,\mu_x\}=0 \\ \min\left\{y,\mu_y\right\}=0 \end{array}$$

b.

$$\begin{array}{l} +4-\lambda_1+\mu_x=0 \\ -2y+6-\lambda_2+\mu_y=0 \\ \min\{-y+6,\lambda_1\}=0 \\ \min\{2-y,\lambda_2\}=0 \\ \min\{0,\mu_x\}=0 \\ \min\left\{y,\mu_y\right\}=0 \end{array}$$

Since $y \leq 2$, we get $-y+6>0$ and therefore $\lambda_1 = 0$. But then $\mu_x = -4$, which contradicts the Kuhn-Tucker conditions above.

c.

$$\begin{array}{l} -4+4-\lambda_1+\mu_x=0 \\ -4+6-\lambda_2+\mu_y=0 \\ \min\{-4+6,\lambda_1\}=0 \\ \min\{2-2,\lambda_2\}=0 \\ \min\{2,\mu_x\}=0 \\ \min\{2,\mu_y\}=0 \end{array}$$

$$\begin{array}{l} -\lambda_1+\mu_x=0 \\ +2-\lambda_2+\mu_y=0 \\ \lambda_1=0 \\ \min\{0,\lambda_2\}=0 \\ \mu_x=0 \\ \mu_y=0 \end{array}$$

$$\begin{array}{l} \mu_x=0 \\ +2-\lambda_2=0 \\ \lambda_1=0 \\ \min\{0,\lambda_2\}=0 \\ \mu_x=0 \\ \mu_y=0 \end{array}$$

$$\begin{array}{l} \mu_x=0 \\ \lambda_2=2 \\ \lambda_1=0 \\ \mu_x=0 \\ \mu_y=0 \end{array}$$

Therefore $\left(x^*,y^*,\lambda_1^*,\lambda_2^*,\mu_x^*,\mu_y^*\right)=(2,2,0,2,0)$ is a solution to the Kuhn-Tucker conditions

5 .[2]

5. a.

i. Canonical form.

For given $\delta \in (0,1), e \in \mathbb{R}_{++}$,

$$\begin{array}{lll} \max_{(c_1,c_2,k)\in\mathbb{R}^3} & u(c_1)+\delta u(c_2) & \\ & s.t. & \\ & e-c_1-k\geq 0 & \lambda_1 \\ & f(k)-c_2\geq 0 & \lambda_2 \\ & c_1\geq 0 & \mu_1 \\ & c_2\geq 0 & \mu_2 \\ & k\geq 0. & \mu_3 \end{array}$$

ii. The set X and the functions f and g (X open and convex; f and g at least differentiable).

$X=\mathbb{R}^3$ is open and convex. Let's compute the gradient and the Hessian matrix of the second constraint:

	c_1	c_2	k
$f(k)-c_2$	0	-1	$f'(k)$

	c_1	c_2	k
0	0	0	0
-1	0	0	0
$f'(k)$	0	0	$f''(k)<0$

[2]Exercise 7 and 8 are taken from David Cass' problem sets for his Microeconomics course at the University of Pennsylvania.

Therefore the second constraint function is C^2 and concave; the other constraint functions are affine.

Let's compute the gradient and the Hessian matrix of the objective functions:

	c_1	c_2	k
$u(c_1) + \delta u(c_2)$	$u'(c_1)$	$\delta u'(c_2)$	0

	c_1	c_2	k
$u'(c_1)$	$u''(c_1)$	0	0
$\delta u'(c_2)$	0	$\delta u''(c_2)$	0
0	0	0	0

Therefore the objective function is C^2 and concave.

iii. Existence.

The objective function is continuous on $\mathbb{R}^3$.

The constraint set is closed because inverse image of closed sets via continuous functions. It is bounded below by 0. It is bounded above: suppose not then

if $c_1 \to +\infty$, then from the first constraint it must be $k \to -\infty$, which is impossible;

if $c_2 \to +\infty$, then from the second constraint and the fact that $f' > 0$, it must be $k \to +\infty$, violating the first constraint;

if $k \to +\infty$, then the first constraint is violated.

Therefore, as an application of the Extreme Value Theorem, a solution exists.

iv. Number of solutions.

Since the objective function is concave and the constraint functions are either concave or affine, uniqueness is not insured on the basis of the sufficient conditions presented in the notes.

v. Necessity of K-T conditions.

The constraints are affine or concave. Take $\left(c_1^{++}, c_2^{++}, k^{++}\right) = \left(\frac{e}{4}, \frac{1}{2} f\left(\frac{e}{4}\right), \frac{e}{4}\right)$. Then the constraints are verified with strict inequality.

vi. Sufficiency of K-T conditions.

The objective function is concave and therefore pseudo-concave. The constraint functions are either concave or affine and therefore quasi-concave.

vii. K-T conditions.

$\mathcal{L}(c_1, c_2, k, \lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3) := u(c_1) + \delta u(c_2) + \lambda_1 (e - c_1 - k) + \lambda_2 (f(k) - c_2) + \mu_1 c_1 + \mu_2 c_2 + \mu_3 k.$

$$\begin{cases} u'(c_1) - \lambda_1 + \mu_1 = 0 \\ \delta u'(c_2) - \lambda_2 + \mu_2 = 0 \\ -\lambda_1 + \lambda_2 f'(k) + \mu_3 = 0 \\ \min\{e - c_1 - k, \lambda_1\} = 0 \\ \min\{f(k) - c_2, \lambda_2\} = 0 \\ \min\{c_1, \mu_1\} = 0 \\ \min\{c_2, \mu_2\} = 0 \\ \min\{k, \mu_3\} = 0 \end{cases}$$

viii. Solve the K-T conditions.

Since we are looking for positive solution we get

$$\begin{cases} u'(c_1) - \lambda_1 = 0 \\ \delta u'(c_2) - \lambda_2 = 0 \\ -\lambda_1 + \lambda_2 f'(k) = 0 \\ \min\{e - c_1 - k, \lambda_1\} = 0 \\ \min\{f(k) - c_2, \lambda_2\} = 0 \\ \mu_1 = 0 \\ \mu_2 = 0 \\ \mu_3 = 0 \end{cases}$$

$$\begin{cases} u'(c_1) - \lambda_1 = 0 \\ \delta u'(c_2) - \lambda_2 = 0 \\ -\lambda_1 + \lambda_2 f'(k) = 0 \\ e - c_1 - k = 0 \\ f(k) - c_2 = 0 \end{cases}$$

Observe that from the first two equations of the above system, $\lambda_1, \lambda_2 > 0$.

5. b.

i. Canonical form.

For given $p > 0,\ w > 0$ and $\bar{l} > 0$,

$$\begin{array}{ll} \max_{(x,l)\in\mathbb{R}^2} & u(x,l) \\ & s.t. \\ & -px - wl + w\bar{l} \geq 0 \\ & \bar{l} - l \geq 0 \\ & x \geq 0 \\ & l \geq 0 \end{array}$$

ii. The set X and the functions f and g (X open and convex; f and g at least differentiable).

$X = \mathbb{R}^2$ is open and convex.

The constraint functions are affine and therefore C^2. The objective function is C^2 and differentiably strictly quasi concave by assumption.

iii. Existence.

The objective function is continuous on $\mathbb{R}^3$.

The constraint set is closed because inverse image of closed sets via continuous functions. It is bounded below by 0. It is bounded above: suppose not then

if $x \to +\infty$, then from the first constraint ($px + wl = w\bar{l}$),it must be $l \to -\infty$, which is impossible. Similar case is obtained, if $l \to +\infty$.

Therefore, as an application of the Extreme Value Theorem, a solution exists.

iv. Number of solutions.

The budget set is convex. The function is differentiably strictly quasi concave and therefore strictly quasi-concave and the solution is unique.

v. Necessity of K-T conditions.

The constraints are pseudo-concave $(x^{++}, l^{++}) = (\frac{w\bar{l}}{3p}, \frac{\bar{l}}{3})$ satisfies the constraints with strict inequalities.

vi. Sufficiency of K-T conditions.

The objective function is differentiably strictly quasi-concave and therefore pseudo-concave. The constraint functions are quasi-concave

vii. K-T conditions.

$\mathcal{L}\left(c_1, c_2, k, \lambda_1, \lambda_2, \lambda_3, \lambda_4; p, w, \bar{l}\right) := u(x,l) + \lambda_1\left(-px - wl + w\bar{l}\right) + \lambda_2\left(\bar{l} - l\right) + \lambda_3 x + \lambda_4 l.$

$$\begin{array}{l} D_x u - \lambda_1 p + \lambda_3 = 0 \\ D_l u - \lambda_1 w - \lambda_2 + \lambda_4 = 0 \\ \min\left\{-px - wl + w\bar{l}, \lambda_1\right\} = 0 \\ \min\left\{\bar{l} - l, \lambda_2\right\} = 0 \\ \min\{x, \lambda_3\} = 0 \\ \min\{l, \lambda_4\} = 0 \end{array}$$

viii. Solve the K-T conditions.

Since we are looking for solutions at which $x > 0$ and $0 < l < \bar{l}$, we get

$$\begin{array}{l} D_x u - \lambda_1 p + \lambda_3 = 0 \\ D_l u - \lambda_1 w - \lambda_2 + \lambda_4 = 0 \\ \min\left\{-px - wl + w\bar{l}, \lambda_1\right\} = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \\ \lambda_4 = 0 \end{array}$$

$$\begin{array}{l} D_x u - \lambda_1 p = 0 \\ D_l u - \lambda_1 w = 0 \\ \min\left\{-px - wl + w\bar{l}, \lambda_1\right\} = 0 \end{array}$$

and then $\lambda_1 > 0$ and

$$\left\{\begin{array}{l} D_x u - \lambda_1 p = 0 \\ D_l u - \lambda_1 w = 0 \\ -px - wl + w\bar{l} = 0 \end{array}\right. .$$

7.

a.

Let's apply the Implicit Function Theorem (:= IFT) to the conditions found in Exercise 7.(a). Writing them in the usual informal way we have:

$$\begin{array}{lccccccc} & c_1 & c_2 & k & \lambda_1 & \lambda_2 & e & a \\ \\ u'(c_1) - \lambda_1 = 0 & u''(c_1) & & & -1 & & & \\ \delta u'(c_2) - \lambda_2 = 0 & & \delta u''(c_2) & & & -1 & & \\ -\lambda_1 + \lambda_2 f'(k) = 0 & & & \lambda_2 f''(k) & -1 & f'(k) & & \lambda_2 \alpha k^{\alpha-1} \\ e - c_1 - k = 0 & -1 & & -1 & & & 1 & \\ f(k) - c_2 = 0 & & -1 & f'(k) & & & & k^\alpha \end{array}$$

To apply the IFT, we need to check that the following matrix has full rank

$$M := \begin{bmatrix} u''(c_1) & & & -1 & \\ & \delta u''(c_2) & & & -1 \\ & & \lambda_2 f''(k) & -1 & f'(k) \\ -1 & & -1 & & \\ & -1 & f'(k) & & \end{bmatrix}$$

.

Suppose not then there exists $\Delta := (\Delta c_1, \Delta c_2, \Delta k, \Delta\lambda_1, \Delta\lambda_2) \neq 0$ such that $M\Delta = 0$, i.e.,

$$\left\{\begin{array}{llllll} u''(c_1)\Delta c_1 + & & & -\Delta\lambda_1 & & = 0 \\ & \delta u''(c_2)\Delta c_2 + & & & -\Delta\lambda_2 & = 0 \\ & & \lambda_2 f''(k)\Delta k + & -\Delta\lambda_1 + & f'(k)\Delta\lambda_2 & = 0 \\ -\Delta c_1 + & & -\Delta k & & & = 0 \\ & -\Delta c_2 + & f'(k)\Delta k + & & & = 0 \end{array}\right.$$

Recall that

$$[M\Delta = 0 \Rightarrow \Delta = 0] \quad iff \quad M \text{ has full rank.}$$

The idea of the proof is either you prove directly $[M\Delta = 0 \Rightarrow \Delta = 0]$, or you 1. assume $M\Delta = 0$ and $\Delta \neq 0$ and you get a contradiction.

If we define $\Delta c := (\Delta c_1, \Delta c_2)$, $\Delta\lambda := (\Delta\lambda_1, \Delta\lambda_2)$, $D^2 := \begin{bmatrix} u''(c_1)\Delta c_1 & \\ & \delta u''(c_2)\Delta c_2 \end{bmatrix}$, the above system can be rewritten as

$$\left\{\begin{array}{llll} D^2\Delta c + & & -\Delta\lambda & = 0 \\ & \lambda_2 f''(k)\Delta k + & [-1, f'(k)]\Delta\lambda & = 0 \\ -\Delta c + & \begin{bmatrix} -1 \\ f'(k) \end{bmatrix}\Delta k & & = 0 \end{array}\right. .$$

$$\left\{\begin{array}{lllll} \Delta c^T D^2\Delta c + & & -\Delta c^T\Delta\lambda & = 0 & (1) \\ & \Delta k\lambda_2 f''(k)\Delta k + & \Delta k[-1, f'(k)]\Delta\lambda & = 0 & (2) \\ -\Delta\lambda^T\Delta c + & \Delta\lambda^T\begin{bmatrix} -1 \\ f'(k) \end{bmatrix}\Delta k & & = 0 & (3) \end{array}\right.$$

.

$$\Delta c^T D^2\Delta c \overset{(1)}{=} \Delta c^T\Delta\lambda = -\Delta\lambda^T\Delta c \overset{(3)}{=} \Delta\lambda^T\begin{bmatrix} -1 \\ f'(k) \end{bmatrix}\Delta k \overset{(2)}{=} \Delta k[-1, f'(k)]\Delta\lambda =$$

$= \Delta k \lambda_2 f''(k) \Delta k > 0,$

while $\Delta c^T D^2 \Delta c = (\Delta c_1)^2 u''(c_1) + (\Delta c_2)^2 \delta u''(c_1) < 0$. since we got a contradiction, M has full rank.

Therefore, in a neighborhood of the solution we have

$$D_{(e,a)}(c_1, c_2, k, \lambda_1, \lambda_2) = - \begin{bmatrix} u''(c_1) & & & -1 & \\ & \delta u''(c_2) & & & -1 \\ & & \lambda_2 f''(k) & -1 & f'(k) \\ -1 & & -1 & & \\ & -1 & f'(k) & & \end{bmatrix}^{-1} \begin{bmatrix} & \\ & \\ & \lambda_2 \alpha k^{\alpha-1} \\ 1 & \\ & k^\alpha \end{bmatrix}.$$

To compute the inverse of the above matrix, we can use the following fact about the inverse of partitioned matrix (see Goldberger, (1964), page 27:

Let A be an $n \times n$ nonsingular matrix partitioned as

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

where $E_{n_1 \times n_1}$, $F_{n_1 \times n_2}$, $G_{n_2 \times n_1}$, $H_{n_2 \times n_2}$ and $n_1 + n_2 = n$. Suppose that E and $D := H - GE^{-1}F$ are non singular. Then

$$A^{-1} = \begin{bmatrix} E^{-1}\left(I + FD^{-1}GE^{-1}\right) & -E^{-1}FD^{-1} \\ -D^{-1}GE^{-1} & D^{-1} \end{bmatrix}.$$

In fact, using Maple, with obviously simplified notation, we get

$$\begin{bmatrix} u_1 & 0 & 0 & -1 & 0 \\ 0 & \delta u_2 & 0 & 0 & -1 \\ 0 & 0 & \lambda_2 f_2 & -1 & f_1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & f_1 & 0 & 0 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} \frac{1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{\delta u_2 f_1^2+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\delta u_2 \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} \\ -\frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \frac{f_1^2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -f_1 \frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{u_1+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} \\ -\frac{1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \frac{1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \delta u_2 \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} \\ -\frac{\delta u_2 f_1^2+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -f_1 \frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -u_1 \frac{\delta u_2 f_1^2+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\delta u_2 f_1 \frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} \\ -\delta u_2 \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\frac{u_1+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & \delta u_2 \frac{f_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\delta u_2 f_1 \frac{u_1}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} & -\delta u_2 \frac{u_1+\lambda_2 f_2}{u_1+\delta u_2 f_1^2+\lambda_2 f_2} \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} D_e c_1 & D_a c_1 \\ D_e c_2 & D_a c_2 \\ D_e k & D_a k \\ D_e \lambda_1 & D_a \lambda_1 \\ D_e, \lambda_2 & D_a, \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{\delta u_2 f_1+\lambda_2 f_2}{u_1+\delta u_2 f_1+\lambda_2 f_2} & \frac{\lambda_2 \alpha k^{\alpha-1}+\delta u_2 f_1 k^\alpha}{u_1+\delta u_2 f_1+\lambda_2 f_2} \\ f_1 \frac{u_1}{u_1+\delta u_2 f_1+\lambda_2 f_2} & \frac{-f_1 \lambda_2 \alpha k^{\alpha-1}+k^\alpha u_1+k^\alpha \lambda_2 f_2}{u_1+\delta u_2 f_1+\lambda_2 f_2} \\ \frac{u_1}{u_1+\delta u_2 f_1+\lambda_2 f_2} & -\frac{\lambda_2 \alpha k^{\alpha-1}+\delta u_2 f_1 k^\alpha}{u_1+\delta u_2 f_1+\lambda_2 f_2} \\ u_1 \frac{\delta u_2 f_1+\lambda_2 f_2}{u_1+\delta u_2 f_1+\lambda_2 f_2} & u_1 \frac{\lambda_2 \alpha k^{\alpha-1}+\delta u_2 f_1 k^\alpha}{u_1+\delta u_2 f_1+\lambda_2 f_2} \\ \delta u_2 f_1 \frac{u_1}{u_1+\delta u_2 f_1+\lambda_2 f_2} & \delta u_2 \frac{-f_1 \lambda_2 \alpha k^{\alpha-1}+k^\alpha u_1+k^\alpha \lambda_2 f_2}{u_1+\delta u_2 f_1+\lambda_2 f_2} \end{bmatrix}.$$

Then $D_e c_1 = \frac{\delta u_2 f_1+\lambda_2 f_2}{u_1+\delta u_2 f_1+\lambda_2 f_2} := \frac{\overset{+}{\delta}\, \overset{-}{u''}(c_2) \overset{+}{f'} + \overset{+}{\lambda_2} \overset{-}{f''}}{\underset{-}{u''}(c_1) + \underset{+}{\delta}\, \underset{-}{u''}(c_2) \underset{+}{f'} + \underset{+}{\lambda_2} \underset{-}{f''}}" = \frac{-}{-}" > 0$

$D_e c_2 = f_1 \frac{u_1}{u_1+\delta u_2 f_1+\lambda_2 f_2} := \frac{\overset{+}{f'} \overset{-}{u''}(c_1)}{\underset{-}{u''}(c_1) + \underset{+}{\delta}\, \underset{-}{u''}(c_2) \underset{+}{f'} + \underset{+}{\lambda_2} \underset{-}{f''}}" = \frac{-}{-}" > 0$

$D_a k = -\frac{\lambda_2 \alpha k^{\alpha-1}+\delta u_2 f_1 k^\alpha}{u_1+\delta u_2 f_1+\lambda_2 f_2} := -\frac{\overset{+}{\lambda_2} \alpha \overset{+}{k^{\alpha-1}} + \overset{+}{\delta}\, \overset{-}{u''}(c_2) \overset{+}{f'} \overset{+}{k^\alpha}}{\underset{-}{u''}(c_1) + \underset{+}{\delta}\, \underset{-}{u''}(c_2) \underset{+}{f'} + \underset{+}{\lambda_2} \underset{-}{f''}},$

which has sign equal to *sign* $\left(\overset{+}{\lambda_2} \alpha \overset{+}{k^{\alpha-1}} + \overset{+}{\delta}\, \overset{-}{u''}(c_2) \overset{+}{f'} \overset{+}{k^\alpha} \right)$.

b.

Let's apply the Implicit Function Theorem to the conditions found in a previous exercise. Writing them in the usual informal way we have:

$$
\begin{array}{lcccccc}
 & x & l & \lambda_1 & p & w & \bar{l} \\
D_x u - \lambda_1 p = 0 & D_x^2 & D_{xl}^2 & -p & -\lambda_1 & & \\
D_l u - \lambda_1 w = 0 & D_{xl}^2 & D_l^2 & -w & & -\lambda_1 & \\
-px - wl + w\bar{l} = 0 & -p & -w & & -1 & \bar{l} - l & w
\end{array}
$$

To apply the IFT, we need to check that the following matrix has full rank

$$
M := \begin{bmatrix} D_x^2 & D_{xl}^2 & -p \\ D_{xl}^2 & D_l^2 & -w \\ -p & -w & \end{bmatrix}
$$

.

Defined $D^2 := \begin{bmatrix} D_x^2 & D_{xl}^2 \\ D_{xl}^2 & D_l^2 \end{bmatrix}$, $q := \begin{bmatrix} -p \\ -w \end{bmatrix}$, we have $M := \begin{bmatrix} D^2 & -q \\ -q^T & \end{bmatrix}$.

Suppose not then there exists $\Delta := (\Delta y, \Delta\lambda) \in \left(\mathbb{R}^2 \times \mathbb{R}\right) \setminus \{0\}$ such that $M\Delta = 0$, i.e.,

$$
\begin{cases} D^2 \Delta y & -q\Delta\lambda & = 0 & (1) \\ -q^T \Delta y & & = 0 & (2) \end{cases}
$$

We are going to show

Step 1. $\Delta y \neq 0$; Step 2. $Du \cdot \Delta y = 0$; Step 3. It is not the case that $\Delta y^T D^2 \Delta y < 0$.

These results contradict the assumption about u.

Step 1.

Suppose $\Delta y = 0$. Since $q \gg 0$, from (1), we get $\Delta\lambda = 0$, and therefore $\Delta = 0$, a contradiction.

Step 2.

From the First Order Conditions, we have

$Du - \lambda_1 q = 0 \quad (3)$.

$Du\Delta y \overset{(3)}{=} \lambda_1 q \Delta y \overset{(2)}{=} 0.$

Step 3.

$\Delta y^T D^2 \Delta y \overset{(1)}{=} \Delta y^T q \Delta\lambda \overset{(2)}{=} 0.$

Therefore, in a neighborhood of the solution we have

$$
D_{(p,w,\bar{l})}(x, l, \lambda_1) = -\begin{bmatrix} D_x^2 & D_{xl}^2 & -p \\ D_{xl}^2 & D_l^2 & -w \\ -p & -w & \end{bmatrix}^{-1} \begin{bmatrix} -\lambda_1 & & \\ & -\lambda_1 & \\ -1 & \bar{l} - l & w \end{bmatrix}.
$$

Unfortunately, here we cannot use the formula seen in the Exercise 4 (a) because the Hessian of the utility function is not necessarily nonsingular. We can invert the matrix using the definition of inverse. (For the inverse of a partitioned matrix with this characteristics see also Dhrymes, P. J., (1978), *Mathematics for Econometrics*, 2nd edition, Springer-Verlag, New York, NY, Addendum pages 142-144.

With obvious notation and using Maple, we get

$$
\begin{bmatrix} d_x & d & -p \\ d & d_l & -w \\ -p & -w & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{w^2}{d_x w^2 - 2dpw + p^2 d_l} & -p\frac{w}{d_x w^2 - 2dpw + p^2 d_l} & -\frac{-dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} \\ -p\frac{w}{d_x w^2 - 2dpw + p^2 d_l} & \frac{p^2}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} \\ -\frac{-dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x d_l + d^2}{d_x w^2 - 2dpw + p^2 d_l} \end{bmatrix}
$$

Therefore,

$D_{(p,w,\bar{l})}(x, l, \lambda_1) =$

$$
-\begin{bmatrix} \frac{w^2}{d_x w^2 - 2dpw + p^2 d_l} & -p\frac{w}{d_x w^2 - 2dpw + p^2 d_l} & -\frac{-dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} \\ -p\frac{w}{d_x w^2 - 2dpw + p^2 d_l} & \frac{p^2}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} \\ -\frac{-dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-d_x d_l + d^2}{d_x w^2 - 2dpw + p^2 d_l} \end{bmatrix} \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & 0 \\ -1 & \bar{l} - l & w \end{bmatrix} =
$$

$$
= \begin{bmatrix} -\frac{-w^2\lambda_1 - dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} & -\frac{pw\lambda_1 - ldw + lpd_l}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-dw + pd_l}{d_x w^2 - 2dpw + p^2 d_l} w \\ \frac{-pw\lambda_1 - d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} & \frac{p^2\lambda_1 - ld_x w + ldp}{d_x w^2 - 2dpw + p^2 d_l} & -\frac{-d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l} w \\ -\frac{-\lambda_1 dw + \lambda_1 pd_l + d_x d_l - d^2}{d_x w^2 - 2dpw + p^2 d_l} & \frac{-\lambda_1 d_x w + \lambda_1 dp - ld_x d_l + ld^2}{d_x w^2 - 2dpw + p^2 d_l} & -\frac{-d_x d_l + d^2}{d_x w^2 - 2dpw + p^2 d_l} w \end{bmatrix}
$$

$$D_p l = \frac{-pw\lambda_1 - d_x w + dp}{d_x w^2 - 2dpw + p^2 d_l}$$

$$D_w l = \frac{p^2\lambda_1 - l d_x w + l dp}{d_x w^2 - 2dpw + p^2 d_l}.$$

The sign of these expressions is ambiguous, unless other assumptions are made.
7.
[⇒]
Since f is concave, then

$$f\left(\frac{1}{2}x + \frac{1}{2}y\right) \geq \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

Since f is homogenous of degree 1, then

$$f\left(\frac{1}{2}(x+y)\right) = \frac{1}{2}f(x+y).$$

Therefore,

$$f(x+y) \geq f(x) + f(y).$$

[⇐]
Since f is homogenous of degree 1, then for any $z \in \mathbb{R}^2$ and any $a \in \mathbb{R}_+$, we have

$$f(az) = af(z). \tag{21.12}$$

By assumption, we have that

$$\text{for any } x, y \in \mathbb{R}^2,\ \ f(x+y) \geq f(x) + f(y). \tag{21.13}$$

Then, for any $\lambda \in [0,1]$,

$$f((1-\lambda)x + \lambda y) \overset{(21.13)}{\geq} f((1-\lambda)x) + f(\lambda y) \overset{(21.12)}{=} (1-\lambda)f(x) + \lambda f(y),$$

as desired.

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